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Solving Nonlinear Aeronautical Problems Using the Carleman Linearization Method

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SOLVING NONLINEAR AERONAUTICAL PROBLEMS USING THE CARLEMAN LINEARIZATION METHOD

by

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THE MASTER'S THESIS SOLVING NONLINEAR AERONAUTICAL PROBLEMS USING THE CARLEMAN LINEARIZATION METHOD

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This Master's Thesis was prepared under the direction of the candidate's Thesis Committee Chair, Dr. Richard S. Baty, Assistant Professor of Aeronautical Science, the candidate's Thesis Committee Members, Dr. Roy S. Baty, Assistant Adjunct Professor of Aeronautical Science, and Dr. John C. Hogan, Assistant Adjunct Professor of Aeronautical Science.

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ABSTRACT

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Many problems in aeronautics can be described in terms of nonlinear systems of equations. Carleman developed a technique to linearize such equations that could lead to analytical solutions of nonlinear problems. Nonlinear problems are difficult to solve in closed form and therefore the construction of such solutions is usually nontrivial. This thesis will apply the Carleman linearization technique to three model problems: a two-degree-of-freedom (2DOF) ballistic trajectory, Blasius' boundary layer, and Van der Pol's equation and evaluate how well the technique can adequately approximate the solutions of these ordinary differential equations.

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CHAPTER I

INTRODUCTION

The major objective of this thesis applies and assesses the use of the Carleman linearization scheme for the approximation of solutions of nonlinear ordinary differential equations (ODEs). This thesis will show that the Carleman linearization technique has utility in solving a broad class of nonlinear aeronautical problems and more specifically, the nonlinear two-degree-of-freedom ballistic trajectory problem.

This research project investigated the flight characteristics of objects re-entering the earth's atmosphere. The work was initiated by developing a two-degree-of-freedom (2DOF) numerical model to study the effects of mass perturbations on the trajectory of a reentry vehicle (RV). The simulation of reentry flight paths with 2DOF models required the use of several specific models and different physical assumptions of atmospheric re-entry. All of the 2DOF trajectory models had closed form solutions. Unfortunately, these simple flight mechanics descriptions are not typical of realistic RV flight mechanics problems, which usually have non-trivial reentry angles. No simple analytical solutions were found for RV flight mechanics problems with non-trivial reentry angles. Few closed form solutions are known because the aerodynamic drag term is a nonlinear function, which is proportional to the square of the velocity. Even though analytical solutions have been found for some special cases of the nonlinear equations of motion, the equations are so complicated that no general closed-form solutions are known.

Most engineers studying RV flight mechanics apply numerical methods to estimate reentry motion. Numerical methods essentially approximate the behavior of an RV over a series of fixed time intervals or cells. The equations of motion are computed over the entire length of the trajectory by passing from one time cell to the next. The resulting computed trajectories are not exact solutions but only approximations. The accuracy of such approximations depends on the size of the time interval or time-step. The larger the time-step, the worse the approximation can be, while, the smaller the time-step the better the approximation may be.

Computing trajectories using numerical methods is common in flight mechanics. Because of the power of the numerical techniques to simulate complicated problems, most reentry trajectory models are solved computationally. The down side of such numerical models is an over-reliance on the computer and a de-emphasis on the underling physics of RV kinematics. The goal of this thesis was to apply an analytical method to approximate RV motion, which would hopefully lead to more insight into the physics of reentry flight mechanics. This thesis applies the analytical linearization method developed by Carleman [9] to approximate solutions of example nonlinear problems in aeronautics. To date, the Carleman linearization has not been applied to simulate the aeronautics problems considered here. Moreover, the Carleman technique has not been applied to concrete examples of inhomogeneous nonlinear problems. In this study, the Carleman linearization method is applied to three model aeronautical problems: 1) a two-degree-of-freedom problem from flight mechanics, 2) Blasius' boundary layer from incompressible flow, and 3) Van der Pol's equation from guidance and control. The approximate analytical solutions obtained for these nonlinear problems using the Carleman linearization are then compared with numerical solutions of high resolution.

In this thesis, Chapter II reviews the historical development and relevant literature of the Carleman linearization or embedding technique. Chapter III develops the Carleman methodology for a system of inhomogeneous ordinary differential equations by deriving the Carleman embedding in the context of a two-degree-of-freedom flight mechanics problem. Chapter IV applies the Carleman method to the three model problems: the trajectory problem, the boundary layer problem, and the Van der Pol oscillator. Each of these problems is derived explicitly using the global linearization method. The results of the numerical experiments are presented in plot format, showing the Carleman solutions in contrast to the high-resolution numerical solutions for several different approximations. Chapter V then discussed the results. A converging Taylor series expansion of a logarithmic function is compared to the convergence of the Carleman scheme. Chapter VI completes the thesis with conclusions and recommendations. Conclusions are made as to the utility of this method and suggestions are made for future research.

CHAPTER II

HISTORY OF THE CARLEMAN LINEARIZATION METHOD

Anderson's text (1989) describes simple equations of motion for atmospheric reentry. It was the current author's interest in atmospheric reentry problems that lead to the study of the equations considered here. The equations of motion discussed by Anderson show that the velocity of the reentry body goes to zero before the vehicle reaches the ground. These equations of atmospheric reentry are a special case of more general equations. The specific example was developed to uncouple the drag force terms and has utility because it leads to a mathematical problem that has a closed form solution. This case models the motion of a body, horizontally through a resistive medium with no gravity force. Reagan and Anandakrishnan (1993) also described two similar cases: vertical reentry and steep vertical reentry.

These examples are further special cases of the flight mechanics equations. The vertical reentry problem uncouples the horizontal component of the drag allowing construction of an analytical solution. The second case with a steep reentry angle also uncouples the horizontal and vertical components so that a solution may be found for the vertical velocity. Reagan and Anandakrishnan address other angles of attack and acknowledge that because of the nature of the coupled nonlinear system of equations, there has been no closed form solution discovered.

Engineers and mathematicians usually solve flight mechanics problems with non-trivial reentry angles with numerical methods. In computational schemes, the trajectories are held constant on very small time steps. By doing this, nonlinear equations are simplified via local linear approximations.

A theoretical technique was developed in the 1930s by the mathematician Torsten Carleman (1932) to globally linearize systems of nonlinear polynomial equations. His article, which introduced the linearization method was entitled "Application De La Théorie Des Équations Intégrales Linéaires Aux Systèmes D'Équations Différentielles Non Linaires" which loosely translated means: "The Application of the Theory of Linear Integral Equations to Systems of Non-Linear Differential Equations." Carleman's ideas were born out of remarks made by Henri Poincaré. Poincaré is known for his studies in celestial mechanics and studying oscillatory motion in celestial bodies. Among other things, Poincaré also discovered the theory of special relativity and helped lay the foundation for modern algebraic topology. Poincaré remarked at a 1908 conference in Rome, that one should be able to apply the theory of linear integral equations to the study ordinary non-linear differential equations. From that remark, Carleman worked on an approach to embed a system of non-linear differential equations in to an infinite set of linear equations. The history relating Poincaré and Carleman is reviewed by Montroll and Helleman, (1976). The rest of the history of the development of the Carleman linearization method is outlined from the introduction to the text by Kowalski and Steeb (1991). The Carleman technique essentially remained unused for a little over thirty years before Bellman and Richardson (1963) applied the method to approximate solutions of a nonlinear ODE. Thirteen years later Montroll and Helleman studied the embedding technique in relation to small denominators and secular terms. Then in 1980, Steeb and Wilhelm used Carleman embedding to approximate solutions of the Lotka-Volterra problem. The Lotka-Volterra model is represented by a system of nonlinear equations that have periodic solutions. The Carleman technique was successfully applied to solve the Lotka-Volterra problem.

In 1981, Kerner studied the technique for embedding nonlinear systems into polynomial systems. Also, in 1981, Andrade and Rauh, and Brenig and Fairen studied the Lorenz model and power series expansions for nonlinear systems, respectively, using the Carleman embedding technique. In 1982, Wong demonstrated that a linear operator acting on Banach space could be related to analytic vector fields. This became known as the Carleman linearization or transformation of a vector field. Moreover, a number of other results were discovered about the linearization: 1) Andrade (1982) calculated Lyapunov exponents, 2) Kus (1983) discovered a class of explicitly time-dependent first integrals for the Lorenz model, 3) Steeb (1983) demonstrated that a matrix could be written in terms of Bose operators, and 4) Ermakov (1984) constructed an approximate Monte-Carlo-like solution to nonlinear integral equations via Carleman embedding. In 1986, Esperidiao and Andrade revisited the study of secular terms in Carleman embedding. In 1987, Kowalski related finite dimensional nonlinear systems to problems in Hilbert space. Tsiligiannis and Lyberatos (1987) studied steady state bifurcation and exact multiplicity conditions using the Carleman method. Finally, by 1989, Steeb showed that there is a one-to-one correspondence between solutions of the infinite linear system and solutions of the associated nonlinear finite system for the analytic solutions. Fortunately, Kowalski and Steeb summarized a large portion of this work into one book. This book is the main reference from which most of the history of the Carleman method is outlined.

The Carleman embedding technique is the theoretical method used to approximate solutions of the nonlinear aeronautical problems studied in this thesis. Kowalski and Steeb's book was used extensively to derive the linearization.

In addition to the references on the Carleman embedding technique, the texts by White (1974) and Schlichting (1979), and the NACA Technical Memorandum 1256 by Blasius (1950) were referenced for background information on the boundary layer problem. Lastly, Van der Pol's equation was developed from Bellman's book (1970) and its solutions studied via the Carleman technique.

CHAPTER III

THE CARLEMAN LINEARIZATION METHOD

Research in mathematics often includes thought experiments and as such has an experimental facet. Almost all current technology, from aircraft to computers, was developed using mathematical ideas. Mathematicians take existing tools and apply them in experimental ways to further their understanding. In this way, new mathematical tools are discovered and developed. These new tools can then be applied to engineering and physics problems.

For this study, the Carleman method was applied to linearize nonlinear aeronautical problems. Model problems were posed to minimize many of the mathematical complexities, while retaining the basic physics of the nonlinearity. The simple 2DOF problem captures the basic features of such nonlinear problem. This chapter is devoted to explaining systematically how to derive the Carleman linearization for systems of inhomogeneous ODEs. Once that is done, a wide range of nonlinear problems can analyzed with this method, to see what practical utility the Carleman linearization technique has.

Consider the following problem. Approximate the two-dimensional ballistic trajectory of a bowling ball pitched off the Eiffel Tower at 1 meter per second (m/s) using the Carleman linearization technique. This problem will be used to illustrate the derivation of the Carleman method.

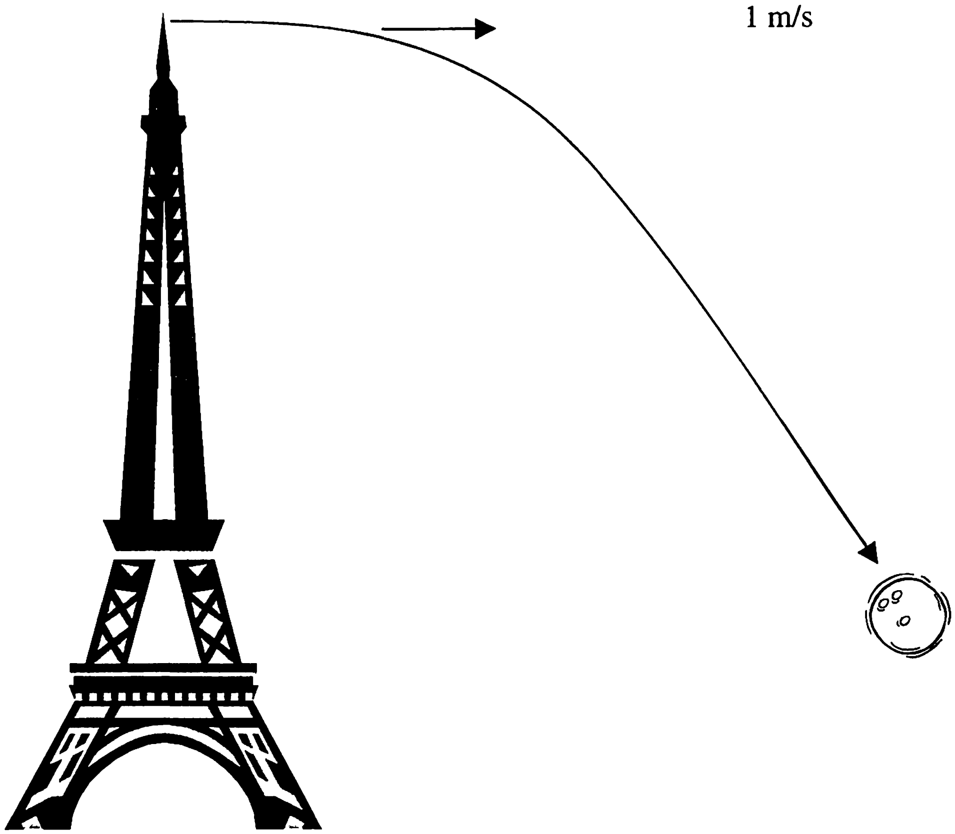

Fig. 1 Bowling ball pitched off the Eiffel Tower

Marion [16] gives the equations of motion for a particle. The equation of motion for a particle falling in a constant gravitation field with a resistive medium is

$$\bar{F} = \bar{F}_g + \bar{F}_r \tag{1}$$

where F_g is the force due to gravity and F_r is the retarding force in the resistive medium. This can be rewritten as

$$\bar{F} = m\bar{g} + \bar{F}_r(v) \tag{2}$$

It sufficient to consider that $F_r(v)$ is proportional to some power of the velocity. This type of approximation can be written as

$$\bar{F} = m\bar{g} - mkv^n \frac{\bar{v}}{v} \tag{3}$$

where k is a positive constant that specifies the strength of the retarding force and $\bar{v}$ is a unit vector in the direction of $\frac{\bar{v}}{v}$ where $\bar{v}$ is the velocity of the relative wind with respect to the bowling ball.

The equations of motion for the trajectory of the bowling ball are developed to determine the bounds of the horizontal and vertical motion. From those bounds, a representation of the speed of the bowling ball, in the form of a fractional power, can be developed within the domain of the bounds of the horizontal and vertical motion. A least squares method is applied to discrete values on the "speed" surface to obtain a polynomial fit of the data. Equations of motion can then be developed in a polynomial form that allows the application of the Carleman technique. As a check, a comparison should be made of the polynomial's positive agreement to the original function to ensure the fit's accuracy.

Horizontal Equation of Motion

Next look at the example of horizontal motion of a particle in a resisting medium. This problem will be used to motivate the equation of motion for the bowling ball. In this case, Newton's second law $F = ma$ gives:

$$m\bar{a} = m\frac{d\bar{v}}{dt} = -km\bar{v} \tag{4}$$

The magnitude of the resisting force is the norm of $-kmv$ where k is a constant. Now multiply both sides of the equation by dt and divide by v. The mass m cancels out. Integrate both sides to solve for v:

$$\int \frac{d\bar{v}}{v} = -k\int dt \tag{5}$$

$$\ln\, \bar{v} = -kt + C \tag{6}$$

To evaluate C define v at time (t) equal to 0 (written as $v[t = 0] = v_0$). The constant C then becomes

$$C = \ln\, v_0 \tag{7}$$

Now solve for v.

$$\bar{v} = \bar{v}_0 e^{-kt} \tag{8}$$

The same approach can be used to solve for horizontal and vertical velocity of the bowling ball. To do this, use the aerodynamic equation of the drag force derived from Anderson [1]. In addition, define the notation u to be the horizontal velocity, v for vertical velocity, and S for the resultant speed.

For the horizontal component of velocity for the bowling ball, the gravity is neglected. The only force considered is the resistive force on the bowling ball as it moves through the air. Note in Marion's example, the right-hand of equation (4) mass is included in the resistive force. Aerodynamic drag, which is the resistive force for our bowling ball, is independent of the mass of the ball. Aerodynamic drag is defined as

$$\vec{F}_r = \vec{D} \tag{9}$$

$$F_r = D = \frac{1}{2}\rho C_d A u^2 \tag{10}$$

$$F_r = D = \frac{1}{2}\rho C_d A u^2 = ku^2 \tag{11}$$

where D is drag, ρ is atmospheric density, C_d is the drag coefficient, A is the surface area of the bowling ball u is the horizontal velocity. The constant k is equal to $\frac{1}{2}\rho C_d A$.

Now, to solve for the horizontal velocity of our bowling ball, substitute the right-hand side of equation (11) for the right-hand side of equation (4) to get equation (12).

$$m\vec{a} = m\frac{d\vec{u}}{dt} = -k\vec{u}^2 \tag{12}$$

The minus sign in front of the k is due to the drag force acting in the opposite direction from the trajectory.

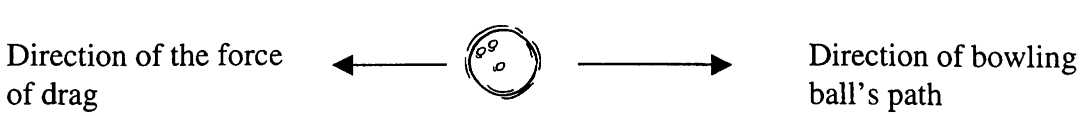

Fig. 2 The force of drag

Divide both sides of equation (12) by m and u^2 then multiply through by dt. Equation (12) can now be integrated.

$$\frac{du}{u^2} = \frac{(-k)dt}{m} \tag{13}$$

Next integrate both sides of (13).

$$\int \frac{1}{u^2} du = \int \frac{-k}{m} dt \tag{14}$$

$$-\frac{1}{u} = -\frac{kt}{m} + c \tag{15}$$

To evaluate c define u at time $t = 0$ at u_0. The variable u_0 is the initial horizontal velocity of our bowling ball. The constant c then becomes

$$c = -\frac{1}{u_0} \quad \text{for } u(t)\big|_{t=0} \tag{16}$$

Insert equation (16) in to equation (15) to get

$$-\frac{1}{u} = -\frac{1}{u_0} - \frac{kt}{m} \tag{17}$$

Solving for u (horizontal velocity) then yields

$$u \to \frac{mu_0}{m + ktu_0} \tag{18}$$

As an example consider values for m, u_0, and k, and graph u (horizontal velocity) as a function of time and assume time (t) goes from 0 to 10 seconds. Remember k is equal to $\frac{1}{2}\rho C_d A$ where $\rho = 1.225\frac{kg}{m^3}$, $C_d = .5$, $A = .1256\ m^2$, mass $m = 1\ kg$, and $u_0 = 1\frac{m}{s}$.

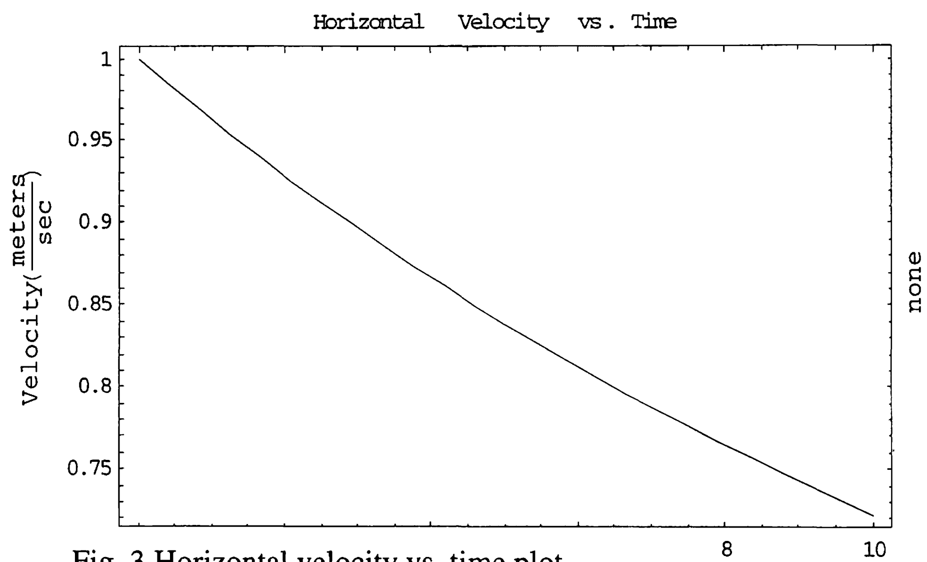

Fig. 3 Horizontal velocity vs. time plot

Figure 3 shows the expected result, that the horizontal velocity decays steadily from an initial velocity. If the bowling ball's initial velocity is a value of $1\frac{m}{s}$ then the horizontal velocity is bounded between 0 and $1\frac{m}{s}$. Knowing the bounds of these functions for a given set of initial conditions will be important later on when these functions are approximated using the Carleman method.

Vertical Equation of Motion

Now that u has been solved for as a component of the speed, next, study and solve for the v (or vertical) component of speed. For the vertical component of velocity for the bowling ball, gravity is important. The concern here is with the resistive force on the bowling ball as it moves through the air as well as the accelerating force of gravity. Now look at equation (4)

$$m\bar{a} = m\frac{d\bar{v}}{dt} = -km\bar{v} \tag{4}$$

The force of gravity, which is $-mg$, has to be added.

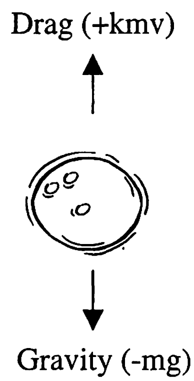

Fig.4 The vertical force drag on our bowling ball

The minus sign indicates a downward direction. The $-kmv$ will be written as $+kmv$ since the resistive force is in the opposite direction. Also, recall that the resistive force is aerodynamic drag, is independent of mass, and is proportional to the square of the velocity. Equation (4) can then be rewritten as equation (19).

$$m\bar{a} = -m\bar{g} + kv^2 \tag{19}$$

or

$$m\frac{d\vec{v}}{dt} = -m\vec{g} + kv^2 \tag{20}$$

Divide both sides of equation (20) by m and multiply through by dt.

$$dv = -\frac{\left(mg + kv^2\right)}{m}dt \tag{21}$$

Divide both sides of equation (21) by $\frac{\left(mg + kv^2\right)}{m}$ to get equation (22).

$$\frac{m}{kv^2 + mg}dv = -dt \tag{22}$$

Integrate both sides of equation (22).

$$\int \frac{m}{kv^2 + mg}dv = -\int dt \tag{23}$$

$$\frac{\sqrt{m}\left(ArcTan\left[\frac{v\sqrt{k}}{\sqrt{g}\sqrt{m}}\right]\right)}{\sqrt{g}\sqrt{k}} = -t + c \tag{24}$$

To evaluate c define v at time (t) equal to 0 (written as $v[t = 0] = v_0$). The variable v_0 is the initial vertical velocity of the bowling ball. The constant c then becomes

$$\frac{\sqrt{m}\left(ArcTan\left[\frac{v_0\sqrt{k}}{\sqrt{g}\sqrt{m}}\right]\right)}{\sqrt{g}\sqrt{k}} = c \tag{25}$$

Insert equation (25) into equation (24) to get equation (26).

$$\frac{\sqrt{m}\left(ArcTan\left[\frac{v\sqrt{k}}{\sqrt{g}\sqrt{m}}\right]\right)}{\sqrt{g}\sqrt{k}} = -t + \frac{\sqrt{m}\left(ArcTan\left[\frac{v_0\sqrt{k}}{\sqrt{g}\sqrt{m}}\right]\right)}{\sqrt{g}\sqrt{k}} \tag{26}$$

Solving for v (vertical velocity) yields

$$v \to -\frac{\sqrt{g}\sqrt{m}\left(Tan\left[\frac{t\sqrt{g}\sqrt{k}}{\sqrt{m}} - ArcTan\left[\frac{v_0\sqrt{k}}{\sqrt{g}\sqrt{m}}\right]\right]\right)}{\sqrt{k}} \tag{27}$$

As an example consider values for m, u_0, and k, and graph u (horizontal velocity) as a function of time and assume time (t) goes from 0 to 10 seconds. Remember k is equal to $\frac{1}{2}\rho C_d A$ where $\rho = 1.225\frac{kg}{m^3}$, $C_d = .5$, $A = .1256\ m^2$, mass $m = 1\ kg$, and $v_0 = 0\frac{m}{s}$.

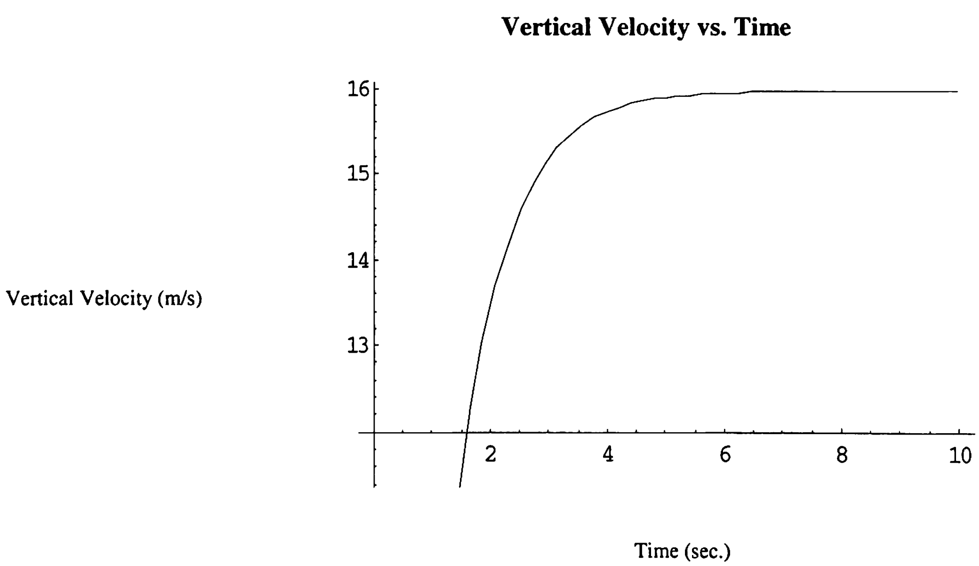

Fig. 5 Vertical velocity vs. time plot

Figure 5 shows the expected result, that vertical velocity increases from an initial velocity, from the force of gravity, to a terminal velocity, the point at which the drag force equals the gravitational force. If the bowling ball's initial velocity is a value of $0\frac{m}{s}$ then, the vertical velocity is bounded between 0 and 16 $\frac{m}{s}$.

Polynomial Representation of Speed

Now that the components of speed for the bowling ball are bounded, the actual speed is as a function of time, can be approximated in a polynomial. The speed is the square root of the sum of the squares of our components. This is derived from the Pythagorean theorem.

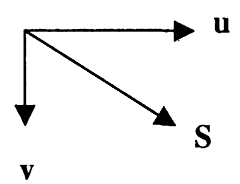

Fig. 6 Vector diagram

From figure 3 it is known that the function for horizontal velocity $\frac{mu_0}{m+ktu_0}$ is continuous for the set of initial conditions and $0 < t < 10$. It is also known from figure 3 that for $0 < t < 10$ u is bounded between 0 and 1. Again, it is known from figure 5 that the function for vertical velocity

$$v \to -\frac{\sqrt{g}\sqrt{m}\left(Tan\left[\frac{t\sqrt{g}\sqrt{k}}{\sqrt{m}} - ArcTan\left[\frac{v_0\sqrt{k}}{\sqrt{g}\sqrt{m}}\right]\right]\right)}{\sqrt{k}}$$

is continuous for the set of initial conditions and $0 < t < 10$, as well as from figure 5, that for $0 < t < 10$, **v** is bounded between 0 and 16. To see what the surface looks like for speed, plot a surface where the x coordinate is u (horizontal velocity), the y coordinate is v (vertical velocity), and the z coordinate is the square root of the sum of the squares of u and v.

$$\left\{u, v, \sqrt{u^2 + v^2}\right\}$$

Now plot $\sqrt{u^2+v^2}$ for u between 0 and 1, and for $0 \le v \le 16$. The plot looks like figure 7.

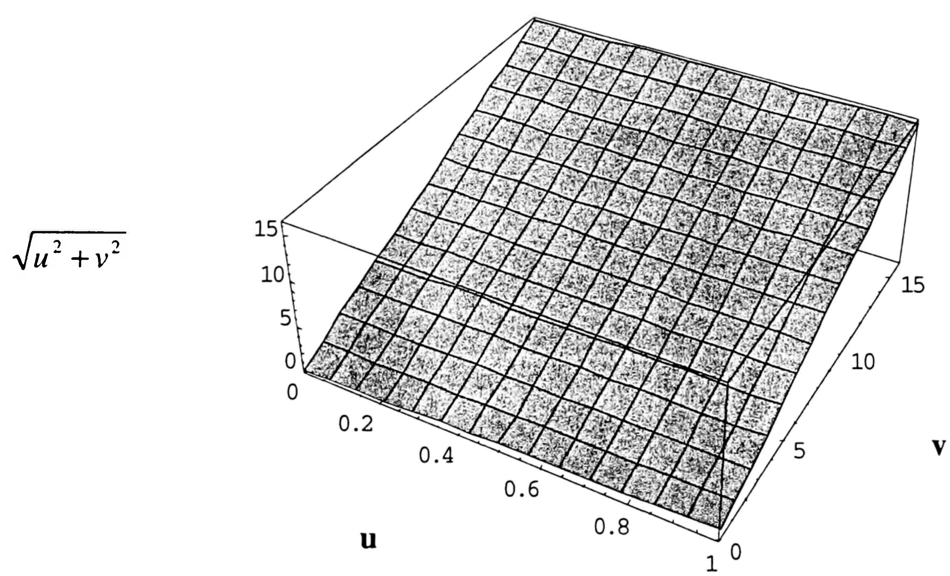

Fig. 7 Surface plot representing the speed

Now that the equations of motion have been derived, the next step is to develop a polynomial approximation. The Carleman method works for analytic ODEs, but in practice the method is applied to polynomial ODEs.

Recall that the bounds of our equation were determined by analyzing them using a given set of initial conditions. The initial horizontal velocity is 1 m/s and the initial vertical velocity is 0 m/s. It was found that the horizontal motion slows down continuously from 1 m/s till it stops at 0 m/s. Therefore, u (horizontal velocity) is bounded between 0 and 1. It was also found that the vertical velocity slows down continuously in a nonlinear fashion until it either impacts the ground or is no longer accelerating due to the force of drag (terminal velocity). It was found that in about 10 seconds the bowling ball reaches a terminal velocity of 16 meters/second, therefore the vertical velocity is bounded between 0 and 16 meters/second. Next, apply those bounds to the equation of motion for our bowling ball $\sqrt{u^2+v^2}$ and plot the surface.

By plotting the surface, a list of points in a plane is defined that represents $\sqrt{u^2+v^2}$. From that list of points, a least squares polynomial fit is computed using Mathematica 3.0.

Depending on what order of polynomial was chosen would yield varying degrees of accuracy for the approximation. Using high order polynomials in Carleman linearization can produce huge matrices. The polynomial approximation used is

$$-0.3819+.3306x+1.0027y-0.0253xy \tag{30}$$

When this function is plotted, the following result is obtained.

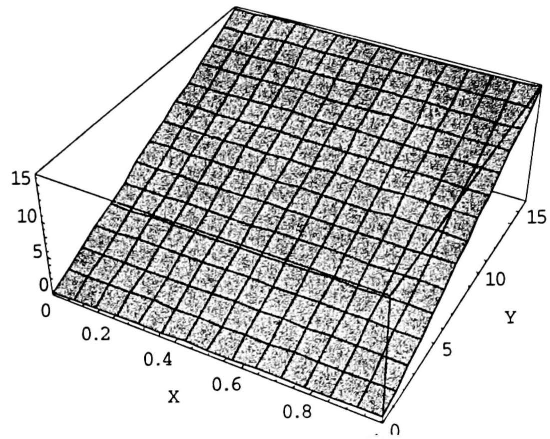

Fig. 8 Surface plot of approximation of speed that converts $\sqrt{u^2+v^2}$ into a polynomial

This plot is very similar to the original function of $\sqrt{u^2+v^2}$. The next step is to see the closeness of fit. The two surfaces look so close that to lay one on top of the other may be difficult to distinguish the differences. To see the differences subtracted the z coordinate from the approximation from the z coordinate in the same (x, y) plane location to see the difference in the 2 planes. Then take the differences and plot the result. The result would show how little difference there is between the two planes. Since the difference between the two plots is small, the approximation is good. The maximum absolute error found was 0.08. Take the result and work backward to get the component equations of motion for out bowling ball. Ultimately the equations will be worked into matrix form that will be well suited to use in the Carleman Linearization technique.

To start to put the equations of motion in vector form, retrace the steps by going back to the equation of the above example

$$\bar{F} = m\bar{a} \tag{31}$$

Recall the equation in the earlier section

$$\bar{F} = m\bar{a} = m\frac{d\bar{v}}{dt} = -kmv^2 \tag{32}$$

Now define the vectors for the system that describes the motion of our bowling ball. Define a vector for the horizontal velocity, the vertical velocity, and a vector for the speed of the bowling ball as a function of both horizontal and vertical velocity.

$$\bar{u} = (u,0) \tag{33}$$
$$\bar{v} = (0,v) \tag{34}$$
$$\bar{s} = (u,v) \tag{35}$$

With these vectors defined, say

$$\bar{s} = \bar{u} + \bar{v} \tag{36}$$

$\|\bar{s}\|$ is then defined as

$$\|\bar{s}\| = \sqrt{u^2 + v^2} \cong a + bu + cv + duv \tag{37}$$

This is where the approximation of the surface comes into play, $a + bu + cv + duv$ is the expression for

$$-0.3819 + .3306x + 1.0027y - 0.0253xy \tag{38}$$

just substitute u and v for x and y respectively.

Now go back and examine $\bar{F} = m\bar{a}$ again. Look at each component of the speed by multiplying it by a unit vector in the direction of the component being studied.

For example, a unit vector in the direction of the horizontal component would be $\frac{\bar{u}}{\|\bar{s}\|}$.

From this it can be written

$$\bar{F} = m\bar{a} = m\frac{d\bar{u}}{dt} = -kms^2\frac{\bar{u}}{\|\bar{s}\|} \tag{39}$$

The "m" and "s" terms cancel and what is left is

$$\frac{d\bar{u}}{dt} = -ksu \tag{40}$$

Again, for the vertical component, write

$$\bar{F} = m\bar{a} = m\frac{d\bar{v}}{dt} = -m\bar{g} - kms^2\frac{\bar{v}}{\|\bar{s}\|} \tag{41}$$

Again the "m" and "s" terms cancel and what is left is

$$\frac{d\bar{v}}{dt} = -\bar{g} - ksv \tag{42}$$

Now combine equation (37) with equations (40) and (42) to put the equations into polynomial form. They look like the following equations.

$$\frac{du}{dt} = -k\left(au + bu^2 + cuv + du^2v\right) \tag{43}$$

$$\frac{dv}{dt} = -k\left(av + buv + cv^2 + duv^2\right) \tag{44}$$

This can also be written in matrix form

$$\begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} = -\begin{pmatrix} ksu \\ g + ksv \end{pmatrix} = -\begin{pmatrix} k\left(au + bu^2 + cuv + du^2v\right) \\ k\left(av + buv + cv^2 + duv^2\right) \end{pmatrix} \tag{45}$$

As a reminder the constants a, b, c, and d are equal to the constants in the polynomial equation (38).

$$\begin{aligned} a &= -0.3819 \\ b &= 0.3306 \\ c &= 1.0027 \\ d &= -0.0253 \end{aligned}$$

This concludes the derivation of the model for two degree-of-freedom ballistic motion. A system of ODEs was found once the speed was approximated as a polynomial that could be applied to the Carleman linearization technique.

Application of the Carleman Method

Briefly, what the Carleman linearization technique does is it converts a system of equations into an infinite system of linear equations. That infinite system of linear equations is truncated and the finite system is solved.

The way the finite system of linear equations is built is developed by Kowalski and Steeb [14]. Assume a system of equations as follows:

$$\frac{dx}{dt} = a + bx + cy + dx^2 + exy + fyx + gy^2 \tag{46}$$

$$\frac{dy}{dt} = h + ix + jy + kx^2 + lxy + myx + ny^2 \tag{47}$$

This system can be rewritten in matrix form as follows:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b & c & d & e & f & g \\ h & i & j & k & l & m & n \end{pmatrix}\begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \tag{48}$$

And also in the form:

$$\frac{dz^T}{dt} = \tilde{M} z^T \tag{49}$$

where $z = (\hat{x}, \hat{y})$ and the "T" means transpose and where

$$\tilde{M} = \begin{pmatrix} a & b & c & d & e & f & g \\ h & i & j & k & l & m & n \end{pmatrix} \tag{50}$$

Now look at a first order system of ordinary differential equations

$$\frac{du}{dt} = A_0(t) + A_1(t)u + \ldots + A_n(t)u^{[n]} \tag{51}$$

such that A_j, $j \in \{0, \cdots, n\}$ is a matrix valued function such that it is a constant of

u except A_0 which is just a matrix of constants, and where $u^{[i]} = \underbrace{u \otimes u \otimes u \otimes u}_{i-times}$.

Where $\otimes$ denotes the Kronecker Product. Let A be a $m \times n$ matrix and let B be a $p \times q$

matrix. The Kronecker Product is defined as

$$A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix} \tag{52}$$

Thus $A \otimes B$ is a $mp \times nq$ matrix. From equation (51), it is found that

$$\frac{du^{[i]}}{dt} = \sum_{v=1}^{i} \left(u \otimes \cdots \otimes \sum_{j=0}^{n} A_j u^{[j]} \otimes \cdots \otimes u \right) \tag{53}$$

holds where $\sum_{j=0}^{n} A_j u^{[i]}$ stands at the $v-th$ place. It also follows that $\frac{du^{[i]}}{dt} = \sum_{j=0}^{n} B_j^i u^{[j+i-1]}$

where

$$B_j^i := \sum_{v=1}^{i} (I \otimes \cdots \otimes I \otimes A_j \otimes I \otimes \cdots \otimes I) \tag{54}$$

and where A_j appears at the $v-th$ site in the $i-fold$ Kronecker Product and I is the $k \times k$ identity matrix. Equation (54) can be expressed as

$$B_j^i = B_j^1 \otimes I^{[i-1]} + I \otimes B_j^{i-1} \tag{55}$$

Now re-examine equations (49) and (46)

$$\frac{dz^T}{dt} = \tilde{M} z^T \tag{49}$$

$$\frac{dx}{dt} = a + bx + cy + dx^2 + exy + fyx + gy^2 \tag{46}$$

The x and y terms in equation (46)can be rewritten as $x_1, x_2, x_3, \ldots x_n$ to what ever order the polynomial is. For example

$$\frac{dx}{dt} = a + bx + cy + dx^2 + exy + fyx + gy^2 \tag{46}$$

can be written as

$$\frac{dx_1}{dt} = a + bx_1 + cx_2 + dx_3 + ex_4 + fx_5 + gx_6 \tag{56}$$

$\tilde{M}$ can be written as

$$\tilde{M} = \begin{pmatrix} B_0^1 & B_1^1 & B_2^1 & \cdots & B_n^1 & 0 & 0 & \cdots \\ 0 & B_0^2 & B_1^2 & \cdots & B_{n-1}^2 & B_n^2 & 0 & \cdots \\ 0 & 0 & B_0^3 & \cdots & B_{n-2}^3 & B_{n-1}^3 & B_n^3 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \end{pmatrix} \tag{57}$$

It then follows that if

$$\frac{dx_1}{dt} = a + bx_1 + cx_2 + dx_3 + ex_4 + fx_5 + gx_6$$

$$\frac{dx_2}{dt} = h + ix_1 + jx_2 + kx_3 + lx_4 + mx_5 + nx_6$$

then

$$B_0^1 = \begin{pmatrix} a \\ h \end{pmatrix} \qquad B_1^1 = \begin{pmatrix} b & c \\ i & j \end{pmatrix} \quad B_2^1 = \begin{pmatrix} d & e & f & g \\ k & l & m & n \end{pmatrix}$$

From equation (4.10)

$$B_0^2 = B_0^1 \otimes I^{[2-1]} + I \otimes B_0^{2-1}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ (Identity matrix)} \qquad (58)$$

$$\text{So } B_0^2 = \begin{pmatrix} a \\ h \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} a \\ h \end{pmatrix}$$

$$B_0^2 = \begin{pmatrix} 2a & 0 \\ h & a \\ h & a \\ 0 & 2h \end{pmatrix}$$

$$B_1^2 = B_1^1 \otimes I^{[2-1]} + I \otimes B_1^{2-1}$$

$$B_1^2 = \begin{pmatrix} b & c \\ i & j \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} b & c \\ i & j \end{pmatrix}$$

$$B_1^2 = \begin{pmatrix} 2b & c & c & 0 \\ i & b+j & 0 & c \\ i & 0 & b+j & c \\ 0 & i & i & 2j \end{pmatrix}$$

$$B_0^3 = B_0^1 \otimes I^{[3-1]} + I \otimes B_0^{3-1}$$

$$B_0^3 = \begin{pmatrix} a \\ h \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 2a & 0 \\ h & a \\ h & a \\ 0 & 2h \end{pmatrix}$$

$$B_0^3 = \begin{pmatrix} 3a & 0 & 0 & 0 \\ h & 2a & 0 & 0 \\ h & a & a & 0 \\ 0 & 2h & 0 & a \\ h & 0 & 2a & 0 \\ 0 & h & h & a \\ 0 & 0 & 2h & a \\ 0 & 0 & 0 & 3h \end{pmatrix}$$

$$\tilde{M} = \begin{pmatrix} a & b & c & d & e & f & g \\ h & i & j & k & l & m & n \\ 0 & 2a & 0 & 2b & c & c & 0 \\ 0 & h & a & i & b+j & 0 & c \\ 0 & h & a & i & 0 & b+j & c \\ 0 & 0 & 02h & 0 & i & i & 2j \end{pmatrix} \tag{59}$$

or

$$\tilde{M} = \begin{pmatrix} B_0^1 & B_1^1 & B_2^1 \\ 0 & B_0^2 & B_1^2 \end{pmatrix} \tag{60}$$

The matrix can be as large as is desired and this can easily be done computationally. Now the truncated infinite linear matrix can be put back into equation (49) yielding

$$\frac{dx^T}{dt} = \tilde{M}x^T \tag{61}$$

Where $x = (x_1, x_2, x_3, x_4, x_5, x_6)$ and the T denotes transpose. Multiply equation (61) through to end up with a system of ordinary differential equations and solve the resulting system for x_1 and x_2 with respect to t. Finally, compare these solutions to ones from the original system of equations and see how well the Carleman Linearization technique works for approximating the original system of equations. This is the approach this research will use to find an analytical solution to the trajectory of the bowling ball.

CHAPTER IV

APPLICATIONS OF THE CARLEMAN LINEARIZATION METHOD TO NONLINEAR PROBLEMS IN AERONAUTICS

2 DOF Ballistic Trajectory

Now that it has been seen how the Carleman linearization technique is derived, it was applied to the problem of throwing the bowling ball off the Eiffel Tower. Chapter III took the equation

$$\sqrt{u^2 + v^2} \tag{62}$$

and approximated it in order to convert it to a polynomial. The result was equation (30). Substituting u and v for x and y, respectively, yields

$$-0.3819 + .3306u + 1.0027v - 0.0253uv \tag{63}$$

Next define

$$\begin{aligned} a &= -0.3819 \\ b &= 0.3306 \\ c &= 1.0027 \\ d &= -0.02530 \end{aligned}$$

$$s \cong a + bu + cv + duv \tag{64}$$

From equations (40), (42), and (64), the following equations can be written

$$\begin{aligned} \dot{u} &= -ksu = -ku(a + bu + cv + duv) = -kau - kbu^2 - kcuv - kdu^2v \\ \dot{v} &= -g - ksv = -g - kv(a + bu + cv + duv) = -kav - kbuv - kcv^2 - kduv^2 \end{aligned}$$

Let

$$\begin{aligned} -ka &= \alpha \\ -kb &= \beta \\ -kc &= \psi \\ -kd &= \Delta \end{aligned}$$

and write

$$\begin{aligned} \dot{u} &= \alpha u + \beta u^2 + \psi uv + \Delta u^2 v \\ \dot{v} &= -g + \alpha v + \beta uv + \psi v^2 + \Delta uv^2 \end{aligned}$$

The following notation is used so that the reader can see how the polynomials $\dot{u}$ and $\dot{v}$ are expanded to capture the zero terms. This is important to build the matrix $\tilde{M}$. All the constants of the matrix need to be included, even the zeros. $\dot{u}$ and $\dot{v}$ are cubic equations that allow all the linear terms, quadratic terms, and cubic terms to be captured. The terms are illustrated in matrix notation below.

$$\begin{bmatrix} u \\ v \end{bmatrix} \underbrace{x_1 \quad x_2}_{u \quad v} \qquad \begin{bmatrix} u \\ v \end{bmatrix}^2 \underbrace{\begin{matrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{matrix}}_{\begin{matrix} u^2 & uv \\ vu & v^2 \end{matrix}} \qquad \begin{bmatrix} u \\ v \end{bmatrix}^3 \underbrace{\begin{matrix} x_{111} & x_{112} & x_{121} & x_{122} \\ x_{211} & x_{212} & x_{221} & x_{222} \end{matrix}}_{\begin{matrix} u^3 & u^2v & u^2v & uv^2 \\ u^2v & uv^2 & uv^2 & v^3 \end{matrix}}$$

$\dot{u}$ and $\dot{v}$ are now expanded and written as

$$\begin{aligned} \dot{u} &= 0 + \alpha u + 0v + \beta u^2 + \psi uv + 0vu + 0u^3 + \Delta u^2 v + 0u^2 v + 0uv^2 + 0u^2 v + 0uv^2 + 0uv^2 + 0v^3 \\ \dot{v} &= -g + 0u + \alpha v + 0u^2 + 0uv + \beta vu + \psi v^2 + 0u^3 + 0u^2 v + 0u^2 v + 0uv^2 + 0u^2 v + 0uv^2 + \Delta uv^2 + 0v^3 \end{aligned}$$

Remember from Chapter III

$$\frac{dx^T}{dt} = \tilde{M} x^T \tag{49}$$

Where $x = (x_1, x_2, x_3, x_4, x_5, x_6)$, the T means transpose, and $\tilde{M}$ is defined as

$$\tilde{M} = \begin{pmatrix} B_0^1 & B_1^1 & B_2^1 & \cdots & B_n^1 & 0 & 0 & \cdots \\ 0 & B_0^2 & B_1^2 & \cdots & B_{n-1}^2 & B_n^2 & 0 & \cdots \\ 0 & 0 & B_0^3 & \cdots & B_{n-2}^3 & B_{n-1}^3 & B_n^3 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \end{pmatrix}$$

where $B_j^i = B_j^1 \otimes I^{[i-1]} + I \otimes B_j^{i-1}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (identity matrix).

From $\dot{u}$ and $\dot{v}$, B_0^1, B_1^1, B_2^1, and B_3^1 are defined by inspection as

$$B_0^1 = \begin{pmatrix} 0 \\ -g \end{pmatrix} \; B_1^1 = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \; B_2^1 = \begin{pmatrix} \beta & \psi & 0 & 0 \\ 0 & 0 & \beta & \psi \end{pmatrix} \; B_3^1 = \begin{pmatrix} 0 & \Delta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \Delta & 0 \end{pmatrix}$$

$\tilde{M}$ will be truncated so that the final matrix will look like

$$\tilde{M} = \begin{pmatrix} B_0^1 & B_1^1 & B_2^1 & B_3^1 \\ 0 & B_0^2 & B_1^2 & B_2^2 \\ 0 & 0 & B_0^3 & B_1^3 \end{pmatrix}$$

B_0^2 through B_1^3 will be defined using the rule $B_j^i = B_j^1 \otimes I^{[i-1]} + I \otimes B_j^{i-1}$

$I^{[i]}$ is defined as $\underbrace{I \otimes I}_{i-times}$ and $I^{[1]}$ is just I.

Writing out B_0^2 through B_1^3 ...

$$B_0^2 = \begin{pmatrix} 0 & 0 \\ -g & 0 \\ -g & 0 \\ 0 & -2g \end{pmatrix} \; B_1^2 = \begin{pmatrix} 2\alpha & 0 & 0 & 0 \\ 0 & 2\alpha & 0 & 0 \\ 0 & 0 & 2\alpha & 0 \\ 0 & 0 & 0 & 2\alpha \end{pmatrix}$$

$$B_2^2 = \begin{pmatrix} 2\beta & \psi & \psi & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta & \beta & 2\psi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\beta & \psi & \psi & 0 \\ 0 & 0 & 0 & 0 & 0 & \beta & \beta & 2\psi \end{pmatrix}$$

$$I \otimes I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_0^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -g & 0 & 0 & 0 \\ -g & 0 & 0 & 0 \\ 0 & -2g & 0 & 0 \\ -g & 0 & 0 & 0 \\ 0 & -g & -g & 0 \\ 0 & 0 & -2g & 0 \\ 0 & 0 & 0 & -3g \end{pmatrix} \quad B_1^3 = \begin{pmatrix} 3\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3\alpha \end{pmatrix}$$

$\tilde{M}$ now becomes

$$\tilde{M} = \begin{pmatrix} 0 & \alpha & 0 & \beta & \psi & 0 & 0 & 0 & \Delta & 0 & 0 & 0 & 0 & 0 & 0 \\ -g & 0 & \alpha & 0 & 0 & \beta & \psi & 0 & 0 & 0 & 0 & 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & 2\alpha & 0 & 0 & 0 & 2\beta & \psi & \psi & 0 & 0 & 0 & 0 & 0 \\ 0 & -g & 0 & 0 & 2\alpha & 0 & 0 & 0 & \beta & \beta & 2\psi & 0 & 0 & 0 & 0 \\ 0 & -g & 0 & 0 & 0 & 2\alpha & 0 & 0 & 0 & 0 & 0 & 2\beta & \psi & \psi & 0 \\ 0 & 0 & -2g & 0 & 0 & 0 & 2\alpha & 0 & 0 & 0 & 0 & 0 & \beta & \beta & 2\psi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g & 0 & 0 & 0 & 0 & 3\alpha & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g & 0 & 0 & 0 & 0 & 0 & 3\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2g & 0 & 0 & 0 & 0 & 0 & 3\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -g & -g & 0 & 0 & 0 & 0 & 0 & 0 & 3\alpha & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3\alpha & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3g & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3\alpha \end{pmatrix}$$

Again, remember

$$\frac{dx^T}{dt} = \tilde{M}x^T \qquad (65)$$

Except this time

$x = (1 \quad u \quad v \quad u^2 \quad uv \quad vu \quad v^2 \quad u^3 \quad u^2v \quad u^2v \quad uv^2 \quad u^2v \quad uv^2 \quad uv^2 \quad v^3)$ or

$x = (1 \quad x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad x_9 \quad x_{10} \quad x_{11} \quad x_{12} \quad x_{13} \quad x_{14})$ where $x_1 = u$, $x_2 = v$, $x_3 = u^2$, etc. Multiply the two matrices together to arrive at a system of 14 ordinary differential equations, which looks like the following…

$$
\begin{gathered}
x_1'[t] = \alpha x_1[t] + \beta x_3[t] + \psi x_4[t] + \Delta x_8[t] \\
x_2'[t] = -g + \alpha x_2[t] + \beta x_5[t] + \psi x_6[t] + \Delta x_{13}[t] \\
x_3'[t] = 2\alpha x_3[t] + 2\beta x_7[t] + \psi x_8[t] + \psi x_9[t] \\
x_4'[t] = -g x_2[t] + 2\alpha x_4[t] + \beta x_8[t] + \beta x_9[t] + 2\psi x_{10}[t] \\
x_5'[t] = -g x_2[t] + 2\alpha x_5[t] + 2\beta x_{11}[t] + \psi x_{12}[t] + \psi x_{13}[t] \\
x_6'[t] = -2g x_3[t] + 2\alpha x_6[t] + \beta x_{12}[t] + \beta x_{13}[t] + 2\psi x_{14}[t] \\
x_7'[t] = 3\alpha x_7[t] \\
x_8'[t] = -g x_4[t] + 3\alpha x_8[t] \\
x_9'[t] = -g x_4[t] + 3\alpha x_9[t] \\
x_{10}'[t] = -2g x_5[t] + 3\alpha x_{10}[t] \\
x_{11}'[t] = -g x_4[t] + 3\alpha x_{11}[t] \\
x_{12}'[t] = -g x_5[t] - g x_6[t] + 3\alpha x_{12}[t] \\
x_{13}'[t] = -2g x_6[t] + 3\alpha x_{13}[t] \\
x_{14}'[t] = -3g x_7[t] + 3\alpha x_{14}[t]
\end{gathered}
$$

Remember from the original problem, the bowling ball was pushed off the Eiffel Tower horizontally at 1 $\frac{m}{s}$. Therefore, the initial conditions are

$$
\begin{gathered}
u[0] = x_1[0] = 1 \\
v[0] = x_2[0] = 0 \\
u^2[0] = x_3[0] = 1 \\
u^3[0] = x_7[0] = 1
\end{gathered}
$$

all the rest of the initial conditions are 0 at $t = 0$. Now the system of 14 differential equations can be solved. To solve this, Mathematica 3.0 was used, which is a symbolic and numerical mathematics software that can be used to solve large systems of equations in which complicated mechanical and numerical operations can be performed.

Next, plots of the original horizontal and vertical velocity functions will be shown again in order to start at the beginning of problem solving. The solution using the Carleman linearization technique will then be shown. Finally, the plots are overlaid so that a comparison can be made. Figures 9 and 10 show the plot for horizontal and vertical velocity over a time interval of 5 seconds.

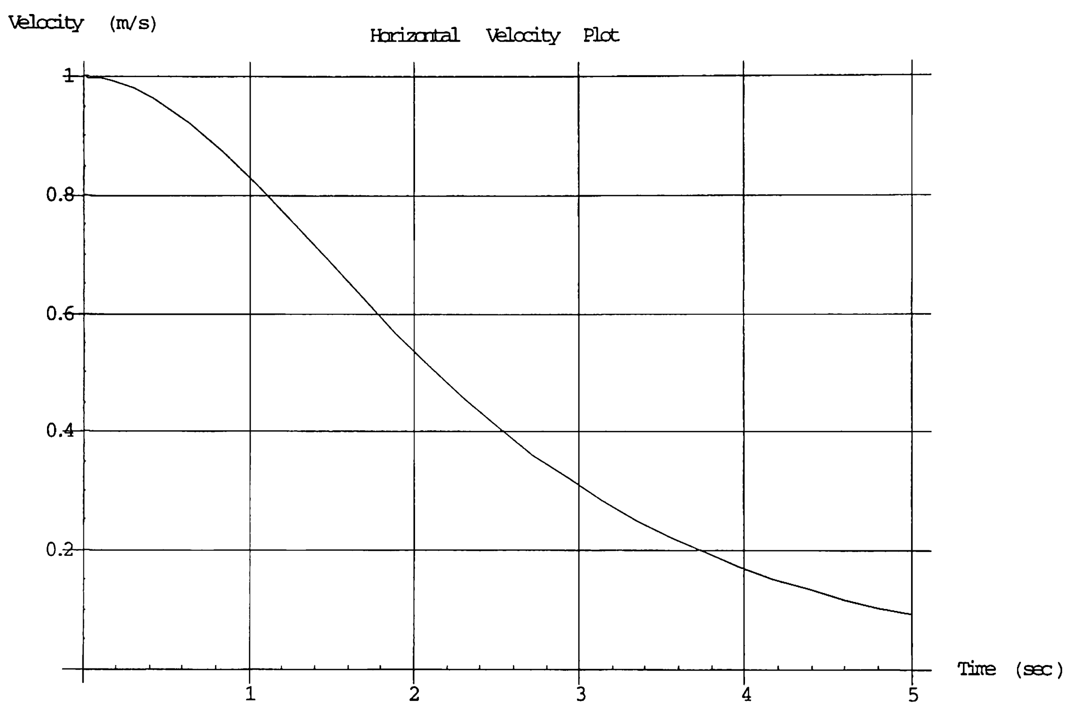

Fig. 9 Horizontal velocity plot of original equation of motion

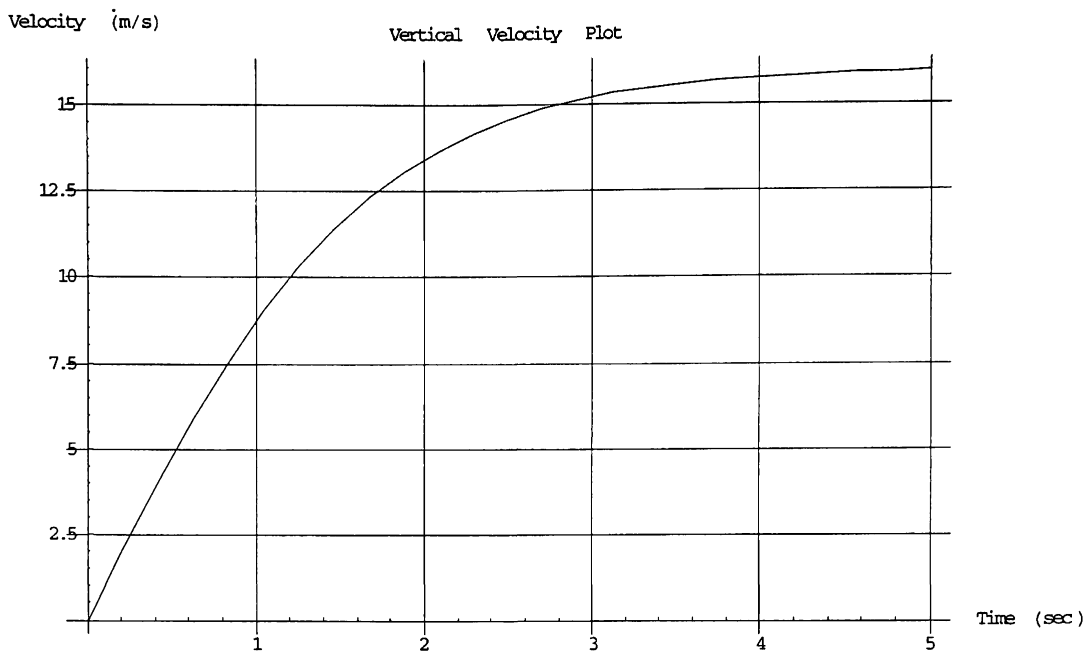

Fig. 10 Vertical velocity plot of original equation of motion

Figures 11 and 12 compare the original function with the Carleman linearization technique over a time interval of 5 seconds.

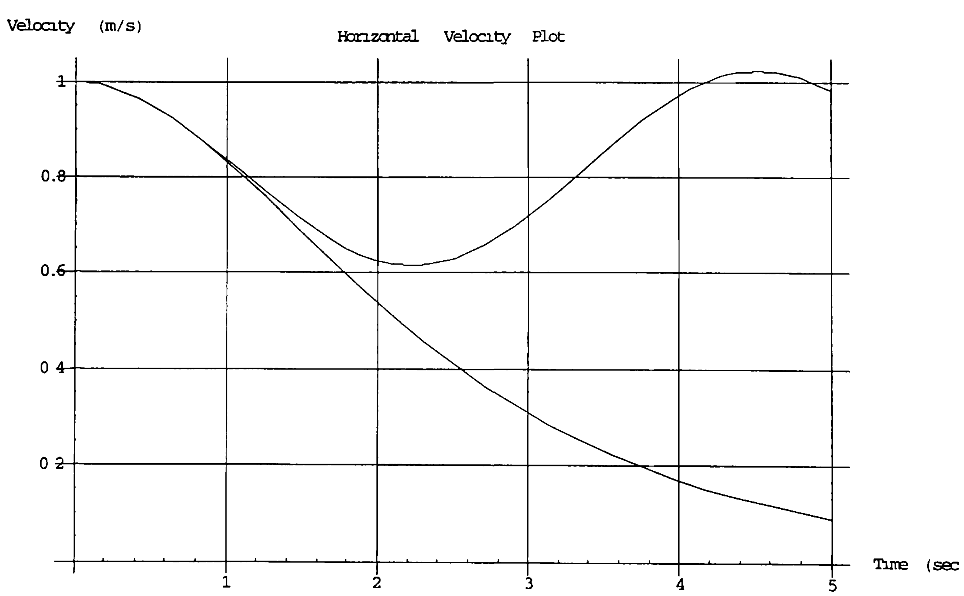

Fig. 11 The original horizontal velocity function compared with the Carleman linearization technique over a time interval of 5 seconds using third order polynomials yielding a system of 14 ODEs

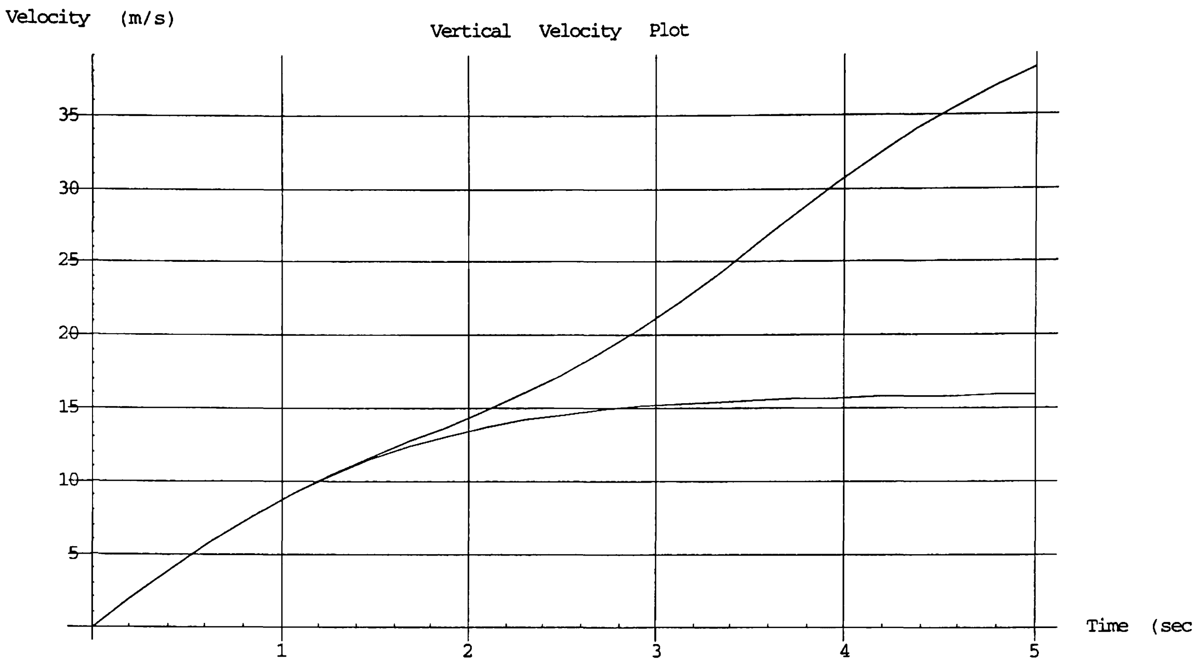

Fig. 12 The original vertical velocity function compared with the Carleman linearization technique over a time interval of 5 seconds using third order polynomials yielding a system of 14 ODEs

The plots show that Carleman's linearization technique works well close to time zero up to about one second. As the bowling ball begins to reach terminal velocity and the function becomes linear, Carleman's approximation begins to diverge rapidly. Thus, an increase in the order of the polynomial should be investigated.

$$\dot{u} = \alpha u + \beta u^2 + \psi uv + \Delta u^2 v$$
$$\dot{v} = -g + \alpha v + \beta uv + \psi v^2 + \Delta uv^2$$

These equations are third order. If the order is increased to fourth order, the resolution of our approximation can be increased. In other words, the length of time the approximation holds before it begins to diverge can be increased. Increasing the order adds another B_n^1 term to the matrix

$$\tilde{M} = \begin{pmatrix} B_0^1 & B_1^1 & B_2^1 & \cdots & B_n^1 & 0 & 0 & \cdots \\ 0 & B_0^2 & B_1^2 & \cdots & B_{n-1}^2 & B_n^2 & 0 & \cdots \\ 0 & 0 & B_0^3 & \cdots & B_{n-2}^3 & B_{n-1}^3 & B_n^3 & \cdots \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \end{pmatrix}$$

Each B_n^1 term added to the matrix causes the matrix to grow in size exponentially. The fourth order polynomial yields a system of 30 ordinary differential equations. Figures 13 and 14 show the plots for the fourth order polynomials.

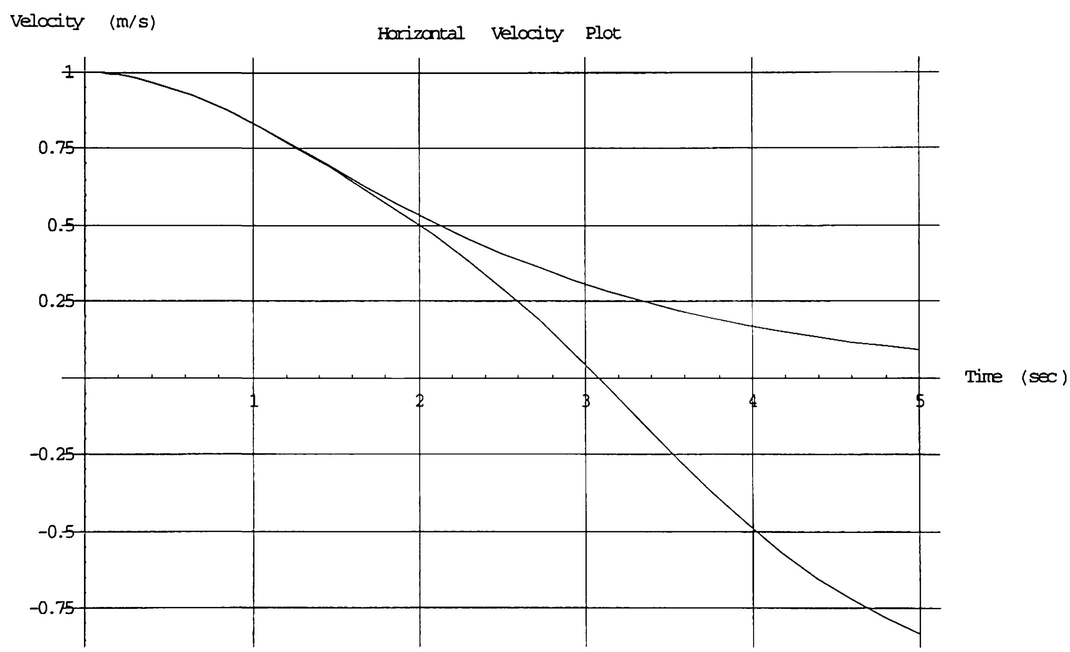

Fig. 13 The original horizontal velocity function compared with the Carleman linearization technique over a time interval of 5 seconds using fourth order polynomials yielding a system of 30 ODEs

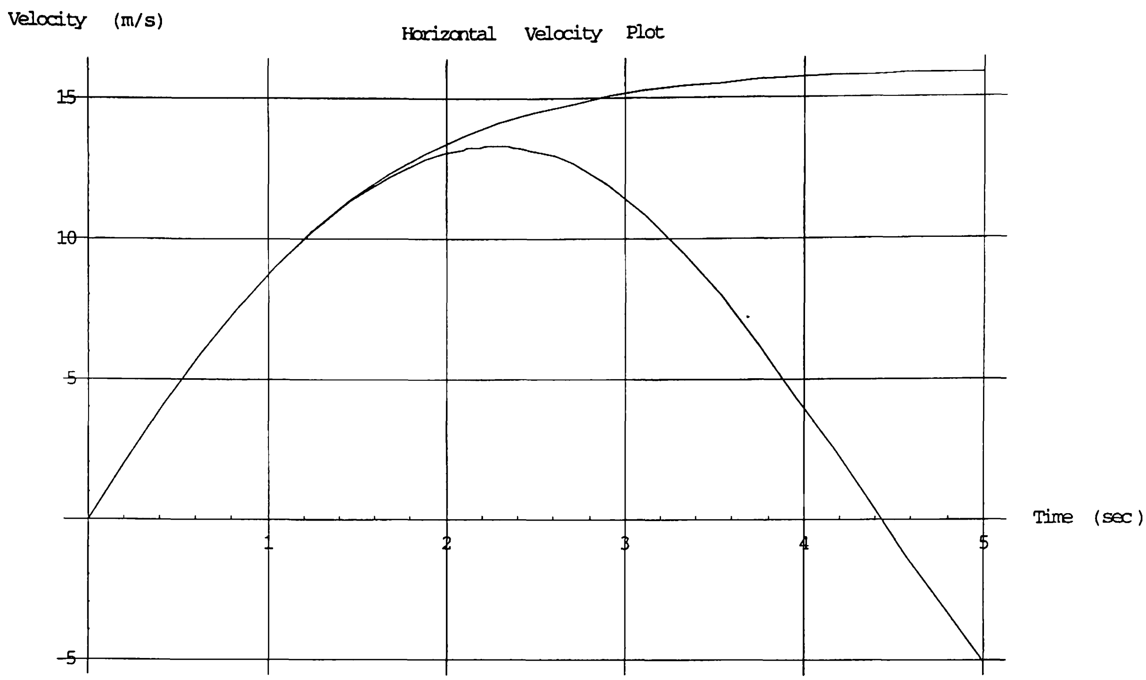

Fig. 14 The original vertical velocity function compared with the Carleman linearization technique over a time interval of 5 seconds using fourth order polynomials yielding a system of 30 ODEs

Figures 15 and 16 show the comparison of a fifth order polynomial, which yields a system of 62 ordinary differential equations.

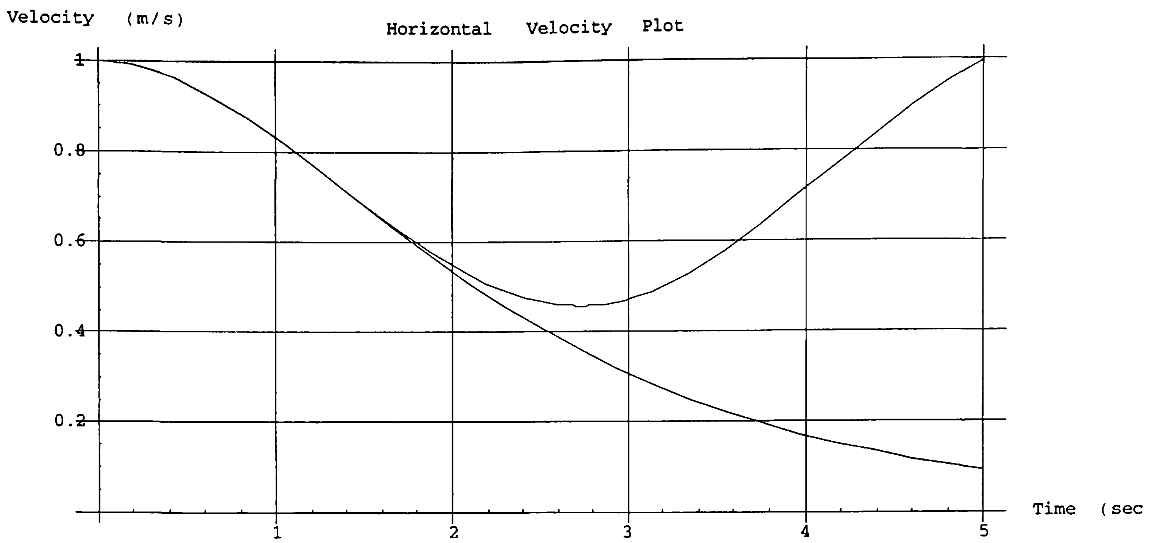

Fig. 15 The original horizontal velocity function compared with the Carleman linearization technique over a time interval of 5 seconds using fifth order polynomials yielding a system of 62 ODEs

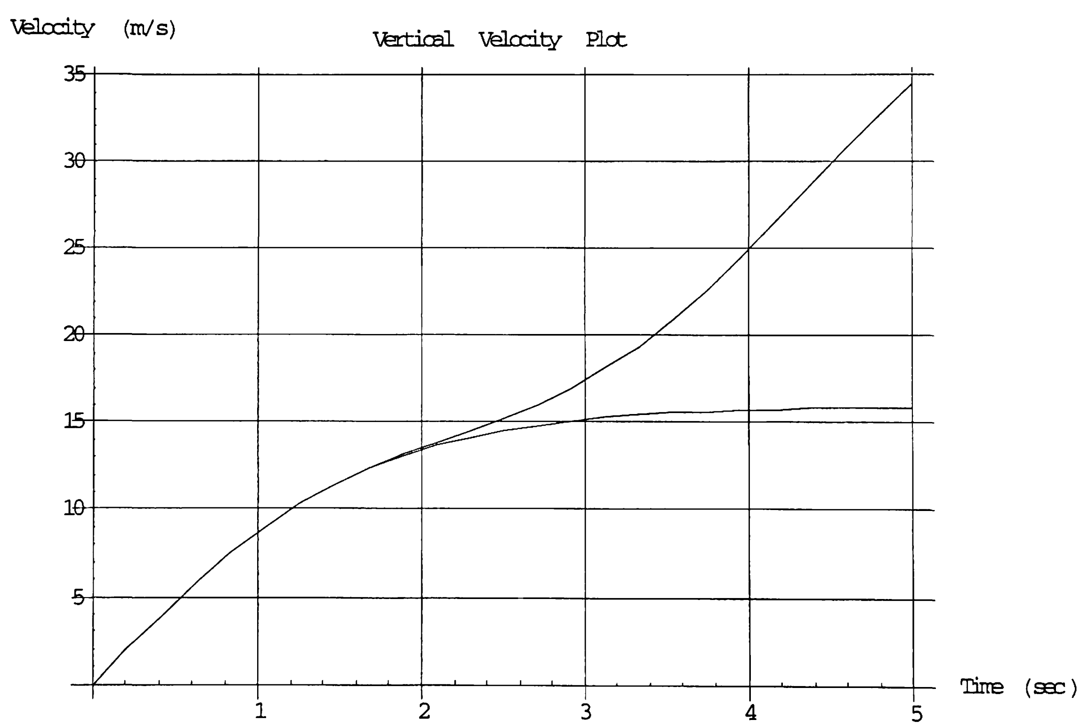

Fig. 16 The original vertical velocity function compared with the Carleman linearization technique over a time interval of 5 seconds using fifth order polynomials yielding a system of 62 ODEs

Figures 17 and 18 show the comparison of a sixth order polynomial, which yields a system of 126 ordinary differential equations.

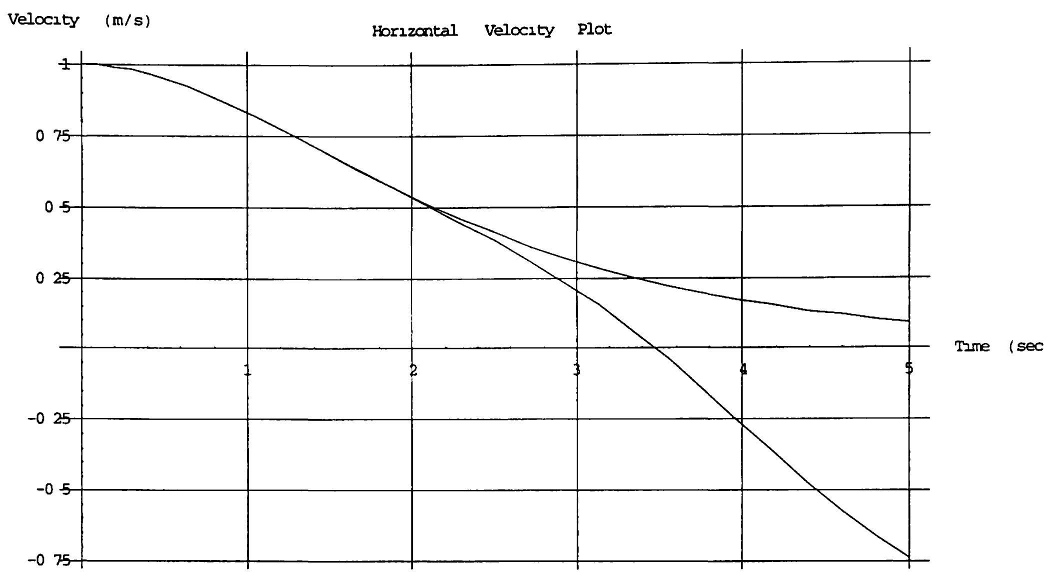

Fig. 17 The original horizontal velocity function compared with the Carleman linearization technique over a time interval of 5 seconds using sixth order polynomials yielding a system of 126 ODEs

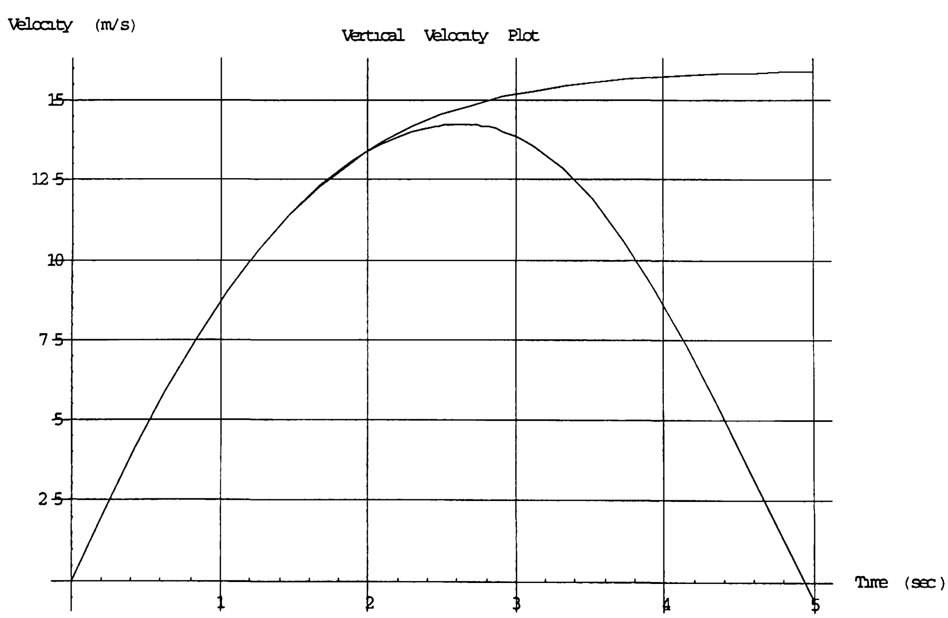

Fig. 18 The original vertical velocity function compared with the Carleman linearization technique over a time interval of 5 seconds using sixth order polynomials yielding a system of 126 ODEs

Figures 19 and 20 show the comparison of a tenth order polynomials, which yields a system of 2046 ordinary differential equations.

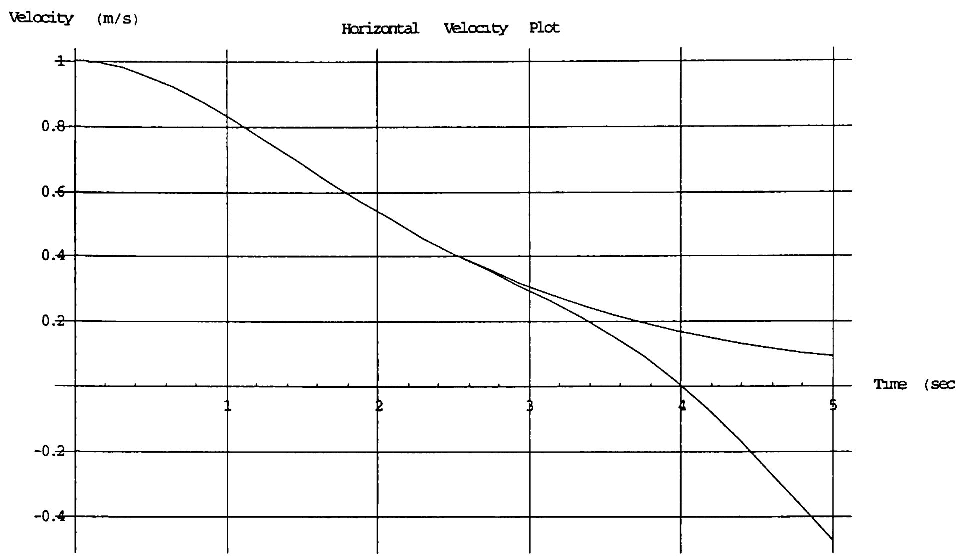

Fig. 19 The original horizontal velocity function compared with the Carleman linearization technique over a time interval of 5 seconds using tenth order polynomials yielding a system of 2046 ODEs

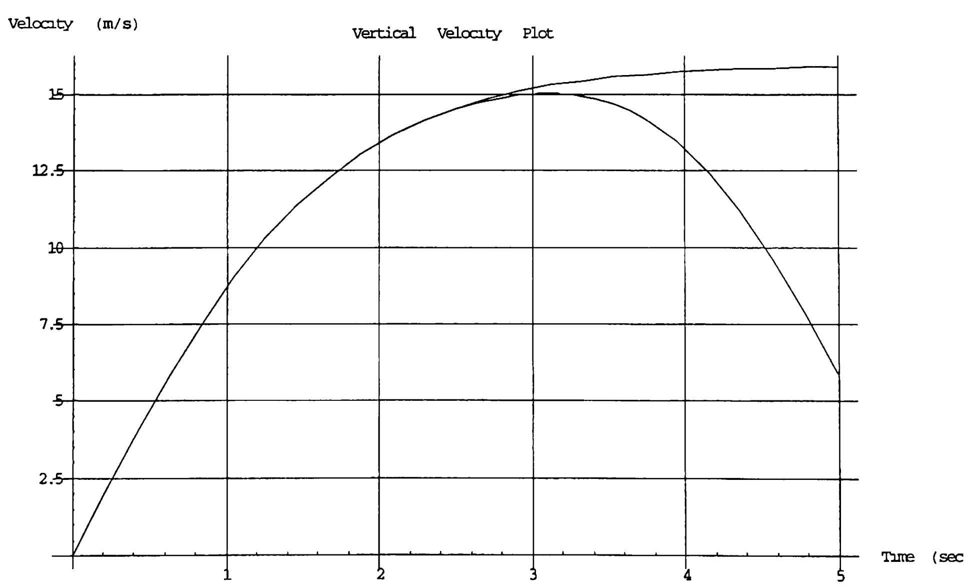

Fig. 20 The original vertical velocity function compared with the Carleman linearization technique over a time interval of 5 seconds using tenth order polynomials yielding a system of 2046 ODEs

Perhaps an easier way to compare the plots is to use an overlay showing the divergence of the plots as a percentage over time (figure 21). Percent divergence is defined as the difference between the original function and the Carleman approximation, divided by the original function and multiplied by one hundred.

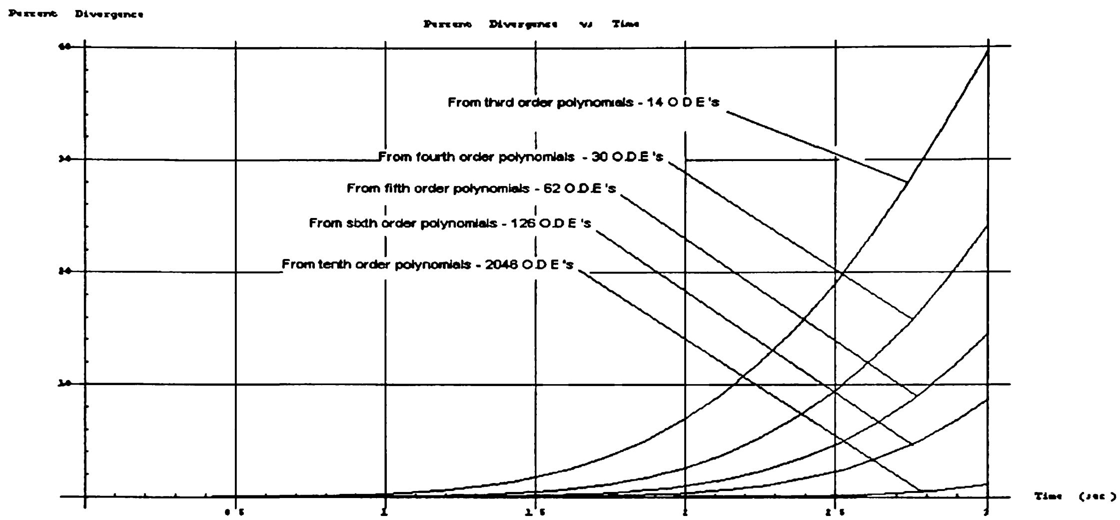

Fig. 21 Percent divergence versus time for varying levels of matrix complexity

The tenth order polynomials (2046 O.D.E's) do indeed capture most of the bend in the trajectory curve. To capture the linear tail of our trajectory approximation (in the vertical velocity), figure 20 shows that with a large enough system of ordinary differential equations it could be done. However, the system would quickly become unmanageable, and a super computer would be needed to compute it. A solution could be captured in perhaps two steps. Step one would be a Carleman linearization technique to approximate the bend of the curve and step two being some other linear approximation of the essentially linear tail. The smaller the system of ordinary differential equations used the more Carleman linearization technique "steps" would have to be used to approximate the bend in the curve.

Blasius' Boundary Layer

Now that a method has been established to approximate polynomial non-linear equations, it can be applied to other mathematical problems. Another such problem that comes up in aeronautics or fluid dynamics is the Blasius boundary layer problem. The problem is governed by equations of a rather simple system of non-linear equations, but to date, no one has ever found an analytical solution. Blasius, made an approximation in 1908, but his approximation is only good locally. The system of equations is given by

$$\begin{aligned} y_1' &= -y_1 y_3 \\ y_2' &= y_1 \\ y_3' &= y_2 \end{aligned}$$

With initial conditions of

$$\begin{aligned} y_1[0] &= .46960 \\ y_2[0] &= 0 \\ y_3[0] &= 0 \end{aligned}$$

From y_1', y_2' and y_3', B_1^1 and B_2^1 are defined by inspection as

$$B_1^1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad B_2^1 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\tilde{M}$ is truncated so that the final matrix will look like

$$\tilde{M} = \begin{pmatrix} B_1^1 & B_2^1 \\ 0 & B_1^2 \end{pmatrix}$$

This matrix can be started with B_1^1 since the system of equations has inhomogeneous terms. Kowalski and Steeb have a simplified version of building the matrix equation when the governing ODE is homogeneous. However, for consistency, the same formulae for building the trajectory matrix equation will be used for all of the examples presented.

B_1^2 will be defined using the rule $B_j^i = B_j^1 \otimes I^{[i-1]} + I \otimes B_j^{i-1}$

$I^{[i]}$ is defined as $\underbrace{I \otimes I}_{i-times}$ and $I^{[1]}$ is just I. Since the original system of equations has three equations I is defined as

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Notice in the trajectory example, where the system of equations consists of two equations, I is defined as

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Writing out B_1^2

$$B_1^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

When all these matrices are put back in to $\tilde{M}$ the result is

$$\tilde{M} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Again, remember

$$\frac{dx^T}{dt} = \tilde{M}x^T$$

Except this time

$$x = \begin{pmatrix} y_1 & y_2 & y_3 & y_1^2 & y_1 y_2 & y_1 y_3 & y_2 y_1 & y_2^2 & y_2 y_3 & y_3 y_1 & y_3 y_2 & y_3^2 \end{pmatrix} \text{ or}$$

$x = (x_1 \quad x_2 \quad x_3 \quad x_4 \quad x_5 \quad x_6 \quad x_7 \quad x_8 \quad x_9 \quad x_{10} \quad x_{11} \quad x_{12})$ where $x_1 = y_1$, $x_2 = y_2$, $x_4 = y_1^2$ and so on and the T means transpose. Multiply the two matrices together to get a system of 12 ordinary differential equations. That system looks like the following

$$
\begin{aligned}
x_1'[t] &= -x_6[t] \\
x_2'[t] &= x_1[t] \\
x_3'[t] &= x_2[t] \\
x_4'[t] &= 0 \\
x_5'[t] &= x_4[t] \\
x_6'[t] &= x_5[t] \\
x_7'[t] &= x_4[t] \\
x_8'[t] &= x_5[t] + x_7[t] \\
x_9'[t] &= x_6[t] + x_8[t] \\
x_{10}'[t] &= x_7[t] \\
x_{11}'[t] &= x_8[t] + x_{10}[t] \\
x_{12}'[t] &= x_9[t] - x_{11}[t]
\end{aligned}
$$

Recall, the Carleman linearization technique matrix can be made as large as needed. A larger the matrix provides better resolution. Figure 22 shows the results of a 12×12 matrix, which translates to a system of 12 ordinary differential equations. The red line depicts the original function and the black line shows the approximation using the Carleman linearization technique. The goal is to make the approximation approach one as the plot moves infinitely to the right.

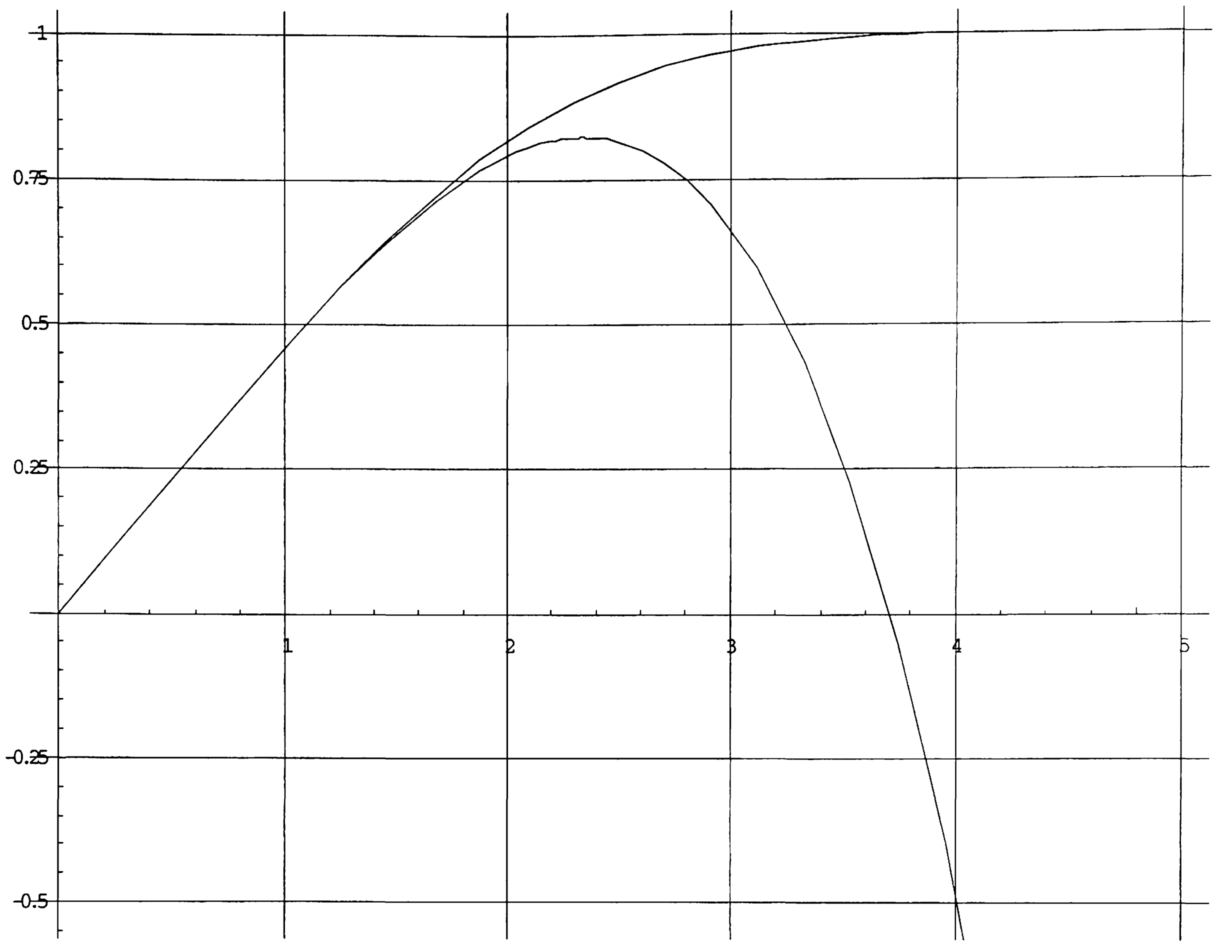

Fig. 22 The original boundary layer function compared with the Carleman linearization technique over a time interval of 5 seconds using quadratic polynomials yielding a system of 12 ODEs

Since this system of equations has three polynomials, the size of the matrix grows very, very quickly as the order of the polynomials is increased. Note that increasing the order of the polynomials means that "zero terms" are added to the equation. For example

$$au + bv$$

can also be written as

$$au + bv + 0u^2 + 0uv + 0vu + 0v^2$$

effectively increasing the order of the polynomial. Next, the equations are increased by four orders so that the result is a system of sixth order equations. This manipulation yields a 1092×1092 matrix, which translates to 1092 ordinary differential equations. Figure 23 depicts the solution.

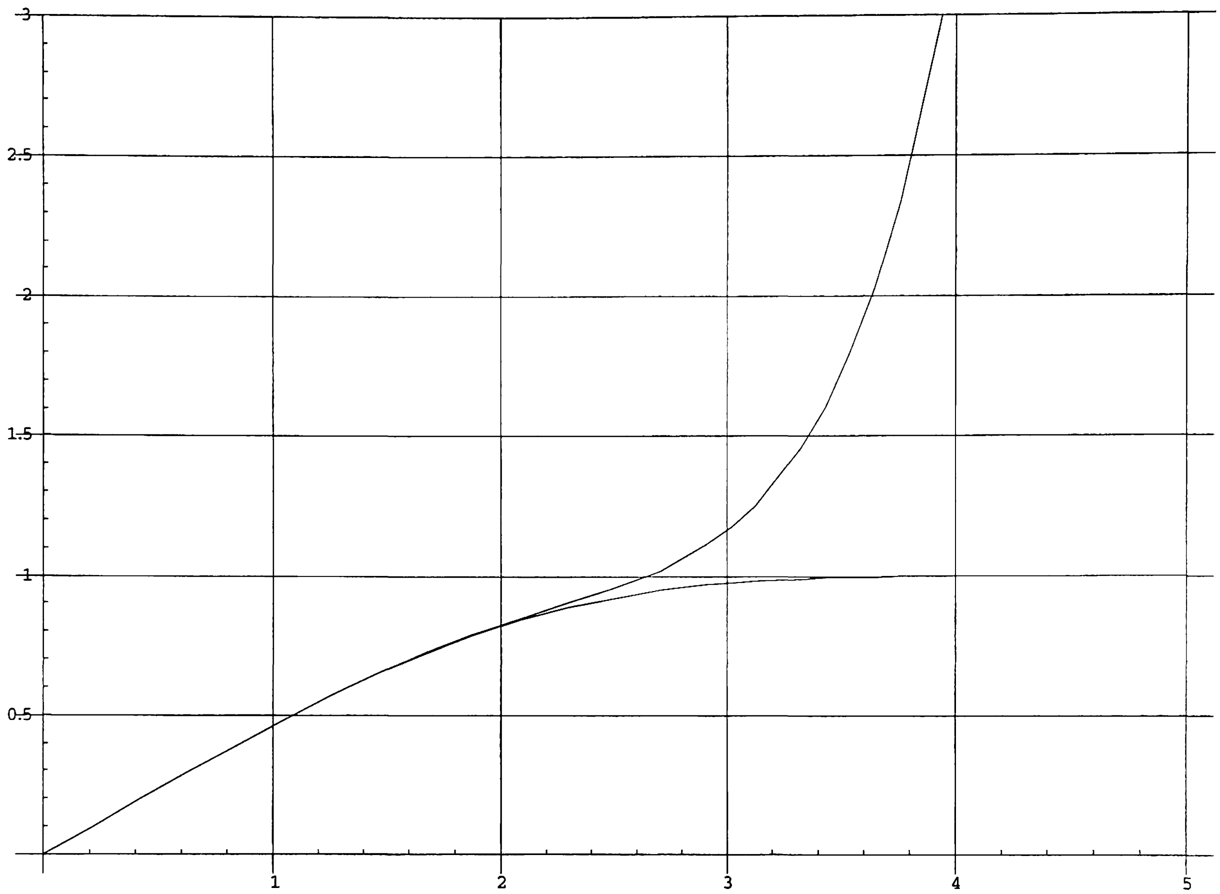

Fig. 23 The original boundary layer function compared with the Carleman linearization technique over a time interval of 5 seconds using sixth order polynomials yielding a system of 1092 ODEs

This is a much better approximation, with the red line being the original function and the black line being the approximation. The approximation gets to about .92 before it starts to diverge. By increasing the order of the polynomial even further, the approximation could probably achieve a 99% "match." However, remember that the matrix that would have to be developed in order to compute the solution will grow extremely fast.

Van der Pol's Equations

The next problem considered was Van der Pol's equation. Although there are analytical solutions to this equation, it was desired to see if the Carleman linearization technique would provide satisfactory results. The example by Kowalski and Steeb [14] of the Lotka Volterra model is periodic. The Carleman linearization technique approximates it quite well over the entire range of the function. Since Van der Pol's equation also has a periodic nature, the Carleman linearization technique was applied to see if it would do as well as the Lotka Volterra model.

Van der Pol's equation looks like the following

$$
\begin{aligned}
&\frac{dx}{dt} = y + \varepsilon\left(x - \frac{1}{3}x^3\right)\\
&\frac{dy}{dt} = -x\\
&x[0] = \alpha\\
&y[0] = \beta\\
&\varepsilon \sim small
\end{aligned}
$$

where α and β are constants. For this example, 1 and 0 were chosen respectively.

From $\dot{x}$ and $\dot{y}$, B_0^1, B_1^1, B_2^1, and B_3^1 are defined by inspection as

$$
B_0^1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\ B_1^1 = \begin{pmatrix} \varepsilon & 1 \\ -1 & 0 \end{pmatrix}\ B_2^1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\ B_3^1 = \begin{pmatrix} -\frac{\varepsilon}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

$\tilde{M}$ will be truncated so that the final matrix will look like

$$
\tilde{M} = \begin{pmatrix} B_0^1 & B_1^1 & B_2^1 & B_3^1 \\ 0 & B_0^2 & B_1^2 & B_2^2 \\ 0 & 0 & B_0^3 & B_1^3 \end{pmatrix}
$$

B_0^2 through B_1^3 will be defined using the rule $B_j^i = B_j^1 \otimes I^{[i-1]} + I \otimes B_j^{i-1}$

$I^{[i]}$ is defined as $\underbrace{I \otimes I}_{i-times}$ and $I^{[1]}$ is just I. I is defined as

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Writing out B_0^2 through B_1^3

$$B_0^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} B_1^2 = \begin{pmatrix} 2\varepsilon & 1 & 1 & 0 \\ -1 & \varepsilon & 0 & 1 \\ -1 & 0 & \varepsilon & 1 \\ 0 & -1 & -1 & 0 \end{pmatrix}$$

$$B_2^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$I \otimes I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_0^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_1^3 = \begin{pmatrix} 3\varepsilon & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 2\varepsilon & 0 & 1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 2\varepsilon & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & \varepsilon & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 2\varepsilon & 1 & 1 & 0 \\ 0 & -1 & 0 & 0 & -1 & \varepsilon & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & 0 & \varepsilon & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \end{pmatrix}$$

When all these matrices are put back in to $\tilde{M}$ you get

$$\tilde{M}=\begin{pmatrix}
0 & \varepsilon & 1 & 0 & 0 & 0 & 0 & -\frac{\varepsilon}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2\varepsilon & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & \varepsilon & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & \varepsilon & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3\varepsilon & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2\varepsilon & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2\varepsilon & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & \varepsilon & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 2\varepsilon & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & \varepsilon & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & \varepsilon & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0
\end{pmatrix}$$

Again, remember

$$\frac{dx^T}{dt}=\tilde{M}x^T$$

Except this time

$x=\begin{pmatrix}1 & x & y & x^2 & xy & yx & y^2 & x^3 & x^2y & x^2y & xy^2 & x^2y & xy^2 & xy^2 & y^3\end{pmatrix}$ or

$x=\begin{pmatrix}1 & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} & x_{13} & x_{14}\end{pmatrix}$ where $x_1=x$, $x_2=y$, $x_3=x^2$ and so on and the T means transpose. Multiply the two matrices together to get a system of 14 ordinary differential equations. That system looks like the following

$$x_1'[t] = \varepsilon x_1[t] + x_2[t] - \frac{\varepsilon}{3} x_7[t]$$
$$x_2'[t] = -x_1[t]$$
$$x_3'[t] = 2\varepsilon x_3[t] + x_4[t] + x_5[t]$$
$$x_4'[t] = -x_3[t] + \varepsilon x_4[t] + x_6[t]$$
$$x_5'[t] = -x_3[t] + x_5[t] + x_6[t]$$
$$x_6'[t] = -x_4[t] - x_5[t]$$
$$x_7'[t] = x_{11}[t] + 3\varepsilon x_7[t] + x_8[t] + x_9[t]$$
$$x_8'[t] = x_{10}[t] + x_{12}[t] - x_7[t] + 2\varepsilon x_8[t]$$
$$x_9'[t] = x_{10}[t] + x_{13}[t] - x_7[t] + 2\varepsilon x_9[t]$$
$$x_{10}'[t] = \varepsilon x_{10}[t] + x_{14}[t] - x_8[t] - x_9[t]$$
$$x_{11}'[t] = 2\varepsilon x_{11}[t] + x_{12}[t] + x_{13}[t] - x_7[t]$$
$$x_{12}'[t] = -x_{11}[t] + \varepsilon x_{12}[t] + x_{14}[t] - x_8[t]$$
$$x_{13}'[t] = -x_{11}[t] + \varepsilon x_{13}[t] + x_{14}[t] - x_9[t]$$
$$x_{14}'[t] = -x_{10}[t] - x_{12}[t] - x_{13}[t]$$

When Carleman's technique was used, it yielded a 14×14 matrix, which translates to 14 ordinary differential equations. The comparison of the solution to the original function is plotted in figure 24 below.

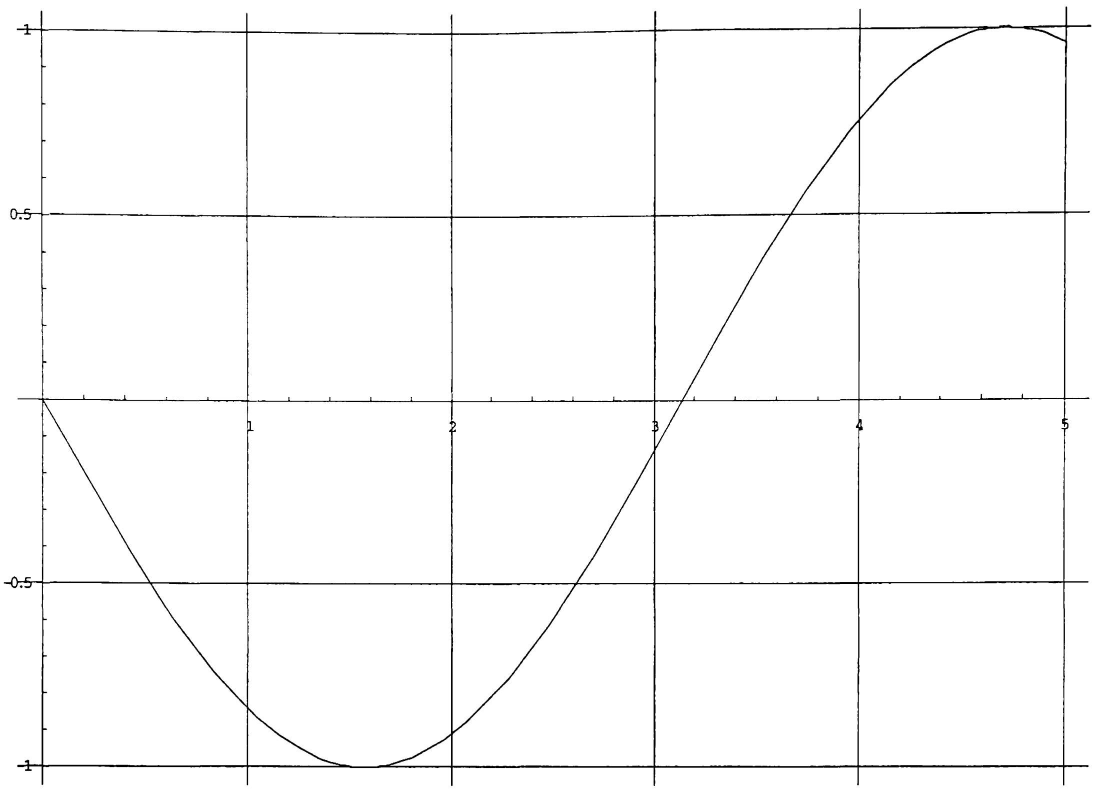

Fig. 24 Van der Pol's equation compared with the Carleman linearization technique over a time interval of 5 seconds using third order polynomials yielding a system of 14 ODEs

There are actually two graphs in figure 24. As the size of ε is increased, the approximation gets worse. For this example an ε of .001 was used, and it can be seen that the approximation is nearly perfect. What this result may indicate is that the Carleman linearization technique may work exceptionally well for non-linear problems that are periodic in nature.

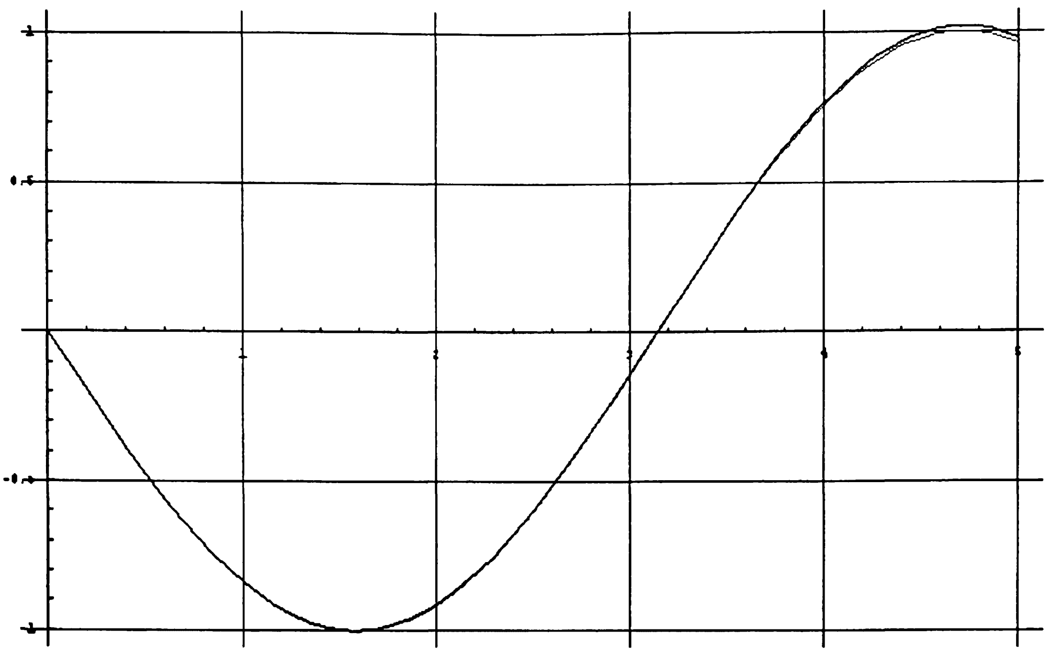

Fig. 25 Van der Pol's equation compared with the Carleman linearization technique over a time interval of 5 seconds. $\varepsilon = 0.01$

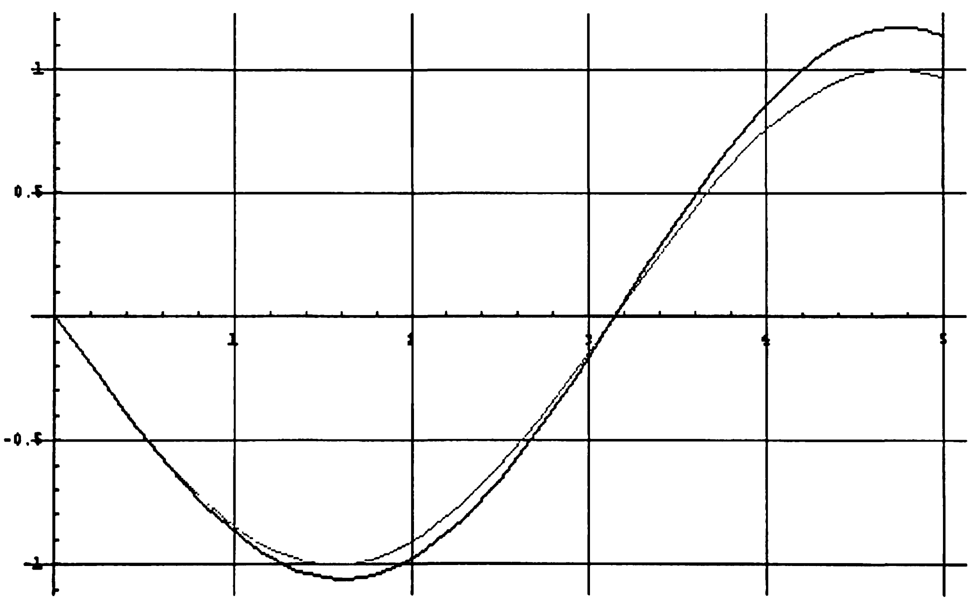

Fig. 26 Van der Pol's equation compared with the Carleman linearization technique over a time interval of 5 seconds $\varepsilon = 0.1$

CHAPTER V

DISCUSSION OF RESULTS

The results obtained from the trajectory problem and the boundary layer problems were less than spectacular. The rate of convergence for these functions was slow at best. It took a large system of ODEs to capture an acceptable portion of the function. For comparison, it is of some interest to expand a log type function in a Taylor series and show the slow convergence. It appears that, even though the rate of convergence for log type functions using the Carleman technique is slow, it converges faster and more accurately than does a Taylor expansion of a log type function.

Below, figure 27 show the plot of function log[x] for x=0 to 3.

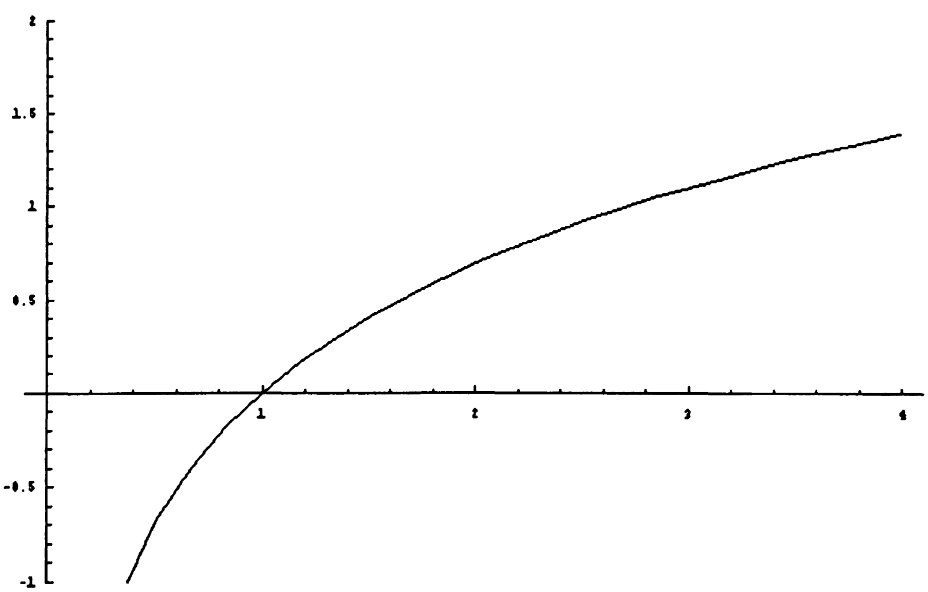

Fig. 27 Plot of Log[x] for x = 0 to 3

The function Log[x] is expanded using a Taylor series for n = 2, n = 4, n = 8, n = 16, and n = 1000. Only the expansion for n = 2, where n is the number of terms and the order of the polynomial, is listed below, though all of the expansions were calculated using Mathematica 3.0 and plotted against the original Log[x] function.

$$-\frac{3}{2}+2x-\frac{x^2}{2}$$

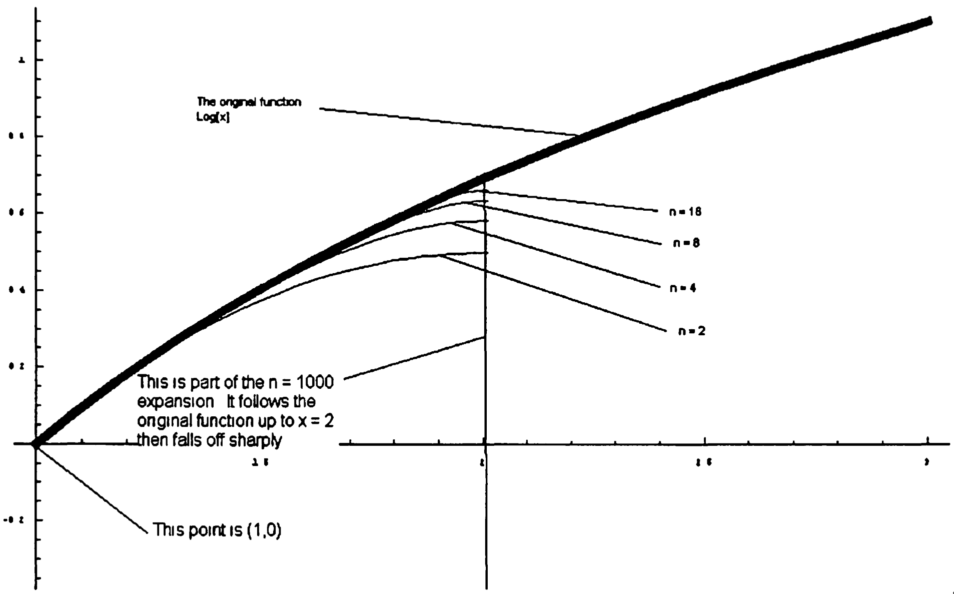

Fig. 28 Plot of Taylor series expansion approximations for different "n" number of terms in the Taylor series expansion compared to the original function Log[x]

Then the difference between the original function and the Taylor series expansion is shown in figure 29 to illustrate the slow rate of convergence. The more terms that are added the better the approximation, but the number of terms grows very rapidly to move farther to the right on the plot. The reason n = 1000 was picked was because it begins to approach the number of terms used in the Carleman technique.

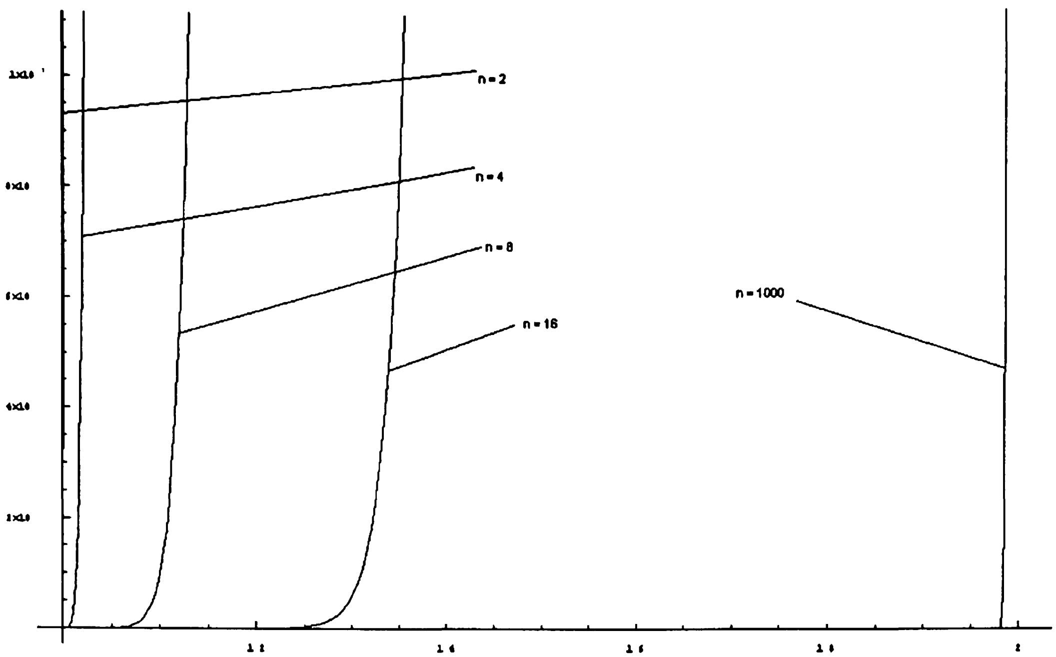

Fig. 29 Shows how much the Taylor series expansion diverges from the function Log[x] as x increases for "n" number of terms in the Taylor series expansion

Figure 30 compares how well the Carleman technique converges in relation not to the number of term but in the number of ODEs, for the trajectory problem.

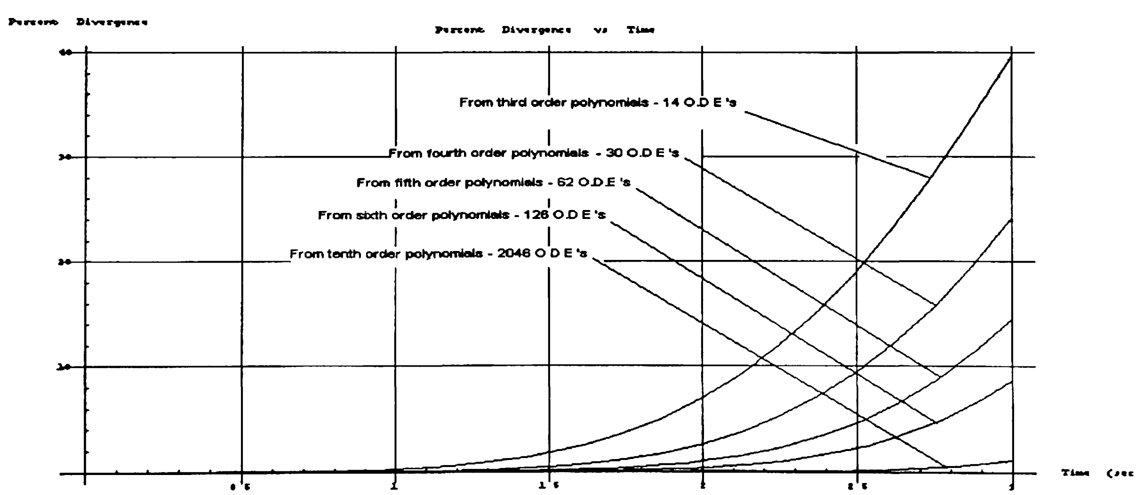

Fig. 30 Percent divergence versus time for varying levels of matrix complexity for the ballistic trajectory problem

Figures 31 and 32 show how well the Carleman technique converges in relation not to the number of term but in the number of ODEs, for the boundary layer problem.

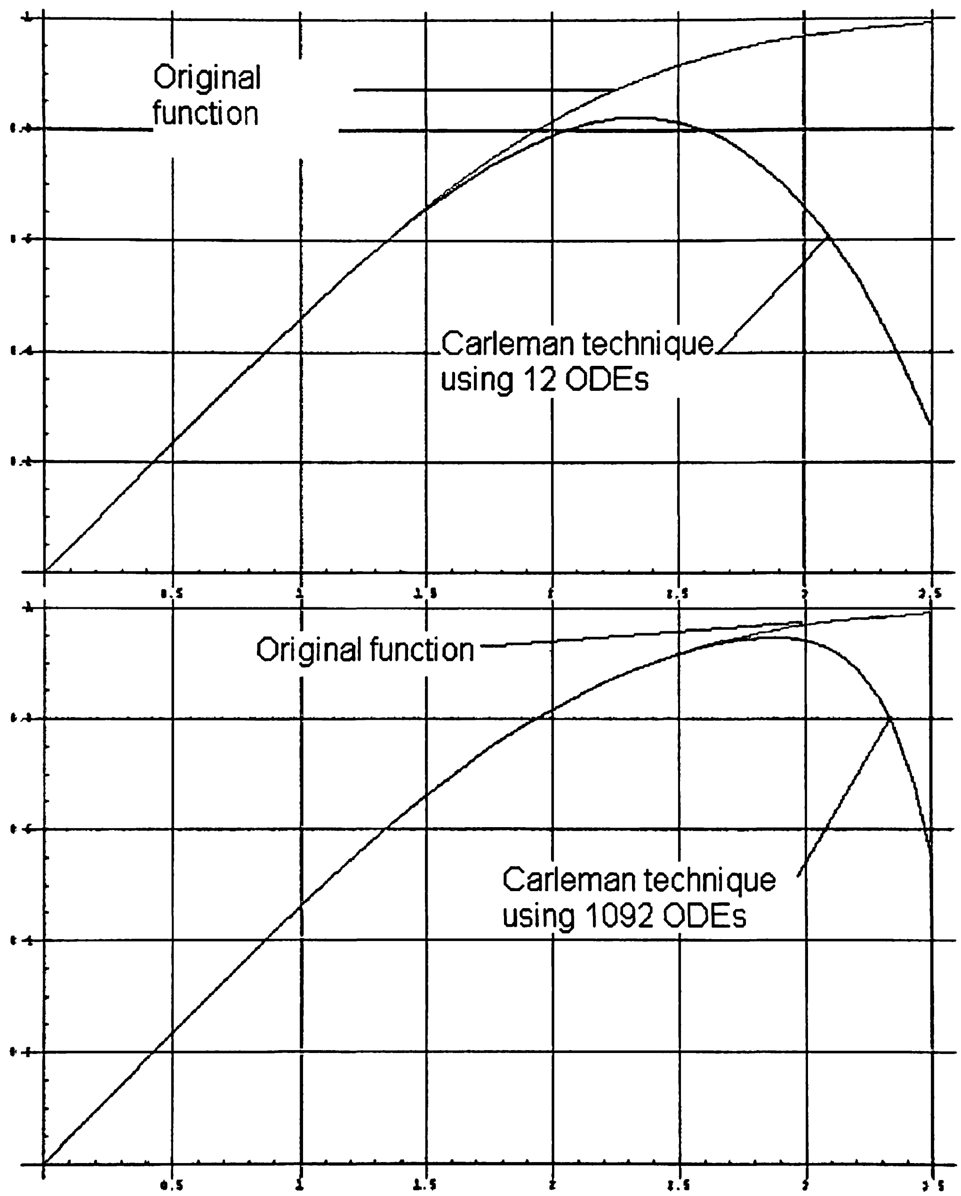

Fig. 31 Convergence of the Carleman technique for the boundary layer problem

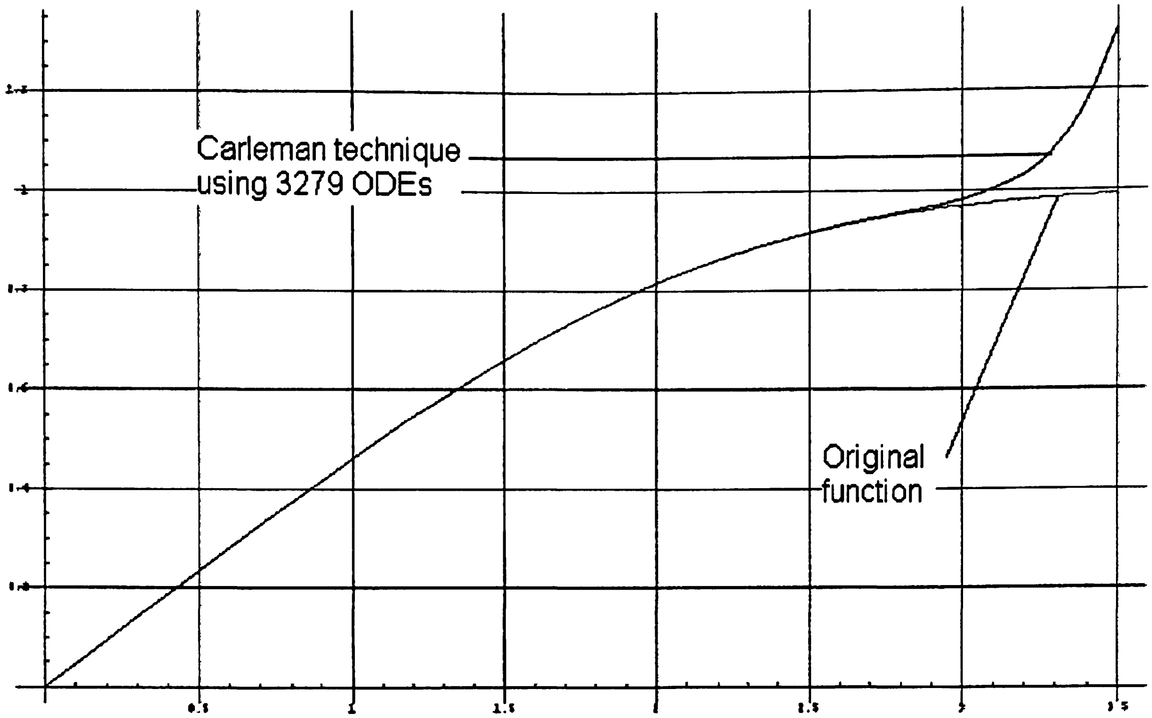

Fig. 32 Convergence of the Carleman technique for the boundary layer problem

The Carleman technique did, however, work very well for Van der Pol's equations for $\varepsilon \sim$ *small* using the smallest system of ODEs generated by the Carleman technique. As ε is increased, the fidelity of the Carleman technique decreases. This was illustrated in figures 33, 34, and 35.

The results, overall, show that the Carleman linearization technique works for functions of a periodic nature such as Van der Pol's equations. It works fine for log type functions, it just converges slowly.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

At the conclusion of this research, it appears that the utility of the Carleman linearization technique for studying trajectories is not very practical. In order to capture adequately the trajectory of the bowling ball, the number of ODEs became impractical for any engineering use. For engineering purposes, such calculations can be just as easily made using numerical methods. There may be minor applications to reentering bodies in the earth's atmosphere that do not reach a terminal velocity. For log type functions, such as the vertical velocity of the bowling ball, the sharper the bends in the log function the more ODEs were required to characterize the trajectory. For objects that do not reach terminal velocity, the bend in this function is slight and the Carleman technique can characterize the trajectory very well with a small number of ODEs.

The boundary layer problem was very similar to that of the trajectory problem in terms of how the function behaved. Therefore, the result that the Carleman linearization technique produced was also similar. By inspection, it did a little better than the trajectory problem but not enough to say that it has any engineering utility. It like the trajectory problem required an extraordinary amount of ODEs to adequately capture the boundary layer function.

There may be some utility in using this technique for theoretical studies, in which case one would not worry about how many terms or numbers of ODEs it takes for the function to converge.

The Carleman linearization method did appear to have some practicality in the study of problems with periodic solutions. Kowalski and Steeb showed that a solution to the Lotka-Volterra model could be found using this method. Even though a solution to Van der Pol's equations already existed, this research showed that a solution to Van der Pol's equations could also be found using the Carleman method. Since the solutions the Carleman technique yielded are periodic, other such problems with periodic solutions should be tried. This could lead to analytical solution of nonlinear problems, which have applications in guidance and control systems for aircraft and missile systems. Anywhere nonlinear but periodic solutions are modeled in nonlinear problems, this technique may have an application.

Recommendations

Future research involving this linearization technique should focus on problems with periodic solutions. It makes sense to go in this direction given that Poincaré and Carleman first thought of this technique to study oscillatory motion. Most of the examples in the review of literature were of a periodic nature. A lot of the application thus far has only been in the area studying circuit and vibration problems. However, many things in engineering are periodic in nature and are worthy of study using this technique. The wave equation to study shocks and vibration is one example. The wave equation would lead to studying how well this technique works with partial differential equations.

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