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Sliding Mode Observers for Distributed Parameter Systems: Theory and Applications

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by

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Abstract

Many processes in nature and industry can be described by partial differential equations. PDEs employ quantities such as density, temperature, velocity, etc. and their partial derivatives to model these phenomena. However, in the case of distributed parameter systems, it is not always possible to have access to the states of the systems due to technical difficulties such as lack of sensors. Therefore, there is the need for state observers to estimate the states of the system only having the output of the system available. In this research, the theory of sliding mode and variable structure systems are employed in order to design observers for different classes of distributed parameter systems such as advection equation, Burgers' equation, Euler equations, etc. Some contributions of this research are: suggesting the state transformation which allows the arbitrary design of sliding manifold in sliding mode observer, developing some formulae for observer gain, discussing the shock wave situation and its properties and solutions, designing sliding mode observer and anomaly detection system for a system of advection equations.

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Chapter 1

Background

1.1 Motivation of the Research

This research discusses the problem of developing state observers for distributed parameter systems. When dealing with systems described by partial differential equations, the access to the states of the system can not be guaranteed, most of the time due to technical difficulties such as lack of sensors.

The motivation of the research came from the lack of enough researches on the subject of designing state observer for systems described by partial differential equations. These types of observers have applications in industry and science. For instance, the motivation of the research done in chapter 4 came from the need to localize the possible leak in the fuel lines of J-2X rocket engine test bed.

Here the problem of designing state observers for distributed parameter systems is attacked using the powerful theory of sliding mode control. This theory allows to design controllers and observers for nonlinear systems in a robust way. Different cases of linear and nonlinear PDEs such as advection equation and Burgers' equation are investigated. In addition, some formulae for designing the observer gain are developed. In designing sliding mode observer, in contrast to sliding mode control, the choice over sliding manifold is not arbitrary. In chapter 2, a novel state transformation is suggested that allows the freedom in designing the desired sliding manifold. In chapter 4, the advection equation is studied and an anomaly detection system is developed that is able to find the parameters of possible anomalies in the system as well as serving as the state observer for the distributed parameter system.

1.2 Outline of the Dissertation

The dissertation begins with an overview on the mathematical background required for the rest of the research in chapter 1. It includes materials on partial differential equations, state observers, and sliding mode theory as the main tool in designing observers in this research. Chapter 1 ends with literature reviews on designing state observers and sliding mode observers for different types of distributed parameter systems. Designing sliding mode observer for a specific class of distributed parameter systems is discussed in chapter 2. A novel state transformation is developed to allow for arbitrary design of sliding manifold in sliding mode observer and a formula is suggested for the observer gain. In chapter 3, the mathematical base for chapters 3 and 4 is provided. The equations describing fluid dynamics and the different variations of them such as Burgers' and advection equations are discussed. A sliding mode observer is designed for the case of Burgers' equation in chapter 3. A sliding mode observer as well as an anomaly detection system for a system of advection equations are developed in chapter 4. The applications of the suggested techniques are simulated in order to predict the behavior of fluid flow in a pipeline and to detect the location and intensity of the possible leakage in it. Each chapter ends with its own conclusion and the suggestions for future work. The overall view of the research and suggestions for future work are provided in chapter 5. Samples of MATALB codes used throughout this research are presented in appendix A.

1.3 Partial Differential Equations

Many natural, biological, chemical, mechanical, and economic phenomena can be described by a set of partial differential equations. These concepts are investigated by employing differential equations, which consist of quantities such as density, pressure, velocity, etc. (Frey & de Buhan, 2008), (Polyanin et al., 2008). Most of the models based on partial differential equations used in practice, have been introduced in the 19^{th} century (Brezis & Browder, 1998).

A **differential equation** is an equation relating an unknown function and its derivatives of different orders. An **ordinary differential equation** (ODE) is a differential equation in which the unknown function depends on a single independent variable. A **partial differential equation** (PDE) is a differential equation in which the unknown function $F : \Omega \to \mathbb{R}$ is a function of two or more independent variables and of their partial derivatives. Let Ω denote an open subset of $\mathbb{R}^d$. Given $F : \mathbb{R}^{d^n} \times \mathbb{R}^{d^{n-1}} \times \cdots \mathbb{R}^d \times \mathbb{R} \times \Omega \to \mathbb{R}$ where $n \geq 1$ and is an integer. The following expression shows a PDE of **order** n

$$F\left(x, v, \frac{\partial v}{\partial x}, \cdots, \frac{\partial^{n-1} v}{\partial x^{n-1}}, \frac{\partial^n v}{\partial x^n}\right) = 0, \quad x \in \Omega, \tag{1.1}$$

where $v(x) : \Omega \to \mathbb{R}$ is the unknown function. A system of partial differential equations is a set of some PDEs for several unknown functions. Solving a PDE means finding all functions v satisfying (1.1) and the additional boundary conditions on some part of the domain boundary $\partial\Omega$.

The PDE (1.1) is called **linear** if it has the form

$$\sum_{|\alpha| \leq n} a_\alpha(x) \frac{\partial^\alpha v}{\partial x^\alpha} = f(x), \tag{1.2}$$

for a given functions f and a_α. Equation (1.2) is called **homogeneous** if $f \equiv 0$. Equation (1.1) is called **semilinear** if

$$\sum_{|\alpha|=n} a_\alpha \frac{\partial^\alpha v}{\partial x^\alpha} + a_0\left(x, v, \frac{\partial v}{\partial x}, \cdots, \frac{\partial^{n-1} v}{\partial x^{n-1}}\right) = 0, \tag{1.3}$$

it is called **quasilinear** if

$$\sum_{|\alpha|=n} a_\alpha\left(x, v, \frac{\partial v}{\partial x}, \cdots, \frac{\partial^{n-1} v}{\partial x^{n-1}}\right) \frac{\partial^\alpha v}{\partial x^\alpha} + a_0\left(x, v, \frac{\partial v}{\partial x}, \cdots, \frac{\partial^{n-1} v}{\partial x^{n-1}}\right) = 0, \tag{1.4}$$

and fully **nonlinear** if it depends nonlinearly upon the highest order derivatives.

A partial differential equation is called **well-posed** if

(a) a solution exists,

(b) the solution is unique,

(c) the solution depends continuously on the information given in the problem.

Otherwise it is **ill-posed**. The well-posedness condition does not define what the unique solution will be and it does not indicate if the solution v is analytic or infinitely differentiable. For a PDE of order n the solution needs to be at least n times continuously differentiable, so all the derivatives in the equation will exist and remain continuous. Such a solution is called a **classical solution** of the PDE. However, not all of the well-posed PDEs have a classical solution, conservation law is considered one of the exceptions. These types of equations develop shock wave situation, which is a discontinuity in the solution. In such cases, a physically meaningful solution known as **weak solution** is introduced, which will be examined in more details in chapter 3. In order to study the properties of solutions for PDEs, let us consider Hilbert spaces H_1, H_2, and an equation as

$$Lv = f \tag{1.5}$$

where $L : H_1 \to H_2$ is a linear operator and $f \in H_2$. The **null space** $N(L)$ of a linear operator is the set $N(L) = \{v \in H_1 : L(v) = 0\}$ and the **range** of the operator is $R(L) = \{w \in H_2 : \exists v \in H_1 \quad \text{such that} \quad L(v) = w\}$. The **existence** of a solution of (1.5) for any right-hand side function $f \in H_2$ is equivalent to the condition $R(L) = H_2$, while the **uniqueness** of the solution is equivalent to the condition $N(L) = \{0\}$.

Given two Banach spaces H_1, H_2, an operator $L = H_1 \to H_2$ is said to be **closed** if for any sequence $(v_n)_{1 \leq n \leq \infty} \subset H_1, v_n \to v$ and $L(v_n) \to w$ imply that $v \in H_1$ and $w = Lv$.

Existence: Let H_1, H_2 be Hilbert spaces and $L : H_1 \to H_2$ be a bounded linear operator. Then $R(L) = H_2$ if and only if $R(L)$ is closed and if $R(L)^\perp = \{0\}$.

Existence and uniqueness: Let H_1, H_2 be Hilbert spaces and $L : H_1 \to H_2$ be a closed linear operator. Suppose that there exists a constant $C > 0$ such that

$$||Lv||_{H_2} \geq C||v||_{H_1}, \quad \text{for all} \quad v \in H_1 \quad \text{(coercivity estimate)} \tag{1.6}$$

If $R(L)^{\perp} = \{0\}$, then the operator equation $Lu = f$ has a unique solution.

When working with ODEs, theorems like Picard-Lindelöf (Lindelöf, 1894) can be applied to determine the existence and uniqueness of the solution. However, it is different for PDEs. Cauchy-Kovalevskaya theorem investigates the existence and uniqueness of the solution of Cauchy problems, although the solution may accompany undesirable properties which will result in weak solutions. For more information on this matter, refer to (Abell & Braselton, 2014), (Egorov & Shubin, 1998).

Functional Analysis

Function spaces are descriptive methods for functions and their norms, in qualitative and quantitative concepts (Frey & de Buhan, 2008), (Tao, 2008), (Showalter, 1994). A **metric space** is a couple (X, d) where X is a set and d is a metric (or a distance) on X that is a function $d : X \to \mathbb{R}^+$ such that

(a) $d(x, y) \geq 0$, non-negativity

(b) $d(x, y) = 0$ if and only if $x = y$, identity

(c) $d(x, y) = d(y, x)$, symmetry

(d) $d(x, z) \leq d(x, y) + d(y, z)$, triangle inequality.

Let (X, d) be a metric space and r a stricly positive scalar value. At any point x in a metric space, we define the **open ball** (closed ball) of radius r about x as the set $B(x,r) = \{y \in X : d(x,y) < r\}$ $(B_c(x,r) = \{y \in X, d(x,y) \leq r\})$. These balls generate a topology on X, making it a **topological space**. A subset Y of X is called **open** if it is a union of open balls, its complement is called a **closed**.

Let us consider a vector space E on $\mathbb{K}$, where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$. A mapping $N : E \to \mathbb{R}^+$ is a **seminorm** on E, if and only if

(a) $N(x+y) \leq N(x) + N(y)$,

(b) for every $\lambda \in \mathbb{K}, N(\lambda x) = |\lambda| N(x)$.

A **norm** is a seminorm with the additional property: $N(x) = 0$ if and only if $x = 0$. Let E and N be a vector space and a norm on E, respectively. The pair (E, N) is called a **normed space**. Let $(E, ||\cdot||)$ be a normed space. The map $E \times E \to \mathbb{R}^+, (x,y) \mapsto ||x-y||$ is a distance on E, called the **distance associated to the norm** $||\cdot||$.

Let (X,d) be a metric space. A **Cauchy sequence** in X is a sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X such that

$$\forall \epsilon > 0, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \forall m \geq n_0, \quad d(x_n, x_m) \leq \epsilon. \tag{1.7}$$

Any Cauchy sequence in a metric space is bounded. A metric space (X,d) in which every Cauchy sequence converges, has a limit in X, is called **complete**.

Let $(E, ||\cdot||)$ be a normed space. $(E, ||\cdot||)$ is a **Banach space** if and only if the metric space (E,d) is a complete space, where d is the distance associated to the norm $||\cdot||$, for instance $d(x,y) = ||x-y||$.

Hilbert spaces, named after the German mathematician David Hilbert (1862-1943), are complete infinitedimensional spaces in which distances and angles can be measured. These spaces provide a convenient and proper setting for the functional analysis of partial differential equations.

Let us define the vector spaces on $\mathbb{K}$. A mapping $f : E \times E \to \mathbb{K}$ is called an **inner product** $(< \cdot, \cdot >)$ on E if and only if it is sesquilinear and is a positive-definite hermitian form satisfying the following axioms

(a) $\forall (x, y) \in E^2, \quad f(y, x) = \overline{f(x, y)}$

(b) $\forall x \in E, \quad f(x, x) \in \mathbb{R}^+$

(c) $\forall x \in E, \quad f(x, x) = 0 \Leftrightarrow x = 0$

A (complex) vector space with an inner product satisfying (a)-(c) is sometimes called a **pre-Hilbert space**. A pre-Hilbert space E is a **Hilbert space** if and only if it is a complete normed space, i.e. a Banach space, under the norm associated with the inner product.

This research concentrates on the specific Hilbert spaces such as the Hilbert spaces in L^2, C^2 or Sobolev space.

Since most of the processes are described as first- or second-order PDEs, they are introduced briefly in the following sections, accompanied by classifications and some examples .

First-Order PDEs

The general form of a first-order PDE with n independent variable, including control input is expressed as

$$F\left(x_1, x_2, \cdots, x_n, v, \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2}, \cdots, \frac{\partial v}{\partial x_n}, u_1, \cdots, u_m\right) = 0 \tag{1.8}$$

where $F(\cdots)$ is a given function, $v(x_1, x_2, \cdots, x_n)$ is the unknown function and $u_i, i = 1, \cdots, m$ are the control inputs. The questions of existence and uniqueness of the solution have to be answered considering the closed-loop system including the feedback control $u_i = u_i(x_1, x_2, \cdots, x_n, v)$. This research is concerned with situations where the independent variables are (t, x), (t, x, y) or (t, x, y, z).

Classification of System of First-Order PDEs

Let us consider a system of PDEs as follows

$$\frac{\partial \Phi}{\partial t} + [A]\frac{\partial \Phi}{\partial x} + [B]\frac{\partial \Phi}{\partial y} + \Psi(x, y, \Phi) = 0 \tag{1.9}$$

where Φ is a vector containing the unknown variables, and the elements of the coefficient matrices $[A]$ and $[B]$ are functions of x, y and t. If the eigenvalues of the matrix $[A]$ (or $[B]$) are all real and distinct, the set of equations are classified as hyperbolic in t and x (or y). If the eigenvalues are complex the system of equations are elliptic in t and x (or y). For instance, for a system of first-order PDEs as

$$\frac{\partial v_1}{\partial t} + a_1\frac{\partial v_1}{\partial x} + a_2\frac{\partial v_2}{\partial x} + a_3\frac{\partial v_1}{\partial y} + a_4\frac{\partial v_2}{\partial y} + \Psi_1 = 0, \tag{1.10}$$

$$\frac{\partial v_2}{\partial t} + b_1\frac{\partial v_1}{\partial x} + b_2\frac{\partial v_2}{\partial x} + b_3\frac{\partial v_1}{\partial y} + b_4\frac{\partial v_2}{\partial y} + \Psi_2 = 0, \tag{1.11}$$

the matrices are defined as follows

$$\Phi = [v_1 \quad v_2]^T, \quad [A] = \begin{bmatrix} a_1 & a_1 \\ b_1 & b_2 \end{bmatrix}, \quad [B] = \begin{bmatrix} a_3 & a_4 \\ b_3 & b_4 \end{bmatrix}, \quad \Psi = [\Psi_1 \quad \Psi_2]^T. \tag{1.12}$$

In the case of steady state form of (1.9)

$$[A]\frac{\partial \Phi}{\partial x} + [B]\frac{\partial \Phi}{\partial y} + \Psi(x, y) = 0, \tag{1.13}$$

the classification is defined based on the sign of H

$$H = R^2 - 4|A||B|, \quad \text{where} \quad R = \begin{vmatrix} a_1 & a_4 \\ b_1 & b_4 \end{vmatrix} + \begin{vmatrix} a_3 & a_2 \\ b_3 & b_2 \end{vmatrix}. \tag{1.14}$$

The set of PDEs is recognized as hyperbolic when $H > 0$, parabolic if $H = 0$ and elliptic when $H < 0$ (Hoffmann & Chiang, 2000).

Second-Order PDEs

The general form of a second-order PDE with n independent variable is given as

$$F\left(x_1, x_2, \cdots, x_n, v, \frac{\partial v}{\partial x_1}, \cdots, \frac{\partial v}{\partial x_n}, \frac{\partial^2 v}{\partial x_1}\partial x_1, \cdots, \frac{\partial^2 v}{\partial x_1 \partial x_n}, \cdots\right) = 0 \tag{1.15}$$

where $v(x_1, x_2, \cdots, x_n)$ is the unknown function and $F(\cdots)$ is a given function.

Classification of Second-Order PDEs

Consider a second-order PDE of the following form

$$C : D^2 v + b \cdot Dv + av = f \tag{1.16}$$

where $\forall x \in \Omega, a(x) \in \mathbb{R}, b(x) \in \mathbb{R}^n, C(x) \in \mathbb{R}^{n\times n}$ are the coefficients of the equation, $A : B = \sum_{i,j=1}^{n} a_{ij}b_{ij}$ and $D = \frac{\partial v}{\partial x}$. Equation (1.16) is called **elliptic** at $x \in \Omega$ if $C(x)$

is positive definite, **parabolic** at $x \in \Omega$ if $C(x)$ is positive semidefinite, **hyperbolic** at $x \in \Omega$ if $C(x)$ has one negative and $n-1$ positive eigenvalues (Frey & de Buhan, 2008).

A linear second-order PDE with two independent variables

$$a\frac{\partial^2 v}{\partial x^2} + b\frac{\partial^2 v}{\partial x \partial y} + c\frac{\partial^2 v}{\partial y^2} + e\frac{\partial v}{\partial y} + fv = g, \quad in \quad \Omega \tag{1.17}$$

is called parabolic if $b^2 - 4ac = 0$, hyperbolic when $b^2 - 4ac > 0$ and elliptic when $b^2 - 4ac < 0$.

Beside the geometric interpretation, classification of PDEs helps to estimate the smoothness of the solution, the speed of information propagation, and the effect of initial and boundary conditions on the solution. Hyperbolic PDEs often describe the phenomena featuring propagation in preferred directions while keeping its strength, the smoothness of the solution depends on the smoothness of initial and boundary conditions. In the case of nonlinear hyperbolic PDEs, discontinuities might occur in the solution even for smooth data, one example is shocks in compressible flow. Elliptic PDEs describe propagation in all directions while decaying in strength, the solution is always smooth independent of smoothness and roughness of the initial and boundary conditions. Parabolic PDEs are a case of hyperbolic PDEs, they are usually time dependent, solutions are smooth in space but may show singularities and the speed of propagation is infinite (Belytschko et al., 2014), (Debnath, 2005), (Manaa et al., 2015).

Some Examples of First- and Second-Order PDEs

Conservation Law	$v_t + \nabla \cdot F(v) = 0$
Transport/Advection Equation	$v_t + v_x = 0$
Inviscid Burgers' Equation	$v_t + vv_x = 0$
Heat Equation	$v_t - v_{xx} = 0$
Wave Equation	$v_{tt} - v_{xx} = 0$
Laplace Equation	$v_{xx} - v_{yy} = 0$
Poisson's Equation	$v_{xx} - v_{yy} = f(x, y)$
Shrödinger's equation	$iv_t + v_{xx} = 0$
Viscous Burgers' Equation	$v_t + vv_x = v_{xx}$
Kolmogorov-Petrovskii-Piskunov Equation	$v_t - av_{xx} = f(v), \quad a > 0$

1.4 State Observer

Following the goal of this dissertation, the next step is to introduce state observers and explore their reason to exist. As discussed previously, partial differential equations render many engineering and scientific inquiries. However, in many practical cases the complete information regarding the states of the system is not available due to the technical difficulties such as lack of sensors. State observer provides an approximation of the internal states of a system, with holding only input and output available. Observers approximate missing state variable $x(t)$ based on the measurements of the system output $y(t)$ and input $u(t)$, Figure 1.1. Different types of observers exist where each possesses advantages for various problems, i.e. systems with disturbances or/and uncertainties in modeling, linear systems, etc. Nevertheless, the idea remains the same and it is based on mimicking the system's behavior and comparing the output with the actual output and minimizing the difference between these two, Figure 1.1, (Luenberger, 1964) and (Luenberger, 1979).

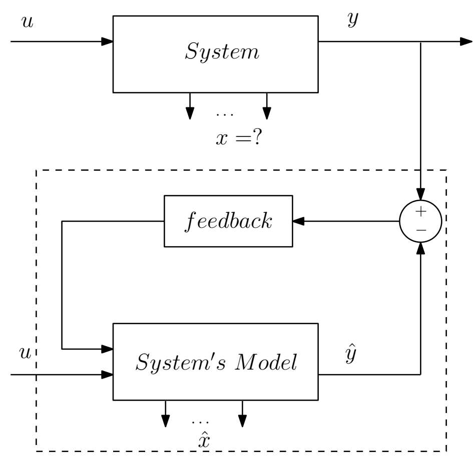

Figure 1.1: Schematic diagram of a state observer.

As an example let us consider a linear time-invariant system as in Figure 1.2

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x \in \mathbb{R}^n, \tag{1.18}$$

In this example $u(t) = -kx(t)$ is the feedback control law. Since the state is not directly measurable, the estimation of the state $\hat{x}(t)$ is used

$$u(t) = -k\hat{x}(t), \quad u \in \mathbb{R}^r. \tag{1.19}$$

The output $y(t)$ is

$$y(t) = Cx(t), \quad y \in \mathbb{R}^m. \tag{1.20}$$

The observer is designed as follows

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + L\left(y(t) - \hat{y}(t)\right), \tag{1.21}$$

where L is the observer gain matrix and $\hat{y}(t) = C\hat{x}(t)$. For the estimation error and its derivative we have

$$\begin{aligned} e(t) &= x(t) - \hat{x}(t), & (1.22) \\ \dot{e}(t) &= (A - LC)e(t). & (1.23) \end{aligned}$$

The error estimation can be driven to zero by selecting proper L (considering the observability conditions). In the case of a deterministic system, with no measurement noises or unmeasured disturbances, the observer is called **Luenberger observer**, Figure 1.2 (Luenberger, 1971). For a linear time-invariant (LTI) system as in (1.18) and (1.20) if matrices A and C are completely observable, L can be assigned in a way that eigenvalues of $A - LC$ locate arbitrarily, notice that complex eigenvalues must appear in complex conjugate pairs.

A system is completely **observable** if every state $x(t_0)$ can be uniquely determined by measuring the output $y(\tau)$ over a finite time interval $\tau \in [t_0, t_1]$. For LTI systems it is equivalent to having a full rank observability matrix O (Luenberger, 1979).

$$O = \begin{bmatrix} C^T & (CA)^T & (CA^2)^T & \cdots & (CA^{n-1})^T \end{bmatrix}, \quad \text{rank}(O) = n. \tag{1.24}$$

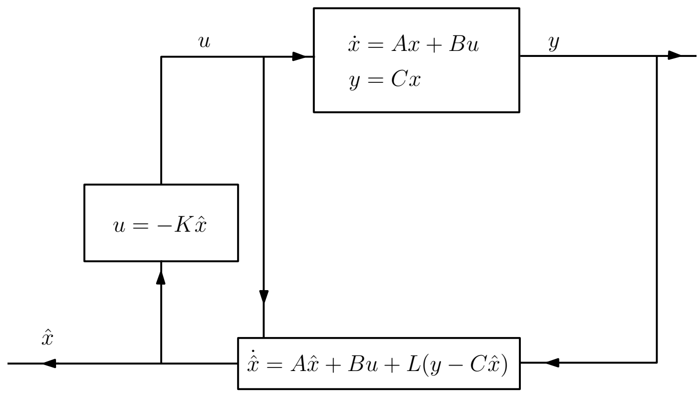

Figure 1.2: Luenberger observer.

For more detailed information on observers refer to (Srivastava et al., 2009), (Bakshi & Bakshi, 2009), (Zabczyk, 2007).

When disturbances and/or measurement noises exist in the system, the **Kalman filter** is considered as an alternative observer, Figure 1.3. This type of filter uses the knowledge of statistical properties of the system in its design. It is an optimal estimate in the sense that the mean value of the sum of the estimation errors gets a minimal value. Refer to (Vaseghi, 2000), (Grewal & Andrews, 2014), (Catlin, 2012) and (Zarchan & Musoff, 2009) for additional information on Kalman filter.

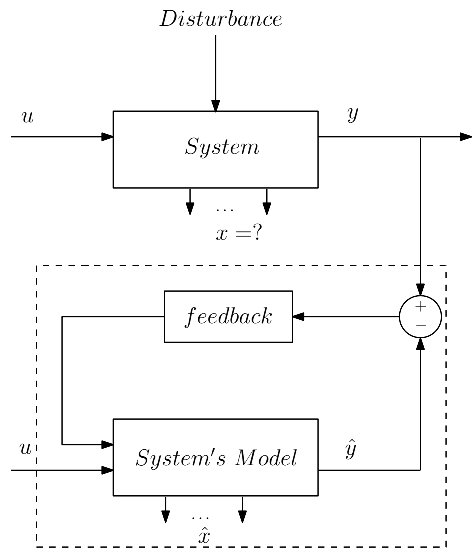

Figure 1.3: Kalman filter demonstration.

1.5 Variable Structure Control

In this research, the sliding mode control theory which is a subset of variable structure control is employed for nonlinear observer design. This section introduces the concept of variable structure control. Variable structure systems maintain varying structures either caused by change in the parameters of the system or by having different inputs as the controller. In variable structure control, the control input varies depending on the state of system, Figure 1.4. The first implementation of variable structure control dates back to 1939 when Irmgard Flügge-Lotz, the German engineer, was working on the automatic control theory and development of a discontinuous, on and off, control system (Flügge-Lotz, 1953). She studied the automatic guidance of the V2 rocket, and the question was to assign parameters β_1 and β_2 in (1.25) to possess a system with the desired behavior, in this case rapid damping of large perturbations (Hájek, 2009)

$$\ddot{x} + \alpha_1\dot{x} + \alpha_2 x = \beta_1 \text{sign}(x + \beta_2\dot{x}). \tag{1.25}$$

The first mention of variable structure theory in literature was by Emelyanov (Emelyanov, 1967).

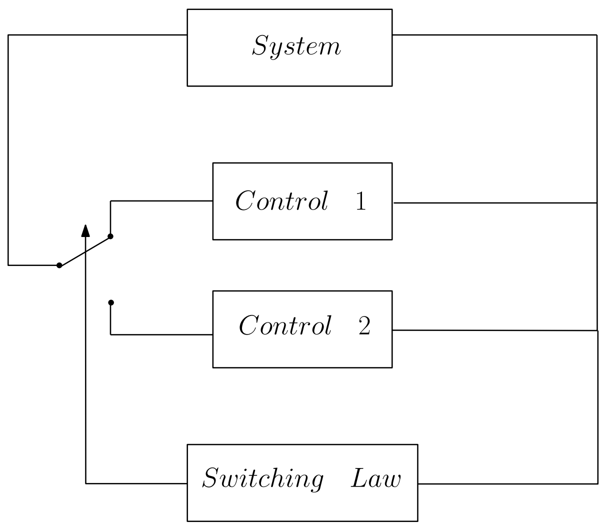

Figure 1.4: Schematic presentation of variable structure control.

To see how a variable structure system works let's consider the following second-order system (Utkin, 1977)

$$\ddot{x} = -\Psi x \tag{1.26}$$

where assigning different positive Ψs results in systems with different behaviors. Figures 1.5 and 1.6 show the state space representation for two cases with constant Ψs that lead to marginally stable systems. However assigning Ψ as in (1.27) results in a system with asymptomatic convergence, Figure 1.7.

$$\Psi = \begin{cases} \alpha_1^2 & x\dot{x} > 0 \\ \alpha_2^2 & x\dot{x} < 0 \end{cases} \qquad \alpha_1^2 > \alpha_2^2 \tag{1.27}$$

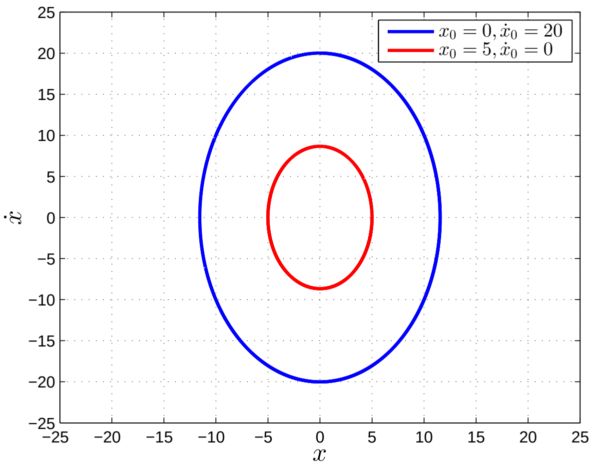

Figure 1.5: System (1.26) trajectories if $\Psi = 3$.

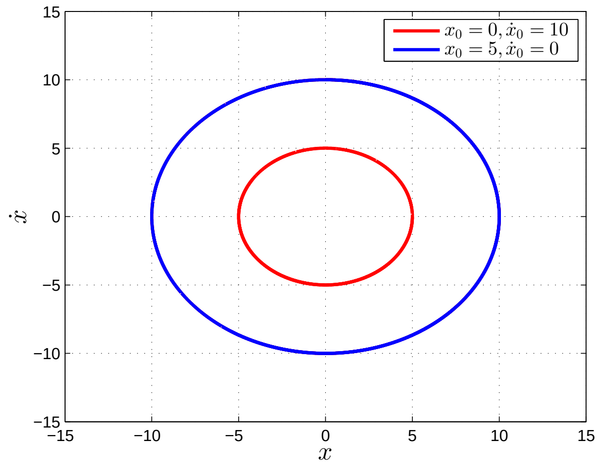

Figure 1.6: System (1.26) trajectories if $\Psi = 1$.

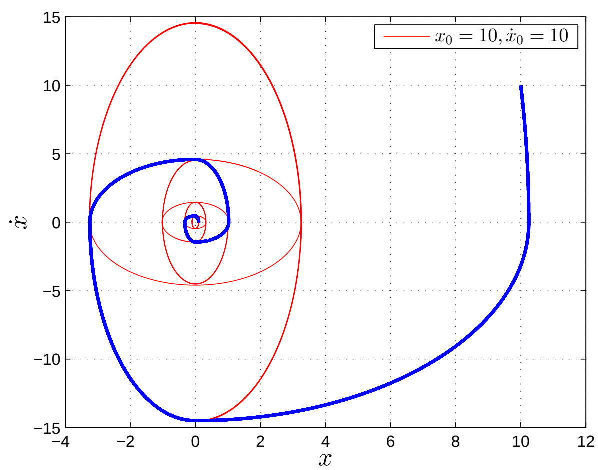

Figure 1.7: Stable variable structure system, switching between $\alpha_1^2 = 20$ and $\alpha_2^2 = 2$.

1.6 Sliding Mode Control

Sliding mode control is a subset of variable structure control, in which the states of the system are guided into a switching surface and then the states slide to the origin, as shown in Figure 1.8. Variable structure system and control were developed by Utkin and sliding mode control was introduced by Utkin as well (Utkin, 1978). For further information on sliding mode control refer to (Utkin, 1993), (Drakunov & Utkin, 1992), (Young et al., 1999) and to further examine sliding mode control design for infinite-dimensional systems consider (Orlov & Utkin, 1987), (Levaggi, 2001) and (Levaggi, 2013).

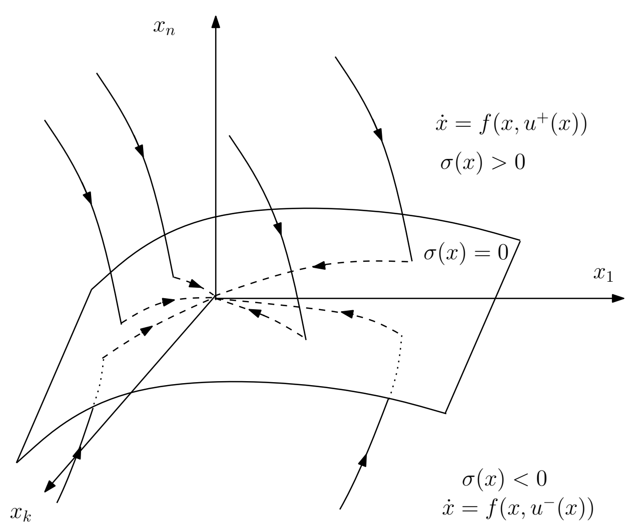

Figure 1.8: Sliding mode demonstration.

As an example let's consider the following system

$$\ddot{x} - \xi\dot{x} + \Psi x = 0, \quad \xi > 0 \tag{1.28}$$

If $\Psi = \alpha$ or $-\alpha$ where $\alpha > 0$ the system is unstable, Figures 1.9 and 1.10. By choosing Ψ as

$$\Psi = \begin{cases} \alpha & x\sigma > 0 \\ -\alpha & x\sigma < 0 \end{cases} \tag{1.29}$$

where $\sigma = cx + \dot{x}$ and $c = -\frac{\xi}{2} \pm \sqrt{\frac{\xi^2}{4} + \alpha}$, the system converges to the origin in a sliding manner (Utkin, 1977). As can be seen in Figure 1.11 for different initial conditions the states of the system are guided to the line $\sigma = 0$ and then they slide into the origin.

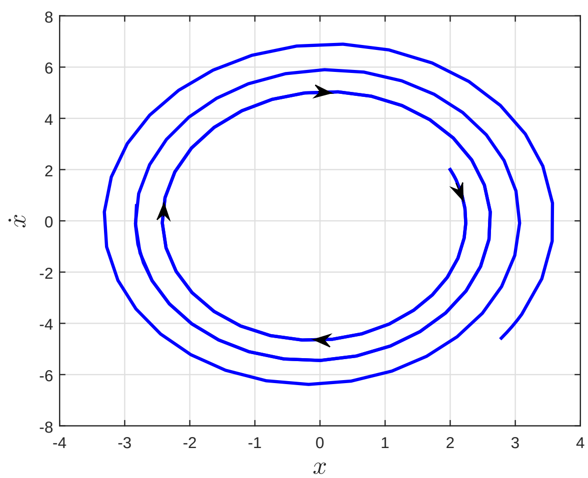

Figure 1.9: System (1.28) if $\Psi = 4$ ($\xi = 0.1, x_0 = 2, \dot{x}_0 = 2$).

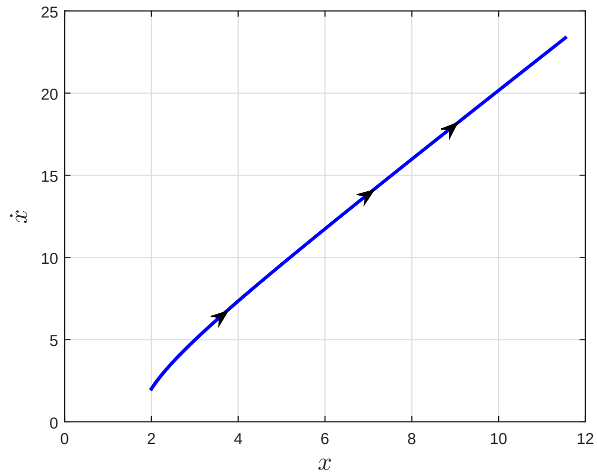

Figure 1.10: System (1.28) if $\Psi = -4$ ($\xi = 0.1, x_0 = 2, \dot{x}_0 = 2$).

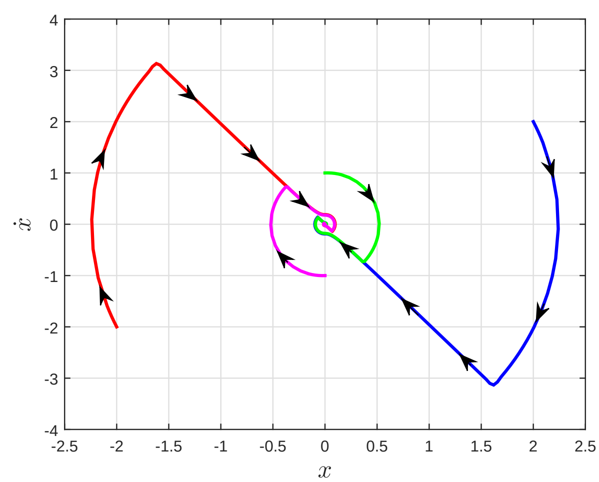

Figure 1.11: System (1.28) for different initial conditions, and Ψ as in (1.29).

The idea of sliding mode control has been applied even prior to the documentation of the concept. Let us consider the circuit with time-varying input voltage $V_{in}(t)$ in the Figure 1.12.

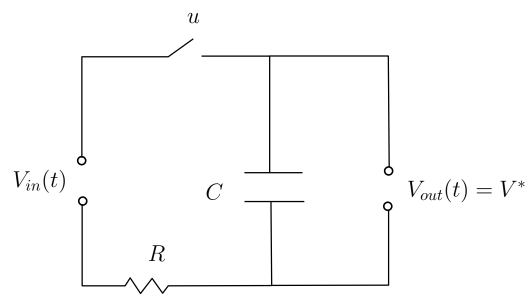

Figure 1.12: Circuit, example.

The desired output is $V_{out}(t) = V^*$ where V^* is a constant. The differential equations for the circuit are [1]

$$RI + \frac{1}{C}\int_0^t I(\tau)d\tau = V_{in}, \tag{1.30}$$

$$V_{out} = \frac{1}{C}\int_0^t I(\tau)d\tau. \tag{1.31}$$

Taking the derivative of (1.31) and substituting in (1.30) we have

$$\dot{V}_{out} + \frac{1}{RC}V_{out} = \frac{1}{RC}V_{in}. \tag{1.32}$$

By designing u as

$$u = \frac{1}{2}\left[1 - \operatorname{sign}(V_{out} - V^*)\right] = \begin{cases} 1 & V_{ou} \leq V^* \\ 0 & V_{ou} > V^* \end{cases} \tag{1.33}$$

the goal of having a constant voltage is achieved by opening and closing the switch u,

[1] We consider the situation when the output resistance is ∞. In the case of presence of a load, the similar analysis can be performed.

and charging and discharging the capacitor C repeatedly, Figure 1.13. The switching law is similar to the one used in sliding mode control.

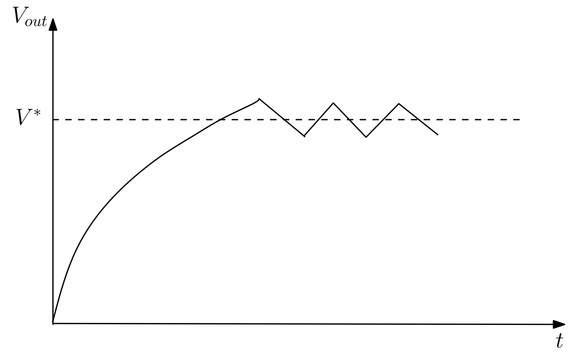

Figure 1.13: Voltage output for the circuit.

Sliding mode uses a discontinuous control law (1.35) to steer the states of the system (1.34) from any initial condition to a manifold, and then to slide them to the origin on the manifold. This manifold σ is called sliding manifold or switching manifold. There are two phases in the sliding mode control. First is the **reaching phase** where the trajectory is steered into sliding manifold σ in finite time, and second is the **sliding phase** in which the trajectory approaches the origin asymptotically, Figure 1.8. Some of the advantages of sliding mode control and observer are their simple implementation, the insensitivity to the parameter uncertainty and external disturbances (robustness), and order reduction (during sliding mode the trajectory dynamics has lower order than the original system).

$$\dot{x} = f(t, x, u), \quad x(0) = x_0 \tag{1.34}$$

$$u = \begin{cases} u^+(t,x) & \sigma(x) > 0 \\ u^-(t,x) & \sigma(x) < 0 \end{cases} \tag{1.35}$$

To demonstrate the sliding mode control idea let us consider the mathematical model describing an inverted pendulum

$$\ddot{\theta} = \sin\theta + u, \tag{1.36}$$

where θ is the inclination from the vertical axis and u is the control input. Writing the system in the state space form

$$x_1 = \theta, \quad \dot{x}_1 \;=\; x_2, \tag{1.37}$$

$$x_2 = \dot{\theta}, \quad \dot{x}_2 \;=\; \sin x_1 + u, \tag{1.38}$$

and designing the discontinuous control law as

$$u = -k\mathrm{sign}\sigma, \tag{1.39}$$

where σ is the sliding manifold

$$\sigma = x_2 + \lambda x_1, \tag{1.40}$$

result in the convergence of the states of the system into origin. For instance for the nominal values $\theta_0 = 2$, $\dot{\theta}_0 = 2$, $\lambda = 1$ and $k = 1.5$, Figure 1.14 shows how the inclination angle and the velocity converge to zero. Figure 1.15 represents the sliding mode happening on the manifold σ, note that the **chattering effect** is visible in this plot, which basically is the result of switching the controller values in order to maintain the states on the sliding surface. Chattering is the result of implementation of the signum function and not the signum function itself. In the actual systems, the swift switching devices are not available due to imperfections such as delay, hysteresis, etc. Reaching and sliding phases are presented in Figure 1.16.

In case of a system with bounded disturbance d

$$\ddot{\theta} = \sin\theta + u + d, \quad d < |d_1| \tag{1.41}$$

everything will remain the same, the only difference is appointing k in (1.39) large enough to compensate for the disturbance.

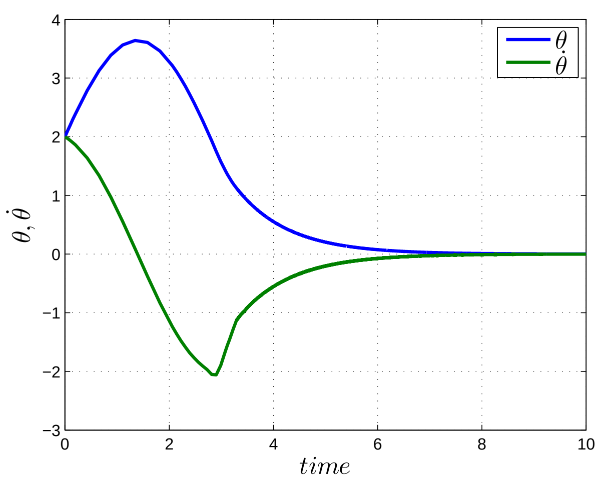

Figure 1.14: Inverted pendulum, states convergence.

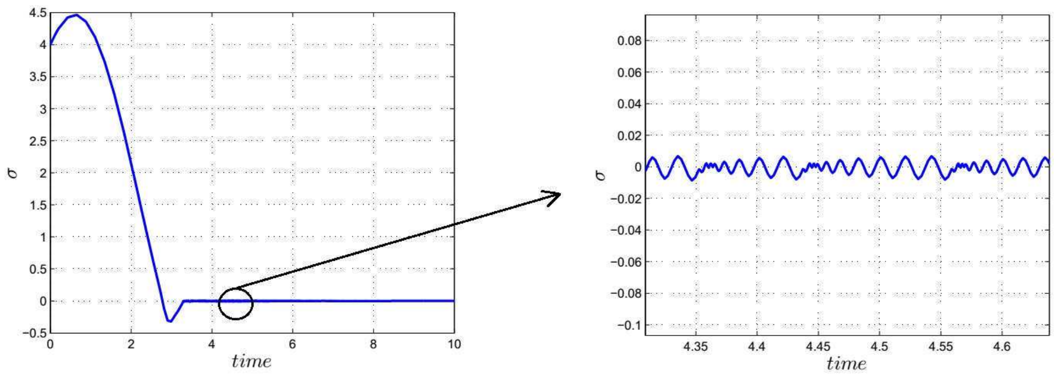

Figure 1.15: Inverted pendulum, sliding manifold and the chattering effect.

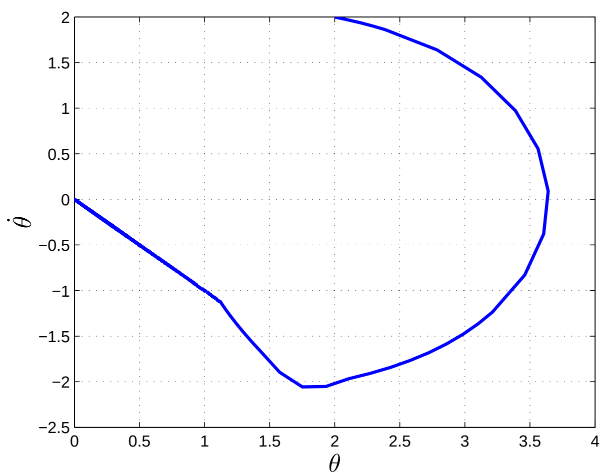

Figure 1.16: Inverted pendulum, reaching and sliding phases.

In order to investigate the convergence of sliding mode, the following quadratic Lyapunov candidate is introduced (Lyapunov techniques and theory are explained in section 1.7)

$$V(\sigma) = \frac{1}{2}\sigma^2. \tag{1.42}$$

For the time derivative we have

$$\dot{V} = \sigma\dot{\sigma}, \tag{1.43}$$

where

$$\dot{\sigma} = \dot{x}_2 + \lambda\dot{x_1} = \sin x_1 + u + \lambda x_2 = \sin x_1 - k\text{sign}\sigma + \lambda x_2$$

By choosing $k > |\sin x_1 + \lambda x_2|$ the term $-k\text{sign}\sigma$ will be the dominant term and $\dot{\sigma} = -k\text{sign}\sigma$, so we have

$$\begin{cases} \sigma > 0 \rightarrow \dot{\sigma} < 0 \\ \sigma < 0 \rightarrow \dot{\sigma} > 0 \end{cases} \Rightarrow \dot{V} < 0 \tag{1.44}$$

which is a desired result based on the Lyapunov method and guarantees the convergence of the sliding mode control. In the case of presence of a bounded disturbance, the controller gain k has to compensate for the disturbance term as well, so having $k > |\sin x_1 + \lambda x_2 + d|$ guarantees the sliding mode convergence. Note that in both cases, the region of attraction is not the entire space, although by assigning the controller gain as a function of the states $k(x_1, x_2)$, we are able to adjust the region of attraction.

Sliding Mode Control Continuation

According to Drakunov & Utkin (1992) the properties of group and semigroup are employed to further describe the sliding mode. Let us start with some definitions.

The general solution of $\dot{x} = f(t, x), x(t_0) = x_0$ is in the form of $x(t) = g(t_0, x_0, t)$ where g is a transformation of $\mathbb{R}^n \to \mathbb{R}^n$ that means by considering a fixed t and t_0 each $x(t) = g(t_0, x_0, t)$ maps $x_0 \to x$. Consider two transformations one from t_0 to t_1: $g_1(x) = g(t_0, x, t_1)$ and the other from t_1 to t_2: $g_2(x) = g(t_1, x, t_2)$. To have a map from t_0 to t_2, the operator $\circ$ is used

$$g_1 \circ g_2 = g_2(g_1(x)) = g(t_1, g(t_0, x, t_1), t_2). \tag{1.45}$$

Let G be a set of all such transformations of $\mathbb{R}^n \to \mathbb{R}^n$, with the operator o, then G is a **group** if it satisfies the following properties

1. $\forall g_1, g_2, g_3 \in G \quad g_1 \circ (g_2 \circ g_3) = (g_1 \circ g_2) \circ g_3$, associativity.

2. $\exists e(x) \in G$ such that $e \circ g = g \circ e \equiv g$ for all $g \in G$. e is called identical transformation (unit element of the group) and $e(x) = g(t_0, x, t_0) = x$.

3. $\forall g \in G, \exists g^{-1} \in G$ such that $g \circ g^{-1} = g^{-1} \circ g = e$. For instance $g(x) = g(t_0, x, t_1)$ results in $g^{-1}(x) = g(t_1, x, t_0)$ and $g(g^{-1}(x)) \equiv x$ and g^{-1} is called an inverse of g.

A **semigroup** only needs to satisfy the associativity property.

Studying the classical differential equations, they can be described using group definition. However, the discontinuity in the right-hand side of the equation for sliding mode results in the families of state space transformations representing closed-loop systems to be semigroups rather than groups. In the sliding manifold, the inverse transformations for states in the sliding manifold are not unique due to the discontinuity on the manifold. The families of transformations

$$F(t, t_0, \cdot) : \varkappa \to \varkappa \tag{1.46}$$

with $t_0, t \in T, t_0 \leq t$, T represents continuous or discrete-time cases, transformation (1.46) is the most general description of dynamic systems in metric space $\varkappa$. F is a continuous function of x satisfying semigroup condition $F(t, t_1, F(t_1, t_0, x_0)) = F(t, t_0, x_0)$ for every $t_0 \leq t_1 \leq t, x_0 \in \varkappa$ and $F(t, t, x) = x$ for every $t \in T, x \in \varkappa$. However, if F corresponds to the system of ODEs with the existence and uniqueness of the Cauchy problem's solution, then for every $t_0 \leq t, x \in \varkappa$ the transformation F is invertible. This means that the family $\{F(t_0, t, x)\}_{t_0, t \in T}$ is a group.

A point x in the state space $\varkappa$ of a dynamic system with a family of semigroup transformations $\{F(t, t_0, \cdot)\}_{t_0 \leq t}$ is considered a **sliding mode point** at the moment $t \in T$, if for every $t_0 \in T, t_0 < t$, the transformation $F(t, t_0, \cdot)$ is not invertible at this point and an equation $F(t, t_0, \xi) = x$ has more than one solution ξ. A set $\Sigma \subset T \times \varkappa$ in the state space is a **sliding mode set**, if for every $(t, x) \in \Sigma$, the point x is a sliding mode point at the moment t.

The manifold $\sigma = 0$ in the domain $\mathcal{D}$ is called a **sliding mode domain** if $\forall \epsilon > 0$ $\exists \delta > 0$, such that any motion starting in the vicinity δ of $\mathcal{D}$ may leave ϵ on $\mathcal{D}$ only through the ϵ vicinity of the boundaries of $\mathcal{D}$, Figure 1.17.

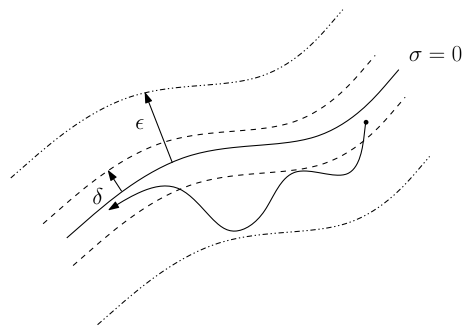

Figure 1.17: Sliding mode domain.

An **ideal sliding mode** exists only when the state trajectory $x(t)$ reaches the manifold $\sigma(t, x) = 0$ in the finite time. Due to imperfections of switching devices, this condition results in fast switching actuators that cause the chattering effect. Note that sliding mode does not necessarily consist of chattering. Let us consider a system including coulomb friction as

$$m\ddot{x} = F - k_f \text{sign}\dot{x} \tag{1.47}$$

where F is an impulse force that makes the object move with some initial velocity, in this case the sliding manifold will be $\sigma = \dot{x}$. As can be seen in Figure 1.18 the object stops as soon as the velocity reaches zero.

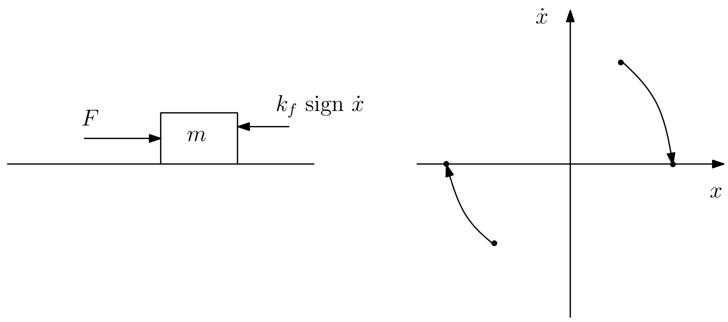

Figure 1.18: Non-chattering sliding mode.

A sliding mode exists if in the vicinity of the switching surface $\sigma = 0$, the tangent vector of the state trajectory, the velocity vector of the state trajectory, always point toward the $\sigma = 0$. For **existence** of a sliding mode, after some finite time t_1 the state of the system $x(t)$ must be in some neighborhood of σ: $\{x| \quad ||\sigma|| < \epsilon\}$.

The **region of attraction** is the largest subset of the state space from which sliding is achievable, as shown in the inverted pendulum example. A sliding mode is globally reachable if the domain of attraction is the entire state space (DeCarlo et al., 2011).

In the classical case, since the right-hand side of the differential equation is continuous with respect to x, the solution exists

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, \quad x \in \mathbb{R}^n, \forall t \in \mathbb{R}, f(t, \cdot) \in C^\infty(\mathbb{R}^n). \tag{1.48}$$

One of the most well-known conditions for the uniqueness of the solution is described by **Lipschitz condition**

$$||f(t, x_1) - f(t, x_2)|| \leq L||x_1 - x_2||, \tag{1.49}$$

where L is called Lipschitz constant.

Variable structure systems as a result of possessing discontinuous function at the right-hand side require special consideration. Filippov suggested one of the first propositions in order to define the control input on the sliding manifold (Filippov, 1988). Let us consider the system and control law as

$$\dot{x} = f(t, x, u), \quad x(0) = x_0 \tag{1.50}$$

$$u = \begin{cases} u^+(t, x) & \sigma(x) > 0 \\ u^-(t, x) & \sigma(x) < 0 \end{cases} \tag{1.51}$$

As it can be seen, the dynamics of the system is not defined on $\sigma = 0$. **Filippov definition** expresses that the state trajectories of (1.50) with control law (1.51) on $\sigma = 0$, are the solution of

$$\dot{x}(t) = \alpha f^+ + (1 - \alpha) f^- = f_{Filippov}, \quad 0 \leq \alpha \leq 1 \tag{1.52}$$

where $f^+ = f(t, x, u^+)$ and $f^- = f(t, x, u^-)$ and $f_{Filippov}$ is the resulting velocity vector of the trajectory while on sliding mode, Figure 1.19. α is determined by the solution of $< \nabla\sigma, f_{Filippov} >= 0$ ($f_{Filippov}$ is the tangential vector to the sliding manifold)

$$\alpha = \frac{< \nabla\sigma, f^- >}{< \nabla\sigma, (f^- - f^+) >} \tag{1.53}$$

where $< \nabla\sigma, f^- > \geq 0$, $< \nabla\sigma, f^+ > \geq 0$ and $< \nabla\sigma, (f^- - f^+) > \neq 0$, (DeCarlo et al., 2011), (Perruquetti & Barbot, 2002).

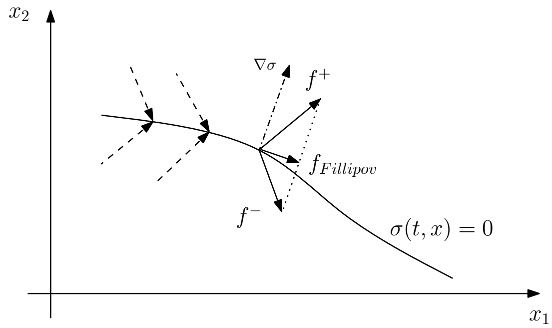

Figure 1.19: Filippov definition.

For a multi-dimensional case, the definitions of differential inclusions and convex hull are employed to introduce the **generalized Filippov definition** as

$$\dot{x} \in F(t,x) = \lim_{\epsilon\to 0} co\left\{f\left(t,x,U(x)\right)\right\}, \tag{1.54}$$

where $u \in U(x)$ is a set of all possible control inputs [2].

Depending on the behavior of the switching system such as delay, hysteresis, etc., Figure 1.20, Filippov definition might not result in the correct solution, so the **equivalent control** was introduced by Utkin, Figure 1.21, (Utkin, 1992). On the sliding manifold $\sigma = 0$ therefore $\dot{\sigma} = 0$, so the equivalent control is the solution of

$$L_f\sigma = \frac{d}{dt}\sigma(t,x(t)) = 0. \tag{1.55}$$

For example for $\dot{x} = f(t,x,u)$ we have

$$\frac{\partial\sigma(t,x)}{\partial t} + \frac{\partial\sigma(t,x)}{\partial x} f(t,x,u_{eq}) = 0 \Rightarrow u_{eq}(t,x) \Rightarrow f(t,x,u_{eq}(t,x)) \tag{1.56}$$

[2]The convex hull may exclude zero measure sets (Filippov, 1988).

Note that Filippov's method and the equivalent control deliver the same results in the case of control affine systems, $\dot{x} = f(t, x) + B(t, x)u$. Figure 1.22 demonstrates $F_{Filippov}$ and F_{eq} when the boundary layer around the sliding manifold is approaching zero.

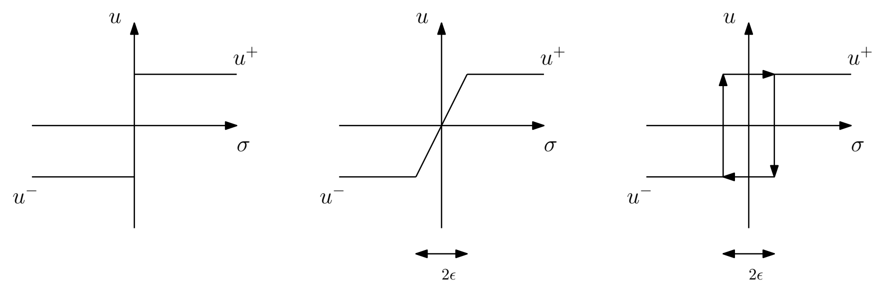

Figure 1.20: Ideal and nonideal switching controllers.

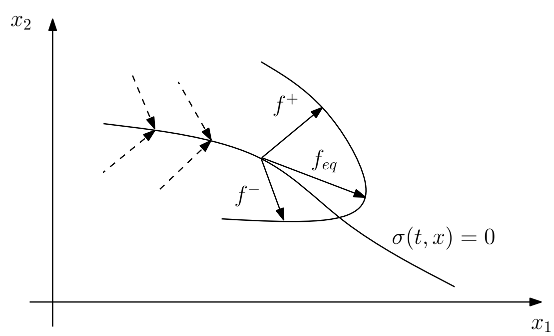

Figure 1.21: Equivalent control.

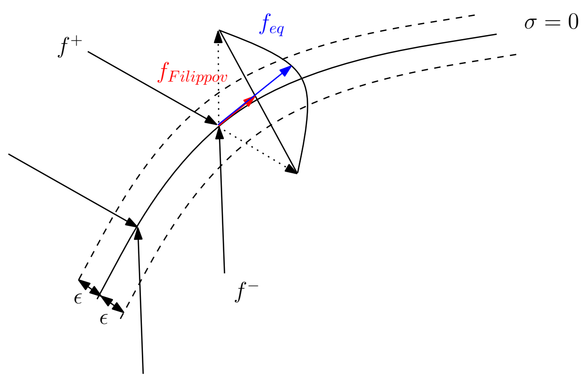

Figure 1.22: Filippov and equivalent control demonstration.

Lyapunov theory is applied in order to investigate convergence of the reaching phase and stability of the sliding phase. First, a positive definite Lyapunov candidate $V(\sigma)$ is introduced and having the total time derivative of $V(\sigma)$ negative definite, guarantees the asymptotic convergence of the reaching phase. However, for the case of sliding mode control, we want the trajectory to converge in finite time so we need to establish $\frac{d}{dt}V(\sigma) \leq \mathcal{G}(V)$ where $|\mathcal{G}(V)| \leq CV^{\alpha}$ and $0 < \alpha < 1$. For the sliding phase to converge to zero, eigenvalues of the Jacobian of the system at the steady-state region need to have negative real parts (for instance by using Routh-Hurwitz stability criterion for the linear sliding manifold).

In the next section the basics of Lyapunov theory are explained briefly.

1.7 Stability Conditions and Lyapunov Theory

In this research, the Lyapunov techniques are applied to investigate the convergence of the designed controller/observer. In this section, the basics of Lyapunov theory are explained without examining the details, for further information on Lyapunov theory refer to (Khalil, 2002).

Let's start with an ordinary differential equation that satisfies the convergence and uniqueness condition (Lipschitz condition)

$$\dot{x} = f(t, x), \quad x(t_0) = x_0, \quad x \in \mathbb{R}^n \tag{1.57}$$

A point $x^* \in \mathbb{R}^n$ is an **equilibrium point** of (1.57) if $f(t, x^*) \equiv 0$. The trajectory x^* is called **stable in Lyapunov sense**, Figure 1.23, if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } ||x(t_0) - x_0|| < \delta \Rightarrow ||x(t) - x^*(t)|| < \epsilon. \tag{1.58}$$

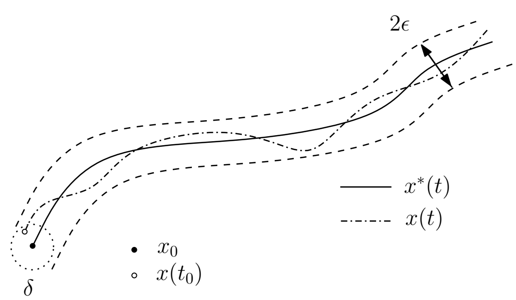

Figure 1.23: Stability definition in Lyapunov sense.

The equilibrium $x^*(t)$ is called **asymptotically stable** if it is stable and

$$\lim_{t \to \infty} ||x(t) - x^*(t)|| = 0. \tag{1.59}$$

In the case of linear systems, stability and asymptotic stability are the same. An equilibrium is **unstable** if it is not stable. Note that asymptotic stability does not quantify the rate of convergence. For that, the **exponential stability** is defined as if there exist constants $m, \alpha > 0$ and $\epsilon > 0$ such that

$$||x(t)|| \leq me^{-\alpha(t-t_0)}||x(t_0)|| \tag{1.60}$$

for all $||x(t_0)|| \leq \epsilon$ and $t \geq t_0$. α is called the **rate of convergence** (Murray et al., 1994). Any of the above definitions are called global if there is no limitation on the location of $x(t_0)$.

Lypunov's direct method or the **second method of Lyapunov** is a technique to determine the stability of a system $\dot{x} = f(x)$ without explicitly solving the differential equation. First we need to introduce a **Lyapunov candidate** $V(x)$ where $x \in \mathbb{R}^n$ and $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(x)$ is differentiable for $x \neq x^*$ and $V(x) > 0$ if $x \neq x^*$ and $V(x^*) = 0$ (V is positive definite). $V(x)$ is some measure of energy in the system, therefore studying the rate of energy change gives us ideas about the behavior of the system. The Lyapunov candidate is called a **Lyapunov function** if the derivative of V in the direction of f is not positive

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}f(x) = L_f V \leq 0. \tag{1.61}$$

If for x^* in (1.57), there is a Lyapunov function $\dot{V}(x) \leq 0$ for some vicinity of x^*, then x^* is **stable**. If $\dot{V}(x) < 0, x \neq x^*$ then x^* is **asymptotically stable**. If x^* is asymptotically stable and $V(x)$ is radially bounded ($V(x) \to \infty$ if $||x|| \to \infty$), then x^* is **globally asymptotically stable** equilibrium. If $\dot{V}(x) < CV^\alpha, 0 < \alpha < 1$ the equilibrium of the system is **exponentially stable**.

The **indirect method of Lyapunov** uses the linearization of a system to determine the local stability of the original system. Let $x^* = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(x), \tag{1.62}$$

having

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x^*}, \tag{1.63}$$

then the origin is asymptotically stable if $Re\lambda_i < 0$ where λ_i are eigenvalues of A and it is unstable if $Re\lambda_i > 0$ for at least one λ.

1.8 State Observers for Distributed Parameter Systems, Literature Review

In this section the different methods exercised to design observer for distributed parameter systems are investigated.

Distributed parameter systems (DPSs), or infinite-dimensional systems, are systems described by partial differential equations. In contrast with distributed parameter systems, there are lumped parameter systems which are described by ordinary differential equations.

Hidayat et al. (2011) provided a survey on designing observers for linear distributed parameter systems. Their research investigated early-lumping, late-lumping, and adaptive methods to design the observer for the first-order PDEs, in addition, it examined the second-order distributed parameter systems. In general, the application of early- or late-lumping methods might possibly lead to the loss of some properties of the system and therefore unsatisfactory results. Demetriou (2004) presented a natural second-order observer for second-order distributed parameter systems, a parameter dependent Lyapunov function used to show the asymptotic convergence. Demetriou & Rosen (2005) suggested an unknown input observer for a class of infinite-dimensional systems. The idea was to decouple the disturbances from the observer and it guaranteed exponential convergence of the state observation to zero. Bitzer & Zeitz (2002) designed a nonlinear observer using a late-lumping approach to estimate the temperature and pressure profile of an oxygen production plant. The observer design procedure was based on the physical and dynamical interpretations of the correction function. The correction function was constructed based on the difference between the measured and estimated values, and their connection with the equation of observer. Vries et al. (2007) designed a Luenberger-type observer for a model of a UV disinfection process with boundary inputs and boundary outputs. In Pourkargar & Armaou (2013) an output feedback control was designed for distributed parameter systems with limited number of sensors. The controller design combined a robust state controller with a dynamic observer of the states of the system. Using the method of weighted residuals, the PDE was approximated into a system of ODEs and the principles of Luenberger observer were corporated in order to design the observer. An infinite-dimensional observer-based controller for partial differential systems was developed in Gahlawat & Peet (2011). The one-dimensional heat equation was expressed as an ODE in the Hilbert space by sum-of-squares method. As can be seen, in the aforementioned researches, Luenberger type of observers had been implemented. In Nguyen (2008), a second-order observer for the second-order DPSs using output injection terms was proposed, the observer was exponentially stable, and different cases for the damping coefficient were investigated. Without going into the details, for the dynamical system and measurements as

$$\rho w_{tt} + \mathcal{C} w_t + \mathcal{K} w = \mathcal{B} u, \quad (t, x) \in \mathbb{R}^+ \times \Omega \tag{1.64}$$

$$y_i(t, x) = w_t \xi_i(x) \tag{1.65}$$

where $\xi_i : \Omega_i \to \mathbb{R}^+$ are given smooth distribution functions. The observer, with $H_j > 0$ as the observer gain, will be

$$\rho \hat{w}_{tt} = -\mathcal{C} \hat{w}_t - \mathcal{K} \hat{w} + \mathcal{B} u - \sum_{i=1}^{N} H_i (\hat{w}_t \xi_i - y_i). \tag{1.66}$$

Xu & Schuster (2009) examined the stabilization problem of an unstable parabolic partial differential equation with constant diffusion coefficient using Sturm-Liouville theory and numerical spectral analysis of differential operators. Designing a state observer based on a boundary measurement was also considered. Meglio et al. (2013) and Smyshlyaev & Krstic (2005), considered backstepping observer design for a class of linear first-order hyperbolic and a class of parabolic PDEs. Backstepping is a robust extension of the feedback linearization approach for nonlinear finite-dimensional systems (Krstic & Smyshlyaev, 2008). In Krstic et al. (2007) and Krstic et al. (2011) Schrödinger equation was considered as heat equation with imaginary diffusion coefficient and backstepping method was utilized to design the observer. In backstepping method the nonlinearity does not necessarily get canceled, however, it might be kept if it is useful or might be dominated if it is potentially uncertain and harmful. In this method an invertible change of variables is used such that the system appears linear in the new variables; except for a nonlinearity, which is in the span of the control input vector. For example in the case of the unstable reaction-diffusion equation

$$u_t = u_{xx} + \lambda u \tag{1.67}$$

$$u(t,0) = 0, \quad u(t,1) = U(t) = control \tag{1.68}$$

since term λu is the source of instability, the natural objective for a boundary feedback is to eliminate this term. The following state transformation can be applied

$$w(t,x) = u(t,x) - \int_0^x k(x,y)u(t,y)dy, \tag{1.69}$$

with the feedback control

$$u(t,1) = \int_0^1 k(1,y)u(t,y)dy, \tag{1.70}$$

to have the target system in the form of

$$w_t = w_{xx} \tag{1.71}$$

$$w(t,0) = 0, \quad w(1,t) = 0. \tag{1.72}$$

The goal will be finding the gain kernel $k(x,y)$, which makes the plant (1.67)-(1.68) with the controller (1.70) equivalnet to the target system (1.71)-(1.72). This is done using the Volterra integral transformation of (1.69), for further details refer to (Krstic & Smyshlyaev, 2008).

1.9 Sliding Mode Observers for DPSs, Literature Review

In this section the history of the sliding mode observer is briefly presented and some researches that employed sliding method in designing observer for distributed parameter systems are introduced. In addition, the advantage of the current work to other researches is explained.

Sliding mode observer follows the same ideas as the sliding mode control. Figure 1.24 shows the schematic diagram of the sliding mode observer for a linear system

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad x \in \mathbb{R}^n, \quad y \in \mathbb{R}^m \tag{1.73}$$

$$\dot{\hat{x}} = A\hat{x} + Bu + L\text{sign}(y - \hat{y}), \quad \hat{y} = C\hat{x}. \tag{1.74}$$

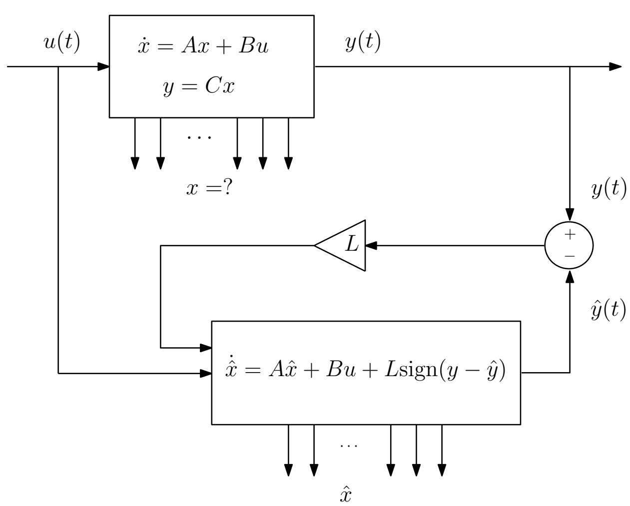

Figure 1.24: Sliding mode observer for a linear system.

In the sliding mode observer the difference between the outputs of the system and the observer in Luenberger observer is replaced by a discontinuous function of the difference. Utkin and Drakunov introduced sliding mode observer for linear systems (Drakunov, 1983) (Drakunov & Utkin, 1995). Later on, Drakunov (1992) developed sliding mode observer for nonlinear systems, that was a research goal of many control theorists for a long time (Krener & Respondek, 1985), (Walcott et al., 1987), (Misawa & Hedrick, 1989) and (Slotine et al., 1987).

Edwards et al. (2000) presented a sliding mode observer for linear system including certain faults. The equivalent output injection concept was obtained to explicitly reconstruct fault signals. This research continued for the linear uncertain systems and developed for the nonlinear case in Spurgeon (2008). Efe et al. (2005) proposed a reduced order and infinite dimensional forms of observers for viscous Burgers' equation. Efe (2008) suggested a finite-dimensional sliding mode observer for a second-order PDE, heat equation, which undergoes an order reduction into a lumped system. Sliding mode theory and backstepping method were practiced in Miranda et al. (2010) to design an observer with a finite time convergence for a class of parabolic PDEs. The output's error injection functions were designed by employing a backstepping procedure introduced by Smyshlyaev & Krstic (2005). Orlov (2000b) presented a model reference adaptive control for distributed parameter systems described by second-order partial differential equations of parabolic and hyperbolic types. In the design process of the controller, a sliding mode-based state derivative observer was constructed which estimated the derivative of the spatial variable.

Drakunov & Barbieri (1997) examined designing sliding mode control complications for a PDE, which includes diffusion with multidimensional spatial variable. The standard technique of separation of variables was employed in the research and the problem of the special case of a diagonal system matrix was solved. Barbieri et al. (2000) expanded the result from the previous research and suggested sliding mode controller and observer for a specific class of distributed parameter systems, heat equation for a robotic arc-welding application, that was written in the Jordan canonical form. The manifold design was based on the desired closed-loop characteristics polynomial evaluated at the known open-loop eigenvalues, developed by Ackermann & Utkin (1998). The transformation examined in chapter 2 of the at hand research, can be used for nonlinear partial differential equations in contrast with the suggestion in Barbieri et al. (2000) .

Chapter 2

SMO for DPS, Sliding Manifold Design, Formula for Observer Gain

2.1 Introduction

In this chapter designing nonlinear observer, and developing formula for the observer gain for a specific class of distributed parameter systems are discussed (Kamran & Drakunov, 2015). The technique suggested in this chapter can be used for hybrid systems, such as systems including the observer dynamics.

The chapter is organized as follows. A general representation of the distributed parameter system is provided in section 2.2. Using the separation of variables the spatial (orthonormal basis) and time (modes) components of the state are separated, and considering the properties of the operator, we end up with a system in the form of ordinary differential equation. Sliding mode observer is developed for the system of ODEs in finite-dimensional space in section 2.4. In the design process, the system in diagonal form is transformed into a new format with the state matrix in the controllable canonical form. Using the freedom provided by the defined transformation, we are able to design the observer based on the desired polynomial coefficients. In addition, a novel formula for the observer gain is developed based on the properties of the Vandermonde matrix. The distributed parameter observer is formulated in section 2.5. The technique is simulated for diffusion equation in section 2.6. The chapter ends at section 2.7 with conclusion and suggestions for future research.

2.2 Problem Statement

The distributed parameter system that is in our interest belongs to the class of systems governed by the following partial differential equation

$$\frac{\partial Q(t,x)}{\partial t} = \mathcal{A}Q(t,x) + \mathcal{B}u(t) \tag{2.1}$$

where $Q(t,x)$ is the state, $t \geq 0$ is time, and $x \in \mathbb{R}^p$ is the spatial variable. For fixed t and x: $Q \in \mathbb{R}^N$. We assume $x \in \Omega$ where $\Omega \subset \mathbb{R}^p$ is a one-component domain in p-dimensional space with a smooth, C^1, boundary $\partial\Omega$. $\mathcal{A}$ is a closed, linear differential operator which is a infinitesimal generator of an exponentially stable semigroup $T_{\mathcal{A}}(t)$ on $\mathcal{H}_1$, $\mathcal{H}_1 = L_2(\Omega)$ is a Hilbert space (Russell, 2010), (Orlov, 2000a), (Curtain & Zwart, 1995).

By assumption, operator $\mathcal{A}$ has all distinct eigenvalues $\lambda_i \in \mathbb{C}$

$$\mathcal{A}\phi_j(x) = \lambda_j\phi_j(x). \tag{2.2}$$

For instance $\mathcal{A}$ could be a linear differential operator of the form

$$\mathcal{A} = A^{(0)}(x) + \sum_{\nu=1}^{\tilde{N}} \sum_{i_1,\cdots,i_\nu=1}^{p} A^{(\nu)}_{i_1,i_2,\ldots,i_\nu}(x) \frac{\partial^\nu}{\partial x_{i_1} \ldots \partial x_{i_\nu}}, \tag{2.3}$$

with corresponding boundary condition, where $A^{(0)}(x)$ and $A^{(k)}_{i_1,i_2,\ldots,i_\nu}(x)$ are $N \times N$ matrix-valued $C^1(\Omega)$ functions of x.

If $\mathcal{A}$ is a self-adjoint operator, then all the eigenvalues $\lambda_j \in \mathbb{R}$ are real and the eigenvectors $\phi_j(x) \in \mathcal{H}_1$ correspond to distinct eigenvalues are orthogonal (Hanson & Yakovlev, 2002).

$\mathcal{B}$ maps the space of the controls into the state space $\mathcal{B} \in L(\mathcal{H}_2, \mathcal{H}_1)$ [1] (Glowinski et al., 2008). Here $\mathcal{B} = B(x)$ is considered. $B(x)$ belongs to the class $C^1(\Omega)$ of matrix-valued functions of appropriate dimensions. The process is controlled by a finite number of inputs, the control is finite-dimensional $u \in \mathbb{R}^m$, and it is a function of time but not the spatial variable, $u(t)$.

In order to define the solution of (2.1) uniquely, one needs to specify a set of boundary conditions on the boundary $\partial\Omega$ of the domain Ω in addition to appropriate initial conditions. Our development does not require specific form of these boundary conditions, the only assumption we will make is that the corresponding solution of boundary value problem is unique and well-posed, which is satisfied for many important cases. For instance, if the differential operator has the second-order spatial derivative, we consider the following general type of homogeneous boundary

[1] For Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, L(\mathcal{H}_2, \mathcal{H}_1)$ denotes the Hilbert space of bounded linear operators from $\mathcal{H}_2$ to $\mathcal{H}_1$.

conditions

$$\nu_0(x)Q(t,x) + \nu_1(x)\,\frac{\partial Q}{\partial \bar{n}}(t,x)\bigg|_{x\in\partial\Omega} = 0, \tag{2.4}$$

where the matrix-valued functions of appropriate dimensions $\nu_0(x)$ and $\nu_1(x)$ are defined on $\partial\Omega$ and belong to the class $C^1(\partial\Omega)$ with respect to the spatial variables and $\bar{n}$ is a normal vector to $\partial\Omega$. The initial condition is

$$Q(0,x) = Q_0(x), \tag{2.5}$$

where $Q_0(x) \in C^1(\Omega)$. If $Q_0(x) \in L_0(\Omega)$ then for any $u(t) \in L_0[0,T], \forall T > 0$. The problem (2.1), (2.4) and (2.5) is known to be well-posed, having a unique generalized solution $Q(t,x)$ (Drakunov & Reyhanoglu, 2010).

The output is a scalar variable $y(t) \in \mathbb{R}$, it is assumed to be measurable and it is a linear functional of the state of the system represented as

$$y(t) = \int_\Omega c^T(x)Q(t,x)dx, \tag{2.6}$$

where $c(x) \in L_2(\Omega, \mathbb{R}^N)$.

Note that the operator $\mathcal{A}$, the control gain $B(x)$ and $c(x)$ have to satisfy the boundary conditions corresponded to the state $Q(t,x)$, and they need to be twice diffrentiable with respect to the spatial variable.

2.3 Separation of Variables

Our goal is to design an observer for estimation of $Q(t,x)$ from data provided by $y(t)$. Using the standard technique of separation of variables we have

$$Q(t,x) = \sum_{k=1}^{\infty} z_k(t)\varphi_k(x), \quad B(x) = \sum_{k=1}^{\infty} b_k\varphi_k(x), \quad c(x) = \sum_{k=1}^{\infty} c_k\varphi_k(x), \tag{2.7}$$

where $z_k(t)$ is scalar function of time known as mode and $\varphi_k(x)$ is orthonormal basis on spatial variable. Equations (2.7) converge in $L_2(\Omega)$ for any $t \geq 0$.

As an example, $\mathcal{A}$ can be the Strum-Liouville operator

$$\begin{aligned} \mathcal{A} &= r(x) + \frac{\partial}{\partial x}\left(p(x)\frac{\partial}{\partial x}\right) \\ &= r(x) + s(x)\frac{\partial}{\partial x} + p(x)\frac{\partial^2}{\partial x^2}, \quad s(x) = \frac{\partial p(x)}{\partial x}, \end{aligned} \tag{2.8}$$

where $r(x), p(x) > 0$ and $r(x), p(x), s(x) \in C^0(x)$, along with the homogeneous boundary condition similar to (2.4).

Applying separation of variables on (2.1) under the assumptions is section 2.2 and using (2.7) we have

$$\sum_{k=1}^{\infty} \dot{z}_k(t)\varphi_k(x) = \sum_{k=1}^{\infty} \left[\lambda_k z_k(t) + b_k u(t)\right]\varphi_k(x), \tag{2.9}$$

The relation (2.9) must be true for every $\varphi_k(x)$ so

$$\dot{z}_k(t) = \lambda_k z_k(t) + b_k u(t), \quad k = 1, 2, \cdots . \tag{2.10}$$

In the same way the output (2.6) is written as

$$\begin{aligned} y(t) &= \int_0^l \sum_{k=1}^{\infty} c_k \varphi_k(x) \sum_{m=1}^{\infty} z_m(t) \varphi_m(x) dx \\ &= \sum_{k,m}^{\infty} c_k z_m(t) \int_0^l \varphi_k(x) \varphi_m(x) dx \\ &= \sum_{k=1}^{\infty} c_k z_k(t). \end{aligned} \tag{2.11}$$

Let $\phi_k(x)$ be (possibly complex valued) normalized eigenvectors ($\|\phi_k\| = 1$) in $L_2(\Omega, \mathbb{R}^N)$ and λ_k denote the corresponding eigenvalues of the associated boundary value problem.

Remark: The described class of systems cover two important cases of DPSs: the diffusion equation

$$\frac{\partial Q(t,x)}{\partial t} = a \frac{\partial^2 Q(t,x)}{\partial x^2} + b(x)u,$$

and the wave equation

$$\frac{\partial^2 \xi(t,x)}{\partial t^2} = a \frac{\partial^2 \xi(t,x)}{\partial x^2} + b(x)u. \tag{2.12}$$

The operator $\mathcal{A}$ for the wave equation, can be represented in the form of (2.3) by defining

$$Q(t,x) = \begin{bmatrix} Q_1(t,x) \\ Q_2(t,x) \end{bmatrix} = \begin{bmatrix} \xi(t,x) \\ \frac{\partial \xi(t,x)}{\partial t} \end{bmatrix} \tag{2.13}$$

as

$$\mathcal{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix} \frac{\partial^2}{\partial x^2}. \tag{2.14}$$

Our class of models allows to consider systems which are combination of a distributed parameter system described by PDE and a linear finite-dimensional sensor dynamics. For instance considering the wave equation given by (2.12) and assuming the variable y is observed that satisfies

$$\dot{y}(t) = a_0 y(t) + d_0 z(t), \tag{2.15}$$

where

$$z(t) = \int_\Omega c^T(x)\xi(t,x)dx. \tag{2.16}$$

The system in (2.12), (2.15) and (2.16) can be represented as (2.1). By introducing the variable $\eta(t,x)$ satisfying

$$\frac{\partial \eta(t,x)}{\partial t} = a_0 \eta(t,x) + d_0 \xi(t,x), \tag{2.17}$$

the state Q of the combined process-sensor system can be chosen as

$$Q(t,x) = \begin{bmatrix} Q_1(t,x) \\ Q_2(t,x) \\ Q_3(t,x) \end{bmatrix} = \begin{bmatrix} \eta(t,x) \\ \xi(t,x) \\ \frac{\partial \xi(t,x)}{\partial t} \end{bmatrix}. \tag{2.18}$$

The operator $\mathcal{A}$ is

$$\mathcal{A} = \begin{bmatrix} a_0 & d_0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a & 0 \end{bmatrix} \frac{\partial^2}{\partial x^2}.$$

The output equation is

$$y(t) = \int_{\Omega} C^T(x) Q(t,x) dx,$$

where

$$C = \begin{bmatrix} c(x) & 0 & 0 \end{bmatrix}^T. \tag{2.19}$$

Similar representation can be obtained for the systems with multidimensional sensor dynamics (Drakunov & Reyhanoglu, 2010).

Our goal is to design an observer to estimate $Q(t,x)$ employing the observation $y(t)$. Using the separation of variables, the original system (2.1) and the output (2.6) are replaced by the ordinary differential equation (2.10) along with the observation (2.11). In the matrix representation we have

$$\dot{Z} = \Lambda Z + bu(t), \tag{2.20}$$

where

$$Z = \begin{bmatrix} z_1 & z_2 & \cdots \end{bmatrix}^T, \quad \Lambda = diag\{\lambda_1, \lambda_2, \cdots\}, \quad b = \begin{bmatrix} b_1 & b_2 & \cdots \end{bmatrix}^T,$$

and

$$y = c^T Z, \quad c = \begin{bmatrix} c_1 & c_2 & \cdots \end{bmatrix}^T. \tag{2.21}$$

To demonstrate our technique, at this point we assume $b_k = 0$ for $k = n+1, n+2, \cdots$, and only a finite number of modes are excited $z_k(0) = 0, k = n+1, n+2, \cdots$. These assumptions are not necessary for the actual proof and they are used to show the method in a clear way. So for the system $k = 1, \cdots, n$ we have

$$\dot{Z}(t) = \Lambda Z(t) + bu(t) \tag{2.22}$$

where $u(t) \in \mathbb{R}$ and

$$Z(t) = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix}^T, \quad b = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T, \quad \Lambda = diag\{\lambda_1, \cdots, \lambda_n\}.$$

The output (2.21) will be

$$y(t) = c^T Z(t), \quad c = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}^T. \tag{2.23}$$

2.4 Observer Design

Let us introduce the sliding mode observer for the system (2.22) as

$$\dot{\widehat{Z}}(t) = \Lambda \widehat{Z}(t) + bu(t) + L\text{sign}(y - \hat{y}), \tag{2.24}$$

where $L = diag\{L_1, \cdots, L_n\}$ and $\hat{y} = c^T \widehat{Z}(t)$. The goal is to design the gain matrix $L \in C^n$ such that $\widehat{Z} \to Z$ as $t \to \infty$. Obviously there are many Ls that guarantee the convergence of the finite dimensional sliding observer, however we are specifically looking for the one that leads to convergence when $n \to \infty$.

In order to introduce the freedom in designing manifold for the sliding mode observer as well as developing some formulae for the observer gain, the system in diagonal form (2.22) is transformed into a system with the state matrix in controllable canonical form, by defining the following transformation

$$X = V\beta^{-1}Z, \tag{2.25}$$

where V is the Vandermonde matrix, using the λ from matrix Λ, as

$$V(\lambda_1, \cdots, \lambda_n) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix},$$

and β is a diagonal matrix with free parameters $\beta_1, \cdots, \beta_n$

$$\beta = diag\{\beta_1, \cdots, \beta_n\}. \tag{2.26}$$

Applying the transformation (2.25) on (2.22) and (2.23) we have

$$\dot{X} = AX + \tilde{b}u, \tag{2.27}$$

and

$$y(t) = \tilde{c}^T X, \tag{2.28}$$

where

$$A = V\beta^{-1}\Lambda\beta V^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -a_1 & -a_2 & -a_3 & \cdots & -a_n \end{bmatrix},$$

and

$$\tilde{b} = V\beta^{-1}b = \begin{bmatrix} \tilde{b}_1 & \tilde{b}_2 & \cdots & \tilde{b}_n \end{bmatrix}^T, \tag{2.29}$$

$$\tilde{c}^T = c^T\beta V^{-1} = \begin{bmatrix} \tilde{c}_1 & \tilde{c}_2 & \cdots & \tilde{c}_n \end{bmatrix}. \tag{2.30}$$

Let's design observer for the system (2.27) as follows

$$\dot{\widehat{X}} \quad = \quad A\widehat{X} + \tilde{b}u + L_0 e_n \text{sign}\sigma, \tag{2.31}$$

where L_0 is a scalar, $e_n = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}^T$ and

$$\sigma = y - \hat{y} = \tilde{c}\overline{X}, \quad \overline{X} = X - \widehat{X}. \tag{2.32}$$

Writing the observer (2.31) in the estimation error $\overline{X}$ we have

$$\dot{\overline{X}} = A\overline{X} - L_0 e_n \text{sign}\sigma. \tag{2.33}$$

Equation (2.33) can be written as

$$
\begin{aligned}
\dot{\bar{x}}_1 &= \bar{x}_2 \\
\dot{\bar{x}}_2 &= \bar{x}_3 \\
&\vdots \\
\dot{\bar{x}}_{n-1} &= \bar{x}_n \\
\dot{\bar{x}}_n &= -\sum_{k=1}^{n} a_k \bar{x}_k - L_0 \mathrm{sign}\sigma.
\end{aligned} \tag{2.34}
$$

For sufficiently large L_0 sliding mode exists on the manifold $\sigma = 0$ in (2.34) (Utkin, 1978). Setting sliding surface equal to zero $\sigma = 0$ we end up with

$$\bar{x}_n = -\bar{c}_{n-1}\bar{x}_{n-1} - \cdots - \bar{c}_1\bar{x}_1, \tag{2.35}$$

where

$$\bar{c}_j = \frac{\tilde{c}_j}{\tilde{c}_n}, \quad j = 1, \cdots, n-1. \tag{2.36}$$

Substituting (2.35) into (2.34) we have the following reduced order system

$$
\begin{aligned}
\dot{\bar{x}}_1 &= \bar{x}_2 \\
\dot{\bar{x}}_2 &= \bar{x}_3 \\
&\vdots \\
\dot{\bar{x}}_{n-1} &= -\bar{c}_{n-1}\bar{x}_{n-1} - \cdots - \bar{c}_1\bar{x}_1
\end{aligned} \tag{2.37}
$$

or in the compact form

$$\dot{\overline{X}}_{red.order} = \bar{A}\overline{X}_{red.order}, \quad \overline{X}_{red.order} \in \mathbb{R}^{n-1}, \tag{2.38}$$

where

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -\bar{c}_1 & -\bar{c}_2 & -\bar{c}_3 & \cdots & -\bar{c}_{n-1} \end{bmatrix}. \tag{2.39}$$

The reduced order system (2.38) needs to be stabilized. Assigning the desired roots as $\mu_1, \cdots, \mu_{n-1}$ the desired polynomial is

$$\begin{aligned} P_{des.}(\lambda) &= (\lambda - \mu_1) \cdots (\lambda - \mu_{n-1}) & (2.40) \\ &= \lambda^{n-1} + \bar{c}^d_{n-1}\lambda^{n-2} + \cdots + \bar{c}^d_2\lambda + \bar{c}^d_1, & (2.41) \end{aligned}$$

and in the form of desired matrix

$$\bar{A}^d = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ -\bar{c}^d_1 & -\bar{c}^d_2 & -\bar{c}^d_3 & \cdots & -\bar{c}^d_{n-1} \end{bmatrix}. \tag{2.42}$$

In order to reach a stable system, (2.39) is set to be equal to (2.42). $\tilde{c}^T$ from (2.30) can be written in the new form as

$$\tilde{c}^T = \beta_{row} c_{diag} V^{-1}, \tag{2.43}$$

where

$$\beta_{row} = \begin{bmatrix} \beta_1 & \cdots & \beta_n \end{bmatrix}, \quad c_{diag} = diag\{c_1, \cdots, c_n\}.$$

Solving (2.43) for β and using relation (2.36) we have

$$\beta_{row} = \tilde{c}_n \bar{A}^d_{row} V c^{-1}_{diag}, \tag{2.44}$$

where

$$\bar{A}^d_{row} = \begin{bmatrix} \bar{c}^d_1 & \cdots & \bar{c}^d_{n-1} & 1 \end{bmatrix}. \tag{2.45}$$

For the elements of β we have

$$\beta_k = \frac{\tilde{c}_n}{c_k} \left(\bar{c}^d_1 + \bar{c}^d_2 \lambda_k + \bar{c}^d_3 \lambda^2_k + \cdots + \bar{c}^d_{n-1} \lambda^{n-2}_k + \lambda^{n-1}_k \right). \tag{2.46}$$

By comparing (2.46) and the desired polynomial (2.41) we have

$$\beta_k = \tilde{c}_n \frac{P_{des.}(\lambda_k)}{c_k}. \tag{2.47}$$

The observer gain for the system with diagonal representation (2.22), is found by comparing (2.31) and (2.24)

$$L \;=\; L_0 \beta V^{-1} e_n = \begin{bmatrix} L_1 & L_2 & \cdots & L_n \end{bmatrix}^T. \tag{2.48}$$

Note that $V^{-1} e_n$ represents the last column of the matrix V^{-1}, so the elements of the observer gain will be

$$L_k = L_0 \beta_k V^{-1}(k,n), \quad k = 1, \cdots, n. \tag{2.49}$$

where $V^{-1}(k,n)$ is the k-th row in the last column (n) of the inverse of the Vandermonde matrix. The k-th row of the last column of V^{-1} is

$$V^{-1}(k,n) = \frac{adjV(k,n)}{|V|} = \frac{(-1)^{k-1} \prod_{\substack{1 \le i < j \le n \\ i,j \ne k}} (\lambda_i - \lambda_j)}{\prod_{1 \le i < j \le n} (\lambda_j - \lambda_i)} = \frac{(-1)^{k-1}}{\prod_{\substack{i=1,\cdots,n. \\ i \ne k}} (\lambda_k - \lambda_i)}. \tag{2.50}$$

Using (2.47), (2.49) and (2.50) the k-th element of the matrix L will be

$$L_k = \tilde{L}_0 \frac{(-1)^{k-1}}{c_k} \frac{P_{des.}(\lambda_k)}{\prod_{\substack{i=1,\cdots,n. \\ i \ne k}} (\lambda_k - \lambda_i)}, \tag{2.51}$$

where $\tilde{L}_0 = L_0 \tilde{c}_n$.

In order to develop a more straightforward formula for the observer gain, let us assign the desired roots in (2.40) as follows

$$\begin{aligned} \mu_k &= \lambda_{m+1}, \quad \text{for} \quad k = 1, 2, \cdots, m. \\ \mu_{m+1} &= \lambda_{m+2} \\ &\vdots \\ \mu_{n-1} &= \lambda_n. \end{aligned}$$

The desired polynomial will be

$$\begin{aligned} P_{des.}(\lambda) &= (\lambda-\mu_1)\cdots(\lambda-\mu_m)(\lambda-\mu_{m+1})\cdots(\lambda-\mu_{n-1}) \\ &= (\lambda-\lambda_{m+1})^m(\lambda-\lambda_{m+2})\cdots(\lambda-\lambda_n). \end{aligned} \tag{2.52}$$

By employing the observer gain formula (2.51) and the new desired polynomial (2.52), for $k = 1, 2, \cdots, m$ the elements of observer gain will be

$$L_k = \tilde{L}_0 \frac{(-1)^{k-1}}{c_k} \frac{(\lambda_k-\lambda_{m+1})^{m-1}}{\prod_{\substack{i=1,\cdots,m. \\ i\neq k}}(\lambda_k-\lambda_i)}. \tag{2.53}$$

For $k = m+1, m+2, \cdots, n$ considering any k, we end up with zero for the desired polynomial (2.52) and as a result zero for the observer gain $L_k = 0$.

In the summary, the following formula represents the observer gain

$$L_k = \tilde{L}_0 \begin{cases} \frac{(-1)^{k-1}}{c_k} \frac{(\lambda_k-\lambda_{m+1})^{m-1}}{\prod_{\substack{i=1,\cdots,m. \\ i\neq k}}(\lambda_k-\lambda_i)}, & k = 1, \cdots, m \\ 0 & k = m+1, \cdots, n. \end{cases} . \tag{2.54}$$

2.5 Observer for Distributed Parameter System

To obtain the observer gain for the distributed parameter system (2.1), the limit of the observer gain for the system in diagonal form (2.54) when $n \to \infty$ is considered

$$\begin{aligned} L(x) &= \tilde{L}_0 \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{(-1)^{k-1}}{c_k} \frac{(\lambda_k-\lambda_{m+1})^{m-1}}{\prod_{\substack{i=1,\cdots,m. \\ i\neq k}}(\lambda_k-\lambda_i)} \varphi_k(x) \right) \\ &= \tilde{L}_0 \sum_{k=1}^{m} \left(\frac{(-1)^{k-1}}{c_k} \frac{(\lambda_k-\lambda_{m+1})^{m-1}}{\prod_{\substack{i=1,\cdots,m. \\ i\neq k}}(\lambda_k-\lambda_i)} \varphi_k(x) \right). \end{aligned} \tag{2.55}$$

Based on the assumptions made in the previous section, we end up with a finite sum for the observer gain. Finally, the observer for the original distributed parameter system will be

$$\frac{\partial \widehat{Q}}{\partial t} = \mathcal{A}\widehat{Q} + B(x)u(t) + L(x)\text{sign}\left(y(t) - \int_0^l c^T(x)\widehat{Q}(t,x)dx\right). \quad (2.56)$$

2.6 Diffusion Equation and Simulation Results

Here a one-dimensional diffusion equation with homogeneous boundary condition is considered, the differential operator is $\mathcal{A} = \frac{\partial^2}{\partial x^2}$

$$\frac{\partial Q(t,x)}{\partial t} = \frac{\partial^2 Q(t,x)}{\partial x^2} + b(x)u, \quad (2.57)$$

where $0 \leq x \leq l, t \geq 0$, the diffusivity is assumed to be equal to one. Let us consider Dirichlet boundary conditions

$$Q(t,0) = Q(t,l) = 0, \quad (2.58)$$

and the initial condition as

$$Q(0,x) = Q_0(x).$$

Applying separation of variables technique, we end up with the following ordinary differential equation

$$\dot{Z}(t) = \Lambda Z(t) + bu(t), \quad (2.59)$$

where

$$Z(t) = \begin{bmatrix} z_1 & z_2 & \cdots \end{bmatrix}^T, \quad \Lambda = diag\{-\omega_1^2, -\omega_2^2, \cdots\}, \quad b = \begin{bmatrix} b_1 & b_2 & \cdots \end{bmatrix}^T.$$

The observer for the distributed parameter system will be

$$\frac{\partial \widehat{Q}(t,x)}{\partial t} = \frac{\partial^2 \widehat{Q}(t,x)}{\partial x^2} + b(x)u + L(x)\mathrm{sign}(y - \hat{y}).$$

where the observer gain is designed using the proposed formula. If the roots are evenly spread on the negative part of the real axis: $\lambda_k = -k\omega^2,\ k = 1, 2, \cdots, n$ for the observer gain from (2.55) we have

$$L(x) = \tilde{L}_0 \sum_{k=1}^{m} \left(\frac{(-1)^{k-1}}{c_k} \frac{(k-(m+1))^{m-1}}{(-1)^{m-k}(k-1)!(m-k)!} \varphi_k(x) \right). \tag{2.60}$$

For diffusion equation the orthonormal basis are $\varphi_k(x) = sin(\frac{k\pi}{l}x)$.

For the simulation, the PDE in (2.57) with $b(x) = 0$, Dirichlet boundary conditions in (2.58) and the initial condition as

$$Q(0,x) = \frac{2x}{1+x^2}.$$

are considered. For the observer we have

$$\frac{\partial \widehat{Q}(t,x)}{\partial t} = \frac{\partial^2 \widehat{Q}(t,x)}{\partial x^2} + L(x)\mathrm{sign}(y - \hat{y}),$$

along with the boundary and initial conditions as

$$\widehat{Q}(t,0) = \widehat{Q}(t,l) = 0, \quad \widehat{Q}(0,x) = x.$$

Figure 2.1 shows the behavior of the system over the time and length. Figure 2.2 represents the performance of the observer. Figure 2.3 shows the absolute difference between the state of the system and the observer. Figure 2.4 demonstrates the convergence of the sliding mode over the time. For the MATLAB code refer to A.1.

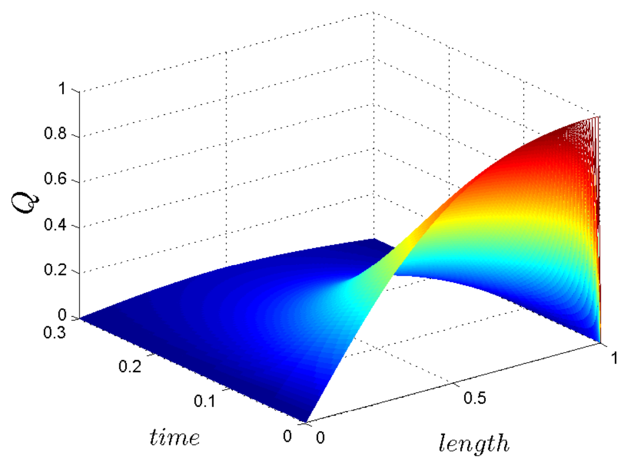

Figure 2.1: Diffusion equation solution.

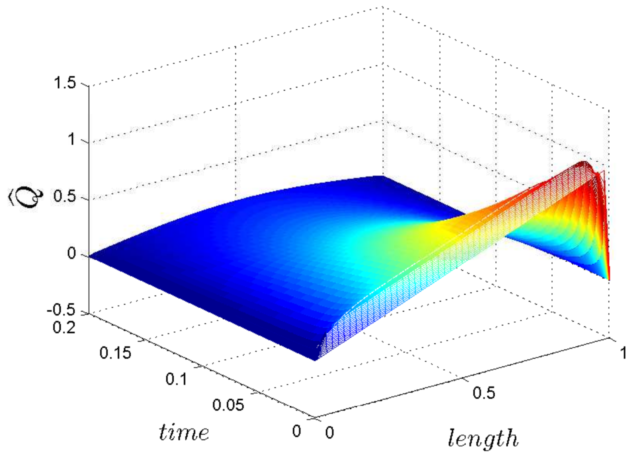

Figure 2.2: Distributed parameter observer for diffusion equation.

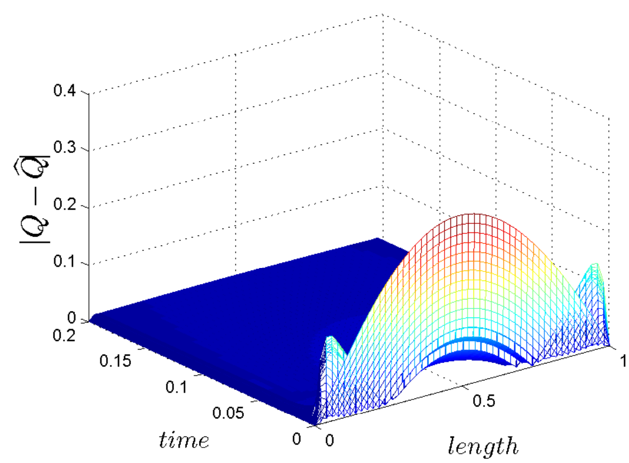

Figure 2.3: Difference between system and observer.

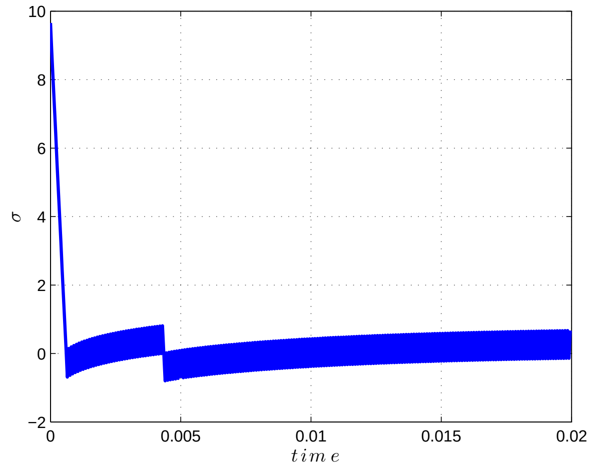

Figure 2.4: Sliding mode convergence.

2.7 Conclusion and Future Work

In this chapter the sliding mode observer for a specific class of distributed parameter systems was designed. The suggested state transformation allows the arbitrary design of the sliding manifold. A formula for the observer gain was obtained that guarantees stability and convergence of the distributed observer to the actual system. The observer and the observer gain design can be extended to be used for hybrid systems employing the same technique. Applying the suggested ideas on different systems such as fluid flows and quantum systems, can be considered in the future work. Another extension of this research will be eliminating the assumptions on the differential operator.

Chapter 3

Background on Fluid Dynamics, SMO for Burgers' Equation

This chapter is devoted to Burgers' equation, the literature review, mathematical background, shock wave situation and its solution. The problem of designing a state observer for Burgers' equation is studied at the end. Note that the mathematical background in this chapter serves as the basic mathematics for chapter 4 as well.

3.1 Burgers' Equation, Literature Review

Stabilizing the unstable shock-liked equilibrium profiles of the viscous Burgers' equation using control at the boundaries was studied in Krstic et al. (2008). In a follow up paper, more advanced problems like trajectory generation, trajectory tracking, nonlinear observer and output feedback stabilization were investigated (Krstic et al., 2009). Two finite element methods were applied on the viscous Burgers' equation in Atwell & King (2000) and a standard LQR controller was employed to optimize the cost function. Sliding mode control of the forced generalized Burgers' equation was considered in Smaoui et al. (2006), Karhunen-Loéve Galekrin method was practiced to decompose the original equation into a set of ODEs that mimics the dynamics of the forced generalized Burgers' equation.

Aubin et al. (2005) investigated the problem of controlling Burgers' equation by employing the general framework of viability theory, and constructed the controlled entropy solutions. The problem of stabilization of the inviscid Burgers' equation using boundary actuation was explored in Blandin et al. (2010). By applying a Lyapunov approach, it was shown that this equation is stabilized around a constant uniform state under appropriate boundary control.

Shock Wave

Shock waves are the result of sudden release of energy in a very small spatial region. The energy released by shock waves can be used in many innovative applications. For further information about shock wave theory and the history behind it refer to Zel'dovich (1967), Rathakrishnan (2006), Krehl (2009) and Salas (2006).

A survey including different topics related to shock wave such as hyperbolic conservation laws, well-posedness theory, shock and radiation-diffusion wave, etc. was presented in Razani (2007). Boundary value problems for Burgers' equation through nonstandard analysis was investigated in Bendaas (2015) and the confluence and interacting shocks were considered. Solovchuk & Sheu (2011) practiced a Mott-Smith distribution function for the Maxwell molecules in order to predict the structure of shock wave in a neutral monatomic gas. The results showed agreement with Monte-Carlo simulation at different Mach numbers. Regulation of an inviscid Burgers' equation using an averaged or low-pass filtered velocity in order to avoid shock wave situation was investigated in Mohseni et al. (2006). Norgard & Mohseni (2008) applied a convectively filtered Burgers' equation in order to model and regulate Burgers' equation. This model is also employed to investigate the shock behavior, shock thickness and kinetic energy decay. In Zhang et al. (2012) the nature of the shock wave in inviscid Burgers' equation was studied and it has been proven that there is a thin spatial zone that a saddle-node bifurcation happens. It was shown that by introducing viscosity the discontinuity resulting from saddle-node bifurcation disappears. Pironneau (2003) examined the sensitivity of the shock wave position with respect to the domain occupied by the fluid. The problem has applications in minimizing the sonic boom of airplanes and the stability of the stream in fast-flowing canals. In Bardos & Pironneau (2003) the solution of Burgers' equation was derived using the weak solution and the initial condition data. In addition in order to control shocks an optimal control was designed. Pironneau (2002) showed how the shock wave position in a nozzle can be controlled using the optimal control theory and the transonic equation. Marchesin & Paes-Leme (1983) considered shocks in gas pipelines. By applying numerical method for the one-dimensional laws of conservation of mass, conservation of momentum and a constitutive equation of state, the authors showed the effects of the Moody friction term in resolving shocks whenever they were present. Marchesin & Plohr (2001) investigated the theory of mixed-type systems of conservation laws with small diffusive terms and the application of the theory to increase the rate of oil recover. They showed that in addition to the classical shock and reflection waves, there are two other features: the first one is a new type of shock wave with intermediate speed and the second is a fast, decaying, oscillatory injection wave. The Saint-Venant equation written in prismatic is practiced to model the flow. The behavior of shock wave propagation of circular dam break problems was investigated in Mungkasi (2014). Three approximate Riemann solver scheme were presented by Zhao et al. (1996) in order to solve two-dimensional shallow water equations for modeling shock waves. In Onizuka & Odai (1998) Burgers' equation was employed as an approximation for Saint-Venant equations to simulate slow transient in wide rectangular open channels of finite length.

3.2 Compressible Fluid Dynamics

The mathematical background starts with deriving the equation for compressible fluid. A single-phase homogeneous fluid is completely described if the velocity $\vec{u}$, any two thermodynamics variables and an equation of state are known (Lomax et al., 2001). In the classical Gibbs axiomatic formulation, the equation of state is

$$e = e(V, s), \tag{3.1}$$

where $V = \frac{1}{\rho}$ is the specific volume, ρ is the density, and s is the specific entropy. The pressure and temperature are defined as

$$p = -\frac{\partial e}{\partial V}, \quad T = \frac{\partial e}{\partial s}. \tag{3.2}$$

Using the above relations the fundamental thermodynamics relation is derived

$$Tds = de + pdV. \tag{3.3}$$

The specific total energy is given by

$$E = e + \frac{1}{2}(\vec{u} \cdot \vec{u}). \tag{3.4}$$

Another important positive quantity is the speed of sound, which is the traveling speed of sound waves in the fluid

$$c^2 = \left.\frac{\partial p}{\partial \rho}\right|_s. \tag{3.5}$$

A thermally perfect gas, ideal gas, is a fluid that obeys

$$p = \rho \mathcal{R} T \tag{3.6}$$

where $\mathcal{R}$ is the gas constant and is defined as the ratio of the universal gas constant to the effective molecular weight of the particular gas.

The Navier-Stokes equations are the differential form of conservation laws and they govern the motion in time for classical fluid. They include, conservation of mass or continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0, \tag{3.7}$$

conservation of momentum

$$\frac{\partial (\rho \vec{u})}{\partial t} + \nabla \cdot (\rho \vec{u} \vec{u}^T) + \nabla p = \nabla \cdot \tau, \tag{3.8}$$

and conservation of energy

$$\frac{\partial(\rho E)}{\partial t} + \nabla \cdot (\rho E \vec{u}) + \nabla \cdot (p\vec{u}) = \nabla \cdot \vec{q} + \nabla \cdot (\tau \vec{u}), \tag{3.9}$$

where τ is the viscous stress tensor, $\vec{q} = -\kappa \nabla T$ is the heat flux and κ shows the thermal conductivity. The Navier-Stokes equations have to be supplemented with an equation of state, for instance the relation for the ideal gas.

When considering the volumetric forces $\vec{f}$, the conservation of momentum and energy become balance laws and we end up with the more general case of the Navier-Stokes equations as follows

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0, \tag{3.10}$$

$$\frac{\partial(\rho \vec{u})}{\partial t} + \nabla \cdot (\rho \vec{u}\vec{u}^T) + \nabla p = \nabla \cdot \tau + \vec{f}, \tag{3.11}$$

$$\frac{\partial(\rho E)}{\partial t} + \nabla \cdot (\rho E \vec{u}) + \nabla \cdot (p\vec{u}) = \nabla \cdot \vec{q} + \nabla \cdot (\tau \vec{u}) + \vec{f} \cdot \vec{u}. \tag{3.12}$$

Compressible Euler equations are the specific case of the Navier-Stokes equations when the Reynolds number $Re \to \infty$ or shear (dynamic) viscosity $\mu \to 0$, as a result all the terms at the right-hand side are vanished

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0, \tag{3.13}$$

$$\frac{\partial(\rho \vec{u})}{\partial t} + \nabla \cdot (\rho \vec{u}\vec{u}^T) + \nabla p = 0, \tag{3.14}$$

$$\frac{\partial(\rho E)}{\partial t} + \nabla \cdot (\rho E \vec{u}) + \nabla \cdot (p\vec{u}) = 0. \tag{3.15}$$

The compressible Euler equations in one dimension can be written as a general hyperbolic conservation law

$$\vec{v}_t + \nabla_x \cdot \vec{f}(\vec{v}) = 0, \tag{3.16}$$

where

$$\vec{v} = (\rho, \rho u, E)^T, \quad \vec{f}(\vec{v}) = \left(\rho u, \rho u^2 + p, (E+p)u\right)^T, \tag{3.17}$$

here ρ is the density, u is the velocity, ρu is the momentum, E is the energy and p is the pressure given as a function of other state variables (Qiu, 2013).

In the following sections some modified models for fluid flow are developed based on the compressible Euler equations and the problem of designing sliding mode observer for them is studied in this and the next chapter.

3.3 Burgers' Equation

In order to derive the Burgers' equation, the one-dimensional ($\vec{u} = u$) form of the first two Euler equations (3.13), (3.14) are employed

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \tag{3.18}$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}(\rho u u + p) = 0. \tag{3.19}$$

Conservation of momentum (3.19) can be written as

$$\rho\frac{\partial u}{\partial t} + u\frac{\partial \rho}{\partial t} + u\frac{\partial}{\partial x}(\rho u) + (\rho u)\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0, \tag{3.20}$$

rearranging (3.20) we have

$$u\frac{\partial\rho}{\partial t} + u\frac{\partial}{\partial x}(\rho u) + \rho\frac{\partial u}{\partial t} + (\rho u)\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0, \tag{3.21}$$

as can be seen in (3.21) the first two terms are the same as the equation for conservation of mass (3.18), therefore they equal to zero and by neglecting the pressure gradient we have

$$\rho\frac{\partial u}{\partial t} + \rho u\frac{\partial u}{\partial x} = 0, \tag{3.22}$$

and finally

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{3.23}$$

which is the one-dimensional Euler equation of motion or **inviscid Burgers' equation**. In the case of viscous fluid we have **viscous Burgers' equation** as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{3.24}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity (also called momentum diffusivity) and μ is dynamic viscosity. Viscosity in fluid is equivalent to friction in solids. Dynamic viscosity is the relation between the stress and strain tensor, while the kinematic viscosity is the dynamic viscosity divided by the density.

Burgers' equation is named after the Dutch physicist Johannes Martinus Burgers (1895-1981). It has application in various areas of applied mathematics, such as modeling of gas dynamics, traffic flow, etc. In this chapter the term Burgers' equation is used for inviscid version of Burgers' equation.

3.4 Solution of Viscous Burgers' Equation, The Effect of Viscosity

In this section the solution of the viscous Burgers' equation and the effect of decreasing viscosity are investigated. Consider the following viscous Burgers' equation (Cameron, 2011)

$$u_t + uu_x = \nu u_{xx}. \tag{3.25}$$

The solution will be the propagation wave type $u(t,x) = w(x-st) = w(y)$, $y = x-st$. For the derivatives we have

$$u_t = -sw', \quad u_x = w', \quad u_{xx} = w'', \quad w' = \frac{\partial w}{\partial y}. \tag{3.26}$$

Substituting (3.26) into (3.25)

$$-sw' + ww' = \nu w'', \tag{3.27}$$

$$-sw' + \left(\frac{w^2}{2}\right)' = \nu w'', \tag{3.28}$$

and taking the first integral of (3.28)

$$-sw + \frac{w^2}{2} = \nu w' + C, \tag{3.29}$$

and imposing the conditions $w(-\infty) = u_l, w(\infty) = u_r, u_l > u_r$ and $w'(\pm\infty) = 0$, we have

$$-su_l + \frac{u_l^2}{2} = -su_r + \frac{u_r^2}{2} = C. \tag{3.30}$$

In order to keep the equality valid, and for the shock speed to be the same as in the inviscid Burgers' equation, $s = \frac{u_l+u_r}{2}$, we have $C = -\frac{u_l u_r}{2}$. Substituting these values into (3.29)

$$\nu w' = \frac{w^2}{2} - \frac{u_l + u_r}{2} w + \frac{u_l u_r}{2}, \tag{3.31}$$

and rearranging (3.31) by $w' = \frac{dw}{dy}$ we have

$$\frac{dy}{2\nu} = \frac{dw}{(w - \frac{(u_l+u_r)}{2})^2 - \frac{(u_l-u_r)^2}{4}}. \tag{3.32}$$

Using the integral formula

$$\int \frac{dw}{(w-a)^2 - b^2} = \frac{1}{2b} \log\left|\frac{w-a-b}{w-a+b}\right|, \tag{3.33}$$

for (3.32) we have

$$\frac{y}{2\nu} + C = \frac{1}{u_l - u_r} \log \frac{u_l - w}{w - u_r}, \qquad u_l > w > u_r. \tag{3.34}$$

Defining $A = \frac{y(u_l-u_r)}{2\nu} + C$, for w we have

$$w = u_r + \frac{u_l - u_r}{2} \frac{2}{e^A + 1}. \tag{3.35}$$

Multiplying and dividing $\frac{2}{e^A+1}$ by $e^{-\frac{A}{2}}$ and using the identity

$$\frac{2e^{\frac{A}{2}}}{e^{\frac{A}{2}} + e^{\frac{-A}{2}}} = 1 - \frac{e^{\frac{A}{2}} - e^{\frac{-A}{2}}}{e^{\frac{A}{2}} + e^{\frac{-A}{2}}} = 1 - tanh\frac{A}{2}, \tag{3.36}$$

the solution for w will be

$$w(y) = u_r + \frac{u_l - u_r}{2} tanh\left(\frac{y(u_l - u_r)}{4\nu} + C\right), \tag{3.37}$$

finally for $u(t, x)$ we have

$$u(t, x) = u_r + \frac{u_l - u_r}{2} tanh\left(\frac{(x - x_0 - st)(u_l - u_r)}{4\nu}\right). \tag{3.38}$$

As $\nu \to 0$, $u(x, t)$ tends to the step function for every t, which is the unique weak solution of the Burgers' equation. Figure 3.1 shows how reducing viscosity leads to a sharp solution for the viscous Burgers' equation. For the code refer to A.3.

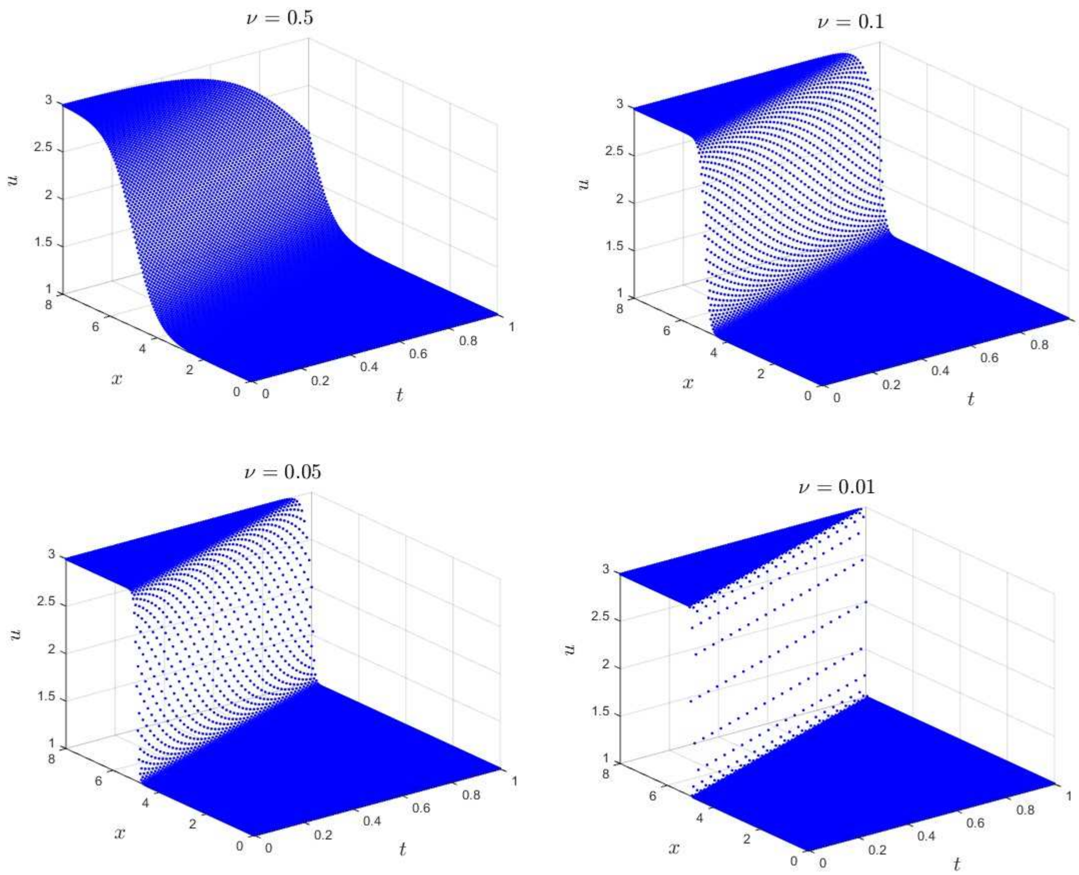

Figure 3.1: Solution of viscous Burgers' equation for diffrent ν.

3.5 Conservation Law

Consider the evolution of the density v of a substance, the total amount inside a set Ω at time t is

$$\int_{\Omega} v(t,x)dV, \tag{3.39}$$

assuming change only happens as the substance goes through the boundary, that is quantified by flux F

$$\frac{d}{dt}\int_{\Omega} v(t,x)dV = -\int_{\partial\Omega} F \cdot ndS, \tag{3.40}$$

where n is the outer normal. Using Gauss theorem, the right-hand side of (3.40) will be

$$\int_{\partial\Omega} F \cdot ndS = \int_{\Omega} \nabla \cdot FdV, \tag{3.41}$$

so we have

$$\frac{d}{dt}\int_{\Omega} v(t,x)dV + \int_{\Omega} \nabla \cdot FdV = 0, \tag{3.42}$$

that can be written as

$$\int_{\Omega} \left(v_t + \nabla \cdot F\right) dV = 0. \tag{3.43}$$

Since Ω is arbitrary we have the following differential equation

$$v_t + \nabla \cdot F = 0. \tag{3.44}$$

A conservation law is obtained when F is a function of v only (Yu, 2012)

$$v_t + \nabla \cdot F(v) = 0, \quad v(0,x) = v_0(x). \tag{3.45}$$

3.6 Advection Equation

Advection equation is a specific case of conservation law. Considering the one-dimensional conservation law

$$\frac{\partial v}{\partial t} + \frac{\partial f(v)}{\partial x} = 0, \tag{3.46}$$

where $v(t, x)$ is an unknown conserved quantity and $f(v)$ is the flux. Equation (3.46) can be written as

$$\frac{\partial v}{\partial t} + a(v)\frac{\partial v}{\partial x} = 0. \tag{3.47}$$

where $a(v) = \frac{df}{dv}$. In the case of flux function depending on x

$$\frac{\partial v}{\partial t} + a(v)\frac{\partial v}{\partial x} = g(v), \tag{3.48}$$

where $g(v) = -\frac{\partial f}{\partial x}$ shows the source term.

Assigning the conserved quantity by ρ and the velocity vector field by $\vec{u}$ we end up with the **advection equation**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0. \tag{3.49}$$

By assuming an incompressible flow, $\nabla \cdot \vec{u} = 0$, we have

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \nabla \rho = 0. \tag{3.50}$$

In the case of constant velocity $\vec{u} = a$ we end up with the **linear advection equation**

$$\frac{\partial \rho}{\partial t} + a\frac{\partial \rho}{\partial x} = 0, \tag{3.51}$$

which describes the flux of a substance in the flow passing some point in the stream. If there is no diffusion in the flow, the concentration profile will convect downstream with the velocity a. Linear advection equation is a hyperbolic equation. Hyperbolic

PDEs usually describe propagation in preferred direction, while keeping its strength.

Considering molecular diffusion and turbulence the **advection-diffusion equation** is introduced, which includes the effect of molecular diffusion by applying the diffusive flux from Fourier's law of heat conduction $-D\frac{\partial \rho}{\partial x}$, where D is diffusivity,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(a\rho - D\frac{\partial \rho}{\partial x}) = 0, \tag{3.52}$$

in this case, the flux depends on $\frac{\partial \rho}{\partial x}$ as well as ρ. Equation (3.52) is a parabolic second-order PDE (Khoo et al., 2003).

3.7 Method of Characteristics

A common method for solving first-order PDEs is method of characteristics and in this section the basics of it is explained. For further information on the history of method of characteristics refer to (Middendorp & Verbeek, 2006). Let us consider a general first-order quasi-linear PDE

$$a(x, y, v)v_x + b(x, y, v)v_y = c(x, y, v). \tag{3.53}$$

Equation (3.53) can be written in the following form

$$(a(x, y, v), b(x, y, v), c(x, y, v)) \cdot (v_x, v_y, -1) = 0, \tag{3.54}$$

which means $(a(x,y,v), b(x,y,v), c(x,y,v))$ and $(v_x, v_y, -1)$ are perpendicular. Having the solution as $v(x,y)$ and introducing the new function G we have

$$G(x,y,v) = v(x,y) - v, \tag{3.55}$$

using (3.54) and (3.55)

$$(v_x, v_y, -1) = (G_x, G_y, G_v) = \nabla G, \tag{3.56}$$

where ∇G is a normal vector of the surface $G = 0$. Using the definition (3.55), $G = 0$ gives us $v = v(x,y)$. Therefore $(v_x, v_y, -1)$ is perpendicular to the surface solution $v = v(x,y)$. It was shown $(a(x,y,v), b(x,y,v), c(x,y,v))$ is perpendicular to $(v_x, v_y, -1)$ as a result $(a(x,y,v), b(x,y,v), c(x,y,v))$ has to be tangent to the surface $v = v(x,y)$. Thus the quasi-linear PDE is equivalent to the geometrical requirement in the x-y-v space that the vector $(a(x,y,v), b(x,y,v), c(x,y,v))$ is tangent to the solution surface $v = v(x,y)$ (Yu, 2012). Therefore the following conditions have to be satisfied

$$\frac{dx}{ds} = a(x,y,v) \tag{3.57}$$
$$\frac{dy}{ds} = b(x,y,v) \tag{3.58}$$
$$\frac{dv}{ds} = c(x,y,v). \tag{3.59}$$

Note that the independent variables x, and y are used to illustrate the method of characteristics and they can be replaced with any other variables like time t, as in the following sections.

Solving Advection Equation Using Method of Characteristics

For the case of linear advection equation

$$\rho_t + a\rho_x = 0, \quad \rho(0,x) = \rho_0(x), \tag{3.60}$$

where $\rho = \rho(t,x)$ is the density. Employing (3.57)-(3.59) we have

$$\frac{dt}{ds} = 1 \quad \rightarrow \quad t = s, \tag{3.61}$$
$$\frac{dx}{ds} = a \quad \rightarrow \quad x = at + x_0 \rightarrow x_0 = x - at, \tag{3.62}$$
$$\frac{du}{ds} = 0 \quad \rightarrow \quad \rho = c = u_0(x_0) \Rightarrow \rho(t,x) = \rho_0(x - at). \tag{3.63}$$

As an example, Figure 3.2 shows the characteristics for the linear advection equation for time between 0 and 10 and for $x_0 = 0, 2, 4, 6, 8, 10$ as the initial conditions. Figure 3.3 indicates the movement of the current over $x = 0 - 50$, for different times and when $\rho_0(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

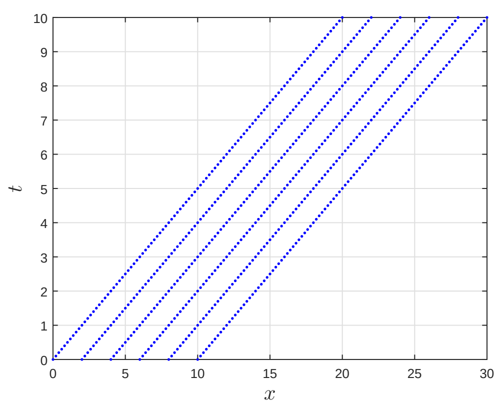

Figure 3.2: Characteristics for linear advection equation.

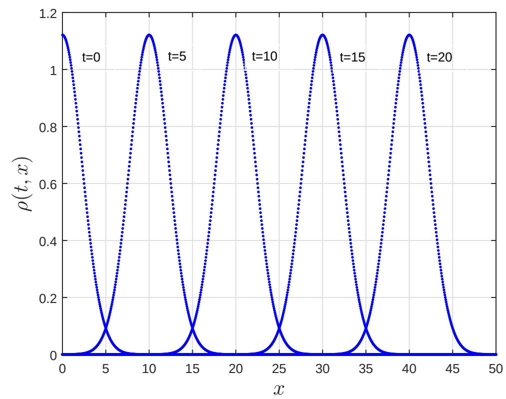

Figure 3.3: Current profiles for linear advection equation in different times.

3.8 Shock Wave

Let us apply method of characteristics on the Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{3.64}$$

with the following initial and boundary conditions

$$\begin{aligned} u(0,x) &= u_0(x), & (3.65)\\ u(t,0) &= u_b(t). & (3.66) \end{aligned}$$

Rewriting (3.64)

$$\frac{1}{u}\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \tag{3.67}$$

and considering the characteristics starting on the initial condition, the new variable $\varphi(x,\xi) = u(t(x,\xi), x+\xi)$ is introduced, where x a parameter and $t(x,\xi)$ is the characteristic for $\xi \geq 0$. For the characteristics originated on the initial condition

$t(x,0)=0$, we have

$$\varphi(x,0)=u(t(x,0),x)=u(0,x)=u_0(x). \tag{3.68}$$

Employing method of characteristics for the characteristics initiating over the spatial variable

$$\frac{d}{d\xi}t(x,\xi) = \frac{1}{\varphi(x,\xi)}, \tag{3.69}$$

$$\frac{d}{d\xi}\varphi(x,\xi) = 0 \tag{3.70}$$

Equation (3.70) gives

$$\varphi(x,\xi)=\varphi(x,0)=u_0(x), \tag{3.71}$$

and substituting (3.71) in (3.69) we have

$$\frac{d}{d\xi}t(x,\xi)=\frac{1}{u_0(x)}. \tag{3.72}$$

Relation (3.72) shows $\frac{1}{u_0(x)}$ has to be decreasing to avoid the shock wave situation, intersecting characteristics, which means $u_0'(x) \geq 0$ leads to the absence of shock wave.

Figures 3.4 and 3.5 demonstrate examples for decreasing and increasing initial conditions, respectively.

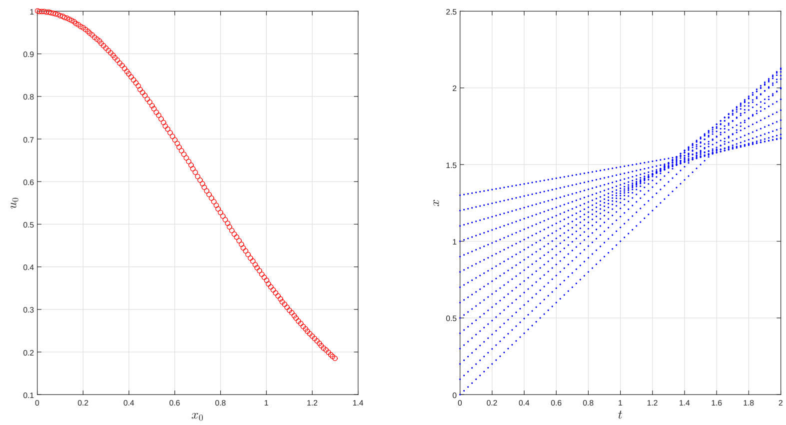

Figure 3.4: Initial condition and corresponding characteristics for $u_0 = e^{-x_0^2}$.

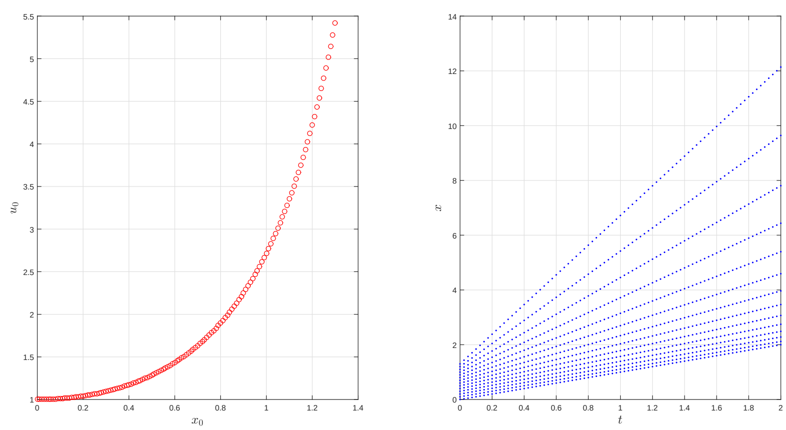

Figure 3.5: Initial condition and corresponding characteristics for $u_0 = e^{x_0^2}$.

Having Burgers' equation in the original form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \tag{3.73}$$

and considering the characteristics starting on the boundary condition, the new variable $\psi(t,s) = u(t+s, x(t,s))$ is introduced, where t is a parameter and $x(t,s)$ is the characteristic for $s \geq 0$. For the characteristics originated on the boundary condition we have

$$\psi(t,0) = u(t, x(t,0)) = u(t,0) = u_b(t). \tag{3.74}$$

Employing method of characteristics for the characteristics initiating at the boundary

$$\begin{aligned} \frac{d}{ds}x(t,s) &= \psi(t,s), && (3.75)\\ \frac{d}{ds}\psi(t,s) &= 0. && (3.76) \end{aligned}$$

Solving for (3.76)

$$\psi(t,s) = \psi(t,0) = u_b(t), \tag{3.77}$$

and substituting (3.77) in (3.75) we end up with the following characteristics equation

$$\frac{d}{ds}x(t,s) = u_b(t). \tag{3.78}$$

From (3.78), $u_b(t)$ has to be decreasing to avoid the shock wave situation which means as long as $\dot{u}_b \leq 0$ shock wave will not occur in the system.

Figures 3.6 and 3.7 show two examples of characteristics originated on the boundary, for decreasing and increasing boundary conditions.

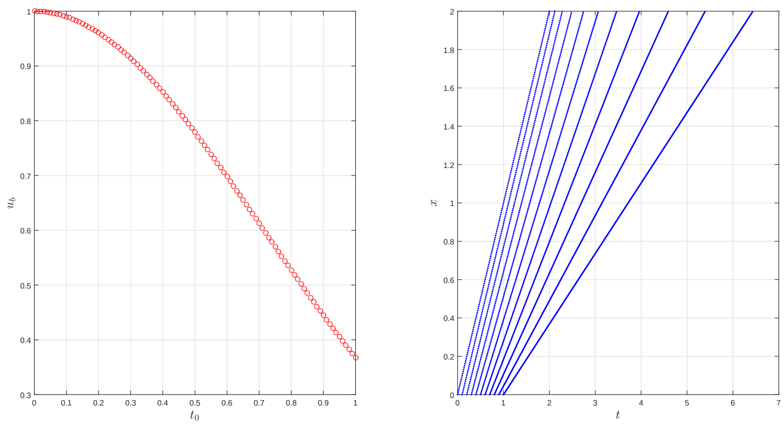

Figure 3.6: Boundary condition and corresponding characteristics for $u_b = e^{-t_0^2}$.

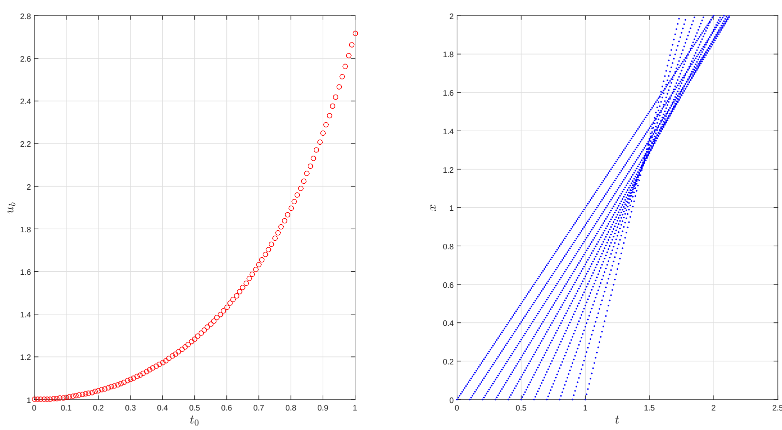

Figure 3.7: Boundary condition and corresponding characteristics for $u_b = e^{t_0^2}$.

In the summary to avoid shock wave, the following conditions have to be satisfied

$$u_0'(x) \geq 0 \tag{3.79}$$

$$\dot{u}_b(t) \leq 0 \tag{3.80}$$

In order to find a meaningful solution in the case of shock wave, one needs to know the following concepts

• Weak solution

• Jump condition

• Entropy solution

These concepts explain when shock wave occurs and how to determine the reasonable solution.

3.8.1 Weak Solution

It was shown that the solution of the Burgers' equation can become discontinuous even if the initial and boundary data are smooth. The concept of weak solution was introduced to allow discontinuous solutions for differential equations and it satisfies the following conditions:

• a smooth function is a weak solution if and only if it is a regular solution,

• a discontinuous function can be a weak solution,

• only those discontinuous functions which satisfy the associated integral equation can be weak solutions.

In order to find the weak solution, the conservation law is multiplied by a test function $\phi \in C^1$ and integrated by parts as if v is in C^1

$$[v_t + f(v)_x]\,\phi(t,x) = 0 \tag{3.81}$$

$$-\int\int_{\Omega} (v\phi_t + f(v)\phi_x)\,dxdt + \int_{\partial\Omega} \phi(t,x)[vn_t + f(v)n_x]dS = 0 \tag{3.82}$$

$$\int_{t>0}\int_{\Omega} (v\phi_t + f(v)\phi_x)\,dxdt + \int_{\mathbb{R}} v_0\phi dx = 0. \tag{3.83}$$

Equation (3.83) is the weak solution of the Burgers' equation. Note that v no longer needs to be C^1 to make the above integral meaningful. The only requirement for ϕ on C^1 is that v and $f(v)$ are measures. It means it is OK for v to be piecewise continuous (Cameron, 2011), (Yu, 2012).

3.8.2 Jump Condition

We can assess what a weak solution would be like considering piecewise C^1 solutions. It means v has discontinuities along some curves but is C^1 everywhere else. Consider such a curve: Γ, and let $\phi \in C_0^1$ be supported in a small ball centering on Γ. The ball is so small that it does not intersect with the x-axis and v is C^1 everywhere in the ball, D, except along Γ. Divide the ball D into two parts D_1 and D_2 by Γ. The weak solution is continuously differentiable in two parts D_1 and D_2 of the domain D. v has a jump discontinuity, shock, along the dividing smooth curve Γ. v, v_t and v_x are continuous in D_1 and D_2. For more details refer to (Yu, 2012), (Zauderer, 2006), (LeVeque, 1992), (Strang, 2006) and (Bhamra, 2010).

ϕ is the test function with compact support in D, and it does not necessarily vanish along Γ. ϕ is zero along the x-axis so for the definition of weak solutions we have

$$\begin{aligned} \int\int_D (v\phi_t + f(v)\phi_x)\, dxdt &= 0, \\ \int\int_{D_1} (v\phi_t + f(v)\phi_x)\, dxdt + \int\int_{D_2} (v\phi_t + f(v)\phi_x)\, dxdt &= 0. \end{aligned} \tag{3.84}$$

$v\phi_t + f(v)\phi_x$ can be written as $(v\phi)_t + (f(v)\phi)_x$, so (3.84) can be written as

$$\int\int_{D_1} ((v\phi)_t + (f(v)\phi)_x)\, dxdt + \int\int_{D_2} ((v\phi)_t + (f(v)\phi)_x)\, dxdt \quad = \quad 0, \quad (3.85)$$

using divergence theorem, $\int_V (\nabla \cdot F) dV = \int_{\partial V} F \cdot da$, for (3.85) we have

$$\int\int_{\partial D_1} \phi\,(vn_t + f(v)n_x)\, dxdt + \int\int_{\partial D_2} \phi\,(vn_t + f(v)n_x)\, dxdt = 0.$$

Let us define $s = -\frac{n_t}{n_x}$ and since ϕ vanishes on ∂D except along Γ, we have

$$\int\int_{\partial D} \phi\,(-s[v] + [f(v)])\, dxdt \quad = \quad 0$$

where $[v]$ is jump of v across Γ. Considering ϕ is arbitrary, the weak solution must satisfy

$$[f(v)] = s[v], \quad \text{or} \quad s = \frac{[f(v)]}{[v]}. \quad (3.86)$$

This is called **jump condition** or **Rankine-Hugoniot jump condition**, where s is the speed of discontinuity.

Considering the specific case of Burgers' equation $f(u) = \frac{1}{2}u^2$ we have

$$s = \frac{f(u)_{x_l} - f(u)_{x_r}}{u(x_l) - u(x_r)} = \frac{\frac{1}{2}u_l^2 - \frac{1}{2}u_r^2}{u_l - u_r} = \frac{1}{2}(u_l + u_r) \quad (3.87)$$

3.8.3 Entropy Solution

We have observed that the classical/strong solution might not exist for conservation laws. In addition, the weak solution does not give a unique solution. Entropy condition is introduced to make the solution unique. The solution satisfying the entropy condition is called an **entropy solution**. Entropy solution is the unique and physically relevant solution among weak solutions (Qiu, 2013). Let us introduce some entropy conditions that can be used in problems

Olenik entropy condition

$$\frac{f(v)-f(v_l)}{v-v_l} \geq s \geq \frac{f(v)-f(v_r)}{v-v_r} \tag{3.88}$$

Lax entropy condition

$$f'(v_l) > s > f'(v_r) \tag{3.89}$$

where $s = \frac{[f(v)]}{[v]}$ is the speed of propagation of discontinuity given by the Rankine-Hugoniot jump condition, v is between v_l and v_r, and v_l and v_r are the left and right states along the discontinuity, respectively.

It can be seen that Oleinik entropy condition implies Lax entropy condition but not the other way around. Lax entropy condition is a necessary but not sufficient condition to single out the entropy condition. In the case of having strictly convex or strictly concave $f(v)$, the Lax entropy condition is equivalent to the Olenik entropy condition and it will be sufficient to single out the entropy condition.

3.9 Riemann Problem

Burgers' equation with the following initial condition is called Riemann problem (Cameron, 2011)

$$u_t + uu_x = 0, \quad u(0,x) = \begin{cases} u_l & x < a \\ u_r & x \geq a \end{cases} \tag{3.90}$$

and has the following unique weak solutions:

Shock wave when $u_l > u_r$

$$u(t,x) = \begin{cases} u_l & x < st + a \\ u_r & x \geq st + a \end{cases} \qquad s = \frac{u_l + u_r}{2} \tag{3.91}$$

Rarefaction wave when $u_l < u_r$

$$u(t,x) = \begin{cases} u_l & x < u_l t \\ \frac{x}{t} & u_l t \leq x \leq u_r t \\ u_r & x > u_r t \end{cases} \tag{3.92}$$

Let us consider the following example

$$u_t + uu_x = 0, \quad u(0,x) = u_0(x_0) = \begin{cases} u_l & x < a \\ u_r & x \geq a \end{cases} \tag{3.93}$$

where $u_l > u_r$, using method of characteristics for characteristics we have

$$x(t) = \begin{cases} u_l t + x_0 & x < a \\ u_r t + x_0 & x \geq a \end{cases}$$

plotting the characteristics, for the nominal values: $u_l = 4, u_r = 2, a = 3$, we have the intersecting characteristics as in Figure 3.8. Defining $s = \frac{u_l + u_r}{2}$ as the speed at intersection and keeping the slope the same before and after the intersection, the characteristics and the solution for $u(t, x)$ using the Riemann problem (3.91) are as in Figure 3.9. In the case of rarefaction wave, $u_l < u_r$, the characteristics and the solution for $u(t, x)$ are given in Figure 3.10.

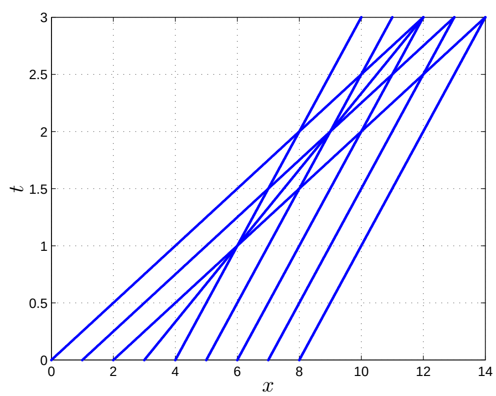

Figure 3.8: Intersecting charactersitics for Burgers' equation.

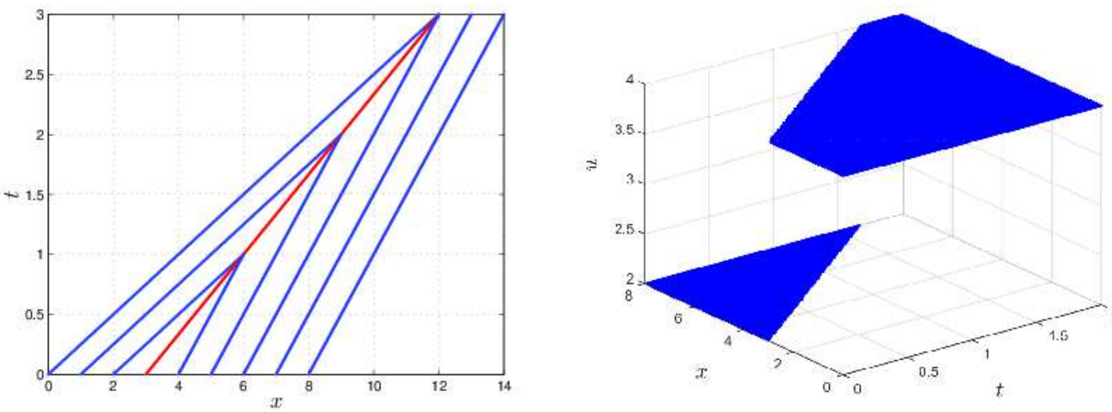

Figure 3.9: Charactersitics and shock wave solution for Reimann problem.

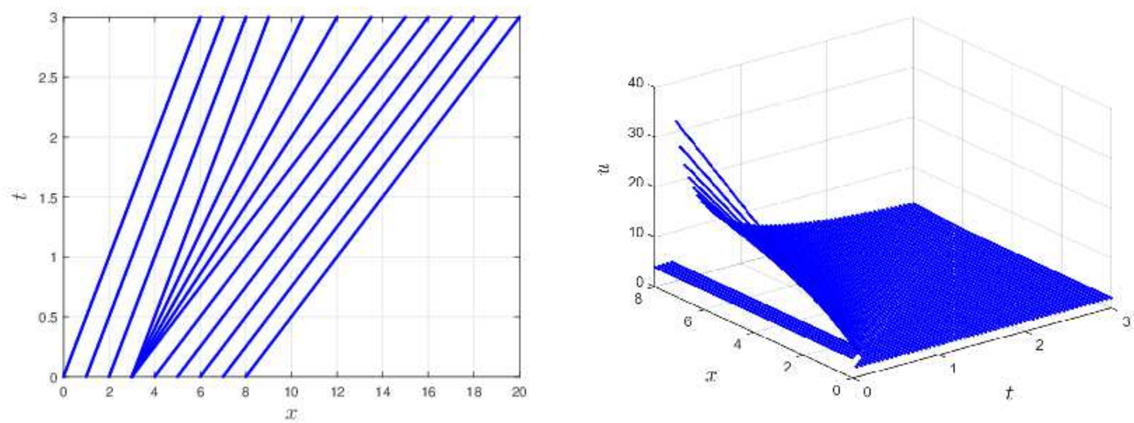

Figure 3.10: Characteristics and rarefaction solution for Reimann problem.

In the second example the Burgers' equation including two shock waves is considered

$$u_t + uu_x = 0, \quad u(0,x) = u_0(x) = \begin{cases} u_l & x < a \\ u_m & a \leq x < b \\ u_r & x \geq b \end{cases} \tag{3.94}$$

where $u_l > u_m > u_r$ and $b > a$. For nominal values of $u_l = 2, u_m = 1, u_r = 0$ and $a = 1, b = 2$ the characteristics are depicted in Figure 3.11. After the shock waves intersect, a new combined shock wave that has the speed as the average speed of the two initial shock waves is generated, for the code refer to A.2 .

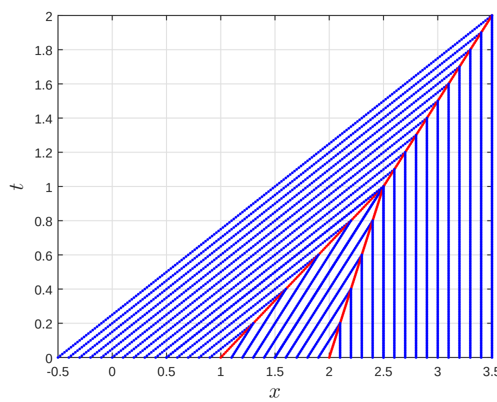

Figure 3.11: Characteristics for Burgers' equation with two shock waves.

3.10 Sliding Mode Observer for Burgers' Equation

Let's consider a Burgers' equation including disturbance at the right-hand side

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = f(t,x,u), \quad u(t,0) = y_0(t), \tag{3.95}$$

and with the discontinuous measurements over spatial variable as follows

$$u(t, x_k) = y_k(t), \quad k = 1, \cdots, m. \tag{3.96}$$

where x_k shows the location of the sensors. Applying method of characteristics, the new variables $\Psi(t, s) = u(t+s, X(t, s))$ and $X(t, s)$ are introduced, such that

$$\frac{d}{ds} X(t, s) = \Psi(t, s), \tag{3.97}$$

where t is a parameter. The derivative of Ψ, using (3.95), will be

$$\begin{aligned} \frac{d}{ds}\Psi(t, s) &= \frac{\partial u}{\partial t}(t+s, X(t, s)) + \frac{d}{ds} X(t, s) \frac{\partial u}{\partial t}(t+s, X(t, s)) \\ &= f(t+s, X(t, s), \Psi(t, s)). \end{aligned} \tag{3.98}$$

Equations (3.97) and (3.98) called characteristic equations and their initial conditions correspond to the boundary condition of (3.95) as

$$X(t, 0) = 0 \tag{3.99}$$

$$\Psi(t, 0) = y_0(t) \tag{3.100}$$

The measurements in (3.96) translated into the characteristics are

$$X(t, s_k) = x_k \tag{3.101}$$

$$\Psi(t, s_k) = y_k(t_k) \tag{3.102}$$

where $s_k = t_k - t$ and s_k is the time of characteristic reaching a sensor position $X(t, s_k) = x_k$, see Figure 3.12.

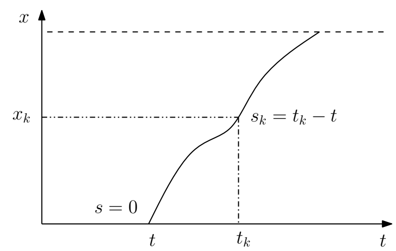

Figure 3.12: Characteristic and the relation between t, s and t_k.

For each characteristic the assumption of $f \geq 0$ needs to be held to make the characteristic meet the sensors position.

The characteristic equations for the observer are as follows

$$\begin{aligned}\frac{d}{ds}\hat{X}(t,s) &= \hat{\Psi}(t,s), && (3.103)\\ \frac{d}{ds}\hat{\Psi}(t,s) &= f(t+s,\hat{X}(t,s),\hat{\Psi}(t,s)) && (3.104)\\ &+ \sum \left[y_k(t+s) - \hat{\Psi}(t,s)\right]\delta(s-s_k).\end{aligned}$$

with the initial conditions as

$$\hat{X}(t,0) = 0 \quad (3.105)$$

$$\hat{\Psi}(t,0) = y_0(t) \quad (3.106)$$

By changing the argument of the δ-function from s to $\hat{X}(t,s)$, using (3.103), (3.104) can be written as

$$\begin{aligned} \frac{d}{ds}\hat{\Psi}(t,s) &= f(t+s,\hat{X}(t,s),\hat{\Psi}(t,s)) \\ &+ \sum \left[y_k(t+s)-\hat{\Psi}(t,s)\right]\hat{\Psi}(t,s)\delta(\hat{X}(t,s)-x_k). \end{aligned} \tag{3.107}$$

Writing the equation for the observer in the integral form and replacing the δ-function by the discontinuous function, that leads to $\hat{u}(t,x_k)=y_k(t)$ after some finite time but not instantly as in (3.107), we have

$$\begin{aligned} \hat{\Psi}(t,s) &= \int_{s_k-\varepsilon}^{s} f(t+v,\hat{X}(t,v),\hat{\Psi}(t,v))dv \\ &+ \int_{s_k-\varepsilon}^{s} L(v-s_k)\mathrm{sign}(y_k(t+v)-\hat{\Psi}(t,v))dv \end{aligned} \tag{3.108}$$

where

$$L(v-s_k)=\begin{cases} L_{max} & v-s_k<0 \\ 0 & v-s_k>0 \end{cases} \tag{3.109}$$

where L_{max} is big enough to guarantee sliding mode existence at $s=s_k$ and therefore $y_k(t_k)-\Psi(s_k,t)=0$. The observer gain $L(v-s_k)$ can be replaced by $\tilde{L}(\hat{X}(v,t)-x_k)$ for the measurements on spatial variable, and for the state observer we have

$$\begin{aligned} \hat{\Psi}(t,s) &= \int_{s_k-\varepsilon}^{s} f(t+v,\hat{X}(t,v),\hat{\Psi}(t,v))dv \\ &+ \int_{s_k-\varepsilon}^{s} \tilde{L}(\hat{X}(t,v)-x_k)\mathrm{sign}(y_k(t+v)-\hat{\Psi}(t,v))dv. \end{aligned} \tag{3.110}$$

The distributed observer in the PDE form will be

$$\frac{\partial \hat{u}}{\partial t} + \hat{u}\frac{\partial \hat{u}}{\partial x} = f(t, x, \hat{u}) + \sum_k \tilde{L}(x - x_k)\text{sign}(y_k(t_k) - \hat{u}(t, x_k)). \tag{3.111}$$

where the schematic representation of $\tilde{L}(x)$ is depicted in the Figure 3.13.

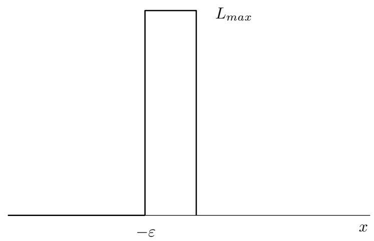

Figure 3.13: Demonstration of obserever gain.

The simulation results for a case of increasing step function as boundary condition are presented in Figures 3.14 and 3.10. Figure 3.14 shows the convergence of the sliding mode and Figure 3.10 depicts the performance of the observer over time.

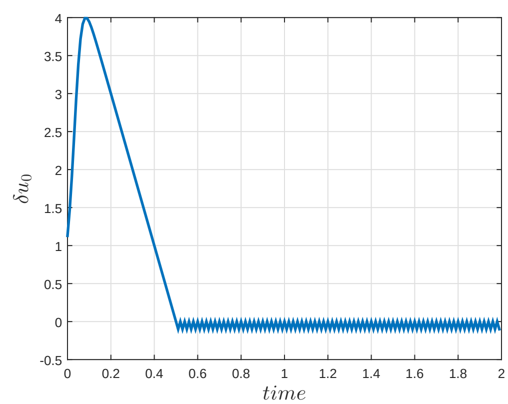

Figure 3.14: Sliding mode for Burgers' equation observer.

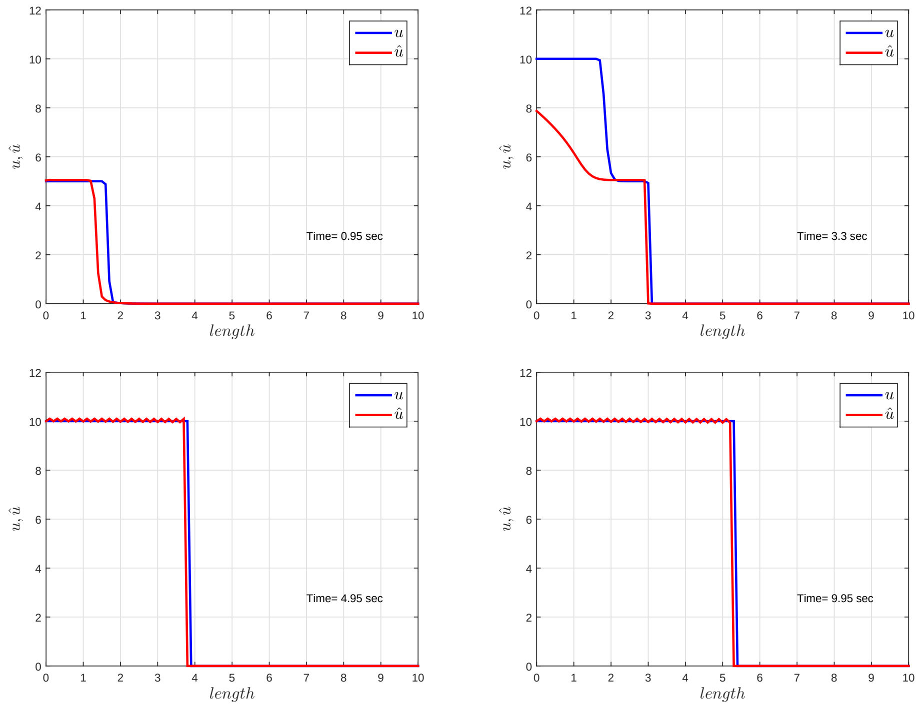

Figure 3.15: Sliding mode observer performance for Burgers' equation.

3.11 Conclusion and Future work

In this chapter, Burgers' equation was introduced and its properties and solutions in the presence of shock wave were studied. At the end, a sliding mode observer was developed for Burgers' equation. One extension for this chapter will be considering designing observer for Burgers' equation in the presence of shock wave and predicting the behavior of shock wave for different cases. In addition, this chapter can be extended to cover different variations of fluid flow equations such as the situations of having more realistic models of the systems.

Chapter 4

SMO and Anomaly Detection System for Advection Equation

4.1 Introduction

In this chapter, the nonlinear observer is designed for a system of advection equations based on the idea of structure variable systems with sliding mode control (Kamran et al., 2015). The observer algorithm is designed in such a way that the output of the model coincides with the output of the system, in spite of the possible mismatches between the model and the actual system.

The initial motivation for this research has come from the need to localize possible leak in the fuel lines of J-2X rocket engine test bed. The J-2X is a liquid-oxygen/liquid-hydrogen fueled rocket engine that is designed to start at altitude as part of a second or third stage of large, multi-stage launch vehicle (Drakunov & Solano, 2012), (NASA, 2011).

Here the focus is on estimating the states of the system and detecting possible anomalies for a class of first order partial differential equations, known as advection equation, only having boundary measurements available. Employing the mathematical theory of variable structure systems with sliding mode, the observer algorithm is designed in such a way that it steers the output of the model to the output of the system, in the presence of the possible differences between the model and the actual system. The properties of sliding mode make it possible to steer the sate of observer to the states of real-life system, as well as to identify the parameters of anomalies that may occur in the actual system.

The chapter is organized as follows. In section 4.2 the advection equation is introduced and the system is transformed into a set of scalar equations using the appropriate transformation and next the system is written in the characteristic form. Section 4.3 represents the design process for the observer based on sliding mode method using only boundary measurements, and the proof of existence and convergence of the proposed observer are provided. Section 4.4 concentrates on designing the anomaly detection system and its proof of convergence. Sections 4.5 and 4.6 demonstrate applications of the suggested nonlinear observer and anomaly detection system. The corresponding simulation results can be found in section 4.7. The chapter ends with the conclusion and suggestions for future work in section 4.8.

4.2 Advection Equation, Problem Statement

The distributed parameter system under consideration governed by partial differential equation of the form

$$\frac{\partial Q(t,x)}{\partial t} + \mathcal{A}Q(t,x) = f(t,x,Q), \tag{4.1}$$

where $Q(t,x)$ is the state, $\mathcal{A}$ is a linear differential operator, $f(t,x,Q) \in C^1(\Omega)$ is the disturbance vector, continuous in t and continuous differentiable function of $x \in \Omega$ where $\Omega \in \mathbb{R}^3$ is spatial region with a smooth boundary $\partial\Omega$. The standard restrictions on $\mathcal{A}$ state that it is a closed, linear, differential operators, that generates a semigroup of strongly continuous bounded operators $e^{\mathcal{A}t}$ defined for $t \geq 0$ (Russell, 2010). In the case of advection equation the operator $\mathcal{A}$ is $A(t,x,Q)\frac{\partial}{\partial x}$ so

$$\frac{\partial Q(t,x)}{\partial t} + A(t,x,Q)\frac{\partial Q(t,x)}{\partial x} = f(t,x,Q), \tag{4.2}$$

where $0 \leq x \leq l$, $t \geq 0$, $Q \in \mathbb{R}^n$, and $A : \mathbb{R}^{n+2} \to \mathbb{R}^{n\times n}$. Such equations play an important role in modeling gas dynamics, flood waves in canals and rivers, transport of pollutant, traffic flow and many other areas.

In order to define the solution uniquely the initial and boundary conditions are needed. The initial condition is

$$Q(0,x) = \Phi(x), \tag{4.3}$$

and the boundary condition is

$$Q(t,0) = Y_0(t). \tag{4.4}$$

Based on the properties of the matrix $A(t,x,Q)$ there might be the need of having the boundary condition at the end of the spatial variable, $Q(t,l) = Y_l(t)$ in (4.4). The equation including boundary condition at the end, needs to be solved backward in time.

Let us consider a new state $\tilde{Q}(t,x) = G(Q(t,x))$, where $G(Q)$ is a diffeomorphism, i.e. continuously differentiable map $G : \mathbb{R}^n \to \mathbb{R}^n$ such that there exist $Q = G^{-1}(\tilde{Q})$. Differentiating $\tilde{Q}(t,x)$ with respect to time and spatial variable we have

$$\frac{\partial}{\partial t}\tilde{Q}(t,x) = \frac{\partial G(Q)}{\partial Q}\frac{\partial}{\partial t}Q(t,x), \tag{4.5}$$

and

$$\frac{\partial}{\partial x}\tilde{Q}(t,x) = \frac{\partial G(Q)}{\partial Q}\frac{\partial}{\partial x}Q(t,x). \tag{4.6}$$

Using (4.2) the following equation is obtained

$$\frac{\partial}{\partial t}\tilde{Q}(t,x) + \tilde{A}(t,x,\tilde{Q})\frac{\partial}{\partial x}\tilde{Q}(t,x) = \tilde{f}(t,x,\tilde{Q}), \tag{4.7}$$

where the matrix $\tilde{A}$ is a similarity transformation of the matrix A

$$\tilde{A} = \frac{\partial G(G^{-1}(\tilde{Q}))}{\partial Q}A(t,x,G^{-1}(\tilde{Q}))\left[\frac{\partial G(G^{-1}(\tilde{Q}))}{\partial Q}\right]^{-1}, \tag{4.8}$$

and the disturbance f at the right-hand side is transformed into $\tilde{f}$ as

$$\tilde{f}(t,x,\tilde{Q}) = \frac{\partial G(Q)}{\partial Q}f(t,x,G^{-1}(\tilde{Q})). \tag{4.9}$$

Here a class of systems with diagonalizable matrix A is considered. So employing the state transformation matrix A is transformed into a diagonal form. It means the transformation decouples the original system into a set of scalar equations of the form

$$\frac{\partial \tilde{q}_j}{\partial t} + \tilde{a}_j(t, x, \tilde{Q}) \frac{\partial \tilde{q}_j}{\partial x} = \tilde{f}_j(t, x, \tilde{Q}), \tag{4.10}$$

where $\tilde{a}_j$ is the jth element of the diagonal matrix $\tilde{A}$

$$\tilde{A} = \begin{bmatrix} \tilde{a}_1 & 0 & 0 & \cdots & 0 \\ 0 & \tilde{a}_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \tilde{a}_n \end{bmatrix}.$$

Each $j = 1, ..., n$ shows different parameters in the system such as pressure, velocity, temperature, etc. Assuming each $\tilde{a}_j$ and $\tilde{f}_j$ only include the corresponding $\tilde{q}_j$, $\tilde{a}_j(t, x, Q) = \tilde{a}_j(t, x, \tilde{q}_j)$ and $\tilde{f}_j(t, x, \tilde{Q}) = \tilde{f}_j(t, x, \tilde{q}_j)$, the system of decoupled advection equations along with the corresponding initial and boundary conditions are obtained as

$$\frac{\partial \tilde{q}_j}{\partial t} + \tilde{a}_j(t, x, \tilde{q}_j) \frac{\partial \tilde{q}_j}{\partial x} = \tilde{f}_j(t, x, \tilde{q}_j), \tag{4.11}$$

$$\tilde{q}_j(0, x) = \tilde{\phi}_j(x), \tag{4.12}$$

$$\tilde{q}_j(t, 0) = \tilde{y}_{j0}(t) \quad \text{or} \quad \tilde{q}_j(t, l) = \tilde{y}_{jl}(t), \tag{4.13}$$

where $\tilde{\phi}_j(x)$, $\tilde{y}_{j0}(t)$ and $\tilde{y}_{jl}(t)$ are the transformed initial and boundary conditions.

Our goal is to design a nonlinear observer as well as an anomaly detection system for the system described by (4.11)-(4.13), having only boundary measurements available. The solution of (4.11) can be determined by applying method of characteristics. Here method of characteristics is practiced in order to obtain a clear view of the design outline and to understand the conditions and restrictions on the design process.

Let us introduce the new variable $\Psi_j(t,s) = \tilde{q}_j(t+s, X_j(t,s))$, where t is a parameter and $X_j(t,s)$ satisfies ordinary differential equation

$$\dot{X}_j(t,s) = \frac{d}{ds}X_j(t,s) = \tilde{a}_j(t+s, X_j(t,s), \Psi_j(t,s)). \tag{4.14}$$

Differentiating Ψ_j we have

$$\dot{\Psi}_j = \frac{d}{ds}\Psi_j(t,s) = \tilde{f}_j(t+s, X_j(t,s), \Psi_j(t,s)). \tag{4.15}$$

So the system of ordinary differential equations, also known as characteristic equations, is obtained as

$$\dot{X}_j(t,s) = \tilde{a}_j(t+s, X_j(t,s), \Psi_j(t,s)), \tag{4.16}$$

$$\dot{\Psi}_j(t,s) = \tilde{f}_j(t+s, X_j(t,s), \Psi_j(t,s)). \tag{4.17}$$

Equations (4.16) and (4.17) are equivalent to the partial differential equation (4.11).

The initial and boundary conditions are needed to be rewritten in the the characteristic form as well. Ignoring the characteristics originated on the x-axis, because of their transient effect, the boundary conditions in (4.13) serve as the initial conditions for the characteristic equations.

$$X_j(t,0) = 0, \quad \Psi_j(t,0) = \tilde{y}_{j0}(t) \quad \text{or} \quad \Psi_j(t,s_l) = \tilde{y}_{jl}(t). \tag{4.18}$$

such that $X_j(t,s_l) = l$. Figure 4.1 shows characteristics and locations of the sensors for characteristics originated on the boundary conditions. Base on section 3.8, the non-increasing boundary conditions lead to the absence of the shock wave which is one of our assumptions in this chapter.

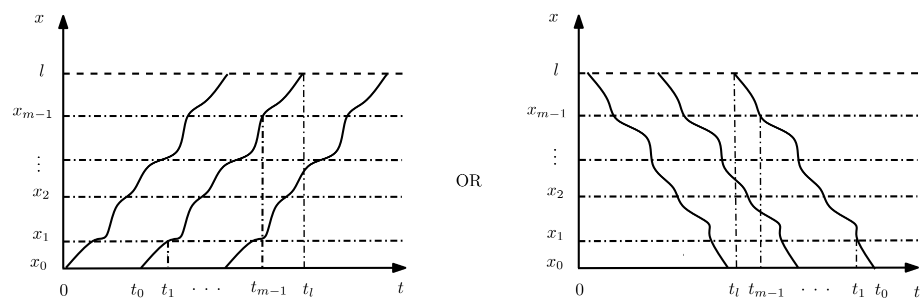

Figure 4.1: Demostration of the characteristics and locations of the sensors.

4.3 Designing Sliding Mode Observer Using Boundary Measurement

In this section our goal is to design state observer for the system (4.11) using discontinuous boundary measurements as

$$\tilde{q}_j(t,x_k) = [\tilde{y}_{jk}(t)], \tag{4.19}$$

where $j = 1, \cdots, n$. shows different variables and $k = 0, ..., m-1$ shows positions of the sensors along the spatial variable: $0 = x_0 < x_1 < ... < x_{m-1} = l$. As it will be shown in the example, just one measurement for each variable would be enough as the minimum required number of the measurements. The distributed measurements in (4.19) are translated into characteristic equations as

$$\begin{aligned} X_j(t, s_k) &= x_k, && (4.20) \\ \Psi_j(t, s_k) &= \tilde{y}_{jk}(s_k), && (4.21) \end{aligned}$$

where $s_k = t_k - t$ is time of characteristic reaching a measurement point x_k (4.20).

Characteristic equations for the observer are written as

$$\begin{aligned} \dot{\hat{X}}_j(t, s) &= \tilde{a}_j(t + s, \hat{X}_j(t, s), \hat{\Psi}_j(t, s)), && (4.22) \\ \dot{\hat{\Psi}}_j(t, s) &= \tilde{f}_j(t + s, \hat{X}_j(t, s), \hat{\Psi}_j(t, s)) && (4.23) \\ &+ \sum_{s_k} [\tilde{y}_{jk}(t + s) - \hat{\Psi}_j(t, s)]\delta(s - s_k). \end{aligned}$$

The initial conditions for the observer are considered as $\hat{X}_j(t, 0) = 0$ and $\hat{\Psi}_j(t, 0) = 0$. By changing the argument of the δ-function from $s - s_k$ to $\hat{X}_j(t, s) - x_k$ in (4.23), using (4.22), we have

$$\begin{aligned} \dot{\hat{\Psi}}_j(t, s) &= \tilde{f}_j(t + s, \hat{X}_j(t, s), \hat{\Psi}_j(t, s)) && (4.24) \\ &+ \sum_{x_k < x} [\tilde{y}_{jk}(t + s) - \hat{\Psi}_j(t, s)]\tilde{a}_j(t + s, \hat{X}_j(t, s), \hat{\Psi}_j(t, s))\delta(\hat{X}_j(t, s) - x_k). \end{aligned}$$

This observer works as follows: at each measurement point x_k the output of the observer $\hat{\Psi}_j$ is set to the measured value $\tilde{y}_{jk}(s_k)$. In other words, the interval between available measurements is treated as a new observer with the corresponding boundary measurements.

Writing observer (4.24) in the distributed form we have

$$\begin{aligned} \frac{\partial \hat{\tilde{q}}_j(t,x)}{\partial t} + \tilde{a}_j(t,x,\hat{\tilde{q}}_j)\frac{\partial \hat{\tilde{q}}_j(t,x)}{\partial x} &= \tilde{f}_j(t,x,\hat{\tilde{q}}_j) \\ &+ \sum_{k=1}^{m-1}(\tilde{y}_{jk}(t) - \hat{\tilde{q}}_j(t,x_k))\tilde{a}_j(t,x,\hat{\tilde{q}}_j)\delta(x-x_k), \end{aligned} \tag{4.25}$$

The same can be achieved by sliding mode using a discontinuous function. Defining $\tilde{L}_{jk}(x,\hat{\tilde{q}}_j(t,x)) = \tilde{a}_j(t,x,\hat{\tilde{q}}_j)\delta(x-x_k)$ and replacing $\tilde{y}_{jk}(t)-\hat{\tilde{q}}_j(t,x_k)$ with $\text{sign}(\tilde{y}_{jk}(t)-\hat{\tilde{q}}_j(t,x_k))$ we have

$$\begin{aligned} \frac{\partial \hat{\tilde{q}}_j(t,x)}{\partial t} + \tilde{a}_j(t,x,\hat{\tilde{q}}_j)\frac{\partial \hat{\tilde{q}}_j(t,x)}{\partial x} &= \tilde{f}_j(t,x,\hat{\tilde{q}}_j) \\ &+ \sum_{k=1}^{m-1}\tilde{L}_{jk}(x,\hat{\tilde{q}}_j(t,x))\text{sign}(\tilde{y}_{jk}(t) - \hat{\tilde{q}}_j(t,x_k)). \end{aligned} \tag{4.26}$$

Equation (4.26) gives robustness in the presence of possible disturbances and has better filtering property in comparison with (4.25). This observer is designed to steer the state of the system to the measured value at any point that information is available. δ-function can be approximated by Gaussian curve as

$$\delta(x-x_k) \simeq \frac{1}{\varepsilon\sqrt{2\pi}}e^{-\frac{(x-x_k)^2}{2\varepsilon^2}},$$

where ε is a small constant, so

$$\tilde{L}_{jk}(x, \hat{\tilde{q}}_j(t,x)) = \tilde{a}_j(t,x,\hat{\tilde{q}}_j)\frac{1}{\varepsilon\sqrt{2\pi}}e^{-\frac{(x-x_k)^2}{2\varepsilon^2}}. \tag{4.27}$$

According to (4.27), in the vicinity of $x = x_k$, $\tilde{L}_{jk}$ is large so the sliding mode exists on the manifold $\sigma_k = \tilde{y}_{jk}(t) - \hat{\tilde{q}}_j(t, x_k) = 0$. Having large $\tilde{L}_{jk}$ helps to suppress for the possible disturbances and makes the observer more effective in the case of big difference between the predicted value and the actual system.

In order to investigate the existence of the sliding mode let us introduce the following quadratic Lyapunov candidate

$$V = \frac{1}{2}\sigma_k^2 \geq 0. \tag{4.28}$$

For the existence purpose the derivative of the Lyapunov candidate needs to be $\dot{V} = \dot{\sigma}_k\sigma_k < 0$, refer to section 1.7. For σ_k and $\dot{\sigma}_k$ we have

$$\begin{aligned}
\sigma_k(t) &= \tilde{y}_{jk}(t) - \hat{\tilde{q}}_j(t, x_k) && (4.29)\\
\dot{\sigma}_k(t) &= \dot{\tilde{y}}_{jk}(t) - \dot{\hat{\tilde{q}}}_j(t, x_k), && (4.30)\\
&= \dot{\tilde{y}}_{jk}(t) + \tilde{a}_j(t, x_k, \hat{\tilde{q}}_j)\frac{\partial \hat{\tilde{q}}_j}{\partial x}(t, x_k) - \tilde{f}_j(t, x_k, \hat{\tilde{q}}_j(t, x_k)) - \sum_{k=1}^{m-1} \tilde{L}_{jk}(x_k, \hat{\tilde{q}}_j(t, x_k))\mathrm{sign}(\sigma_k),
\end{aligned}$$

having $|\tilde{L}_{jk}(x_k, \hat{\tilde{q}}_j(t, x_k))| > |\dot{\tilde{y}}_{jk}(t) + \tilde{a}_j(t, x_k, \hat{\tilde{q}}_j)\frac{\partial \hat{\tilde{q}}_j}{\partial x}(t, x_k) - \tilde{f}_j(t, x_k, \hat{\tilde{q}}_j(t, x_k))|$, guarantees the existence of the sliding mode.

By combining all the variables of system in the matrix form, relation (4.26) for the system with diagonal matrix $\tilde{A}$, will be

$$\frac{\partial\hat{\tilde{Q}}}{\partial t} + \tilde{A}(t,x,\hat{\tilde{Q}})\frac{\partial\hat{\tilde{Q}}}{\partial x} = \tilde{f}(t,x,\hat{\tilde{Q}}) + \sum_{k=1}^{m-1}\tilde{L}_k(x,\hat{\tilde{Q}})\text{sign}\left(\tilde{Y}_k(t) - \hat{\tilde{Q}}(t,x_k)\right), \tag{4.31}$$

where

$$\hat{\tilde{Q}}(t,x) = [\hat{\tilde{q}}_1(t,x)\cdots\hat{\tilde{q}}_n(t,x)]^T,$$

$$\hat{\tilde{Q}}_k(t) = [\hat{\tilde{q}}_1(t,x_k)\cdots\hat{\tilde{q}}_n(t,x_k)]^T,$$

$$\tilde{Y}_k(t) = [\tilde{q}_1(t,x_k)\cdots\tilde{q}_n(t,x_k)]^T,$$

$$\tilde{L}_k(x,\hat{\tilde{Q}}) = diag[\tilde{L}_{1k}(x,\hat{\tilde{q}}_1(t,x)),\cdots,\tilde{L}_{nk}(x,\hat{\tilde{q}}_n(t,x))].$$

4.4 Designing Anomaly Detector

Consider a system in the original variable $Q(t,x)$ including a disturbance depending on the unknown vector parameter $d \in \mathbb{R}^n$ as

$$\frac{\partial Q}{\partial t} + A(t,x,Q)\frac{\partial Q}{\partial x} = d(t,x), \tag{4.32}$$

with the measurements as

$$y = Q(t,x). \tag{4.33}$$

The goal is to estimate the parameter $d(t,x)$.

Designing the distributed observer as

$$\frac{\partial\hat{Q}}{\partial t} + A(t,x,\hat{Q})\frac{\partial\hat{Q}}{\partial x} = L\text{sign}\sigma, \tag{4.34}$$

where

$$\sigma = Q(t,0) - \hat{Q}(t,0). \tag{4.35}$$

Then the state of the observer converges to the state of the system as $t \to \infty$

$$\hat{Q}(t,x) - Q(t,x) \to 0 \tag{4.36}$$

In order to prove the existence of the sliding mode, let us introduce the following Lyapunov candidate as

$$V = (\text{sign}\sigma)^T \sigma \geq 0. \tag{4.37}$$

For $\dot{\sigma}$ we have

$$\dot{\sigma} = \dot{Q}(t,0) - \dot{\hat{Q}}(t,0) \tag{4.38}$$

$$= A(t,0,Q)\frac{\partial Q}{\partial x}(t,0) - d - A(t,0,\hat{Q})\frac{\partial \hat{Q}}{\partial x}(t,0) - L\text{sign}\sigma \tag{4.39}$$

having $|L| > |A(t,0,Q)\frac{\partial Q}{\partial x}(t,0) - d - A(t,0,\hat{Q})\frac{\partial \hat{Q}}{\partial x}(t,0)|$, guarantees the existence of the sliding mode. The estimate of d is determined by equivalent control law $\hat{d} = \{L\text{sign}\sigma\}_{eq.}$.

4.5 Application of State Observer: Fluid Flow in a Pipe

In this section the focus is on designing the nonlinear observer for fluid flow in a pipe. For more information about pipelines and the related problems refer to Geiger & Werner (2003), Leckerkennung (2003), Matko et al. (2000) and Matko et al.

(2001).

The general model for fluid flow is provided by Euler equations by conservation of mass and conservation of momentum equations as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0, \tag{4.40}$$

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} \right] + \nabla p = 0, \tag{4.41}$$

where ρ is the fluid density, $\vec{u}$ is the velocity vector field and p is the pressure [1].

Assuming the fluid satisfies the ideal gas law: $p = \rho \mathcal{R} T$, (4.40) can be written as

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + p \frac{\partial u}{\partial x} = 0, \tag{4.42}$$

and writing (4.41) in one-dimensional space we have

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} = 0. \tag{4.43}$$

For an ideal gas, pressure, density and the speed of sound c are related through

$$p = c^2 \rho. \tag{4.44}$$

Including disturbances f_1 and f_2 in the right-hand side, (4.42) and (4.43) can be

[1] In this section a one-dimensional flow in pipe is considered so $\nabla = \frac{\partial}{\partial x}$.

written in the matrix form as

$$\frac{\partial}{\partial t}\begin{bmatrix} p \\ u \end{bmatrix} + A(p,u)\frac{\partial}{\partial x}\begin{bmatrix} p \\ u \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \tag{4.45}$$

where

$$A(p,u) = \begin{bmatrix} u & p \\ \frac{c^2}{p} & u \end{bmatrix}. \tag{4.46}$$

In order to decouple the equations, a transformation similar to Aamo et al. (2006) is employed

$$q_1 = c\ln\frac{p}{\bar{p}} + u - \bar{u}, \tag{4.47}$$

$$q_2 = -c\ln\frac{p}{\bar{p}} + u - \bar{u} \tag{4.48}$$

where the point $(\bar{p}, \bar{u})$ corresponds to the nominal values in the new coordinates. The pressure and velocity are transformed into the new variables q_1 and q_2. Taking the time derivative of q_1 and q_2, substituting values of p_t and u_t and using q_{1x} and q_{2x}, we have the following decoupled equations

$$\frac{\partial q_1}{\partial t} + \tilde{a}_1\frac{\partial q_1}{\partial x} = \tilde{f}_1, \tag{4.49}$$

$$\frac{\partial q_2}{\partial t} + \tilde{a}_2\frac{\partial q_2}{\partial x} = \tilde{f}_2, \tag{4.50}$$

where

$$\tilde{a}_1 = u + c, \quad \tilde{a}_2 = u - c, \quad \tilde{f}_1 = \frac{c}{p}f_1 + f_2, \quad \tilde{f}_2 = -\frac{c}{p}f_1 + f_2$$

$$u = \bar{u} + \frac{q_1 + q_2}{2}, \quad p = \bar{p}e^{\frac{q_1+q_2}{2c}}.$$

Linearization around q_1 and q_2, (4.49) and (4.50) will be

$$\frac{\partial q_1}{\partial t} + \tilde{a}_{1l}\frac{\partial q_1}{\partial x} = \tilde{f}_{1l}, \tag{4.51}$$
$$\frac{\partial q_2}{\partial t} + \tilde{a}_{2l}\frac{\partial q_2}{\partial x} = \tilde{f}_{2l}, \tag{4.52}$$

where

$$\tilde{a}_{1l} = \bar{u} + c, \quad \tilde{a}_{2l} = \bar{u} - c, \quad \tilde{f}_{1l} = \frac{c}{\bar{p}}f_1 + f_2, \quad \tilde{f}_{2l} = -\frac{c}{\bar{p}}f_1 + f_2.$$

For the system (4.51), (4.52) the corresponding distributed observers, using the design from section 4.3, are expressed as

$$\frac{\partial \hat{q}_1}{\partial t} + \tilde{a}_{1l}\frac{\partial \hat{q}_1}{\partial x} = \tilde{f}_{1l} + \sum_k L_{1k}(x)\text{sign}\left(y_{1k}(t) - \hat{q}_1(t,x)\right), \tag{4.53}$$
$$\frac{\partial \hat{q}_2}{\partial t} + \tilde{a}_{2l}\frac{\partial \hat{q}_2}{\partial x} = \tilde{f}_{2l} + \sum_k L_{2k}(x)\text{sign}\left(y_{2k}(t) - \hat{q}_2(t,x)\right). \tag{4.54}$$

4.6 Application of Anomaly Detector: Leak Detection in Pipelines

Let us consider a system such as is (4.32) with a specific disturbance as $f(t,x,Q,d) = b(t,x,Q)w\delta(x-x^*)$, where w and x^* represent the intensity and position of the anomaly. This type of disturbance is applicable in estimation the leakage in pipelines. Our goal is to determine these two parameters in the system. Based on (4.34) and (**??**) we have the following observer

$$\frac{\partial \hat{Q}}{\partial t} + A(t,x,\hat{Q})\frac{\partial \hat{Q}}{\partial x} = \hat{b}(t,x,Q)\hat{w}\delta(x-\hat{x}^*), \tag{4.55}$$

$$\dot{\hat{d}} = \begin{bmatrix} \dot{\hat{x}}^* \\ \\ \dot{\hat{w}} \end{bmatrix} = L\text{sign}\sigma. \tag{4.56}$$

Since we are dealing with two unknowns, w and x^*, having a system with two measurable parameters $n = 2$ ($j = 1, 2$), for instance pressure and velocity (p and u) of the fluid in pipeline, is enough to determine the disturbance parameters, so the gain matrix will be $L \in \mathbb{R}^{2\times 2}$ and $\sigma = [\sigma_1 \quad \sigma_2]^T$.

Following the same steps from (**??**) to (**??**) the system (4.55) is written as

$$\frac{\partial \tilde{q}_j}{\partial t} + \tilde{a}_j(t,x)\frac{\partial \tilde{q}_j}{\partial x} = \tilde{b}_j(t,x)w\delta(x - x^*), \tag{4.57}$$

$$\tilde{q}_j(0,x) = \tilde{\phi}_j(x), \quad \tilde{q}_j(t,0) = \tilde{y}_{0j}(t), \tag{4.58}$$

and for the observer

$$\frac{\partial \hat{\tilde{q}}_j}{\partial t} + \tilde{a}_j(t,x)\frac{\partial \hat{\tilde{q}}_j}{\partial x} = \hat{\tilde{b}}_j(t,x)\hat{w}\delta(x - \hat{x}^*), \tag{4.59}$$

where the estimates $\hat{x}^*$ and $\hat{w}$ satisfy the equation (4.56). The initial and boundary conditions for the observer (4.58) are

$$\hat{\tilde{q}}_j(0,x) = 0, \quad \hat{\tilde{q}}_j(t,0) = \tilde{y}_{0j}(t), \quad \hat{w}(0) = \hat{w}_0, \quad \hat{x}^*(0) = \hat{x}^*_0. \tag{4.60}$$

4.7 Simulation

In this section the simulation results for the applications in sections 4.5 and 4.6 are provided.

4.7.1 State Observer

For the system in (4.49) and (4.50) the initial conditions are considered as half-normal distribution

$$p(0,x) = \frac{2}{\sqrt{\pi}}e^{-x^2}, \quad u(0,x) = \frac{2}{\sqrt{\pi}}e^{-(x-0.5)^2}. \tag{4.61}$$

For the boundary condition it is assumed that only measurements at the upstream (or downstream) are available

$$p(t,0) = \sin t, \quad u(t,l) = \cos t. \tag{4.62}$$

The system is described by (4.49), (4.50) and the observer is presented by (4.53) and (4.54). By writing the observer equation using boundary condition at upstream/downstream for pressure/velocity, we have

$$\frac{\partial \hat{p}(t,0)}{\partial t} + a_1 \frac{\partial \hat{p}(t,0)}{\partial x} = L(x)\text{sign}[p(t,0) - \hat{p}(t,0)], \tag{4.63}$$

$$\frac{\partial \hat{u}(t,l)}{\partial t} - a_2 \frac{\partial \hat{u}(t,l)}{\partial x} = L(x)\text{sign}[u(t,l) - \hat{u}(t,l)], \tag{4.64}$$

These observers are designed to construct the sliding mode at the upstream/downstream of the pipeline. The data generated at the upstream/downstream employed to predict the states of the system over the entire spatial variable. $L(x)$ is chosen according to the recommendations in section 4.3.

Figures 4.2 and 4.3 show performance of the observers for the pressure and velocity estimation in the pipe. Figures 4.4 and 4.5 present the sliding mode constructed on the upstream and downstream of the pipe, respectively. As it can be seen they converge to zero and keep chattering around the equilibrium. Figures 4.6 and 4.7 show the differences between the pressure and velocity and their estimates respect to time and the pipe length. For the MATLAB code refer to A.4.

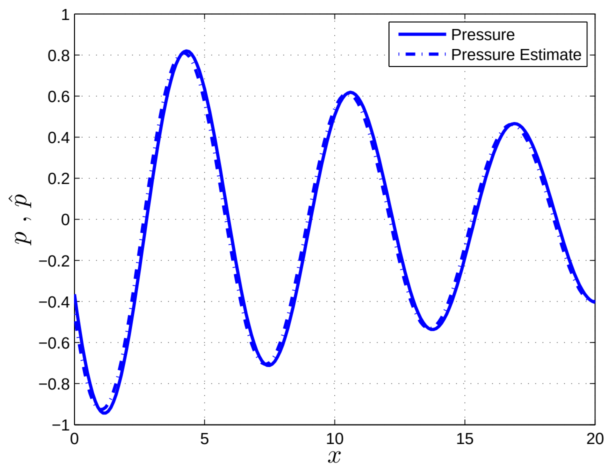

Figure 4.2: Pressure and pressure observer, after 50 seconds over the pipe length.

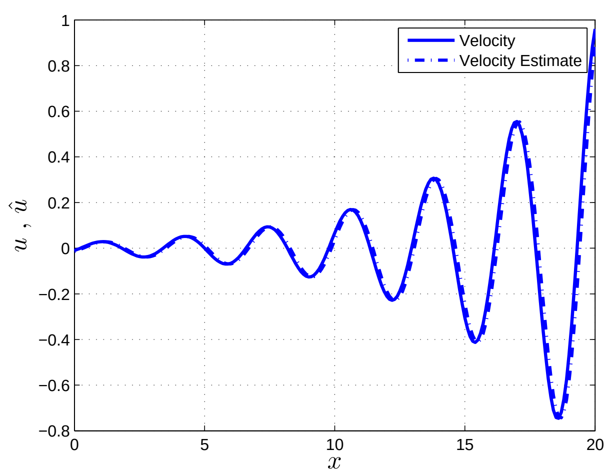

Figure 4.3: Velocity and velocity observer, after 50 seconds over the pipe length.

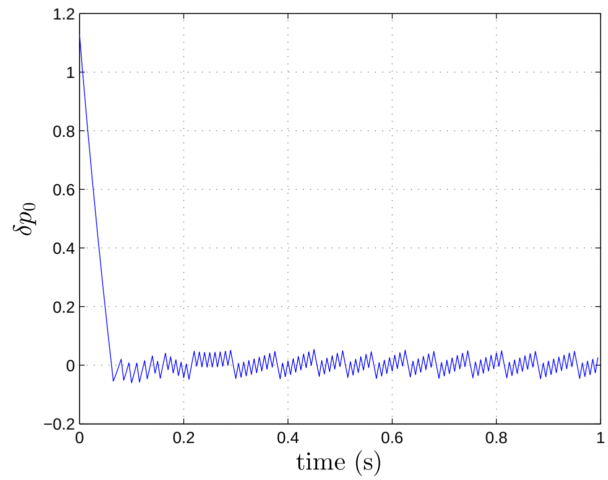

Figure 4.4: Sliding mode for pressure observer at upstream.

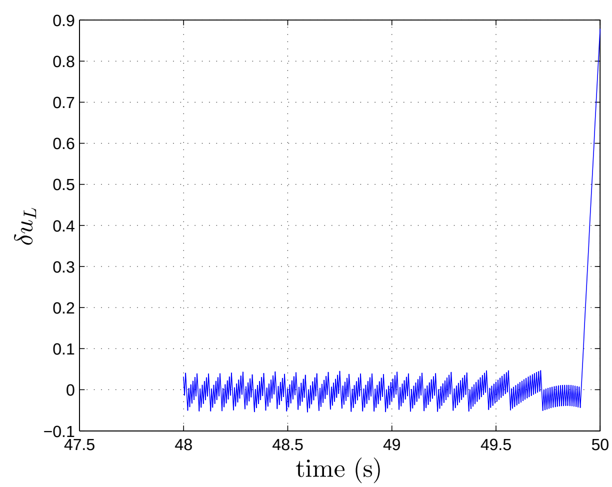

Figure 4.5: Sliding mode for velocity observer at downstream.

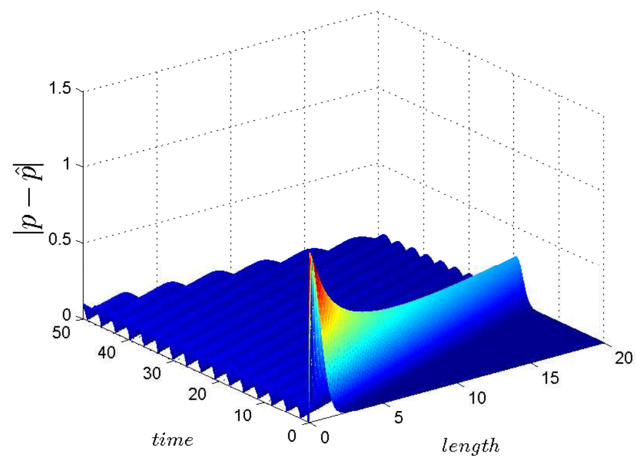

Figure 4.6: Difference between the pressure and pressure estimate.

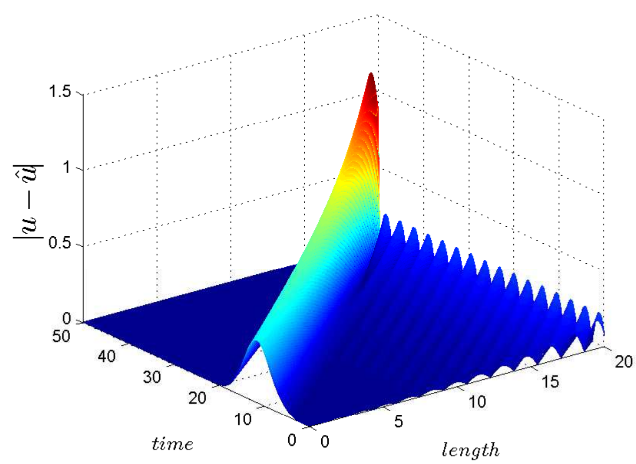

Figure 4.7: Difference between the velocity and velocity estimate.

4.7.2 Leak Detection

In this section the situation of estimating the intensity of leakage only using boundary measurement is simulated. Here the same initial and boundary conditions as in section 4.7.1 are assumed.

Figure 4.8 shows performance of the observer in detecting the pressure drop in the system after 120 seconds. Figure 4.9 represents estimation of the anomaly over time. The disturbance intensity in the system has been set to $w = -3$ and as it could be seen in Figure 4.9, $\hat{w}$ reaches the nominal value of w after few seconds. Figure 4.10 depicts the sliding mode constructed in the observer. For the MATLAB code refer to A.5.

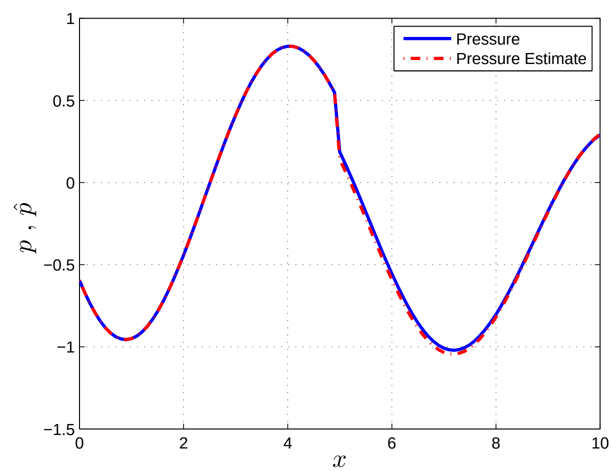

Figure 4.8: Pressure and pressure estimate along the length of the pipe.

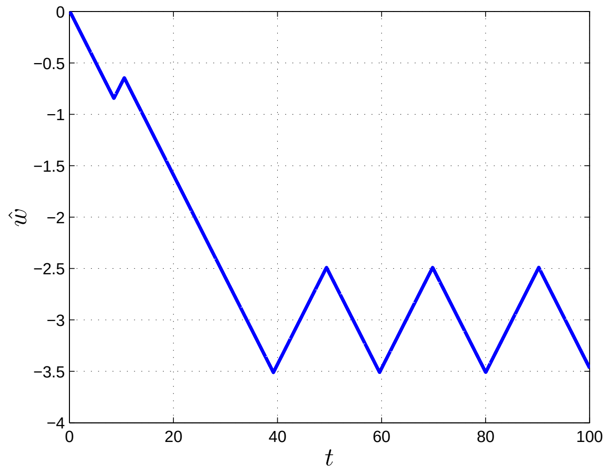

Figure 4.9: Estimation of the leakage intensity.

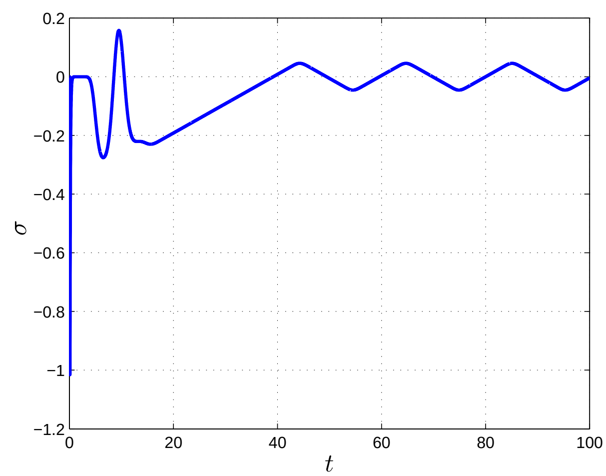

Figure 4.10: Sliding mode for anomaly detection.

4.8 Conclusion and Future Work

In this chapter a nonlinear observer for a class of first-order PDEs known as advection equation is developed. The design which is based on the idea of variable structure systems with sliding mode, leads to a fast converging observer. The boundary measurements are provided as the input of the observer, and the number of boundary measurements could be as small as just one measurement for each variable. In addition, an anomaly detection system is developed which is able to determine the parameters of the possible disturbance in the system. To demonstrate some applications of the suggested methods, the performance of the observer and the anomaly detection system have been simulated for a system of fluid flow pipeline.

As the future work for this chapter, removing restrictions on the differential operator and the disturbance function can be considered. As well as accounting for the situation of having an increasing boundary condition that leads to the presence of shock wave. The suggested observer and anomaly detection system can be applied to different practical cases. In addition, the performance of the observer and anomaly detector can be examined under the situations of having turbulent flow or under the noise condition.

Chapter 5

Conclusion and Future work

This research explored designing sliding mode observer for different classes of distributed parameter systems. The main tool in designing the state observers was sliding mode control theory and the idea of variable structure systems. Different types of systems described by partial differential equations such as advection equation, Burgers' equation, Euler equations, etc. have been studied. In dealing with some first-order PDEs, one might encounter the shock wave situation which is the unwanted discontinuity in the solution in spite of smooth initial and boundary conditions. The shock wave situation, its properties and solutions were discussed in this research. In designing the state observer, by using the theory of sliding mode they are designed to be robust to the mismatches between the model and the system. In addition, an anomaly detection system was developed to estimate the parameters of possible anomaly in the system. Most of the time in the process of designing sliding manifold for sliding observer, the designer does not have the freedom to choose the desired roots. However, this problem has been addressed in chapter 2 by suggesting a novel transformation which allowed to assign the arbitrary roots. In addition, a formula for designing the observer gain was proposed. For each chapter, the conclusion is provided that discussed the suggestions for future work. In general the idea of removing different assumptions and restrictions on the systems, and considering the presence of shock wave could be considered as a general idea to continue the research. In addition, developing sliding mode controller for the mentioned distributed parameter systems are under consideration by the author.

Appendix A

Matlab Code

In this section, MATLAB codes for the examples and simulations in the research are presented. The code used to create the various plots have not been included for brevity.

A.1 SMO for Diffusion Equation

```
clear all; close all; clc
% time variable
T    = 0.3;
dt   = T/6000;
M    = T/dt;  % number of time steps to be iterated over
t(1) = 0;

% spatial variable
length  = 1;
dx = length/100;
N  = (length/dx)+1;    % number of grid points in x
x  = 0:dx:length;      % vector of x values, to be used for plotting

% second derivative ratio
```

```
r = dt/dx^2;

% system boundary conditions
%(any time at beginning and end of spatial variable)
Q(:,1) = 0;
Q(:,N) = 0;

% system initial condition (anywhere at time=0)
for j  = 2:N-1;
Q(1,j) = (2*x(j))/(1+x(j)^2);
end

% observer boundary conditions
Qh(:,1) = 0;
Qh(:,N) = 0;

% observer initial condition
for j   = 2:N-1;
Qh(1,j) = x(j);
end

% observer gain using eigenvalues
m    = 10;
Lbar = 0.000001;%10^13;
L(1) = 0;
c    = ones(1,m);
for k=1:m
    L(k+1) = Lbar * sin(k*pi*x(k)/length)*((-1)^(k-1)*(k-(m+1))^(m-1))
             /(c(k)*(-1)^(m-k)*factorial(k-1)*factorial(m-k));
end
L  = sum(L);

% updating through the time
for i = 1:M,
    t(i+1) = t(i)+dt;

    % updating at each time for spatial variable except
    % the boundary points
    for  j = 2:N-1;
        % outputs
```

```
        y(i,:)      = sum(Q(i,:));
        yh(i,:)     = sum(Qh(i,:));
        sigma(i,:) = y(i,:)-yh(i,:);

        % system
        Q(i+1,j)  = Q(i,j)  + r *( Q(i,j+1)  - 2*Q(i,j)  + Q(i,j-1));

        %observer
        Qh(i+1,j) = Qh(i,j) + r *( Qh(i,j+1) - 2*Qh(i,j) + Qh(i,j-1))
                   + L * sign(sigma(i,:));

    end

end

    figure(1)
    mesh(x,t,Q)
    xlabel('$length$','FontSize',12,'interpreter','latex')
    ylabel('$time$','FontSize',12,'interpreter','latex')
    zlabel('$Q$','FontSize',14,'interpreter','latex')
    ylim([0,0.3])

    figure(2)
    mesh(x,t,Qh)
    xlabel('$length$','FontSize',12,'interpreter','latex')
    ylabel('$time$','FontSize',12,'interpreter','latex')
    zlabel('$\widehat{Q}$','FontSize',14,'interpreter','latex')
    ylim([0,0.3])

    figure(3)
    mesh(x,t,abs(Q-Qh))
    xlabel('$length$','FontSize',12,'interpreter','latex')
    ylabel('$time$','FontSize',12,'interpreter','latex')
    zlabel('$|Q-\widehat{Q}|$','FontSize',14,'interpreter','latex')
    ylim([0,0.3])

    figure(4)
    plot(t(1:400),sigma(1:400));     grid on
    ylabel('$\sigma$','FontSize',12,'interpreter','latex')
    xlabel('$time$','FontSize',12,'interpreter','latex')
```

A.2 Burgers' Equation with Two shock Waves

```
clear all; close all; clc
% ut + u ux = 0
% u(0,x) = u0(x) = ul x<a, um a=<x<b, ur x>=b
% ul>um>ur

ul = 2;
um = 1;
ur = 0;
a  = 1;
b  = 2;

dx0 = 0.1;
dt  = 0.01;

s1 = (ul+um)/2;
s2 = (um+ur)/2;
s3 = (s1+s2)/2;

% intersection of shocks
ts = (b-a)/(s1-s2);
xs = (s1*b-s2*a)/(s1-s2);

x01 = ( (s1-ul)*b+(ul-s2)*a )/(s1-s2);
x02 = ( (s1-ur)*b+(ur-s2)*a )/(s1-s2);

% after shock waves intersection
tf = 2;
c = xs-s3*ts;
x0min = (s3-ul)*tf + c;
x0max = (s3-ur)*tf + c;

for x0 = x01:dx0:a-dx0
    for t = 0:dt:(a-x0)/(ul-s1)
    x = ul*t + x0;
```

```
    plot(x,t,'.'); hold on
    end
end

for x0 = a;
    for t = 0:dt:ts
    x = s1*t + x0;
    plot(x,t,'. r'); hold on
    end
end

for x0 = a+dx0:dx0:xs-um*ts;
    for t = 0:dt:(x0-a)/(s1-um);
    x = um*t + x0;
    plot(x,t,'.'); hold on
    end
end

for x0 = xs-um*ts:dx0:b-dx0;
    for t = 0:dt:(b-x0)/(um-s2)
    x = um*t + x0;
    plot(x,t,'.'); hold on
    end
end

for x0 = b;
    for t = 0:dt:ts
    x = s2*t + x0;
    plot(x,t,'. r'); hold on
    end
end

for x0 = b+dx0:dx0:x02;
    for t = 0:dt:(b-x0)/(ur-s2);
    x = ur*t + x0;
    plot(x,t,'.'); hold on; grid on
    end
end

for x0 = x0min:dx0:x01
```

```
    for t = 0:dt:(c-x0)/(ul-s3);
    x = ul*t + x0;
    plot(x,t,'.'); hold on
    end
end

for x0 = c
    for t = ts:dt:tf;
    x = s3*t + x0;
    plot(x,t,'.r'); hold on
    end
end

for x0 = x02:dx0:x0max;
    for t = 0:dt:(c-x0)/(ur-s3);
    x = ur*t + x0;
    plot(x,t,'.'); hold on; grid on
    end
end

xlabel('$x$','FontSize',16,'interpreter','latex')
ylabel('$t$','FontSize',16,'interpreter','latex')
```

A.3 Viscous Burgers' Equation

```
clear all; close all; clc
uL = 4;
uR = 2;
s  = (uL+uR)/2;
a  = 3;
T  = 1.5;
l  = 8;
nu = 0.01;
x0 = 5;

for t = 0:0.01:T;
    for x = 0:0.05:l;
        u = uR + ((uL-uR)/2)*tanh( (x-x0-s*t)*(uL-uR)/(4*nu));
```

```
        plot3(t,x,u,'b.'); hold on
    end
end

xlabel('$t$','FontSize',16,'interpreter','latex')
ylabel('$x$','FontSize',16,'interpreter','latex')
zlabel('$u$','FontSize',16,'interpreter','latex')
grid on
```

A.4 SMO for System of Advection equations

```
clear all; close all; clc
T    = 50;
dt   = 0.01;
t(1) = 0;
M    = T/dt;

xmin = 0;
xmax = 20;
dx   = 0.1;
x    = [xmin:dx:xmax];
N    = round((xmax-xmin)/dx);

% system parameters
c    = 0.75;
tiu  = 0.25;

ap  = (c+tiu);
ahp = (c+tiu);

% observre gain
L   =  10   ;

% system initial
p(1,:) = (2/sqrt(pi))*exp(-x.^2);

% observer initial
ph(1,:) = 0*x;
```

```
% observer
ph0     = 0;

for i = 1:T/dt,
    t(i+1)=t(i)+dt;
    t;

    % system
    p0 = sin(t(i));
    p(i+1,:) = p(i,:) - (dt*ap/dx) .* [p(i,1)-p0 diff(p(i,:))];

    % observer on the begining of the pipe
    ph0(i+1) = ph0(i)  - dt * L * sign(ph0(i)-p(i,1));

    % predicting system based on boundry condition
    ph(i+1,:) = ph(i,:) - dt * ahp
                .* ([ph(i,1)-ph0(i) diff(ph(i,:))]./dx);

    % checking sliding mode
    deltap0(i) = p(i,1)-ph0(i);

end

    figure(1)
    plot(t(1:200),deltap0(1:200))
    xlabel('time (s)','FontSize',16,'interpreter','latex')
    ylabel('$\delta{p_{0}}$' ,'FontSize',16,'interpreter','latex')
    grid on;

    figure(2)
    plot(x, p(i,:), 'b' , x, ph(i,:), 'b-.','LineWidth',2)
    xlabel('$x$','FontSize',16,'interpreter','latex')
    grid
    ylabel('$p$, $\hat{p}$','FontSize',16,'interpreter','latex')
    legend('Pressure', 'Pressure Estimate')

    figure(3)
    mesh(x,t,abs(p-ph))
    xlabel('$length$','FontSize',12,'interpreter','latex')
```

```
    ylabel('$time$','FontSize',12,'interpreter','latex')
    zlabel('$|p-\hat{p}|$','FontSize',14,'interpreter','latex')
    set(gca,'ylim',[0 50])

xb    = [xmax:-dx:xmin];

au  = - 0.5;
ahu = -0.5;

% system final
u(M+1,:) = (2/sqrt(pi))*exp(-(xb-0.5).^2);

% observer initial
uh(M+1,:) = 0*x;

% observer
uhL(M+1) = 0;

tb(M+1)  = T;

    for ii = M+1:-1:2,
    tb(ii-1)= tb(ii)-dt;
    tb;

    % system
    uL = cos(t(ii));
    u(ii-1,:) = u(ii,:) - (dt*au /dx) .* [diff(u(ii,:)) uL-u(ii,N+1)];

    % observer on the begining of the pipe
    uhL(ii-1) = uhL(ii)  - dt * L * sign(uhL(ii) - u(ii,N+1));

    % predicting system based on boundry condition
    uh(ii-1,:) = uh(ii,:) - (dt*ahu/dx)
                 .* [diff(uh(ii,:)) uhL(ii)-uh(ii,N+1)];

   % checking sliding mode
    deltauL(ii) = -uhL(ii) + u(ii,N+1);

    end
```

```
    figure(4)
    plot(tb(M+1:-1:M+1-200),deltauL(M+1:-1:M+1-200))
    xlabel('time (s)','FontSize',16,'interpreter','latex')
    ylabel('$\delta{u_{L}}$','FontSize',16 'interpreter','latex')
    grid on;

    figure(5)
    plot(x, u(ii,:), 'b', x, uh(ii,:), 'b-.','LineWidth',2)
    xlabel('$x$','FontSize',16,'interpreter','latex')
    grid
    ylabel('$u$, $\hat{u}$','FontSize',16,'interpreter','latex')
    legend('Velocity', 'Velocity Estimate')

    figure(6)
    mesh(x,tb,abs(u-uh))
    xlabel('$length$','FontSize',12,'interpreter','latex')
    ylabel('$time$','FontSize',12,'interpreter','latex')
    zlabel('$|u-\hat{u}|$','FontSize',14,'interpreter','latex')
    set(gca,'ylim',[0 50])
```

A.5 Leak Detection System

```
clear all; close all; clc

T    = 100;
xmin = 0;
xmax = 10;
dt   = 0.01;
dx   = 0.1;
x    = [xmin:dx:xmax];
t(1) = 0;
N    = round((xmax-xmin)/dx);

% System parameters
c    = 0.75;
tiu  = 0.25;
a1   = (c+tiu);
```

```
% system initila condition
p(1,:) = (2/sqrt(pi))*exp(-x.^2);

% Anomaly parametr
w   = -3;
f1  = [zeros(1,N/2) w zeros(1,N/2)];

% Anomaly detector
h1    = xmax/a1;
hmax  = h1;
% Detector gain
L11   = 0.1;
% Detector Initial Conditions
ph(1,:)     = 0*x;
f1h(1,:)    = 0*x;
sigma1(1,:) = 0;
wh(1,:)     =  0;

for i=1:T/dt,
    t(i+1)=t(i)+dt;
    t;

    % system boundary at upstream (measuremnts at upstream)
    p0 =  sin(t(i));
    % system simulation
    p(i+1,:) = p(i,:)-dt*a1.*([p(i,1)-p0 diff(p(i,:))]./dx)+ dt*f1;

    % detector
    if i<= hmax
    ph(i+1,:) = ph(1,:);
    wh(i+1,:) = wh(1,:) ;

    else

    f1h(i,:)  = [zeros(1,N/2) wh(i) zeros(1,N/2)];

    ph(i+1,:) = ph(i,:)-dt*a1.*([ph(i,1)-p0 diff(ph(i,:))]
                ./dx)+dt*f1h(i,:);

    sigma1(i+1,:) = p(i+1,N+1) - p(i+1-h1,1)- ph(i+1,N+1)
```

```
                + ph(i+1-h1,1);

    wh(i+1,:)  = wh(i,:)  + dt *  L11 * sign(sigma1(i,:)) ;

   end

end

    figure(1)
    plot(x, p(i,:) , x , ph(i,:), 'r-.','LineWidth',2)
    xlabel('$x$','FontSize',16,'interpreter','latex')
    grid
    ylabel('$p$, $\hat{p}$','FontSize',16,'interpreter','latex' )
    legend('Pressure', 'Pressure Estimate')

    figure(2)
    plot(t, wh, 'b' ,'LineWidth',2);
    xlim([0 T])
    xlabel('$t$','FontSize',16,'interpreter','latex')
    grid
    ylabel('$\hat{w}$','FontSize',16,'interpreter','latex' )

    figure(3)
    plot(t, sigma1, 'b','LineWidth',2)
    xlim([0 T])
    xlabel('$t$','FontSize',16,'interpreter','latex')
    grid
    ylabel('$\sigma$ ','FontSize',16,'interpreter','latex' )
```

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