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Book Review: Visual Motion of Curves and Surfaces

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BOOK REVIEW

Visual Motion of Curves and Surfaces, by Roberto Cipolla and Peter Giblin, Cambridge University Press, Cambridge 2009, viii+184 pp., ISBN: 978-0-521-63251-5, hardback, ISBN: 978-0-521-11818-7 paperback

1. Introduction

This interesting book is dedicated to the theory of reconstructing surfaces from the outlines they present to the distant viewer, i.e., from their apparent contours. The contour is defined in this book as the projection of the locus of points on the surface which separates visible and occluded parts in the view. The main applications of the theory presented here are part of the Computer Vision science, namely the automatic analysis of sequences of images for the purpose of recovering three-dimensional surface shapes.

The book can be divided in two parts. The first part (about 60% of the book) is formed by Chapters 2, 3 and 4, and it is organized as a differential geometry course. This part contains the introduction and presentation of the main geometrical concepts and the associated calculus details with abundant examples and excellent pedagogical figures that amply fulfill the reader's wants. In these chapters the text is structured in terms of: Definitions, Properties, Remarks and Examples, all labelled in the decimal system. The authors do not provide the complete proofs for all affirmations, but the explanations contain both rigor and motivational power. If proofs are provided they are located in separate terminal sections so the reader can use them (or skip them) which is a function of the needs. Sometimes the definitions and remarks are intertwined along the book, so a second reading of this part may be welcome.

The second part (about 40% of the book) is formed by Chapters 5 and 6 where the differential and projective geometry concepts are put to work for image reconstruction.

The first chapter plays the role of the Introduction where the authors briefly describe what each chapter contains, add one or two basic references for each chapter, and provide a short (motivational) history of the topic discussed in that chapter. The book is intended to be self-contained, so it is essential for the reader to browse this Introduction in order to understand the structure of the book.

In Chapter 2 the authors introduce elements of differential geometry and singularity theory in an prerequisite-free way. The reader equipped with calculus and algebra basic knowledge can navigate through this chapter very easy and pleasurable. The material is organized rather in an vision application perspective, and several examples are provided.

The third chapter is devoted to the introduction of the apparent contours as outlines or profiles of curved surfaces. This chapter produces the two essential geometric relationships for the depth and the curvature of the recovered surface. In Chapter 4 the authors introduce the dynamic contour approach where the apparent contours are obtained from a moving camera, in stereo vision. Here the authors introduce the key concepts of image recovery namely the epipolar geometry and the orthographic and perspective projections. Chapter 5 contains the procedures of implementation of the projective and epipolar geometry algorithms in computer programs of image digitization. The sixth (and last) chapter is an extension of these procedures in the case when the observer's motion is not known, and one still has to obtain the real surface reconstruction from the set of apparent contours. The book is completed with an useful list of references listed alphabetically, and an index.

2. Differential Geometry Elements used in this Book

The geometrical objects described in the book are defined in the three-dimensional Euclidean space $\mathbb{R}^3$. The structure of the chapter is clear and simple: each section begins with a definition, then follow explanations and alternate forms, and then an example. The chapter begins with a brief definition of regular parametrized curves $\mathbf{r}(t)$, the unit tangent $\mathbf{T}$, arclength s, and a helix example. Curves are also mentioned again in Section 2.6. We want to emphasize as a special feature Fig. 2.12 at page 24 which, to our opinion represent one of the best examples in literature for the three fundamental planes of a space curve: osculating, normal and rectifying ones.

Next, in Section 2.2 we are introduced to the parametric form $\mathbf{r}(u, v)$ of surfaces. In the rest of the chapter the surfaces under study are immersions without self-intersections. Examples in parametric form, Monge form and implicit form are provided, including surfaces of revolution and quadric surfaces. The authors introduce the first fundamental form I for surfaces from the expression of the tangent to a curve on the surface. The form is explained also as the magnitude of the tangent, the magnitude of a vector in the tangent plane, the scalar product of tangent vectors, and the matrix form.

The second fundamental form II of a surface is introduced in an original way as the geometrical object encoding all the local variations of the tangent plane when the point of tangency moves around on the surface.This explanation is valuable because it makes a sort of intuitive relation with the connection in differential geometry. The matrix entries of the fundamental forms are denoted E, F, G, L, M, N respectively. Then the authors introduce in a natural way the Gauss map of a surface and the Weingarten map as its differential.

In Section 2.8, for any vector $\mathbf{a}$ of the tangent plane the authors introduce the normal curvature κ, also called the *sectional curvature* of a surface as the ratio between the second and the first fundamental forms valued in the same vector

$$\kappa = \frac{II(\mathbf{a}, \mathbf{a})}{I(\mathbf{a}, \mathbf{a})}. \tag{1}$$

The next definitions introduce the principal directions and principal curvatures, the asymptotic directions, the geodesic curvature and the Gauss and mean curvatures K, H, followed by all their important relationships. The next paragraphs are devoted to the classification of special points of a surface.

The next sections become more technical and applicative. We learn the explicit expressions for the fundamental forms and curvatures in the Monge description of a surface $z = f(x, y)$. Keeping it simple and crystal clear the authors take the reader through all important formulas of surface geometry in three dimensions. The special attention dedicated to the Monge form for surfaces is finally justified in the last section of the chapter where we are introduced to the concept of *contact*. Without using sophisticated topology and geometry tools like jet space and germs of functions, the reader finds a very intuitive way to learn about the k-point contact on a surface as the algebraic multiplicity of the roots of an equation describing the intersection between a line and the surface. Fig. 2.24 at page 49 is very intuitive, too.

A warning for this chapter: all definitions and remarks in this chapter become important further on in the book, so we advise the reader to read them all, even if initially they do not present a direct interest, or they are not among the traditional concepts of differential geometry. For example, in Definition 2.8.2 the authors introduce the *conjugacy* relation between two tangent vectors $\mathbf{a}, \mathbf{b}$ to a surface. Basically, this means that they cancel the second fundamental form $II(\mathbf{a}, \mathbf{b}) = 0$.

If the two coordinate directions in a surface are conjugated the second fundamental form is diagonal, too. Loosely speaking, the conjugate $\mathbf{a}$ to the unit tangent vector $\mathbf{T} = \mathbf{r}'$ to a curve γ on a surface M should be perpendicular to the directional derivative along the curve γ of the unit surface normal $\mathbf{n}$, i.e.,

$$II(\mathbf{r}', \mathbf{a}) = -\mathbf{a} \cdot \mathbf{n}' = 0. \tag{2}$$

Apparently, this is not a key property of a pair of two tangent directions, except if they are the coordinate ones. However, later on in Section 3.5 of Chapter 3 the conjugacy property becomes the essential tool for the orthogonal projection construction and its properties.

As an introduction to the differential geometry of surfaces in three-dimensions Chapter 1 is backed-up with some classical references. Not too many, not too abstract, and not too few: just enough and in the line of the computer vision applications emphasized by the book. Among them the authors rely on [4] for comparisons. We also mention the use of two well known differential geometry monographs [2, 9], of a basic book on applied algebra [10], and the mention of few other interesting geometry books in the same line [1, 3, 6–8].

3. Chapter 3: Views of Curves and Surfaces

In this chapter the authors introduce the two main *projections* of surfaces onto planes: the *orthographic or parallel projection* and the *perspective projection*. The first one is defined by a fixed direction $\mathbf{k}$ while the second one by an optical center, the point $\mathbf{c}$. Curves on the investigated surface M for which the visual rays (either parallel or emerging from the optical center) are tangent to the surface are called *contour generators* and denoted Γ. These generators are formed by the points of M where the surface appears to fold or have an occluding contour. The projections of the contour generators onto the vision plane are called *apparent contours*. Properties of the apparent contours are described in terms of their smoothness, normals, and curvatures. Among these expressions we note an interesting one

$$K_{\mathbf{r}} = \kappa^p \kappa^t \tag{3}$$

where K is the Gauss curvature of M at $\mathbf{r}$, κ^p is the curvature of the apparent contour curve at the image p of the surface point $\mathbf{r}$ in the vision plane, and κ^t is the normal curvature of M at $\mathbf{r}$ along the ray of vision.

In continuation the two types of projections are discussed and exemplified for several surfaces like sphere, cylinder, cusped contours, quadric. The last sections are dedicated to proofs of the previous propositions.

4. Chapter 4: Dynamic Analysis of Apparent Contours

This chapter is the backbone of the book, because it develops the theory of the reconstruction procedures. It is also the longest one. In this chapter the authors show how the information from different camera centers about the apparent contours of a real surface can be combined. Basically, if one uses a moving camera and the perspective projection analysis of the surface, one obtains a moving cone which is tangent to the surface. So, the surface can be expressed as the envelope of these cones, and the uniqueness of its reconstruction results from the fact that there is only one surface which is tangent to all the cones. Consequently the knowledge of the path of the optical center of the camera, and of the set of all apparent contours should provide the exact knowledge of the surface, or at least of those domains where the various cones are tangent.

Two parameters are introduced: t as time which describes the motion of the optical center, and s the arclength along the apparent contours. The authors introduce relations between the position of the point on the surface, the position of the points of the image and the distance λ between these points, called *depth*

$$\lambda = -\frac{\mathbf{c}_t \cdot \mathbf{n}}{\mathbf{p}_t \cdot \mathbf{n}}, \qquad \mathbf{r} = \mathbf{c} - \frac{\mathbf{c}_t \cdot \mathbf{n}}{\mathbf{p}_t \cdot \mathbf{n}}\mathbf{p}. \tag{4}$$

The philosophy of the reconstruction technique from a series of orthographic projections on a *moving viewplane* is the following: we assume that we know the equation of the surface, and we place a certain fixed system of coordinates in space. The points in the surface are described by the vector $\mathbf{r}$. Inspired by this fixed frame, and may be by some symmetries of the surface, we choose a particular system of coordinates in the moving viewplane (time dependent) defined by two orthogonal directions $(\mathbf{u}, \mathbf{v})$. We describe the viewplane motion through some velocity components (rotations of viewplane about a fixed axis, for example). Finally, we can express the positions of all the points on the surface only by using the coordinate system of the viewplane and the view directions $\mathbf{k}$. It is possible to express all the first and second order derivatives of the position vector $\mathbf{r}$ of the points on the surface in terms of the moving triad $(\mathbf{u}, \mathbf{v}, \mathbf{k})$, and consequently to calculate the second fundamental form of the surface relative to the basis $\{\mathbf{r}_s, \mathbf{r}_t\}$ of the tangent plane. Then, according to the *Theorema egregium* the surface is uniquely determined modulo Euclidean motions.

In the followings, the authors derive the formulae for the reconstruction of a surface observed through the perspective projection with a calibrated camera and rotated coordinates. They use a special type of parametrization called *epipolar* in which the vector tangent to the apparent contour curves is in the view direction, namely

$\mathbf{r}_t$ is parallel to $\mathbf{k}$. In the epipolar parametrization the camera receives a spatio-temporal image of the surface.

In this epipolar parametrization the surface geometry from apparent contours is obtained from the differential equation for the position of the image point $\mathbf{p}$ in time $\mathbf{p}(t)$

$$\mathbf{p}_t = \frac{(\mathbf{c}_t \wedge \mathbf{p}) \wedge \mathbf{p}}{\lambda} \tag{5}$$

where λ is the depth (distance object point to image point).

Another useful formula is obtained for the curvature κ^s of the surface M in the direction of the contour generator Γ

$$\kappa^s = \frac{\kappa^p \sin^2 \theta}{\lambda} \tag{6}$$

where θ is the angle between the visual ray $\mathbf{k}$ and the contour generator Γ

$$\theta = \pm \tan^{-1}\left(\frac{\lambda ||\mathbf{p}_s||}{\lambda_s}\right). \tag{7}$$

It is obvious that κ^s does not depend on the dynamic considerations. Finally, the authors express the Gauss curvature of the object surface as a function of the angle θ

$$K = \frac{\kappa^s \kappa^t}{\sin^2 \theta} \tag{8}$$

a relation of a special beautiful symmetry.

The next section describe some limitations of the procedure, namely it is devoted to the analysis of special situations when the epipolar parametrization fails to exist for different reasons. In continuation the authors describe what they call *visual events* on the apparent contours like cusps, swallowtails, lips and beaks. Such situations may occur when, for example, the visual ray is asymptotic at the real surface. Another situation happens if the contour generator becomes singular, or simply cannot form a coordinate grid on the real surface (the so called *frontiers* situation described in Section 4.8). In all these cases different reparation procedures in terms of differential criteria are indicated.

The main messages of this chapter are encapsulated in equations (4) and (8), that is the depth and the surface curvature obtained as functions of the first and second order temporal derivatives of the spatio-temporal apparent curves parameters. The results presented in Chapter 4 rely on eleven research articles published by the two authors in international peer reviewed journals and alphabetically cited by the author names in the book references. For more details on the theory of curves and surface motion we recommend [5] and references herein.

5. The "Second Part" of the Book: Chapters 5 and 6

In the next two chapters the authors describe the implementation of the theory introduced in the "first part" of the book and show how to recover the geometry of the surface from an image sequence of apparent contours from different viewpoints. The language of the book changes a little from differential geometry to computer science, so these two chapters may need some previous prerequisites on edge detection, B-spline theory, convolution and filtering procedures.

The relationship between the vision rays in the three-dimensional Euclidean space and the image pixel coordinates is provided by the projective (*homogeneous*) coordinates formalism. This is realized by the use of a 3×4 projection matrix representing the perspective projection of a point in space onto the digitized image. The epipolar geometry introduced in the previous chapters is recalled, and it plays a key part in the algorithms to recover the geometry of the surface. Basically, the authors show how the use of stereo vision can recover a three-dimensional position by triangulation and using two calibrated viewpoints. In this way the reader is familiarized with the methods of stereo vision algorithms and their close connections to the general structure-from-motion problem, that is to apply stereo vision algorithms to a structure from motion task.

The use of projective geometry techniques in computer vision is natural thanks to important works in projective invariants and reconstruction revelled by Faugeras and Hartley and cited in the book. The application of projective geometry to a stereo view situation finds its natural framework in the epipolar geometry approach. The key concept is the existence of a unique *epipolar plane* for each world point $\mathbf{r}$ (based on a matching constraint called the *epipolar constraint*). This plane is determined by the real position of the point and two optic centers of the camera separated by a time interval. The intersections of this plane with the two image planes existing at the two different moments of time form the epipolar lines.

The key role in the epipolar geometry is played by the *essential matrix* $\mathbf{E}$ which is a 3×3 entity with 7 independent parameters (9 entries minus 2 from the constraint to have rank 2 in order to be able to undergo an arbitrary scale factor). The essential matrix defines the geometry of the correspondences between two views in a compact way, encoding intrinsic camera geometry as well as the extrinsic relative motion between the two cameras. We define it by

$$\mathbf{E} = \mathbf{t} \wedge \mathbf{R} \tag{9}$$

where $\mathbf{t}, \mathbf{R}$ are the displacement vector and rotation matrix of the space separations between the pair of cameras in the stereo view. Subjected to a unitary transformation the essential matrix becomes the so called *fundamental matrix* $\mathbf{F}$, and the whole construction of the epipolar geometry is based on this concept in the followings. Chapter 5 ends with an example of recovery of the depth.

In the last chapter the authors describe how the epipolar geometry (hence the camera motion) can be recovered from the deformation of apparent contours when the viewer motion is not known a priori.The chapter presents the mathematical formalism of the recovery in terms of the projection matrices. In the end of the chapter we find examples of recovery of surfaces from pure translation and circular motion.

In this chapter it is also shown how the recovery theory can be put in practice even in the difficult case of uncalibrated cameras with not fully known motion. However, since the methods presented in this book are basically iterative, the question of uniqueness of the reconstruction from the apparent contours analysis is still open.

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