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Nonlinear Analysis of Composite Beams under Random Excitations

Snorri Gudmundsson

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NONLINEAR ANALYSIS OF COMPOSITE BEAMS UNDER RANDOM EXCITATIONS

by

Snorri Gudmundsson

A Thesis Submitted to the Office of Graduate Programs in
Partial Fulfillment of the Requirements of the Degree of
Master of Science in Aerospace Engineering

Embry-Riddle Aeronautical University

Daytona Beach, Florida

March 1995

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by

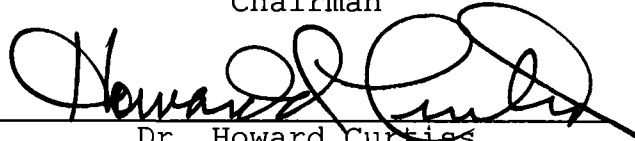
Snorri Gudmundsson

This thesis was prepared under the direction of the candidate's thesis committee chairman, Dr. Habib Eslami, Department of Aerospace Engineering, and has been approved by the members of this thesis committee. It was submitted to the office of the Graduate Programs and was accepted in partial fulfillment of the requirements for the degree of Master of Science in Aerospace Engineering.

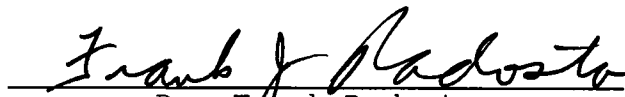
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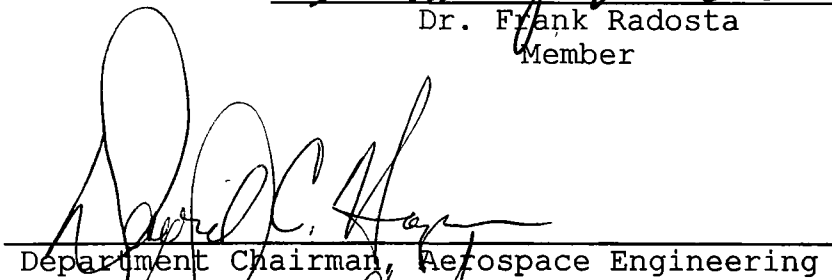
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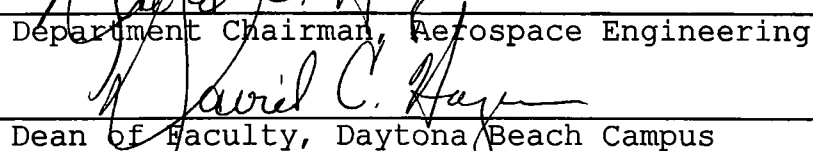
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ABSTRACT

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The vibration responses of three unsymmetrically laminated beams, that are excited with a Gaussian random forcing function, are studied in this thesis. The beams are analyzed nonlinearly and compared to linear results, indicating some important corrections. The solution procedure begins with the derivation of the general equation of motion using Galerkin's method. Then, two approaches are taken in the solution. First, the equation of motion is attacked directly employing a real time Runge-Kutta numerical analysis. Second, the method of equivalent linearization is used. The thesis finds the results from the two approaches to be in a close agreement, although some discrepancies at high loads could be found. However, the most important achievement of the thesis is undoubtedly that the same response equation can be used for any type of laminate layup, and any of the three types of beams. One of

many applications that the solution can be used for is the assessment of fatigue tolerances.

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Finally, the compilation of the work presented in the thesis was a monstrous task and consumed a lot of time that otherwise would have been directed toward my precious daughter, Viktoria. It is to her that this thesis is dedicated.

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LIST OF SYMBOLS

Symbol	Representation	Units
A	Cross-sectional area	m or in
A_{ij} $i, j = 1, 2, 3$	Extensional stiffness of a laminate	N·m or lb _f ·in
a_i	Constant	
B_{ij} $i, j = 1, 2, 3$	Coupling stiffness of a laminate	N or lb _f
b	Width	m or in
D_{ij} $i, j = 1, 2, 3$	Bending stiffness of a laminate	N/m or lb/in
E	Young's modulus	Pa=N/m ² or psi
E_1, E_2, E_3	Young's moduli in 1, 2, and 3 directions	Pa or psi
$E[]$	Mathematical expectation of a random process	Depends on argument
f	Circular frequency	Hz
$F(t)$	A time dependent forcing function	N or lb _f
$f(x)$	A generic function, used in explanation	
$H(w)$	Complex frequency response function	
h	Thickness: [m] or [in]	m or in
G_1, G_2	Shear stiffness moduli in 1 and 2 directions	Pa or psi
G_{ij} $i, j = 1, 2, 3$	Shear stiffness moduli for a laminate	Pa or psi
$g(x)$	A generic function, used in explanation	
I	Moment of inertia	N·m ² or lb·in ²
i	Index	
h	Height	m or in
j	Index	
K_1	Coefficient	
K_2	Coefficient	
k	Index	
L	Length	m or in
L	Differential operator	

M_v	Moment resultant	N·m/m or lb _f ·in/in
M_{vv}	Twisting moment resultant	N·m/m or lb _f ·in/in
M_v	Moment resultant	N·m/m or lb _f ·in/in
m	Mass of a beam	kg or lb _m
m	cos θ	
m	Index	
m_1	Mean of a distribution	Depends on argument
N_v	Force resultant	N/m or lb _f /in
N_{vv}	Shear force resultant	N/m or lb _f /in
N_v	Force resultant	N/m or lb _f /in
n	sin θ	
n	Cycles in a σ - n diagram	
n	Number of values in a domain	
n	Index	
P	Forcing function	N or lb _f
P_{RRP}	Reference pressure, the threshold of hearing = 2×10^{-5} Pa	Pa or psi
PSD	Power Spectral Density	Pa or psi
$p(x)$	A probability distribution	
$Q_{i,j}$ $i, j = 1, 2, 3$	Members of the reduced stiffness matrix	Pa or psi
$\bar{Q}_{i,j}$ $i, j = 1, 2, 3$	Members of the reduced transformed stiffness matrix	Pa or psi
$q(t)$	Deflection response with respect to time	
R	Radius of gyration = $\sqrt{I/A}$	m or in
rms	Root mean square	Depends on argument
S	Nondimensional excitation parameter	
$S_n(f)$	Spectral density function in terms of circular frequency	(N/m) ² /Hz or (lb _f /in) ² /Hz
$S_n(\Omega)$	Spectral density function in terms of linearized frequency	(N/m) ² /Hz or (lb _f /in) ² /Hz
S	Nondimensional excitation parameter	
SSL	Sound Spectrum Level	dB
t	Time	sec
U_i $i=1..5$	Invariants	Pa or psi

u_o	Displacement of midplane along x-axis	m or in
v_o	Displacement of midplane along y-axis	m or in
W	Experimental spectral density	Depends on argument
W_{ext}	External work done by a system	N·m or lb _f ·in
W_{int}	Internal work done by a system	N·m or lb _f ·in
w	Deflection	m or in
X	Random variable	Depends on argument
x_1	Random process	Depends on argument
$x(t)$	Random process	Depends on argument
x, y, z	Displacements along the axes of a Cartesian coordinate system	
α	Coefficient of quadratic nonlinearity	Hz ²
β	Coefficient of cubic nonlinearity	Hz ²
ϵ	Strains	m/m or in/in
κ	Curvatures	
$\nu_{ij}, i, j = 1, 2, 3$	Poisson's ratios for a laminate	
Ω	Equivalent linearized frequency	Hz
ω	Frequency	Hz
ω_o	Natural frequency	Hz
ρ	Mass density of a beam: [kg sec ² /m ⁴] or [lb sec ² /in ⁴]	
σ	Stress	Pa or psi
$\sigma_{ij}, i, j = 1, 2, 3$	Normal stresses for a laminate	Pa or psi
σ_1	Standard deviation of a given distribution	
τ	Time interval, used in autocorrelation functions	sec
$\tau_{ij}, i, j = 1, 2, 3$	Shear stresses for a laminate	Pa or psi
ξ	Damping coefficient = ω/ω_o	
Ψ_i	Functions of x and t	Pa or psi

CHAPTER 1

INTRODUCTION

1.1 Introductory Remarks.

The purpose of this thesis is to study the nonlinear dynamic response of composite beams that are excited with random forcing functions of predominantly Gaussian distribution. Solutions for three types of beams are presented in the thesis; a simply-supported, a clamped-clamped, and a cantilever beam. The random excitation force has many practical analogies, turbulent aerodynamic loads and jet noise from turbomachinery being only two examples of many. The importance of this topic to the aerospace industry, in particular, cannot be overemphasized and it has a wide range of application. For instance, the methods presented in the thesis may aid in determining fatigue life of aerospace vehicles and help in the design of structures that support random loads. Applications outside of the

aerospace industry include wind and gust loads on various structures, such as support structures for electrical cables, tall buildings etc. Within the aerospace industry, modern aircraft design is one of many subjects that the results of the thesis can be applied to. An important task in the design of any aircraft involves the determination of the life-expectancy (or design life) of the airframe. For instance, the Anglo-French Concorde, shown in Figure 1 [36] is a good example of this. The aircraft is a supersonic passenger transport and cruises regularly at an altitude of approximately 18 km (60,000 ft), experiencing cyclic loading¹ unknown to any other present passenger transport.

Several methods are available to determine the life-expectancy, as has been documented by many sources. References [37], [38], and [39] explain and treat typical fatigue problems for isotropic and composite materials. Generally, the lifetime of the airframe is decided by the designers, and it is frequently based on the mission profile. It is expected that an aircraft should tolerate a given number of cycles, where each take-off and a subsequent landing comprises a cycle [37]. The concept of a cycle indicates that a load is applied to the airframe and then removed.

¹ This loading is a result of cabin pressurization.

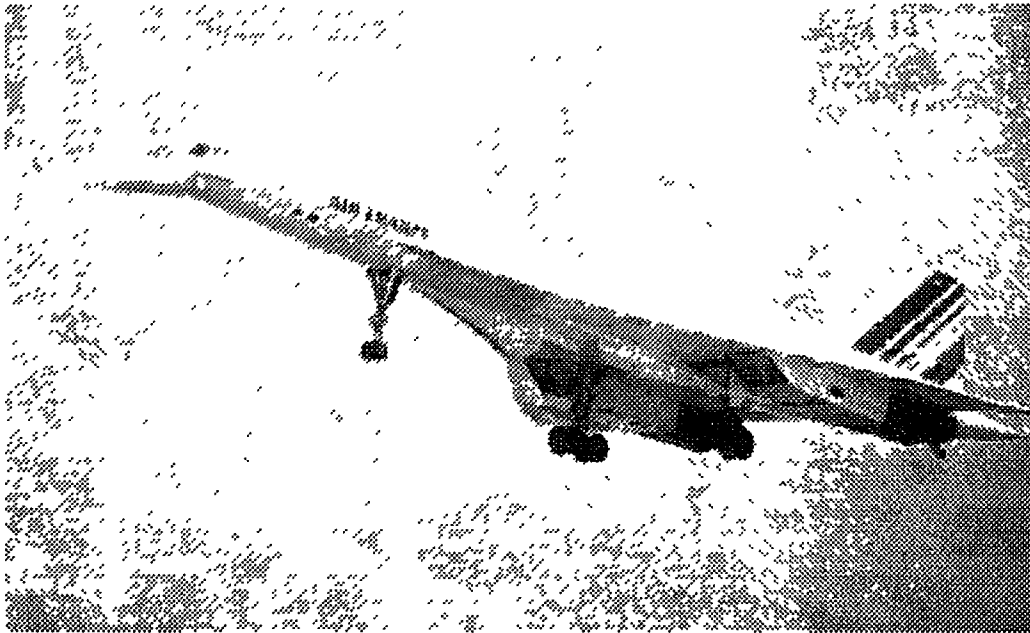


Figure 1. The Anglo-French Concorde is an example of a modern transportation aircraft [36].

Consider the wings, for instance. While on the ground the wings carry their own weight (depending upon wing-undercarriage configuration), but after take-off they are loaded to support almost the entire weight of the aircraft. After landing they again carry their own weight - completing the so-called ground-air-ground (GAG) cycle. Figure 2 [37] shows a typical GAG cycle for a transport aircraft. If the aircraft encounters turbulence the mean-to-mean GAG cycle for that flight increases. If the GAG-cycle remains consistently higher for one of two identical aircraft its life-expectancy will be reduced.

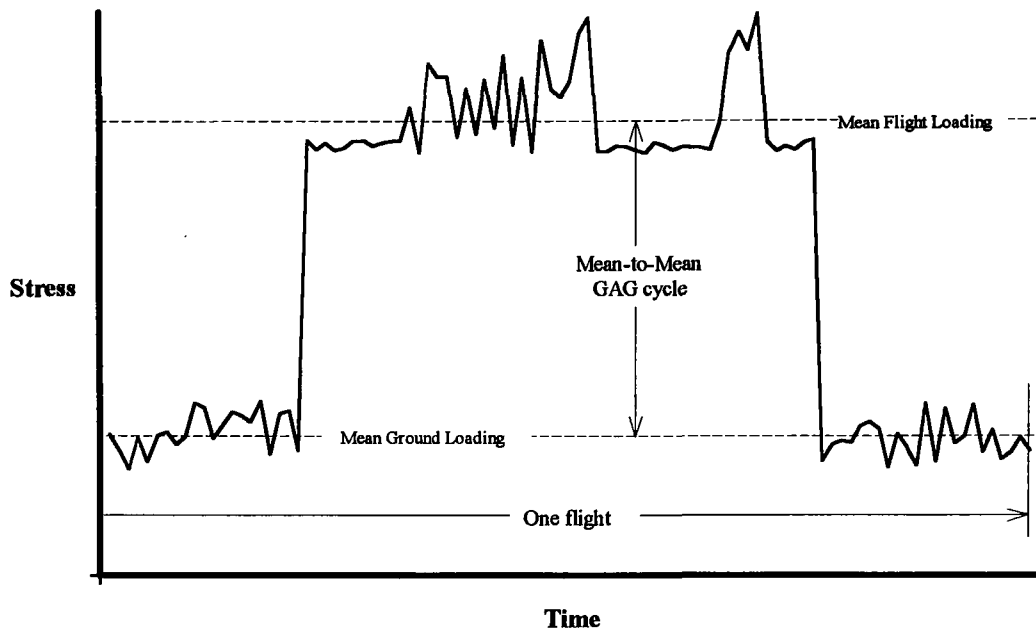


Figure 2. A typical ground-air-ground (GAG) cycle.

It has been found by experiments that most materials, when undergoing cyclic loading, break at a lower strength than what is to be expected from static loading. For instance, a certain type of aluminum might yield at, say, 48,000 psi static loading, but when repeatedly loaded to 10,000 psi and then unloaded, it might break after perhaps 50,000 cycles. This kind of failure is called fatigue failure. By performing many such experiments, where specimens of metals are loaded to a given stress, σ , and then the number of cycles-to-break, n , are recorded, it is possible to plot σ - n diagrams. Figure 3. shows a typical such curve for steel. This figure indicates a large scatter

of data points, which occurs in practice. Of interest is to note the so-called endurance limit for the steel, below which no fatigue is experienced. Aluminum alloys do not have endurance limits.

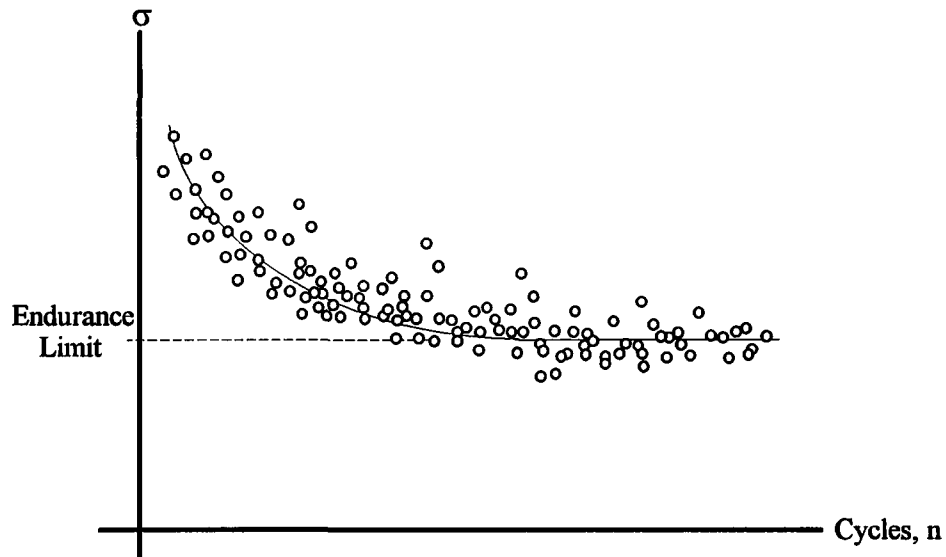


Figure 3. An $\sigma - n$ diagram for steel.

One of the greatest difficulties in assessing the life-expectancy of an airframe is to take into account the various in-flight loads. It can be seen that a typical mission of a transportation jet consists of a take-off, after which a fairly constant loading of $1g$ is applied to the wings. However, changes in loads inevitably occur during the flight, as depicted in Figure 2. Turbulent air may for a short period of time increase the loading - accelerations up

to several times the weight of the aircraft have been recorded, although such events are rare. Clear air turbulence (CAT) is frequently encountered in high-altitude flight and it is known to impose considerable loads on the airframe, in extreme cases causing physical damage to aircraft. On the other hand, a military aircraft whose mission is to fly at tree-top level may encounter turbulence more frequently than a high-flying aircraft. Two identical aircraft flying at two different flight levels throughout most of their operational life show different levels of metal fatigue. The one flying in higher turbulence usually shows the worse signs.

The turbulent behavior of air can be approximated using a random stationary behavior. Therefore, it is the focus of this thesis to introduce solution techniques that best describe the behavior of structures under such a loading. The solution can then be used to assess, with a greater accuracy, the design life of the airframe by enabling a more precise dynamic stress analysis. This thesis deals with the random behavior of composite beams, of which isotropic beams can be considered the simplest case.

Of particular interest is the random response of cantilever beams, which by nature resemble wings flying in turbulent conditions. Figure 4 shows a cantilever wing with a lift distribution described by $P(x, t)$. If P is allowed to

vary randomly with time, the response or the deflection of the beam can be analyzed. Once the deflection curve becomes known, other important parameters can be retrieved, such as bending and shear stresses.

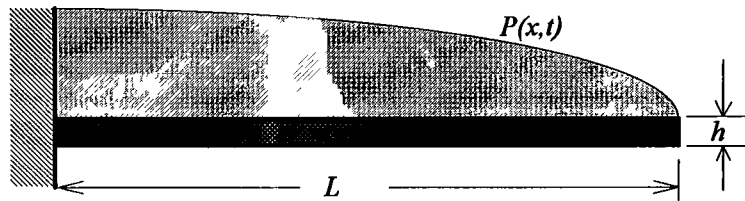


Figure 4. A cantilever beam analogous to a wing.

The general solution procedure presented in the thesis is the following. In Chapter 2 some preliminary concepts used in laminate structural analysis, random analysis, and numerical analysis are introduced. In Chapter 3 the general equation of motion for any laminated beam is derived and pertinent solution techniques are introduced. In Chapter 4, the equation of motion is solved for simply supported, clamped-clamped, and cantilever beams, respectively. In Chapter 5 some numerical examples are presented. Finally, Chapter 6 contains some concluding remarks.

1.2. Literature Survey.

The research on nonlinear random response of laminated beams is very limited, so a thorough survey including the free and harmonic excitation and random excitation is conducted in this thesis. In the first section of this survey, work on nonlinear free vibrations of isotropic and composite beams is cited. In the second section, emphasis is placed on the published research of nonlinear forced vibrations of beams. Finally, the third section entails the publication of research in random analysis.

1.2.1 Nonlinear Free-Vibration.

A large number of references exists on nonlinear free vibration of isotropic beams, e.g. references [1] through [18]. More recently Kapania and Raciti [20] developed a simple one-dimensional finite element for the nonlinear analysis of symmetrically and unsymmetrically laminated composite beams including shear deformation. The formulation of the problem, the solution procedure, and the computer program were developed for a variety of static and nonlinear dynamic problems, including isotropic and symmetrically laminated beams. Raciti found that the nonlinear vibration had, what he called, "soft spring" behavior for certain boundary conditions as opposed to "hard spring" behavior

observed in isotropic and symmetrically laminated beams. The mid-plane boundary conditions were found to affect the nonlinear response significantly. Singh, Rao and Iyengar [21] investigated large-amplitude free vibrations of unsymmetrically laminated composite beams using von Kármán's large deflection theory. They applied a one dimensional finite element model based on the classical lamination theory, first-order shear deformation theory and higher-order shear-deformation theory having 8, 10, and 12 degrees of freedom per node, respectively. This was done to bring out the effects of transverse shear on the large amplitude vibrations. Bangera and Chandrashekhara [22] developed a finite-element model to study the large-amplitude free vibrations of generally-layered laminated composite beams. They considered the effects of Poisson's ratio by including it in the constitutive equations. The direct iteration method was used to solve the nonlinear equation at the point of reversal of motion. The influence of boundary conditions, beam geometry, Poisson's effect, and ply orientation on the nonlinear frequencies and mode shapes were demonstrated.

1.2.2 Nonlinear Forced Vibration.

Even though the present research focuses on nonlinear random vibration of beams, special attention was nonetheless

given to the work of harmonic forced vibrations of isotropic and composite beams. The survey yielded the following.

Tseng and Dugundji [23] applied the harmonic balance method to solve the problem of a straight isotropic beam with fixed ends subjected to a harmonic excitation at its supporting ends. Alturi [24] applied the method of multiple scales [25] to investigate the response of nonlinear forced vibration of a hinged isotropic beam considering nonlinear inertia terms. His conclusion demonstrates nonlinearity effects of the softening type. Srinivasan [6] solved for free and forced responses of isotropic beams subjected to moderately large-amplitude steady-state oscillations by the Ritz-averaging method. The application of this method transforms the partial differential governing equation into a system of nonlinear algebraic equations. Srinivasan then applied the Newton's method to solve those equations. Nayfeh, Mook and Lobitz [26] presented a numerical-perturbation method for the nonlinear analysis of forced vibration of isotropic beams. A multiple-mode expansion in terms of the linear mode shapes was considered. The problem was then solved using the method of multiple scales, considering internal resonance.

Some research on nonlinear forced vibrations of composites beams has also been carried out. Pai and Nayfeh [27] investigated the forced nonlinear vibration of a

symmetrically laminated graphite-epoxy composite beam. Their analysis focused on the case of primary resonance of the first flexural-torsional mode. A combination of the fundamental-matrix method, a Galerkin procedure and the method of multiple scales is used to derive four first-order ordinary-differential equations describing the modulations of the amplitudes and phases of the interacting modes with damping, nonlinearity and resonance. The result shows that the motion was non-planar despite a planar input. It was further concluded that non-planar responses can be periodic motions, as well as amplitude- and phase modulated motions.

Chandrashekhara [28] considered the flexural analysis of fiber-reinforced composite beams based on higher order shear deformation theory. A von Kármán type nonlinearity is incorporated in the formulation of the problem. The finite-element method is used to solve the nonlinear governing equations by a direct iteration. Unlike the conventional beam models, Chandrashekhara took into account the transverse strains and investigated the differences in the solutions for the cross-ply laminates and the angle-ply laminates. He concluded that the solution obtained from the two approaches differ slightly in the case of the cross-ply laminates, but there exists a considerable difference in the case of the angle-ply. Also, Kenareh [29] is working on the response of a symmetrically and unsymmetrically laminated

composite beam subjected to nonlinear forced vibration, using a finite element method.

Some work has also been carried out on nonlinear forced vibrations of isotropic and composite plates. Reddy [30] investigated forced motion of laminated plates using a finite element method that accounts for transverse shear deformation, rotary inertia and large rotation (in von Kármán's sense). In his paper, he presented numerical results for the nonlinear analysis of composite plates, and pointed out the effects of the plate's thickness, boundary conditions and loading on the deflection and stresses. Mei and Decha-Umphai [31] extended the finite element method to determine the response of large-amplitude forced vibrations of thin isotropic plates. A force matrix under uniform harmonic excitations was developed for nonlinear forced vibration analysis. The results obtained were compared with simple elliptic response, perturbation, and other approximation solutions. The method of multiple scales in conjunction with Galerkin's method was used by Eslami and Kandil [32] to analyze the nonlinear forced and damped response of a rectangular orthotropic plate subjected to a uniformly distributed transverse loading. The analysis considered simply-supported as well as clamped panels. By using the method of multiple scales, all possible resonances were investigated, such as the primary resonance,

subharmonic and superharmonic resonance. Hua [33] studied the geometric nonlinear forced flexural vibration of anisotropic symmetrically laminated composite plates under a harmonic force. He presented the effects of angle of orientation of the symmetrically laminated plates on the amplitude-frequency response. Chiang, Xue and Mei [34] presented a finite element formulation for determining the large-amplitude free and steady-state forced vibration response of arbitrarily laminated anisotropic composite thin plates using the Discrete Kirchhoff Theory (DKT) triangular elements. Their work focuses only on primary resonance. The nonlinear stiffness and harmonic force matrices of an arbitrarily laminated composite thin plate element were developed for nonlinear free and forced vibration analyses. The effects of damping were not included. Huang [35] investigated the forced nonlinear axisymmetric vibrations of an orthotropic composite plate with fixed boundary conditions. The governing nonlinear partial differential equations were converted into the corresponding nonlinear ordinary differential equations by the elimination of the time variable with the Kantorovitch time-averaging method. The solutions of the eigenvalue problems were obtained using a Newton iteration technique. The results revealed the effects of finite amplitude and anisotropy of materials upon the fundamental responses.

1.2.3 Randomly Forced Vibration.

Considerable work has been done on random vibrations, especially in the field of isotropic beams and plates. However, due to limitation of space in the thesis, only work that directly pertains to the subject of the thesis is included. Seide [45] studied Gaussian vibration in the seventies and used a numerical scheme to solve for the random response of beams. An important contribution related to the material herein is made by Mei and Prasad [44]. They solved the Duffing equation for isotropic beams and plates (see Section 4.6) using a numerical iteration scheme. Other work on random vibration and fatigue include references [35], [39], and [49]. Practical work helpful in the random analysis of composites include references [37], [46], [47], and [48]. Little work has currently been published on random vibration of unsymmetrically laminated beams. None was found on Gaussian vibration of unsymmetrically laminated beams.

1.3 The Scope of the Thesis.

The purpose of this thesis is to study the nonlinear random vibrations of unsymmetrically laminated beams. This subject has never before been studied as such, to the best knowledge of the author. Considerable amount of work has already been done on random analysis of beams, both linear

and nonlinear, involving both isotropic and symmetrically laminated beams. However, previous work falls short of treating unsymmetrically laminated lay-ups. That, in itself, attests to the urgent need for research of this sort, as does a greater sophistication in the application of composite materials in technology and engineering.

The results of the work presented in this thesis enables an analytical treatment of virtually any kind of laminated lay-up, whether it be symmetrical or unsymmetrical. The solution method treats the governing equations of motion by introducing von Kármán's geometrical nonlinearities. The solution process is implemented by the application of Galerkin's method², which transforms the fourth order partial differential equation into an ordinary differential equation. This equation is usually referred to as the Duffing equation. It is a nonlinear differential equation which, in the thesis, is solved by the method of equivalent linearization. The solution directly yields the deflection response of the beam being studied.

It turns out that the random response of the simply-supported, clamped-clamped, and cantilever beams can be expressed with a single common equation. The only difference

² There are other methods besides Galerkin's method that can be used to perform such a transformation, but this is the simplest one to apply.

among the beams is in the expression of various beam characteristics, such as the natural frequency, mass, and the nonlinearity coefficients. This finding was supported by the application of numerical analysis, the result of which were in good agreement with the analytical method.

Furthermore, since the solution is applicable to any lay-up method, as well as to isotropic material, the solution was compared to the work of Mei and Prasad [44] and Seide [45] and found to be in good agreement.

CHAPTER 2

PRELIMINARIES

In this chapter some important concepts used in random analysis, the structural analysis of composites, and numerical analysis are discussed.

2.1 Concepts in Random Analysis.

The following section is intended to introduce some concepts and definitions that apply to random or stochastic analysis. The definitions and explanations that follow are general and as such can be found in most texts on statistics. In particular, references [40], [41], [42], and [43] were helpful.

2.1.1 Ensembles and Sample Functions.

A random process $x(t)$ is a process that cannot be described with just one time history, but many which are

called a family or an ensemble. Figure 5 shows an ensemble of four processes, but to represent any possible outcome an infinite number of processes $x^{(j)}$ are required. Any individual time history, $x^{(j)}$, belonging to the ensemble is called a sample function.

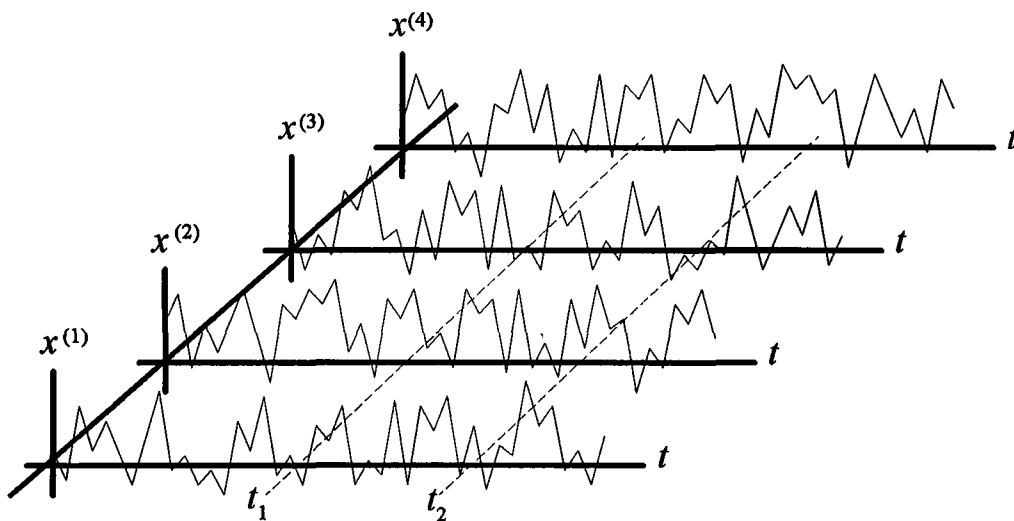


Figure 5. An ensemble of four processes.

2.1.2 Random Variable.

A random variable X is a real-valued function defined on a sample space. If the random variable has a finite number of values it is said to be a discrete random variable. If it has an infinite number of values X is a continuous random variable. Continuous random variables are idealizations, but

they enable full power of mathematical analysis, not always possible with discrete random variables. The random variables (for instance the distribution of the random forcing function) in this thesis are assumed to be continuous.

2.1.3 Stationary Random Process.

A random process is said to be stationary if its probability distribution remains unchanged with time. This implies that all the averages based upon a given probability distribution, $p(x)$, are independent of time. In this thesis the random processes are assumed to be stationary.

2.1.4 Temporal Average.

Given one sample, $x^{(j)}$, a temporal average is the average of that sample along its time-axis.

2.1.5 Ergodic Process.

Within the class of stationary random processes there is a subclass called an ergodic process. An ergodic process is a process in which averages are equal to the corresponding temporal averages taken along any representative sample function. The random processes treated in the thesis are ergodic.

2.1.6 Normal or Gaussian Processes.

Random processes that can be described by a first-order normal or Gaussian probability distribution are called normal or Gaussian processes. These are described by the expression

$$p(x_1) = \frac{1}{\sqrt{2\pi} \sigma_1} \exp\left[-\frac{(x_1 - m_1)^2}{2\sigma_1^2}\right] \quad (2.1)$$

where $x(t)$ is a random process and x_1 is an abbreviation for $x(t_1)$, m_1 is the mean of the distribution, and σ_1 is the standard deviation of the distribution (see Figure 6).

The advantage of considering the random processes in this thesis as Gaussian is primarily two-fold. First, experiments show (see for instance references [42], [48], [49], and [51]) that many processes in nature behave in that manner. This includes wind loading and noise in general. Second, Gaussian random processes allow for some algebraic simplification of the equations describing the behavior. Consequently, the mathematics is more easily manipulated than would be the case for most other random processes.

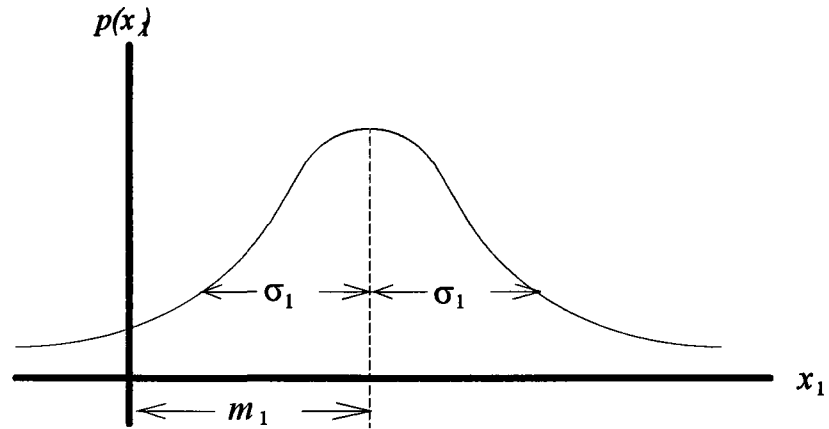


Figure 6. Gaussian distribution.

A convenient way to get a feeling for how a Gaussian process manifests itself is to consider the following example. Say that we obtain 30 readings from an anemometer for wind velocity, each taken 1 second after the previous. Let the results be tabulated as follows.

Table 1. Example readings from an anemometer.

Reading number	Time, sec.	Wind speed, m/s
1	0	5.0
2	1	5.1
3	2	5.3
4	3	4.8
5	4	4.9
6	5	4.6
7	6	5.1
8	7	6.0
9	8	5.8
10	9	5.0
11	10	5.0
12	11	4.2
13	12	4.0
14	13	4.8

15	14	5.2
16	15	4.7
17	16	5.9
18	17	4.3
19	18	4.5
20	19	5.6
21	20	4.9
22	21	5.2
23	22	5.3
24	23	4.9
25	24	4.6
26	25	4.7
27	26	5.2
28	27	5.4
29	28	5.1
30	29	5.5
	Average:	5.02

The average of the wind speed measurements in the table is 5.02 m/s, with the lowest value being 4.0 and the highest value 6.0. Also, it can be seen that six measurements are significantly lower or higher than the average (4.0, 4.2, 4.3, 5.8, 5.9. and 6.0), but the other values are "close" to the average. Also, there are three values of 5.0 m/s. This behavior is typically Gaussian and the resulting frequency distribution is plotted against the Gaussian frequency curve in Figure 7.

The distribution in Figure 7 is characteristic of normal or Gaussian processes. In a stationary process, m_1 and σ_1 remain independent of time (in this case m_1 remains at 5.02 m/s), whereas they depend on time in a non-stationary process.

Frequency of Wind Measurements

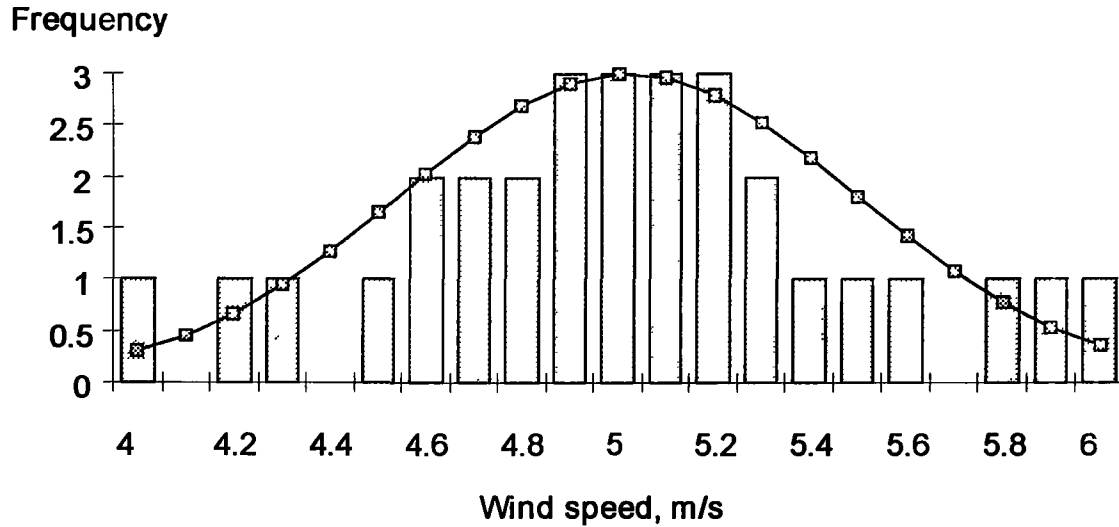


Figure 7. Gaussian distribution for wind speed measurements.

2.1.7 The Mathematical Expectation of a Function.

Consider that, at a fixed time, a domain of n random values of x are associated with a known function $g(x)$. If these n values are assumed to adequately represent the process that yields the x 's, the average of the function g can be determined using

$$\text{Average}[g(x)] = \frac{1}{n} \sum_{j=1}^n g(x^{(j)}) \quad (2.2)$$

An alternative interpretation of Equation (2.2) is to say that it represents a weighted sum of g values, where the weighting factor, $1/n$, gives the fraction of samples having that certain value of g . This interpretation can be used to extend to the theoretical case in which there is a continuous set of infinitely many samples that are described by the first-order probability density $p(x)$ [42]. In this case, the fraction of samples that lie between x and $x+dx$ is $p(x)dx$. The average of g over the continuous set may be inferred from the discrete average of Equation (2.2), or

$$E[g(x)] = \int_{-\infty}^{\infty} g(x)p(x)dx \quad (2.3)$$

This average is called the mathematical expectation of $g(x)$ and the operator E is used³ to denote it. In general, this represents the long-run average value that is to be observed.

2.1.8 Mean and Mean Square.

When $g(x)$ is taken to be x , Equation (2.3) becomes

$$E[x] = \int_{-\infty}^{\infty} xp(x)dx \quad (2.4)$$

³ In addition to $E[g(x)]$ in the literature, one finds $\overline{g(x)}$ and $\langle g(x) \rangle$.

which defines the mean of x or the expected value of x . When $g(x)$ is simply x^2 , Equation (2.3) becomes

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p(x) dx \quad (2.5)$$

and is called the mean square value of x . In addition to this the root mean square value, rms , is defined as

$$rms = \sqrt{E[x^2]} \quad (2.6)$$

2.1.9 Autocorrelation.

Let x_1 and x_2 be abbreviations for $x(t_1)$ and $x(t_2)$, respectively, and $f(x)$ and $g(x)$ be known functions. Then, the ensemble average or mathematical expectation of $f(x_1)g(x_2)$ is given by

$$E[f(x_1)g(x_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1)g(x_2)p(x_1, x_2)dx_1dx_2 \quad (2.7)$$

If $f(x_1) = x_1$ and $g(x_2) = x_2$ then Equation (2.7) yields the average $E[x_1x_2]$, which is called the autocorrelation function

$$E[x_1x_2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1x_2p(x_1, x_2)dx_1dx_2 \quad (2.8)$$

Now, defining $\tau = t_2 - t_1$, the second-order density of a stationary process may be written $p(t, t + \tau)$. With this notation the autocorrelation function becomes

$$E[x_1x_2] = E[x(t)x(t+\tau)] = R(\tau) \quad (2.9)$$

2.1.10 Spectral Density.

A frequency decomposition of $R(\tau)$ can be made by writing

$$R(\tau) = \int_{-\infty}^{\infty} S(\omega)e^{i\omega\tau} d\omega \quad (2.10)$$

where $S(\omega) = (\text{Fourier transform of } R(\tau))/2\pi$, or

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau)e^{-i\omega\tau} d\tau \quad (2.11)$$

$R(0)$ is the mean square of the process. According to Equation (2.10), $R(0)$ equals $\int S(\omega) d\omega$, so $S(\omega)$ can be interpreted as a mean square spectral density. The units of $S(\omega)$ are mean square per unit of circular frequency. The experimental spectral density is denoted by $W(f)$ where f is frequency in

cycles per unit time. The relation between $S(\omega)$ and $W(f)$ is given by

$$W(f) = 4\pi S(\omega) \quad (2.12)$$

2.1.11 Properties of Normal Processes.

A Gaussian process that has a zero mean at all times has important properties that can be stated as follows⁴:

$$E[x_1] = 0 \quad (2.13)$$

$$E[x_1 x_2] = 0 \quad \text{if } x_1 \neq x_2 \quad (2.14)$$

$$E[x^2] \neq 0 \quad (2.15)$$

$$E[x^3] = 0 \quad (2.16)$$

$$E[x_1 x_2 x_3 x_4] = E[x_1 x_2] E[x_3 x_4] + E[x_2 x_3] E[x_1 x_4] + E[x_1 x_3] E[x_2 x_4] \quad (2.17)$$

$$E[x^4] = 3(E[x^2])^2 \quad (2.18)$$

$$E[x^5] = 0 \quad (2.19)$$

$$E[x^6] = 15(E[x^2])^3 \quad (2.20)$$

⁴ These properties are derived in Appendix A.

2.2 The Application of Practical Units and Their Conversions.

The preceding discussion would not have any practical meaning unless it could be applied to real life problems. Thus, the random loading experienced by a structure must be related to measurable terms. If the forcing function is pressure fluctuations due to noise (for instance jet engine noise) or changes in air velocity (turbulences), the loading is represented by a so-called pressure spectral density (PSD). This requires the commonly applied unit of dB (decibels) to be converted to PSD. Figure 8 [36] shows a representation of the intensity of sound level ranging from 0 to 150 dB.

The following explains how to convert sound spectrum level (SSL) given in dB into pressure spectral density. Note that the reference pressure, P_{REF} , is the lowest audible sound sensed by the "average" ear [50]⁵.

Step 1: Let SSL be given in dB. The general relation for SSL is [50]

$$SSL = 10 \log_{10} \left(\frac{PSD}{P_{REF}} \right)^2 = 20 \log_{10} \left(\frac{PSD}{P_{REF}} \right) \quad (2.21)$$

⁵ This reference pressure is used throughout the text.

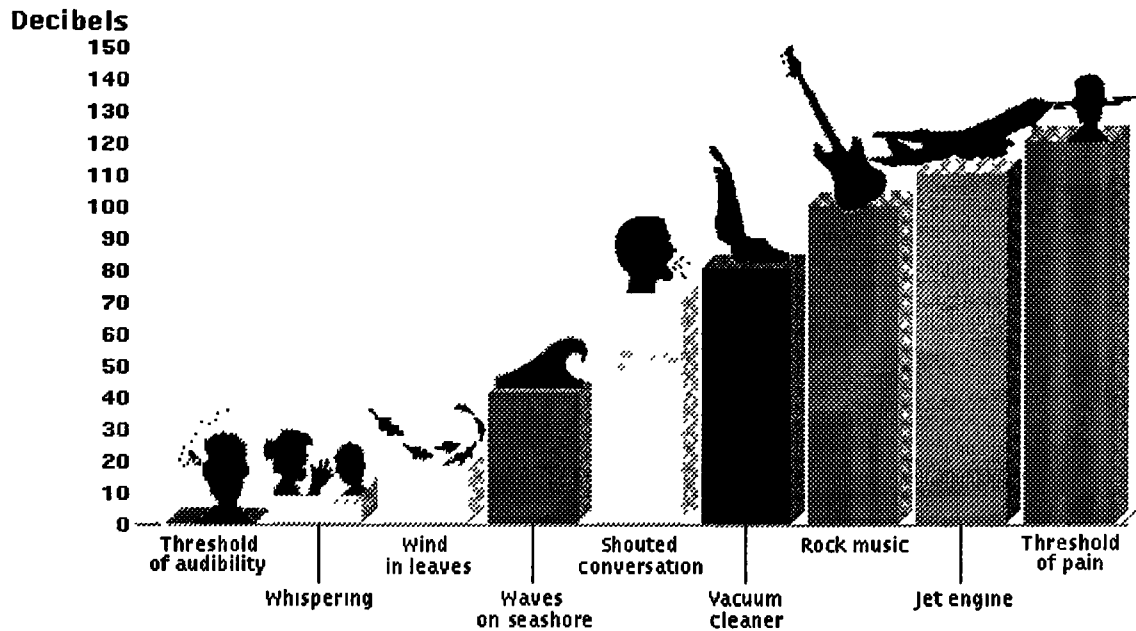


Figure 8. Sound level intensity [36].

where PSD = Pressure Spectral Density, with units $[N/m^2]$ or $[psi]$, and P_{REF} = Reference sound pressure = $2 \times 10^{-5} N/m^2 = 2.9020 \times 10^{-9} psi$.

Step 2: Convert SSL into pressure spectral density (PSD) in terms of circular frequency, by solving Equation (2.21) for PSD as follows. Using the SI-system

$$PSD = P_{REF} 10^{SSL/20} = 2 \times 10^{(SSL/20-5)} \quad (2.22)$$

Using the UK-system

$$PSD = 2.9020 \times 10^{(SSL/20-9)} \quad (2.23)$$

Step 3: Convert PSD into pressure spectral density (S_p), in terms of frequency, as follows

$$S_p(f) = PSD^2 b^2 = \text{Pressure}^2 \times \text{area} \quad (2.24)$$

The units are [Pa^2/Hz] or [psi^2/Hz] for plates, but [$(\text{N/m})^2/\text{Hz}$] or [$(\text{lb/in})^2/\text{Hz}$] for beams. Using equations (2.22) and (2.23) this becomes, in the SI-system

$$PSD^2 = (P_{REF} 10^{SSL/20})^2 = 4 \times 10^{(0.1SSL-10)} \quad (2.25)$$

In the UK-system

$$PSD^2 = (2.9020 \times 10^{(SSL/20-9)})^2 = 8.4216 \times 10^{(0.1SSL-18)} \quad (2.26)$$

It is to be noted that, in addition to the above, the following relation is used to convert pressure spectral density into of circular frequency

$$S_p(\Omega) = \frac{S_p(f)}{2\pi} \quad (2.27)$$

2.3 Concepts in Laminated Structural Analysis.

The following section is intended to introduce some concepts important in the structural analysis of laminate structures. The information that follows is described in greater detail in references [37] and [47].

2.3.1 General on Composites.

A composite material is defined as a combination of two or more constituent materials, such that the resulting combination has characteristics of all constituent materials. In addition, composite materials often exhibit qualities that none of the constituent materials possess. Properties that often are improved by forming a composite material include (not all are improved simultaneously)

- Strength
- Stiffness
- Corrosion Resistance
- Wear Resistance
- Weight
- Fatigue Life
- Temperature-Dependent behavior
- Thermal Insulation
- Thermal Conductivity
- Acoustical Insulation

2.3.2 Types of Composites.

Three types of composites are predominantly mentioned in conjunction with industrial use. Fibrous composites

consist of fibers in a matrix, for instance fiberglass reinforced plastics. Laminated composites consist of layers of various materials, for instance fibrous composites in many layers. Finally, particulate composites are composed of particles in a matrix. Steel reinforced concrete is an example of a particulate composite material. The two first are the subject of the thesis.

2.3.3 Mechanical Behavior of Composite Materials.

Composite materials have many characteristics that are different from more conventional engineering materials. Some characteristics are a modification of conventional behavior, whereas others are entirely new and thus, require new analytical and experimental procedures. Composite materials are often both inhomogeneous (or heterogeneous) and orthotropic (or anisotropic).

2.3.4 Approaches in the Analysis of Composites.

There are generally two approaches taken in the analysis of composite materials. First, there is the micromechanics approach, which is the study of composite material wherein the interaction of the constituent materials (i.e. of the fibers and matrix) is examined on a microscopic level. This type of study predicts the "average" properties (such as strength and stiffness) in terms of the properties and

behavior of the constituent materials. Second, there is the macromechanics approach, which is the study of composite material wherein the material is presumed homogeneous and the effects of the constituent materials are detected only as averaged apparent properties of the composite. The approach in the thesis is of the microscopic nature.

2.3.5 Basic Terminology of Composite Materials.

A lamina is a flat (or curved) arrangement of fibers that can be uni-directional or bi-directional (see Figure 9), embedded in a matrix that maintains a proper predetermined orientation of the fibers.

Fibers are filaments that are the principal reinforcing or load carrying material. The arrangement of the fibers is commonly split into the following categories

- Unidirectional Fibers
- Bi-directional Fibers
- Glass Mats (random fiber orientation)

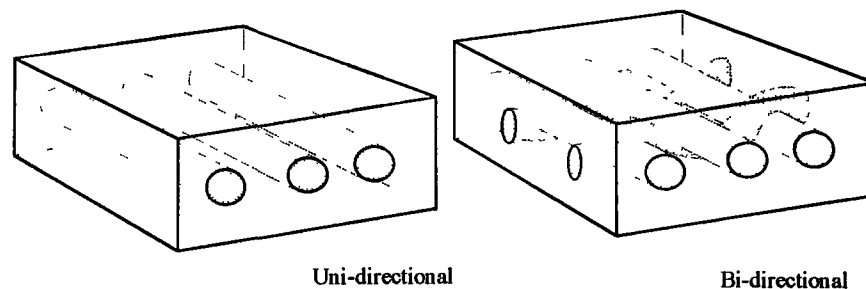


Figure 9. A schematic of typical fiber arrangement.

The matrix can be organic, ceramic, or metallic. Its function is to support and protect the fibers, and to provide a means of distributing the load between the fibers.

A laminate is a stack of laminae with various orientation of principal material directions in the laminae. The layers of a laminate are usually bonded together by the same matrix material that is used in the laminae.

2.3.6 Assumptions and Definitions.

The material is orthotropic and there is no coupling between the normal and shear strains, ϵ and γ , and the normal and shear stresses, σ and τ , respectively. General directions are denoted in Figure 10.

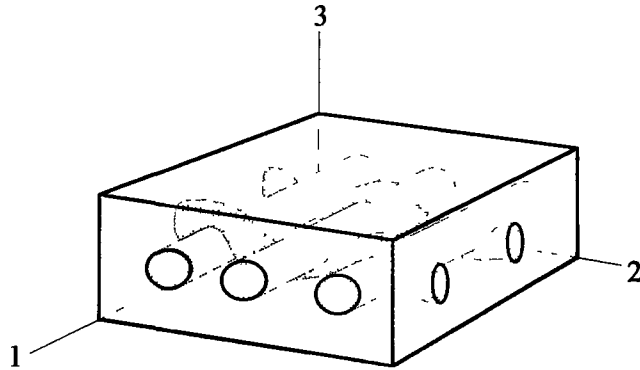


Figure 10. Principal material directions.

Material properties in the principal material directions 1, 2, and 3 are defined as follows

$E_1, E_2,$ and E_3 = Young's stiffness moduli in the
principal material directions

$G_{23}, G_{31},$ and G_{21} = Shear stiffness moduli

ν_{ij} = Poisson's ratio for transverse strain in the j -
direction, when stressed in the i -direction.

2.3.7 Orthotropic Compliance Matrix (Strain-Stress Relations).

Using the above assumptions, the strains are related to the applied stresses as follows

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} \quad (2.28)$$

where the matrix with the material constants is referred to as the orthotropic compliance matrix.

2.3.8 Orthotropic Stiffness Matrix (Stress-Strain Relations).

If the compliance matrix of Equation (2.28) is inverted the resulting matrix becomes

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \frac{(1 - \nu_{23}\nu_{32})E_1}{K} & \frac{(\nu_{21} + \nu_{23}\nu_{32})E_1}{K} & \frac{(\nu_{31} + \nu_{21}\nu_{32})E_1}{K} & 0 & 0 & 0 \\ \frac{(\nu_{12} + \nu_{13}\nu_{32})E_2}{K} & \frac{(1 - \nu_{13}\nu_{31})E_2}{K} & \frac{(\nu_{32} + \nu_{12}\nu_{31})E_2}{K} & 0 & 0 & 0 \\ \frac{(\nu_{13} + \nu_{12}\nu_{23})E_3}{K} & \frac{(\nu_{23} + \nu_{13}\nu_{21})E_3}{K} & \frac{(1 - \nu_{12}\nu_{21})E_3}{K} & 0 & 0 & 0 \\ K & K & K & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & G_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & G_{12} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \gamma_{23} \\ \gamma_{31} \\ \gamma_{12} \end{Bmatrix} \quad (2.29)$$

where $K = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{13}\nu_{31} - \nu_{12}\nu_{23}\nu_{31} - \nu_{13}\nu_{21}\nu_{32}$

The matrix with the material constants is called the orthotropic stiffness matrix. Due to symmetry of the compliance matrix the following relation is useful

$$\frac{\nu_y}{E_i} = \frac{\nu_x}{E_j} \quad i, j = 1, 2, 3 \quad (2.30)$$

2.3.9 Orthotropic Compliance Matrix for Plane Stress.

In this thesis the stress experienced by the beams can be considered one-dimensional. Therefore, two-dimensional

plane stress suffices to describe the state of stress.

Consequently, Equation (2.28) can be reduced to

$$\begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} \quad (2.31)$$

where the matrix with the material constants is referred to as the orthotropic compliance matrix.

2.3.10 Orthotropic Stiffness Matrix for Plane Stress.

Equation (2.31) inverted results in

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \frac{E_1}{1-\nu_{12}\nu_{21}} & \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}} & 0 \\ \frac{\nu_{12}E_1}{1-\nu_{12}\nu_{21}} & \frac{E_2}{1-\nu_{12}\nu_{21}} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} \quad (2.32)$$

The stiffness matrix is also written in the following shorthand notation, which is called the reduced stiffness matrix

$$\{\sigma\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_1 \\ \epsilon_2 \\ \gamma_{12} \end{Bmatrix} \quad (2.33)$$

2.3.11 Stress-Strain Relations for a Lamina of Arbitrary Orientation.

If the lamina is rotated through an angle θ with respect to the x-y axes, as can be seen in Figure 12, the stresses along 1-2 axis are transformed as follows

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} \cos^2 \theta & \sin^2 \theta & 2 \sin \theta \cos \theta \\ \sin^2 \theta & \cos^2 \theta & -2 \sin \theta \cos \theta \\ -\sin \theta \cos \theta & \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad (2.34)$$

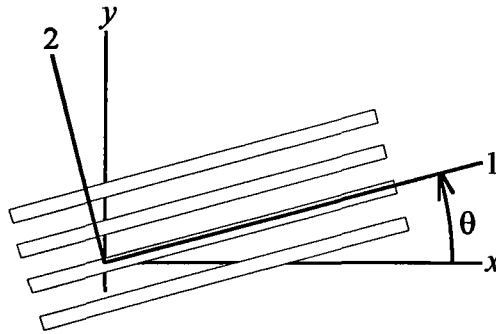


Figure 11. Schematic of fiber angular orientation.

A short hand notation for the transformation matrix is the following, where $\cos \theta$ and $\sin \theta$ are denoted by m and n , respectively

$$\{\sigma\} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2 - n^2 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} \quad (2.35)$$

where the matrix of m and n is called the transformation matrix.

2.3.12 Stress-Strain Relations for x-y Axis of an Arbitrarily Oriented Lamina.

Stresses along x- and y-axes can be determined using the following equation

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} \quad (2.36)$$

where the matrix with the Q-bars is called the reduced transformed stiffness matrix. The coefficients are given by

$$\begin{aligned} \overline{Q}_{11} &= Q_{11}m^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}n^4 \\ \overline{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})m^2n^2 + Q_{12}(m^4 + n^4) \\ \overline{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})m^3n + (Q_{12} - Q_{22} + 2Q_{66})mn^3 \\ \overline{Q}_{22} &= Q_{11}n^4 + 2(Q_{12} + 2Q_{66})m^2n^2 + Q_{22}m^4 \\ \overline{Q}_{26} &= (Q_{11} - Q_{12} + 2Q_{66})mn^3 + (Q_{12} - Q_{22} + 2Q_{66})m^3n \\ \overline{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})m^2n^2 + Q_{66}(m^4 + n^4) \end{aligned} \quad (2.37)$$

Tsai [52] developed a multiple-angle formulation to replace equations (2.37). This formulation is called the invariant property of an orthotropic lamina. They are well suited for computer algorithms because the U's are invariant with respect to the axis of rotation.

$$\begin{aligned}
 \overline{Q}_{11} &= U_1 + U_2 \cos 2\theta + U_3 \cos 4\theta \\
 \overline{Q}_{22} &= U_1 - U_2 \cos 2\theta + U_3 \cos 4\theta \\
 \overline{Q}_{12} &= U_4 - U_3 \cos 4\theta \\
 \overline{Q}_{66} &= U_5 - U_3 \cos 4\theta \\
 \overline{Q}_{16} &= \frac{1}{2} U_2 \sin 2\theta + U_3 \sin 4\theta \\
 \overline{Q}_{26} &= \frac{1}{2} U_2 \sin 2\theta - U_3 \sin 4\theta
 \end{aligned} \tag{2.38}$$

where

$$\begin{aligned}
 U_1 &= \frac{1}{8} [3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66}] \\
 U_2 &= \frac{1}{2} [Q_{11} - Q_{22}] \\
 U_3 &= \frac{1}{8} [Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66}] \\
 U_4 &= \frac{1}{8} [Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66}] \\
 U_5 &= \frac{1}{8} [Q_{11} + Q_{22} - 2Q_{12} + 4Q_{66}]
 \end{aligned} \tag{2.39}$$

2.3.13 Stress-Strain Relations for the k^{th} Lamina.

Consider the laminate in Figure 13. Stresses along x- and y-axes for the k^{th} lamina can be determined using the following relation

$$\{\sigma\}_k = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \left\{ \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z \begin{Bmatrix} \kappa_x \\ \kappa_y \end{Bmatrix} \right\} = [\overline{Q}] \left\{ \begin{Bmatrix} \epsilon^0 \\ \kappa \end{Bmatrix} \right\} \quad (2.40)$$

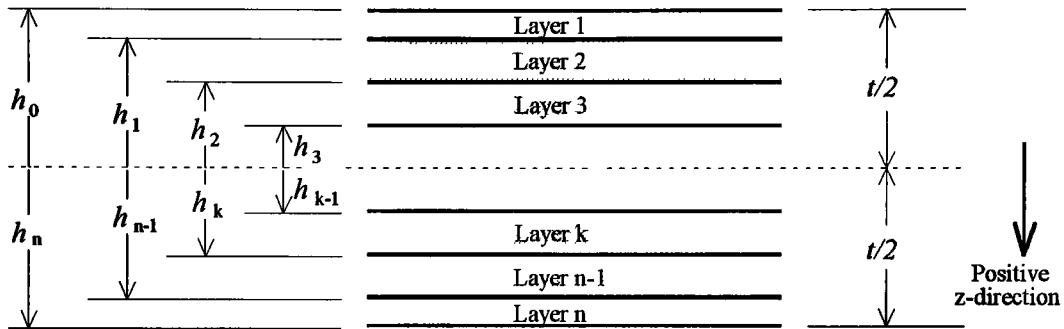


Figure 12. A composite laminate constructed from \$n\$ laminae.

2.3.14 Laminate Constitutive Equations (A-B-D Matrix).

The stress and moment resultants for a laminate of \$n\$ plies are frequently written in the following form, which is called the laminate constitutive equations, or the A-B-D matrix:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{Bmatrix} N_x \\ N_x \\ N_x \\ M_x \\ M_x \\ M_x \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \epsilon^0 \\ \kappa \end{Bmatrix} \quad (2.41)$$

where the coefficients, which are referred to as the laminate stiffness, are given by

Extensional stiffness:

$$A_y = \sum_{k=1}^n (\overline{Q}_y)_k (h_k - h_{k-1}) \quad (2.42)$$

Coupling stiffness:

$$B_y = \frac{1}{2} \sum_{k=1}^n (\overline{Q}_y)_k (h_k^2 - h_{k-1}^2) \quad (2.43)$$

Bending stiffness:

$$D_y = \frac{1}{3} \sum_{k=1}^n (\overline{Q}_y)_k (h_k^3 - h_{k-1}^3) \quad (2.44)$$

Note that $[B] = [0]$ for isotropic materials (which can be considered a special case of composites) and for symmetrically laminated composites.

CHAPTER 3

DERIVATION OF THE NONLINEAR EQUATION OF MOTION FOR AN UNSYMMETRICALLY LAMINATED BEAM

In this chapter the nonlinear equation of motion for an unsymmetrically laminated beam is derived. The final general nonlinear equation of motion includes the effects of any type of boundary conditions and material lay-up. It will be the subject of subsequent chapters to modify that equation to accommodate specific boundary conditions and lamination lay-up methods.

This chapter is also intended to serve as a review for the reader as to how the underlying nonlinear theory of mechanics of materials is used to set up the equation of motion for any beam. The actual derivation of the equation of motion is then implemented towards the end of the chapter. The process is typically as follows. First the A-B-D matrix is modified to entail the two dimensional characteristics of

beams (i.e. deflection along one axis depends on position along another one, and the two axes are mutually perpendicular). Then, the so-called constitutive equations are introduced and modified for beams. Once that step is accomplished, the equation of motion can be derived.

3.1 The General Constitutive Relation.

The stress and moment resultants on an element in any composite material is given by (see also Section 2.3.14)

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{Bmatrix} \epsilon^0 \\ \kappa \end{Bmatrix} \quad (3.1)$$

which is also written in expanded form as follows

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{11} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \epsilon_x^0 \\ \epsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \end{Bmatrix} \quad (3.2)$$

The strain-displacement and curvature-displacement relations are given by von Kärman's geometrical nonlinearities

$$\{\varepsilon^0\} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{1}{2} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \end{Bmatrix} \quad (3.3)$$

$$\{\kappa\} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 w}{\partial y^2} \\ -2 \frac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} \quad (3.4)$$

3.2 The Constitutive Relation Modified for a Beam.

The beam considered here is long in the x-direction compared to dimensions in y- and z-directions. Thus, the displacements u_0 , v_0 , and w are all functions of x only. This renders the analysis one-dimensional in the x-direction. The simplification begins by rewriting Equations (3.3) and (3.4) as follows;

$$\{\varepsilon^0\} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \\ 0 \\ \frac{\partial v_0}{\partial x} \end{Bmatrix} \quad (3.3)$$

and

$$\{\kappa\} = \begin{Bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{Bmatrix} = \begin{Bmatrix} -\frac{\partial^2 w}{\partial x^2} \\ 0 \\ 0 \end{Bmatrix} \quad (3.4)$$

so Equation (3.2) becomes

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{11} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \\ 0 \\ \frac{\partial v_0}{\partial x} \\ -\frac{\partial^2 w}{\partial x^2} \\ 0 \\ 0 \end{Bmatrix} \quad (3.2)$$

Consequently, this can be expanded by writing

$$N_x = A_{11} \left[\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + A_{16} \frac{\partial v_0}{\partial x} - B_{11} \frac{\partial^2 w}{\partial x^2} \quad (3.5)$$

$$N_y = A_{12} \left[\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + A_{26} \frac{\partial v_0}{\partial x} - B_{12} \frac{\partial^2 w}{\partial x^2} \quad (3.6)$$

$$N_{xy} = A_{16} \left[\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + A_{66} \frac{\partial v_0}{\partial x} - B_{16} \frac{\partial^2 w}{\partial x^2} \quad (3.7)$$

$$M_x = B_{11} \left[\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + B_{16} \frac{\partial v_0}{\partial x} - D_{11} \frac{\partial^2 w}{\partial x^2} \quad (3.8)$$

$$M_y = B_{12} \left[\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + B_{26} \frac{\partial v_0}{\partial x} - D_{12} \frac{\partial^2 w}{\partial x^2} \quad (3.9)$$

$$M_{xy} = B_{16} \left[\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 \right] + B_{66} \frac{\partial v_0}{\partial x} - D_{16} \frac{\partial^2 w}{\partial x^2} \quad (3.10)$$

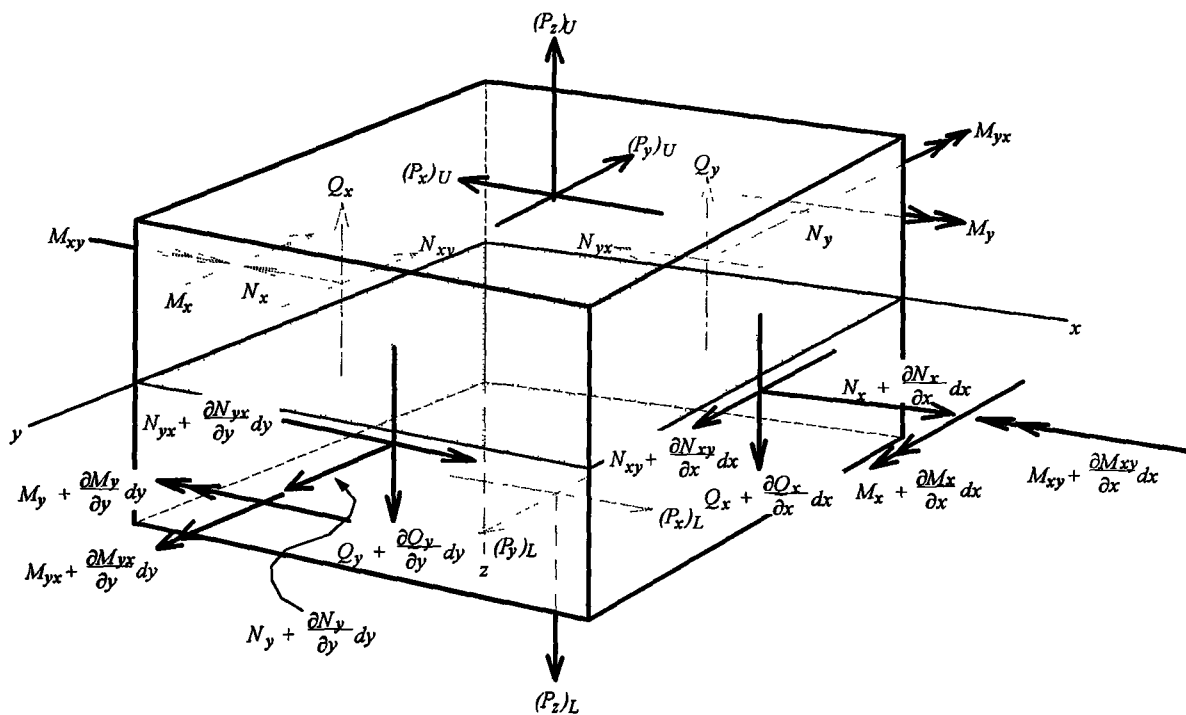


Figure 13. Definition of force and moment resultants.

3.3 The Governing Equations⁶.

The governing equations for the element in Figure 13, are given by

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (3.11)$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \quad (3.12)$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + P = \rho h \frac{\partial^2 w}{\partial t^2} \quad (3.13)$$

where h is the height of the beam, ρ is the density, and P is the forcing function per unit area (usually time dependent).

3.4 The Governing Equations for a Beam in Terms of N_x and M_x .

The governing equations for an element of a beam are to be derived by simplifying the equations above. Again, the variations in the y -direction are negligible, because the

⁶ The derivation of the governing equations of motion appears in Appendix B.

structure is assumed to be essentially one-dimensional in the x-direction. Therefore, Equations (3.11), (3.12), and (3.13) reduce to

$$\frac{\partial N_x}{\partial x} = 0 \quad (3.14)$$

$$\frac{\partial N_{xy}}{\partial x} = 0 \quad (3.15)$$

$$\frac{\partial^2 M_x}{\partial x^2} + N_x \frac{\partial^2 w}{\partial x^2} + P = \rho h \frac{\partial^2 w}{\partial t^2} \quad (3.16)$$

Equations, given by (3.14), (3.15), and (3.16), yield the equation of motion for the beam. The first step in deriving the equation of motion is to determine selected terms of Equation (3.16) by using the two other ones. From Equation (3.14)

$$N_x = \text{Constant} \quad (3.17)$$

From this and Equation (3.5)

$$A_{11} \left[\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \right] + A_{16} \frac{\partial^2 v_0}{\partial x^2} - B_{11} \frac{\partial^3 w}{\partial x^3} = 0 \quad (3.18)$$

From Equations (3.7) and (3.15),

$$A_{16} \left[\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \right] + A_{66} \frac{\partial^2 v_0}{\partial x^2} - B_{16} \frac{\partial^3 w}{\partial x^3} = 0 \quad (3.19)$$

From Equation (3.8)

$$\frac{\partial^2 M_x}{\partial x^2} = B_{11} \left[\frac{\partial^3 u_0}{\partial x^3} + \frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial x} + \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] + B_{16} \frac{\partial^3 v_0}{\partial x^3} - D_{11} \frac{\partial^4 w}{\partial x^4} \quad (3.20)$$

3.5 Equations of Motion⁷ in Terms of Displacements u_0 , v_0 , and w .

The occurrence of the midplane displacements u_0 and v_0 in the above equations requires their determination. This is done in the following manner. Begin by rearranging terms in Equations (3.18) and (3.19):

$$A_{11} \frac{\partial^2 u_0}{\partial x^2} + A_{16} \frac{\partial^2 v_0}{\partial x^2} = B_{11} \frac{\partial^3 w}{\partial x^3} - A_{11} \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \quad (3.21)$$

$$A_{16} \frac{\partial^2 u_0}{\partial x^2} + A_{66} \frac{\partial^2 v_0}{\partial x^2} = B_{16} \frac{\partial^3 w}{\partial x^3} - A_{16} \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \quad (3.22)$$

⁷ The equation of motion for an element is derived in Appendix C.

Solving the two linearly independent equations for $\frac{\partial^2 u_0}{\partial x^2}$ and

$\frac{\partial^2 v_0}{\partial x^2}$ results in

$$\frac{\partial^2 u_0}{\partial x^2} = K_1 \frac{\partial^3 w}{\partial x^3} - \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \quad (3.23)$$

$$\frac{\partial^2 v_0}{\partial x^2} = K_2 \frac{\partial^3 w}{\partial x^3} \quad (3.24)$$

where

$$K_1 = \left(\frac{A_{16}B_{16} - A_{66}B_{11}}{A_{16}^2 - A_{11}A_{66}} \right) \quad (3.25)$$

and

$$K_2 = \left(\frac{A_{16}B_{11} - A_{11}B_{16}}{A_{16}^2 - A_{11}A_{66}} \right) \quad (3.26)$$

The normal force resultant, N_x , is manipulated in a special way, before it can be used in the equation of motion (3.16). This is done as follows. First integrate Equation (3.5) over the length, L , of the beam:

$$\int_0^L N_x dx = A_{11} \int_0^L \frac{\partial u_0}{\partial x} dx + A_{16} \int_0^L \frac{\partial v_0}{\partial x} dx + \frac{A_{11}}{2} \int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx - B_{11} \int_0^L \frac{\partial^2 w}{\partial x^2} dx$$

Carrying out the first three intergrals, noting Equation (3.17) to obtain

$$N_x = \frac{A_{11}}{L} [u_0(L) - u_0(0)] + \frac{A_{16}}{L} [v_0(L) - v_0(0)] + \frac{A_{11}}{2L} \int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx - \frac{B_{11}}{L} \int_0^L \frac{\partial^2 w}{\partial x^2} dx \quad (3.27)$$

The state of the midplane motion is represented by the terms $[u_0(L) - u_0(0)]$ and $[v_0(L) - v_0(0)]$, which depends on the type of inplane boundary conditions. In this thesis, beams with immovable edge conditions are considered. Consequently, the first two terms on the right hand of Equation (3.27) vanish.

Equations (3.23) and (3.24) must be differentiated before they can be inserted into Equation (3.20), which in turn, is used in Equation (3.16). Carrying out this differentiation yields

$$\frac{\partial^3 u_0}{\partial x^3} = K_1 \frac{\partial^4 w}{\partial x^4} - \frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial x} - \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \quad (3.28)$$

and

$$\frac{\partial^3 v_0}{\partial x^3} = K_2 \frac{\partial^4 w}{\partial x^4} \quad (3.29)$$

Substitution of equations (3.20) and (3.27) into equations (3.16) yields

$$\begin{aligned}
& B_{11} \left[\frac{\partial^3 u_0}{\partial x^3} + \frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial x} + \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] + B_{16} \frac{\partial^3 v_0}{\partial x^3} - D_{11} \frac{\partial^4 w}{\partial x^4} + \\
& \left(\frac{A_{11}}{L} [u_0(L) - u_0(0)] + \frac{A_{16}}{L} [v_0(L) - v_0(0)] + \frac{A_{11}}{2L} \int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx + \right. \\
& \left. - \frac{B_{11}}{L} \int_0^L \frac{\partial^2 w}{\partial x^2} dx \right) \frac{\partial^2 w}{\partial x^2} + P = \rho h \frac{\partial^2 w}{\partial t^2}
\end{aligned}$$

Rearranging terms and inserting Equations (3.28) and (3.29) into the above, then expanding and collecting terms the equation of motion of $w(x, t)$ is obtained:

$$\begin{aligned}
& (B_{11}K_1 + B_{16}K_2 - D_{11}) \frac{\partial^4 w}{\partial x^4} + \\
& \left(\frac{A_{11}}{L} [u_0(L) - u_0(0)] + \frac{A_{16}}{L} [v_0(L) - v_0(0)] + \frac{A_{11}}{2L} \int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx + \right. \\
& \left. - \frac{B_{11}}{L} \int_0^L \frac{\partial^2 w}{\partial x^2} dx \right) \frac{\partial^2 w}{\partial x^2} + P - \rho h \frac{\partial^2 w}{\partial t^2} = 0
\end{aligned} \tag{3.30}$$

CHAPTER 4

METHOD OF SOLUTION

In this chapter, a solution to the equation of motion (i.e. Equation (3.30)) will be attempted for three specific cases; simply-supported, clamped-clamped, and cantilevered unsymmetrically laminated beams with immovable edges (see Figure 14).

The solution process is briefly as follows. First, a deflection curve that agrees with the geometric and kinematic boundary conditions is proposed. Then, the equation of motion is simplified to adopt the characteristic of the boundary conditions. Next, the proposed solution is inserted into the modified equation, which, by the use of Galerkin's method, is manipulated to yield the so-called Duffing differential equation. The Duffing equation is a nonlinear differential equation and is solved using the method of equivalent linearization. The solution of this equation directly gives

the deflection response of the beam when excited with a randomly varying forcing function.

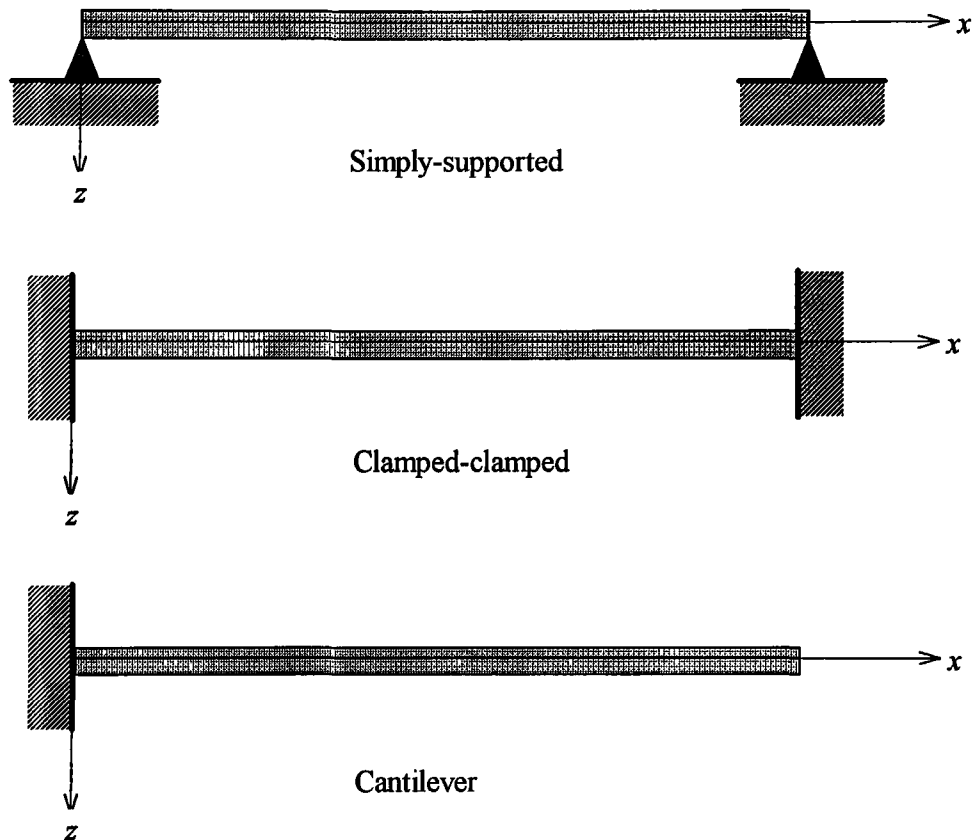


Figure 14. The three boundary conditions considered herein.

The solution of the Duffing equation is made even more specialized by considering three common material properties: an isotropic beam, an unsymmetrical angle-ply, and an unsymmetrical cross-ply. Furthermore, the responses of the beams are compared to that of the linear theory. It is of

importance to note that it does not serve a purpose to present every step of the derivations in this chapter. The interested reader can turn to Appendixes A, D, and E to see the details of selected steps.

4.1 The Galerkin Method.

The solution process for the governing partial differential equation of motion begins with the application of the Galerkin method. The method transforms a partial differential equation into an algebraic equation for a static problem and into an ordinary differential equation for a dynamic problem. For Equation (3.30) this leads to a nonlinear differential (Duffing) equation. The Galerkin method has proven successful in treating linear and nonlinear problems, as well as stability and buckling problems.

Galerkin's method can be described as follows. Assume that the motion of a beam can be described by:

$$L(w, t) - P = 0 \quad (4.1)$$

where L is a differential operator, P is forcing function acting on the structure, w is the displacement function, and t is the time. Let δw be a small arbitrary variation in the

displacement of the beam. Then, the external and internal work done by the system are given by

$$\delta W_{ext} = \int_{Area} (P \delta w) \cdot dA \quad (4.2)$$

$$\delta W_{int} = \int_{Area} [L(w, t) \delta w] \cdot dA \quad (4.3)$$

If w is the solution of equation (4.1), then (4.2) and (4.3) must be identical, so that

$$\int_{Area} [L(w, t) - P] \delta w \cdot dA = 0 \quad (4.4)$$

Now, assuming that w is expressed in the form

$$w = a_1 \psi_1 + a_2 \psi_2 + a_3 \psi_3 + \dots + a_n \psi_n = \sum_{i=1}^n a_i \psi_i \quad (4.5)$$

where the a_i are unknown constants to be determined and ψ_i are functions of x and t the variable w satisfying the boundary conditions. From Equation (4.5) the virtual displacement is given by

$$\delta w = \sum_i \delta a_i \psi_i \quad (4.6)$$

Substituting Equation (4.6) into (4.4) and requiring the equality to be maintained for arbitrary values of δa_i , yields

$$\begin{aligned} \int_{Area} [L(w,t) - P] \cdot \psi_1 \cdot dA &= 0 \\ \int_{Area} [L(w,t) - P] \cdot \psi_2 \cdot dA &= 0 \\ &\vdots \\ \int_{Area} [L(w,t) - P] \cdot \psi_n \cdot dA &= 0 \end{aligned} \quad (4.7)$$

For a uniform beam $dA = \text{width } dx$, so equations (4.7) simplify to the system

$$\int_L [L(w,t) - P] \cdot \psi_i \cdot dx = 0, \quad i = 1, \dots, n \quad (4.8)$$

Substituting

$$w = \sum_{i=1}^n a_i \cdot \psi_i(x) \quad (4.9)$$

into Equations (4.8) yields a set of algebraic equations (for a static problem) or a set of differential equations (for a dynamic problem). The application of this method, in all likelihood, will simplify the analytical solution of a problem.

4.2 The Method of Equivalent Linearization.

As cited above, Galerkin's method transforms the governing partial differential equation of motion into an ordinary differential equation (i.e. the Duffing equation). However, the nonlinearity of the equation, in conjunction with the random nature of the forcing function, presents an obstacle of considerable difficulty. To ease the analytical approach, the equation can be linearized. This is a statistical method, which replaces the nonlinear system by a linear one. The method is adopted from the so-called Krylov-Bogoliubov equivalent linearization technique for deterministic vibration problems. It minimizes the difference between the nonlinear and the linear systems, the error of linearization, to achieve the most reasonable solution. This concept has found considerable popularity in control theory and in the solution of single degree-of-freedom systems.

In the thesis, this step is implemented Section 4.6, and the resulting error, once minimized, is inserted into the linearized equation. Since the Duffing equation is also solved numerically (see Section 4.3), the difference in the above approach can be assessed and in this thesis was found to be acceptable (see Chapter 5).

4.3 A Numerical Solution to the Duffing Equation.

The Duffing equation can also be solved using numerical analysis, primarily for comparison reasons. The numerical approach used is a classical Runge-Kutta fourth order scheme for a system of equations. For this particular problem the system consists of two equations obtained as follows.

Consider a solution to the differential equation

$$\ddot{q} + 2\xi\omega_0\dot{q} + \omega_0^2q + \alpha q^2 + \beta q^3 = \frac{F(t)}{m} \quad (4.10)$$

By replacing $\dot{q} = \frac{dq}{dt}$ by z , this can be rewritten as

$$\dot{z} + 2\xi\omega_0z + \omega_0^2q + \alpha q^2 + \beta q^3 = \frac{F(t)}{m} \quad (4.11)$$

Consequently, Equation (4.10) has been split into two first order differential equations, namely

$$\frac{dq}{dt} = z \quad (4.12)$$

and

$$\frac{dz}{dt} = \frac{F(t)}{m} - 2\xi\omega_0z - \omega_0^2q - \alpha q^2 - \beta q^3 \quad (4.13)$$

Equations (4.12) and (4.13) are the two equations making up the system of equations to which the Runge-Kutta scheme is applied. A more detailed description of how this is exactly implemented is not appropriate here, but the Windows program written to solve this and the source-code can be obtained upon request.

4.4 Boundary Conditions.

The boundary conditions for the three beams are as follows.

4.4.1 Simply Supported Beam:

$$\begin{array}{ll}
 @ x = 0 & w = 0 \\
 & \frac{\partial^2 w}{\partial x^2} = 0 \\
 & u_0 = 0 \\
 & v_0 = 0
 \end{array}
 \qquad
 \begin{array}{ll}
 @ x = L & w = 0 \\
 & \frac{\partial^2 w}{\partial x^2} = 0 \\
 & u_0 = 0 \\
 & v_0 = 0
 \end{array}$$

4.4.2 Clamped-Clamped Beam:

$$\begin{array}{ll}
 @x = 0 & w = 0 \\
 & \frac{\partial w}{\partial x} = 0 \\
 & u_0 = 0 \\
 & v_0 = 0
 \end{array}
 \qquad
 \begin{array}{ll}
 @x = L & w = 0 \\
 & \frac{\partial w}{\partial x} = 0 \\
 & u_0 = 0 \\
 & v_0 = 0
 \end{array}$$

4.4.3 Cantilever Beam:

$$\begin{array}{ll}
 @x = 0: & w = 0 \\
 & \frac{\partial w}{\partial x} = 0 \\
 & u_0 = 0 \\
 & v_0 = 0 \\
 @x = L: & M_x = 0 \Rightarrow \frac{\partial^2 w}{\partial x^2} = 0 \\
 & V_z = 0 \Rightarrow \frac{\partial^3 w}{\partial x^3} = 0
 \end{array}$$

4.5 Assumed Solution.

The relationships used to describe the deflection must obey geometrical, as well as kinematic boundary conditions. These are fulfilled by the following relationships, which are used to describe the deflection for the three beams:

4.5.1 Simply Supported Beam:

$$w(x, t) = \sum_{m=1}^k Rq_m(t) \sin \frac{m\pi x}{L} \quad (4.14)$$

4.5.2 Clamped-Clamped Beam:

$$w(x, t) = \sum_{m=1}^k Rq_m(t) \left(1 - \cos \frac{2m\pi x}{L} \right) \quad (4.15)$$

4.5.3 Cantilever Beam:

$$w(x, t) = \sum_{m=1}^k Rq_m(t) \left(1 - \cos \frac{m\pi x}{2L} \right) \quad (4.16)$$

In the above equations R is radius of gyration of the beam cross-section⁸, $q(t)$ is the non-dimensional time-dependent amplitude of the mode shape, and k is the number of modes to be considered.

4.6 Determining the Duffing Equation for the Beams.

The first step in determining the Duffing equation is to rewrite Equation (3.30), using each of the three boundary conditions, and then, manipulate the result by inserting the appropriate derivatives. This process is partly repeated in the following pages.

⁸ For a rectangular cross-section $R = \sqrt{I/A} = \sqrt{\frac{bh^3}{12bh}} = \frac{h}{\sqrt{12}}$, where A is the cross-sectional area of the beam, I is the moment of inertia of the beam, h is the height of the beam, and b is the width of the beam.

4.6.1 Simply Supported Beam:

It is important to note that the axial displacement $u(x, t)$ is zero since the edges are immovable, i.e. $u(0, t) = u(L, t) = 0$. For a simply supported beam, once the boundary conditions in Section 4.4.1 are applied, Equation (3.30) becomes

$$(B_{11}K_1 + B_{16}K_2 - D_{11}) \frac{\partial^4 w}{\partial x^4} + \left(\frac{A_{11}}{2L} \int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx - \frac{B_{11}}{L} \int_0^L \frac{\partial^2 w}{\partial x^2} dx \right) \frac{\partial^2 w}{\partial x^2} + P - \rho h \frac{\partial^2 w}{\partial t^2} = 0 \quad (4.17)$$

After substituting Equation (4.12) this becomes

$$\begin{aligned} & (B_{11}K_1 + B_{16}K_2 - D_{11}) \sum_{m=1}^k \left(\frac{m\pi}{L} \right)^4 R q_m \sin \frac{m\pi x}{L} + \\ & \left\{ \frac{A_{11}R^2}{2L} \int_0^L \left[\sum_{m=1}^k \sum_{n=1}^k \left(\frac{mn\pi}{L} \right) q_m(t) q_n(t) \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} \right] dx + \frac{B_{11}R}{L} \sum_{m=1}^k \left(\frac{m\pi}{L} \right)^2 q_m \int_0^L \sin \frac{m\pi x}{L} dx \right\} \cdot \\ & \left[-R \sum_{m=1}^k \left(\frac{m\pi}{L} \right)^2 q_m \sin \frac{m\pi x}{L} \right] + P - \rho h R \sum_{m=1}^k \ddot{q}_m \sin \frac{m\pi x}{L} = 0 \end{aligned} \quad (4.18)$$

It is the purpose of this thesis to consider only single-mode solution to beam vibration, i.e. $k = 1$. Thus, Equation (4.18) reduces to

$$\begin{aligned}
& (B_{11}K_1 + B_{16}K_2 - D_{11})\left(\frac{\pi}{L}\right)^4 Rq \sin \frac{\pi x}{L} + \\
& \left\{ \frac{A_{11}R^2}{2L} \left(\frac{\pi}{L}\right)^2 q^2 \int_0^L \cos^2 \frac{\pi x}{L} dx + \frac{B_{11}R}{L} \left(\frac{\pi}{L}\right)^2 q \int_0^L \sin \frac{\pi x}{L} dx \right\} \cdot \\
& \left[-R \left(\frac{\pi}{L}\right)^2 q \sin \frac{\pi x}{L} \right] + P - \rho h R \ddot{q} \sin \frac{\pi x}{L} = 0
\end{aligned}$$

Performing the integration and simplifying results in

$$\begin{aligned}
\left(\frac{\pi}{L}\right)^4 Rq(B_{11}K_1 + B_{16}K_2 - D_{11}) \sin \frac{\pi x}{L} - \left(\frac{\pi}{L}\right)^4 R^2 q^2 \left\{ \frac{A_{11}}{4} Rq + \frac{2B_{11}}{\pi} \right\} \sin \frac{\pi x}{L} + \\
P - \rho h R \ddot{q} \sin \frac{\pi x}{L} = 0
\end{aligned} \tag{4.19}$$

Galerkin's method requires

$$\int_0^L \left[\text{Left - hand side of Equation (4.19)} \right] \cdot \left(Rq(t) \sin \frac{\pi x}{L} \right) dx = 0$$

yielding

$$\begin{aligned}
\left(\frac{\pi}{L}\right)^4 Rq(B_{11}K_1 + B_{16}K_2 - D_{11}) \int_0^L \sin^2 \frac{\pi x}{L} dx - \left(\frac{\pi}{L}\right)^4 R^2 q^2 \left\{ \frac{A_{11}}{4} Rq + \frac{2B_{11}}{\pi} \right\} \int_0^L \sin^2 \frac{\pi x}{L} dx + \\
\int_0^L P(x, t) \sin \frac{\pi x}{L} dx - \rho h R \ddot{q} \int_0^L \sin^2 \frac{\pi x}{L} dx = 0
\end{aligned}$$

Performing the integration and rearranging terms yields the Duffing equation

$$\ddot{q} - (B_{11}K_1 + B_{16}K_2 - D_{11}) \frac{\pi^4}{\rho h L^4} q + A_{11} \frac{\pi^4 R^2}{4 \rho h L^4} q^3 + 2B_{11} \frac{\pi^3 R}{\rho h L^4} q^2 = \frac{2}{\rho h R L} \int_0^L P(x, t) \sin \frac{\pi x}{L} dx \quad (4.20)$$

To simplify the equation make the following definitions:

$$\omega_0^2 = (D_{11} - B_{11}K_1 - B_{16}K_2) \frac{\pi^4}{\rho h L^4} \quad (4.21)$$

$$\alpha = 2B_{11} \frac{\pi^3 R}{\rho h L^4} \quad (4.22)$$

$$\beta = A_{11} \frac{\pi^4 R^2}{4 \rho h L^4} \quad (4.23)$$

$$F(t) = \int_0^L P(x, t) \sin \frac{\pi x}{L} dx \quad (4.24)$$

$$m = \frac{\rho h R L}{2} \quad (4.25)$$

In terms of the above, Equation (4.18) is written

$$\ddot{q} + \omega_0^2 q + \alpha q^2 + \beta q^3 = \frac{F(t)}{m} \quad (4.26)$$

This equation represents a non-damped vibration of the beam. In order to verify the unit consistency of equation (4.26), a unit consistency test is applied, which appears in Appendix D.

Since all structures have internal damping which depends on the time-rate-of-change of the vibration, a Coulomb like damping terms is added to equation (4.26), so that

$$\ddot{q} + 2\xi\omega_0\dot{q} + \omega_0^2q + \alpha q^2 + \beta q^3 = \frac{F(t)}{m} \quad (4.27)$$

4.6.2 Clamped-Clamped Beam:

As for the simply-supported beam, no relations in the axial direction are required for the clamped-clamped beam, because the edges are immovable, or constrained against axial motion. Thus, Equation (3.30) can be rewritten, using the boundary conditions of Section 4.4.2 as follows

$$(B_{11}K_1 + B_{16}K_2 - D_{11})\frac{\partial^4 w}{\partial x^4} + \left(\frac{A_{11}}{2L} \int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx - \frac{B_{11}}{L} \int_0^L \frac{\partial^2 w}{\partial x^2} dx \right) \frac{\partial^2 w}{\partial x^2} + P - \rho h \frac{\partial^2 w}{\partial t^2} = 0 \quad (4.28)$$

Substituting Equation (4.15), performing the integration, and simplifying for a single-mode solution, yields

$$(D_{11} - B_{11}K_1 - B_{16}K_2)R \left(\frac{2\pi}{L} \right)^4 q \cos \frac{2\pi x}{L} + \frac{A_{11}R^3}{4} \left(\frac{2\pi}{L} \right)^4 q^3 \cos \frac{2\pi x}{L} + P - \rho h R \ddot{q} \left(1 - \cos \frac{2\pi x}{L} \right) = 0 \quad (4.29)$$

Galerkin's method requires

$$\int_0^L \left[\text{Left - hand side of Equation (4.29)} \right] \cdot \left[Rq(t) \left(1 - \cos \frac{2\pi x}{L} \right) \right] dx = 0$$

and once the result has been treated similarly as in section 4.6.1, the Duffing equation for the clamped-clamped beam becomes

$$\ddot{q} + (D_{11} - B_{11}K_1 - B_{16}K_2) \frac{16\pi^4}{3\rho hL^4} q + A_{11} \frac{4\pi^4 R^2}{3\rho hL^4} q^3 = \frac{2}{3\rho hRL} \int_0^L P(x,t) \left(1 - \cos \frac{2\pi x}{L} \right) dx \quad (4.30)$$

To make the equation more legible, define the following terms

$$\omega_0^2 = (D_{11} - B_{11}K_1 - B_{16}K_2) \frac{16\pi^4}{3\rho hL^4} \quad (4.31)$$

$$\beta = A_{11} \frac{4\pi^4 R^2}{3\rho hL^4} \quad (4.32)$$

$$F(t) = \int_0^L P(x,t) \left(1 - \cos \frac{2\pi x}{L} \right) dx \quad (4.33)$$

$$m = \frac{3\rho hRL}{2} \quad (4.34)$$

In terms of the above, Equation (4.30) may be written

$$\ddot{q} + \omega_0^2 q + \beta q^3 = \frac{F(t)}{m} \quad (4.35)$$

This equation represents a non-damped vibration of the structure. As before, a Coulomb-like damping term is added to Equation (4.35) to obtain

$$\ddot{q} + 2\xi\omega_0\dot{q} + \omega_0^2 q + \beta q^3 = \frac{F(t)}{m} \quad (4.36)$$

4.6.3 Cantilever Beam:

In the case of a cantilever beam, the axial displacements cannot be neglected, but they are approximated by

$$u_0(x, t) = u(t)R \sin \frac{\pi x}{2L} \quad (4.37)$$

$$v_0(x, t) = v(t)R \sin \frac{\pi x}{2L} \quad (4.38)$$

The second derivatives of these expressions are

$$\frac{\partial^2 u_0}{\partial x^2} = -\left(\frac{\pi}{2L}\right)^2 u(t)R \sin \frac{\pi x}{2L} \quad (4.39)$$

$$\frac{\partial^2 v_0}{\partial x^2} = -\left(\frac{\pi}{2L}\right)^2 v(t)R \sin \frac{\pi x}{2L} \quad (4.40)$$

Equation (3.30) is solved by applying the boundary conditions in Section 4.4.3, where there are four boundary conditions listed at $x = 0$ and at $x = L$. The conditions on u_0 and v_0 at $x = L$ are time dependent and in conjunction with Equations (4.16) and (4.37) through (4.40) are used to derive a relationship between $u(t)$ and $v(t)$ and $q(t)$. Thus, one can write

$$\begin{aligned}
 u(t) \sin \frac{\pi x}{2L} - K_1 \left(\frac{\pi}{2L} \right) \sum_{m=1}^k m^3 q_m(t) \sin \frac{m\pi x}{2L} + \\
 - R \left(\frac{\pi}{2L} \right) \left\{ \sum_{m=1}^k m^2 q_m(t) \cos \frac{m\pi x}{2L} \right\} \left\{ \sum_{m=1}^k m q_m(t) \sin \frac{m\pi x}{2L} \right\} = 0
 \end{aligned}
 \tag{4.41}$$

and

$$v(t) \sin \frac{\pi x}{2L} - K_2 \left(\frac{\pi}{2L} \right) \sum_{m=1}^k m^3 q_m(t) \sin \frac{m\pi x}{2L} = 0
 \tag{4.42}$$

One can determine the amplitudes $u(t)$ and $v(t)$ by applying Galerkin's method to Equations (4.41) and (4.42). Consistent with the previous two boundary conditions, only the first mode, $k = 1$, is considered, in which Equations (4.41) and (4.42) become

$$u(t) \sin \frac{\pi x}{2L} - K_1 \left(\frac{\pi}{2L} \right) q(t) \sin \frac{\pi x}{2L} - \left(\frac{\pi}{2L} \right) R q^2(t) \sin \frac{\pi x}{2L} \cos \frac{\pi x}{2L} = 0
 \tag{4.43}$$

$$v(t) \sin \frac{\pi x}{2L} - K_2 \left(\frac{\pi}{2L} \right) q(t) \sin \frac{\pi x}{2L} = 0 \quad (4.44)$$

Applying Galerkin's method to these equations (recalling Equation (4.16)) requires

$$\int_0^L \left[\text{Left - hand side of Equations (4.43) and (4.44)} \right] \cdot \left[Rq(t) \left(1 - \cos \frac{\pi x}{2L} \right) \right] dx = 0$$

After some simplifications these two equations yield

$$u(t) = \left(\frac{\pi}{6L} \right) Rq^2 + \left(\frac{\pi}{2L} \right) K_1 q \quad (4.45)$$

and

$$v(t) = \left(\frac{\pi}{2L} \right) K_2 q \quad (4.46)$$

Then, substitute Equations (4.45) and (4.46) into Equations (4.37) and (4.38), evaluated at $x = L$, to obtain

$$u_0(L, t) = u(t) R \sin \frac{\pi L}{2L} = \left(\frac{\pi}{6L} \right) R^2 q^2 + \left(\frac{\pi}{2L} \right) K_1 R q \quad (4.47)$$

and

$$v_0(L, t) = v(t) R \sin \frac{\pi L}{2L} = \left(\frac{\pi}{2L} \right) K_2 R q \quad (4.48)$$

Using the boundary conditions in Section 4.4.3, Equation (3.30) can be written

$$\begin{aligned} (B_{11}K_1 + B_{16}K_2 - D_{11}) \frac{\partial^4 w}{\partial x^4} + \left(\frac{A_{11}}{L} u_0(L) + \frac{A_{16}}{L} v_0(L) + \frac{A_{11}}{2L} \int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx + \right. \\ \left. - \frac{B_{11}}{L} \int_0^L \frac{\partial^2 w}{\partial x^2} dx \right) \frac{\partial^2 w}{\partial x^2} + P - \rho h \frac{\partial^2 w}{\partial t^2} = 0 \end{aligned}$$

Substituting Equations (4.16), (4.47), and (4.48), and then expanding, integrating, simplifying, rearranging, and finally applying Galerkin's method, results in

$$\begin{aligned} (D_{11} - B_{11}K_1 - B_{16}K_2) \left(\frac{4 - \pi}{3\pi - 8} \right) \frac{\pi^4}{16\rho h L^4} q + (A_{11}K_1 + A_{16}K_2 - B_{11}) \left(\frac{4 - \pi}{3\pi - 8} \right) \frac{\pi^3}{8\rho h L^4} q^2 + \\ A_{11} \left(\frac{4 - \pi}{3\pi - 8} \right) \left(\frac{1}{3} + \frac{\pi R}{8} \right) \frac{\pi^3}{8\rho h L^4} q^3 + \int_0^L P \left(1 - \cos \frac{\pi x}{2L} \right) dx + \frac{(3\pi - 8)\rho h R L}{2\pi} \ddot{q} = 0 \end{aligned} \quad (4.49)$$

As before, define the following constants

$$\omega_0^2 = (D_{11} - B_{11}K_1 - B_{16}K_2) \left(\frac{4 - \pi}{3\pi - 8} \right) \frac{\pi^4}{16\rho h L^4} \quad (4.50)$$

$$\alpha = (A_{11}K_1 + A_{16}K_2 - B_{11}) \left(\frac{4 - \pi}{3\pi - 8} \right) \frac{\pi^3}{8\rho h L^4} \quad (4.51)$$

$$\beta = A_{11} \left(\frac{4 - \pi}{3\pi - 8} \right) \left(\frac{1}{3} + \frac{\pi R}{8} \right) \frac{\pi^3}{8\rho h L^4} \quad (4.52)$$

$$m = \frac{(3\pi - 8)\rho h R L}{2\pi} \quad (4.53)$$

$$F(t) = \int_0^L P(x,t) \left(1 - \cos \frac{\pi x}{2L}\right) dx \quad (4.54)$$

Equation (4.49) can be written

$$\ddot{q} + \omega_0^2 q + \alpha q^2 + \beta q^3 = \frac{F(t)}{m} \quad (4.55)$$

Adding Coulomb damping to Equation (4.55) yields

$$\ddot{q} + 2\xi\omega_0\dot{q} + \omega_0^2 q + \alpha q^2 + \beta q^3 = \frac{F(t)}{m} \quad (4.56)$$

4.7 The Application of the Method of Equivalent Linearization.

The solution to Equations (4.27), (4.36), and (4.56) yields $q(t)$, which once inserted into Equations (4.15), (4.16), and (4.16), respectively, gives (for $k = 1$) the time-dependent deflection of the beam. The method of equivalent linearization assumes that an approximate solution to the above non-linear differential equation can be obtained from the linearized equation

$$\ddot{q} + 2\xi\omega_0\dot{q} + \Omega^2 q = \frac{F(t)}{m} \quad (4.57)$$

The difference between Equations (4.27), (4.36), or (4.56) and (4.57) is called the error of linearization, and in its most general form is given by

$$err = (\omega_0^2 - \Omega^2)q + \alpha q^2 + \beta q^3 \quad (4.58)$$

The mean-square response of the modal amplitude from Equation (4.57) is given by

$$E[q^2] = \int_0^\infty S_P(\omega) |H(\omega)|^2 d\omega \quad (4.59)$$

where $S_P(\omega)$ is the power spectral density function and $H(\omega)$ is called the complex frequency response function of the system, and is obtained as follows. Substitution of $F = me^{i\omega t}$ and $q(t) = H(\omega)e^{i\omega t}$, into Equation (4.57) results in

$$i^2 \omega^2 H(\omega) e^{i\omega t} + 2\xi \omega_0 \omega H(\omega) e^{i\omega t} + \Omega^2 H(\omega) e^{i\omega t} = e^{i\omega t}$$

or

$$[-\omega^2 + 2\xi \omega_0 \omega + \Omega^2] H(\omega) e^{i\omega t} = e^{i\omega t}$$

Solving for $H(\omega)$ results in

$$H(\omega) = \frac{1}{[\Omega^2 - \omega^2 + 2i\omega_0\omega]} \quad (4.60)$$

The integration of Equation (4.59) can be simplified greatly when the spectral density of the excitation varies slowly in the neighborhood of Ω [53]. In this case the spectral density function $S_p(\omega)$ can be considered a constant, so that the integration in Equation (4.59) yields

$$E[q^2] \cong \frac{\pi S_p(\Omega)}{4m^2\xi\omega_0\Omega^2}$$

In practice, the spectral density function is given in terms of frequency (Hz). Then, the following substitutions can be made

$$\Omega = f/2\pi \quad \Rightarrow \quad S_p(\Omega) = S_p(f)/2\pi$$

The units for $S_p(f)/2\pi$ are Pa^2/Hz for plates and $(\text{N/m})^2/\text{Hz}$ for beams. The above can now be rewritten in the following form

$$E[q^2] \cong \frac{S_p(f)}{8m^2\xi\omega_0\Omega^2} = \frac{S}{\xi\Omega^2} \quad (4.61)$$

where

$$S = \frac{S_p(f)}{8m^2\omega_0} \quad (4.62)$$

The coefficient S has units of $1/\text{sec}^2$. From the above equations and the preceding text it becomes evident that the linearized frequency, Ω , introduces the nonlinear effects to the response. In order to minimize the error that arises from the linearization Equation (4.57) and that of each of the original equations (i.e (4.27), (4.36), or (4.56)), the conditions for the least error must be determined. A necessary condition that includes the effects of both α and β and minimizes the error has not been determined yet. It will, however, be shown in Chapter 5 that the effects of α are small, indicating that Equation (4.59) is acceptably accurate in most instances. Generally, the error is minimized with respect to Ω^2 (note that minimizing the error with respect to $\omega_0^2 - \Omega^2$ yields the same condition)

$$\frac{\partial}{\partial \Omega^2} E[err^2] = 0 \quad (4.63)$$

Inserting Equation (4.58) into (4.63) results in

$$\frac{\partial}{\partial \Omega^2} E[err^2] = \frac{\partial}{\partial \Omega^2} E\left[\left((\omega_0^2 - \Omega^2)q + \alpha q^2 + \beta q^3\right)^2\right]$$

which, expanded and manipulated using the rules of Appendix A, reduces to the following relationship between the equivalent linear frequency and the mean-square displacement⁹

$$\Omega^2 = \omega_0^2 + 3\beta E[q^2] \quad (4.64)$$

Substituting this into Equation (4.61) yields

$$E[q^2] \cong \frac{S}{\xi \Omega^2} = \frac{S}{\xi \{\omega_0^2 + 3\beta E[q^2]\}} \quad (4.65)$$

From which the following relationship is obtained

$$3\beta (E[q^2])^2 + \omega_0^2 E[q^2] - \frac{S}{\xi} = 0 \quad (4.66)$$

whose solution is given by

$$E[q^2] = -\frac{\omega_0^2}{6\beta} + \sqrt{\left(\frac{\omega_0^2}{6\beta}\right)^2 + \frac{S}{3\beta\xi}} \quad (4.67)$$

This equation therefore represents the general random vibration response of any type of laminate lay-up. When this equation is used one should be aware of its limitations, i.e.

⁹ See Appendix E.

that the effects of α in the equation are nonexistent. However, the effects of a nonzero α are implicitly introduced, since any laminate lay-up has its own distinguishable value of ω_0 and β that will affect the final outcome of Equation (4.67). It is the purpose of the following chapter to discuss the true limitation of this presentation by comparing the solution obtained by the Equation (4.67) to the numerical solution of Equation (3.30).

4.8 The Linear Response.

In addition to the numerical comparison, it is common to compare the nonlinear case to the linear case and study the corrections that result. Therefore, well documented results for isotropic, angle-ply, and cross-ply laminates are presented below for the convenience of the reader. These solutions are easily obtained from the linear equations of motion. The general random vibration response for the linear case is given by:

$$E[q^2] = \frac{S}{\xi\omega_0^2} \quad (4.68)$$

where S is given Equation by (4.62) and ξ is the damping coefficient. The following specialized cases are given as examples of how the theory is applied.

4.8.1 Isotropic Simply-Supported Beam:

For isotropic beams, E is the Young's modulus, I is the moment of inertia, A is the cross-sectional area, L is the length, and ρ is the density of the beam. For a simply-supported beam the natural frequency and the mass are given by

$$\omega_0^2 = \frac{\pi^4 EI}{\rho AL^4} \quad (4.69)$$

$$m = \frac{\pi \rho h A}{8\sqrt{3}} \quad (4.70)$$

4.8.2 Generally Laminated Simply-Supported Beam:

For a generally laminated simply-supported beam, h represents its height, L its length, and ρ the density of the beam. The natural frequency and the mass are given by:

$$\omega_0^2 = (D_{11} - B_{11}K_1 - B_{16}K_2) \frac{\pi^4}{\rho h L^4} \quad (4.71)$$

$$m = \frac{\rho h R L}{2} \quad (4.72)$$

4.8.3 Generally Laminated Clamped-Clamped Beam:

For a generally laminated clamped-clamped beam, h represents its height, L its length, and ρ the density of the beam. The natural frequency and the mass are given by:

$$\omega_0^2 = (D_{11} - B_{11}K_1 - B_{16}K_2) \frac{16\pi^4}{3\rho h L^4} \quad (4.73)$$

$$m = \frac{3\rho h RL}{2} \quad (4.74)$$

4.8.4 Generally Laminated Cantilever Beam:

For a generally laminated cantilever beam, h is the height, L is the length, and ρ is the density. The natural frequency and the mass are given by

$$\omega_0^2 = (D_{11} - B_{11}K_1 - B_{16}K_2) \left(\frac{\pi - 4}{3\pi - 8} \right) \left(\frac{\pi^4}{16\rho h L^4} \right) \quad (4.75)$$

$$m = \frac{(3\pi - 8)\rho h RL}{2\pi} \quad (4.76)$$

4.9 The Nonlinear Response.

This section serves partially as a summary of the results obtained so far, and partially to introduce some particular results. Equation (4.67) gives the general response for any of

the three beams. Its coefficients, however, depend on the different boundary conditions.

4.9.1 Simply-Supported Angle-Ply Beam:

An angle-ply is typically laid up in a sequence such as: $[\theta/-\theta/\theta/-\theta/\theta/-\theta/...]$, and if the number of laminae is even, it is a symmetrical laminate. This, regardless of the number of laminae, results in the following material properties being zero

$$A_{16} = A_{26} = 0,$$

$$B_{11} = B_{12} = B_{22} = B_{66} = 0,$$

$$D_{16} = D_{26} = 0,$$

which gives the following coefficients of the Equation (4.27).

From Equations (4.21) through (4.25)

$$\omega_0^2 = (D_{11} - B_{11}K_1 - B_{16}K_2) \frac{\pi^4}{\rho h L^4} \quad (4.77)$$

$$\alpha = 2B_{11} \frac{\pi^3 R}{\rho h L^4} \quad (4.78)$$

$$\beta = A_{11} \frac{\pi^4 R^2}{4\rho h L^4} \quad (4.79)$$

$$m = \frac{\rho h R L}{2} \quad (4.80)$$

These coefficients are then inserted into Equations (4.62) and (4.67) to obtain the dynamic response.

4.9.2 Simply-Supported Cross-Ply Beam:

A cross-ply is typically laid up in the sequence $[90^\circ/0^\circ/90^\circ/0^\circ/\dots]$. This results in the following material properties of the A-B-D matrix being zero

$$A_{16} = A_{26} = 0,$$

$$B_{12} = B_{16} = B_{26} = B_{66} = 0,$$

$$D_{16} = D_{26} = 0,$$

which results in the following simplifications

$$\omega_0^2 = (D_{11} - B_{11}K_1) \frac{\pi^4}{\rho h L^4} \quad (4.81)$$

$$\alpha = 2B_{11} \frac{\pi^3 R}{\rho h L^4} \quad (4.82)$$

$$\beta = A_{11} \frac{\pi^4 R^2}{4\rho h L^4} \quad (4.83)$$

$$m = \frac{\rho h R L}{2} \quad (4.84)$$

4.9.3 A Simply-Supported Isotropic Beam:

The terms of the A-B-D matrix for an isotropic beam, are as follows:

$$\begin{aligned}
 A_{11} = A_{22} &= Eh & D_{11} = D_{22} &= \frac{Eh^3}{12} \\
 A_{12} = A_{16} = A_{26} &= 0 & B_{ij} &= 0 & D_{12} = D_{16} = D_{26} &= 0 \\
 A_{66} &= \frac{A_{11}}{2} = \frac{Eh}{2} & D_{66} &= \frac{D_{11}}{2} = \frac{Eh^3}{24}
 \end{aligned}$$

This gives the following solution for the mean-square response of the modal amplitude¹⁰

$$E[q^2] = 2 \left(\sqrt{\frac{1}{9} + \frac{S}{3\xi\omega_0^2}} - \frac{1}{3} \right) \quad (4.85)$$

where S is given by Equation (4.62), ω_0^2 by Equation (4.69), and m by Equation (4.70).

4.9.4 Clamped-Clamped Angle-Ply Beam:

$$\omega_0^2 = (D_{11} - B_{16}K_2) \frac{16\pi^4}{3\rho hL^4} \quad (4.86)$$

¹⁰ Note that the answer is merely a modification of Equation (4.65).

$$\beta = A_{11} \frac{4\pi^4 R^2}{3\rho h L^4} \quad (4.87)$$

$$m = \frac{3\rho h R L}{2} \quad (4.88)$$

4.9.5 Clamped-Clamped Cross-Ply Beam:

$$\omega_0^2 = (D_{11} - B_{11}K_1) \frac{16\pi^4}{3\rho h L^4} \quad (4.89)$$

$$\beta = A_{11} \frac{4\pi^4 R^2}{3\rho h L^4} \quad (4.90)$$

$$m = \frac{3\rho h R L}{2} \quad (4.91)$$

4.9.6 A Clamped-Clamped Isotropic Beam¹¹:

$$\omega_0^2 = \frac{16\pi^4 EI}{3\rho A L^4} \quad (4.92)$$

$$\beta = \frac{1}{4} \frac{16\pi^4 EI}{3\rho A L^4} = \frac{1}{4} \omega_0^2 \quad (4.93)$$

$$m = \frac{3\rho h A}{4\sqrt{3}} \quad (4.94)$$

¹¹ Use with equation (4.83).

4.9.7 Cantilever Angle-Ply Beam:

$$\omega_0^2 = (D_{11} - B_{16}K_2) \left(\frac{\pi - 4}{3\pi - 8} \right) \frac{\pi^4}{16\rho h L^4} \quad (4.95)$$

$$\alpha = A_{11}K_1 \left(\frac{\pi - 4}{3\pi - 8} \right) \frac{\pi^3}{8\rho h L^4} \quad (4.96)$$

$$\beta = A_{11} \left(\frac{\pi - 4}{3\pi - 8} \right) \left(\frac{1}{3} + \frac{\pi R}{8} \right) \frac{\pi^3}{8\rho h L^4} \quad (4.97)$$

$$m = \frac{(3\pi - 8)\rho h R L}{2\pi} \quad (4.98)$$

4.9.8 Cantilever Cross-Ply Beam:

$$\omega_0^2 = (D_{11} - B_{11}K_1) \left(\frac{\pi - 4}{3\pi - 8} \right) \frac{\pi^4}{16\rho h L^4} \quad (4.99)$$

$$\alpha = (A_{11}K_1 - B_{11}) \left(\frac{\pi - 4}{3\pi - 8} \right) \frac{\pi^3}{8\rho h L^4} \quad (4.100)$$

$$\beta = A_{11} \left(\frac{\pi - 4}{3\pi - 8} \right) \left(\frac{1}{3} + \frac{\pi R}{8} \right) \frac{\pi^3}{8\rho h L^4} \quad (4.101)$$

CHAPTER 5

NUMERICAL EXAMPLES

Once a relationship between the random force exciting a beam and the mean-square response has been found, a number of important characteristics can be determined. Among these are the following:

- Deflection at a given spectral density,
- Stress at a given spectral density,
- Strain at a given spectral density.

These can be found by inserting values for $E[q^2]$ into Equations (4.14), (4.15), and (4.16) for each of the three beams. This yields the deflection equation, which can easily be used to determine stress and strain using the theory of elasticity.

The following relation can be used to determine the stress in the k^{th} layer in a laminated beam¹²:

$$\{\sigma_x\}_k = (\overline{Q}_{11})_k \left[\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 - z \frac{\partial^2 w}{\partial x^2} \right] + (\overline{Q}_{16})_k \frac{\partial v_0}{\partial x} \quad (5.1)$$

The strain at the surface of the composite is given by

$$(\varepsilon_x)_{\pm h/2} = \left[\frac{\partial u_0}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 - z \frac{\partial^2 w}{\partial x^2} \right]_{z=\pm h/2} \quad (5.2)$$

The following problems are meant to demonstrate applications of the preceding theory, as well as to compare it to already established results - such as that of Prasad [44]. Note that the presentation of nonlinear results for unsymmetrically laminated beams are original.

5.1 Problem 1: White Noise Exciting an Isotropic Beam.

5.1.1 Statement of Problem.

An interesting and important problem that can be studied is the deflection caused by noise from a source such as a jet engine. The jet engine emits white noise (which has a

¹² Equations (5.1) and (5.2) are derived from Eq. (2.40).

stationary Gaussian density distribution) which varies in strength dependent upon the thrust produced. It is common to measure such sound in dB (decibels), so one must convert dB into spectral density units, using the method discussed in Section 2.2. This problem is presented here to enable comparison between Prasad's results [44] and the ones obtained by this thesis.

The problem can be stated as follows. Determine, analytically, the value of $E[q^2]$ when an isotropic simply-supported beam is excited by a pressure spectrum level of 100 dB. The beam has the following properties:

Young's modulus	$E = 10.5 \times 10^6$ psi
Damping coefficient	$\xi = 0.01$
Mass density	$\rho = 2.588 \times 10^{-4}$ lb·sec ² /in ⁴
Thickness	$h = 0.064$ in
Width	$b = 2$ in
Length	$L = 12$ in

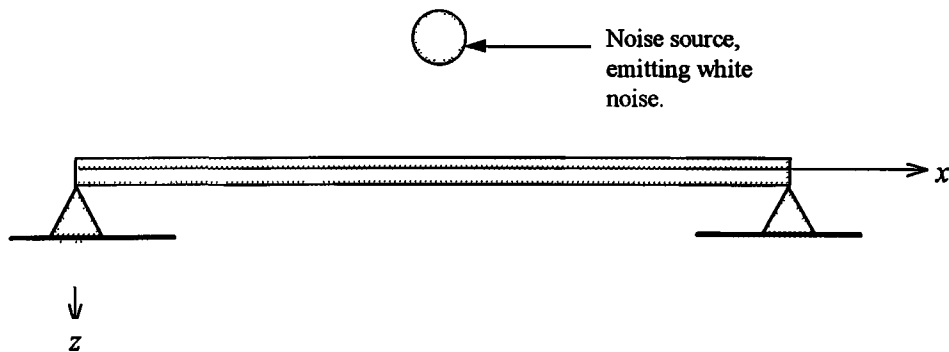


Figure 15. Problem setup.

5.1.2 Solution.

First, the coefficients for the beam are determined, in this case, for a simply-supported isotropic beam. These are found using Equations (4.69) and (4.70)

$$\omega_0^2 = \frac{\pi^4 EI}{\rho AL^4} = \frac{\pi^4 (10.5 \times 10^6) (2 \times 0.064^3 / 12)}{(2.588 \times 10^{-4}) (0.064 \times 2) (12^4)} = 65054 \text{ s}^{-2}$$

$$m = \frac{\pi \rho h A}{8\sqrt{3}} = \frac{\pi (2.588 \times 10^{-4}) (0.064) (2 \times 0.064)}{8\sqrt{3}} = 4.807 \times 10^{-7} \text{ lb s}^2 / \text{in}$$

Then, the spectral density is determined using equations (2.26), (2.24), and $SSL = 100$ dB, respectively

$$PSD^2 = 8.4216 \times 10^{(0.1SSL-18)} = 8.4216 \times 10^{(10-18)} = 8.4216 \times 10^{-8} \text{ (lb}^2 / \text{in}^4)\text{Hz}$$

and

$$S_p(f) = PSD^2 b^2 = (8.4216 \times 10^{-8}) \times (2)^2 = 3.3686 \times 10^{-7} \text{ (lb / in)}^2 / \text{Hz}$$

Then, calculate the coefficient S from Equation (4.62)

$$S = \frac{S_p(f)}{8m^2\omega_0} = \frac{3.3686 \times 10^{-7}}{8 \times (4.807 \times 10^{-7}) (255.1)} = 714.5 \text{ s}^{-2}$$

Finally, this is inserted into Equation (4.83), which was specifically derived for isotropic materials and directly yields the mean square of the response

$$E[q^2] = 2 \left(\sqrt{\frac{1}{9} + \frac{S}{3\xi\omega_0^2}} - \frac{1}{3} \right) = 0.7150$$

The response, therefore, is the root mean square

$$rms = \sqrt{E[q^2]} = 0.8455$$

5.1.3 Accuracy of Solution.

The best way of assessing the accuracy of the method is by a direct comparison to Prasad [44] and Seide [45]. Prasad used equivalent linearization to calculate displacements and stresses of simply-supported and clamped-clamped beams. To accomplish this he used a numerical iteration scheme on Equation (4.65). Seide, also, used a numerical scheme obtaining similar results. This thesis, on the other hand, attacks Equation (4.65) directly, but this allows for a simplified analysis of the mean square response. Additionally, the Duffing equation is solved numerically, which enables a direct comparison between the analytical and numerical results.

Figure 16 shows the response of the above beam for a spectrum sound level (*SSL*) from 70 to 130 dB. The nonlinear results essentially contain both Prasad's results and those obtained by Equation (4.83) (or Equation (4.65) for that matter), but each method yields the same results. The scatter is the result from several thousand runs of the Runge-Kutta code, and underlines the importance of understanding that in reality there is a band of solutions and the analytical should merely be considered an average, rather than a point solution. The figure also identifies an unacceptable over-estimation of the linear response.

Random Response of a Simply-Supported Isotropic Beam

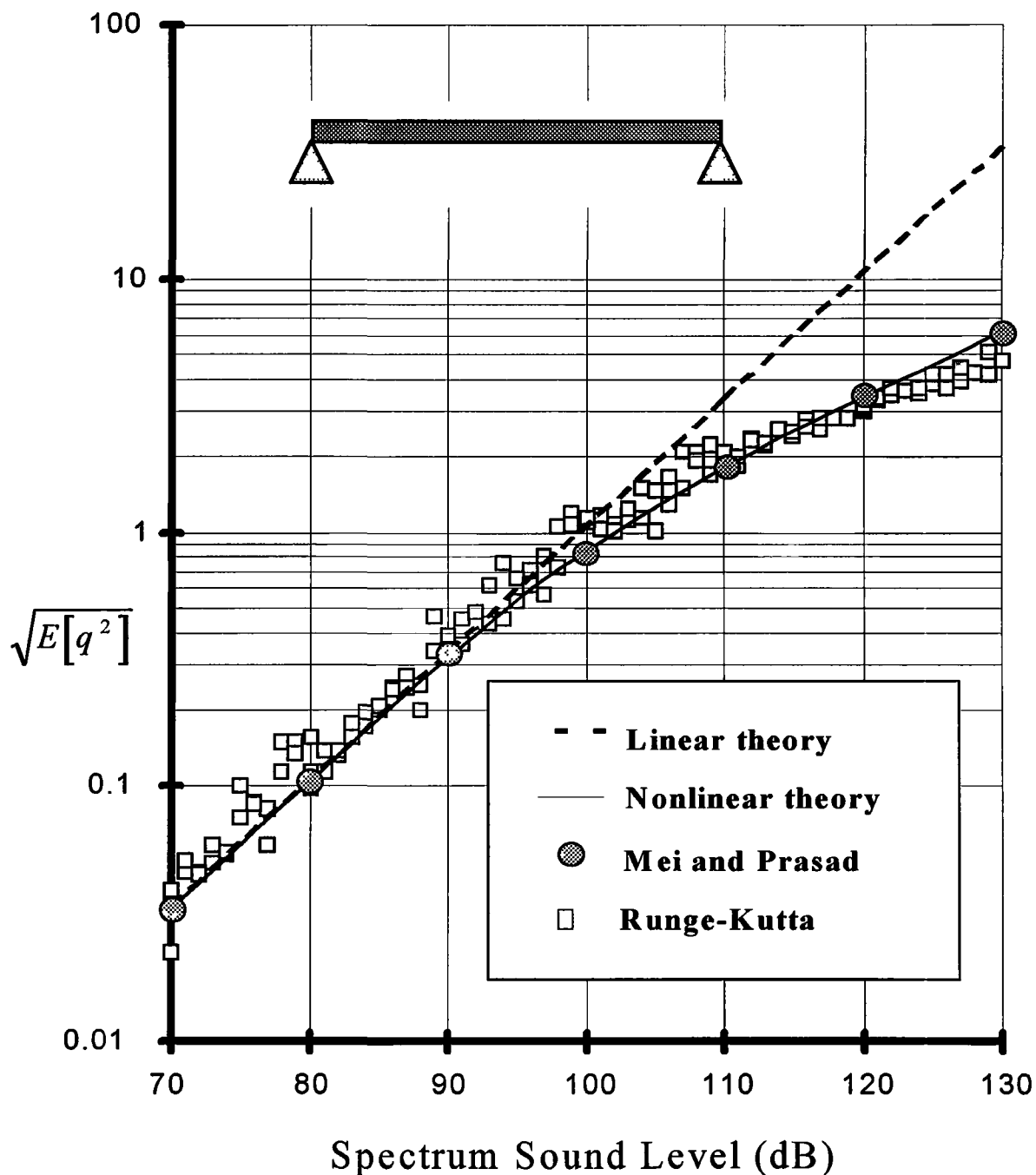


Figure 16. Comparison of $E[q^2]$ obtained by different methods for a 12x2x0.064 inch simply supported isotropic beam. Note that $E[q^2]$ is equivalent to $E[w_{\max}^2]$.

5.2 Problem 2: White Noise Exciting an Angle-Ply Beam.

5.2.1 Statement of Problem.

Another similarly interesting problem is the deflection of a laminated beam laid-up as an angle-ply (see Section 4.9.1). In this problem all three boundary conditions are to be analyzed. The problem can be stated as follows. Determine, analytically, the value of $E[q^2]$ when simply-supported, clamped-clamped, and cantilever beams, fabricated as angle-ply, are excited by pressure spectral level ranging from 70 to 130 dB. The beams have the following properties (note that the lamination sequence is unsymmetrical)

Longitudinal stiffness modulus	$E_1 = 10.5 \times 10^6$ psi
Transverse stiffness modulus	$E_2 = 2.625 \times 10^6$ psi
In-plane shear modulus	$G_{12} = 1.3125 \times 10^6$ psi
Major Poisson's ratio	$\nu_{12} = 0.25$
Damping coefficient	$\xi = 0.01$
Mass density	$\rho = 2.588 \times 10^{-4} \frac{\text{lb} \cdot \text{sec}^2}{\text{in}^4}$
Total thickness	$h = 0.060$ in
Width	$b = 2$ in
Length	$L = 12$ in
Number of layers	$n = 4$
Thickness of a layer	$t = 0.015$ in
Lay-up sequence	$[45^\circ / -45^\circ / 45^\circ / -45^\circ]$

5.2.2 Solution.

As in Section 5.1.2, the solution process begins by calculating beam properties. These are determined using Equations (4.21) through (4.25) for the simply-supported beam, Equations (4.31) through (4.34) for the clamped-clamped beam, and Equations (4.50) through (4.54) for the cantilever beam.

Simply-supported beam:

Natural frequency, $\omega_0 = 160.5$ Hz

Coefficient $\alpha = 0$

Coefficient $\beta = 6778.4$

Moment of inertia, $I = 0.000036$ in⁴

Area, $A = 0.12$ in²

Radius of Gyration, $R = 1.732 \times 10^{-2}$

Mass, $m = 1.614 \times 10^{-6}$

Clamped-clamped beam:

Natural frequency, $\omega_0 = 370.6$ Hz

Coefficient $\alpha = 0$

Coefficient $\beta = 36151$

Mass, $m = 4.841 \times 10^{-6}$

Cantilever beam:

Natural frequency, $\omega_0 = 31.1$ Hz

Coefficient $\alpha = 0$

Coefficient $\beta = 101615$

Mass, $m = 7.319 \times 10^{-7}$

These constants are used with Equation (4.67) to generate Figures 17, 18, and 19. Note that Equation (4.67) is plotted against results from linear analysis, Runge-Kutta analysis, and a curve, based on a best fit correlation of the Runge-Kutta analysis.

Random Response of a Simply-Supported Laminated Angle-Ply Beam

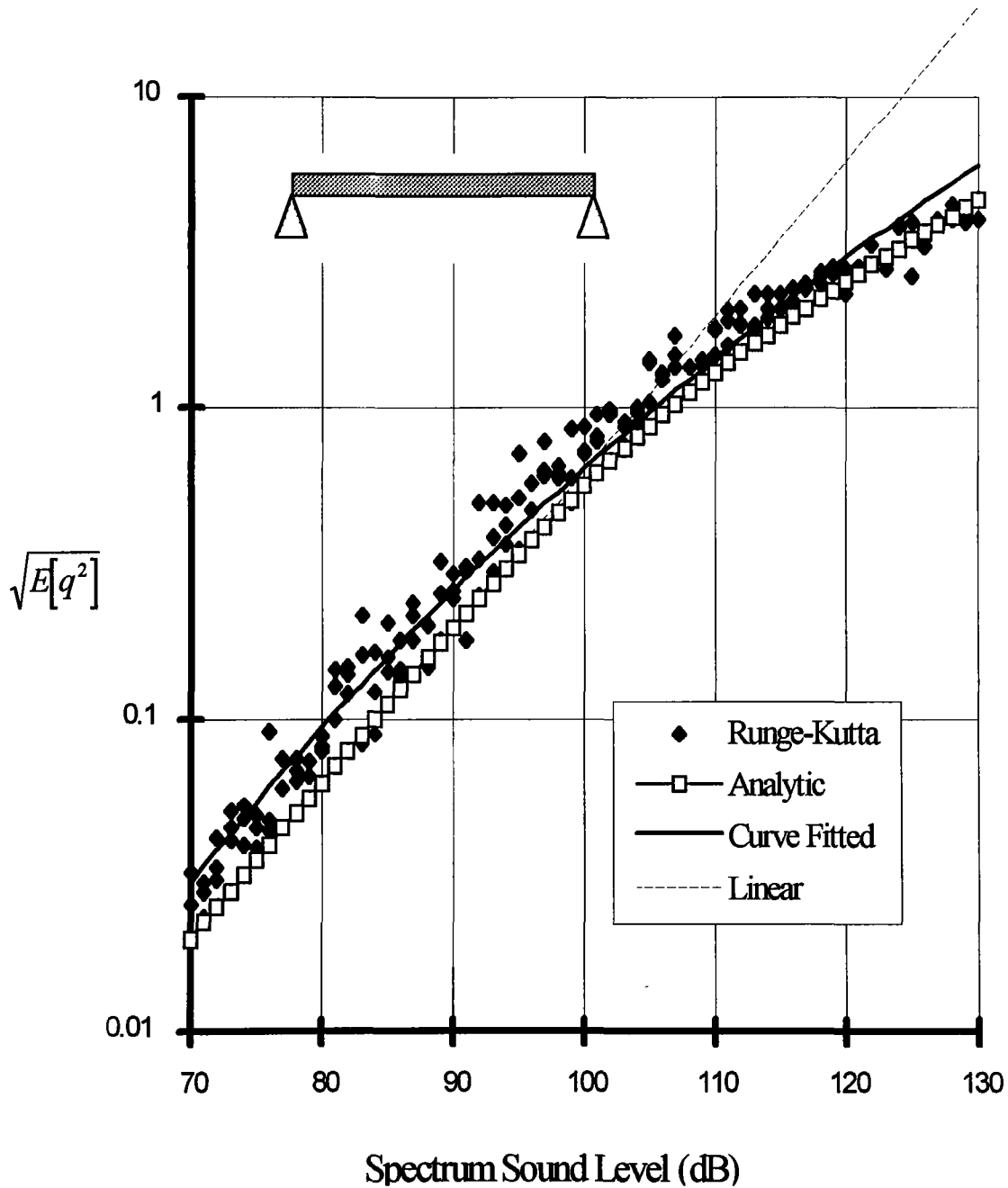


Figure 17. Comparison of $E[q^2]$ obtained by different methods for a 12x2x0.060 inch simply-supported angle-ply beam.

Random Response of a Clamped-Clamped Laminated Angle-Ply Beam

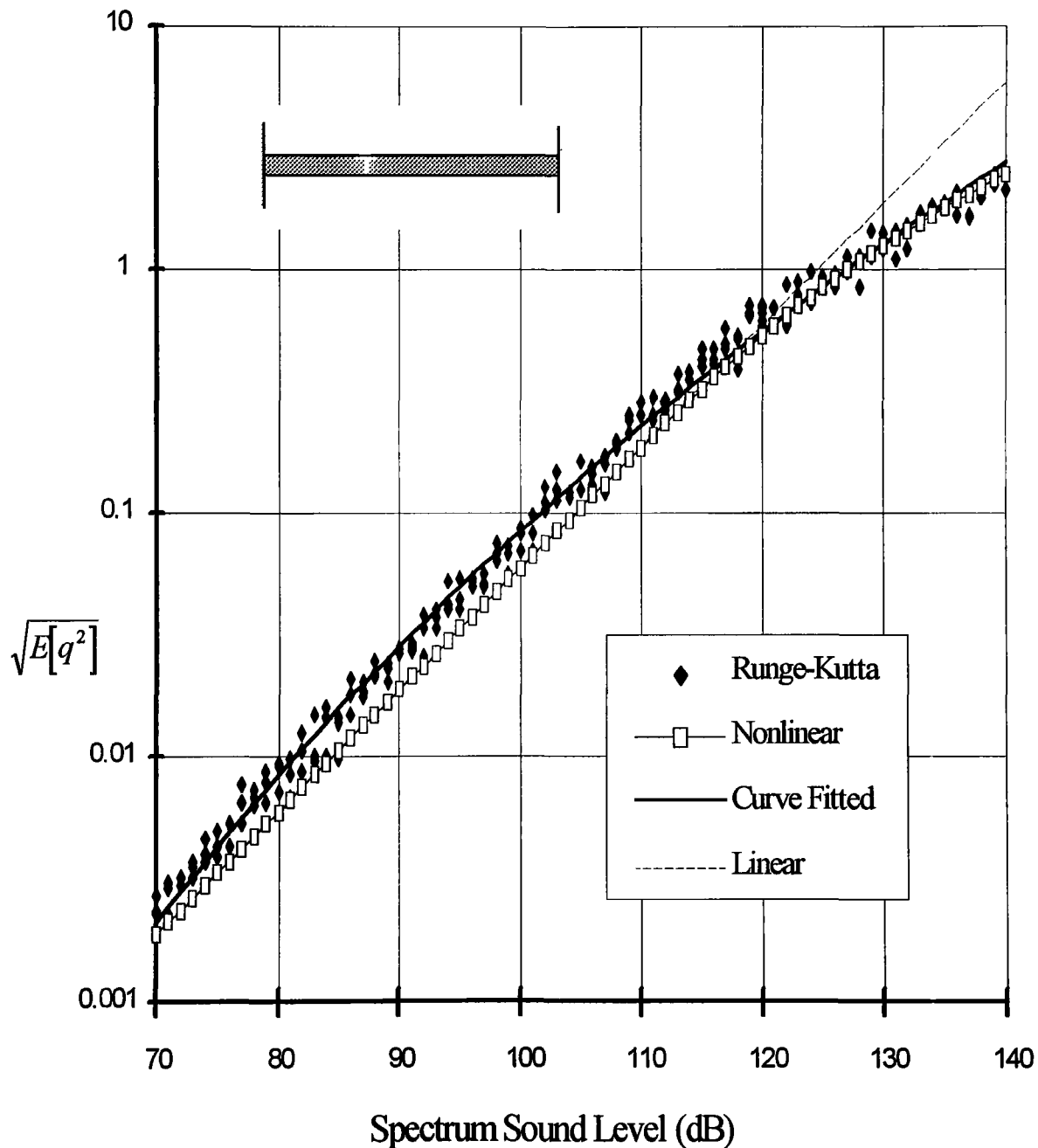


Figure 18. Comparison of $E[q^2]$ obtained by different methods for a 12x2x0.060 inch clamped-clamped angle-ply beam.

Random Response of a Cantilever Laminated Angle-Ply Beam

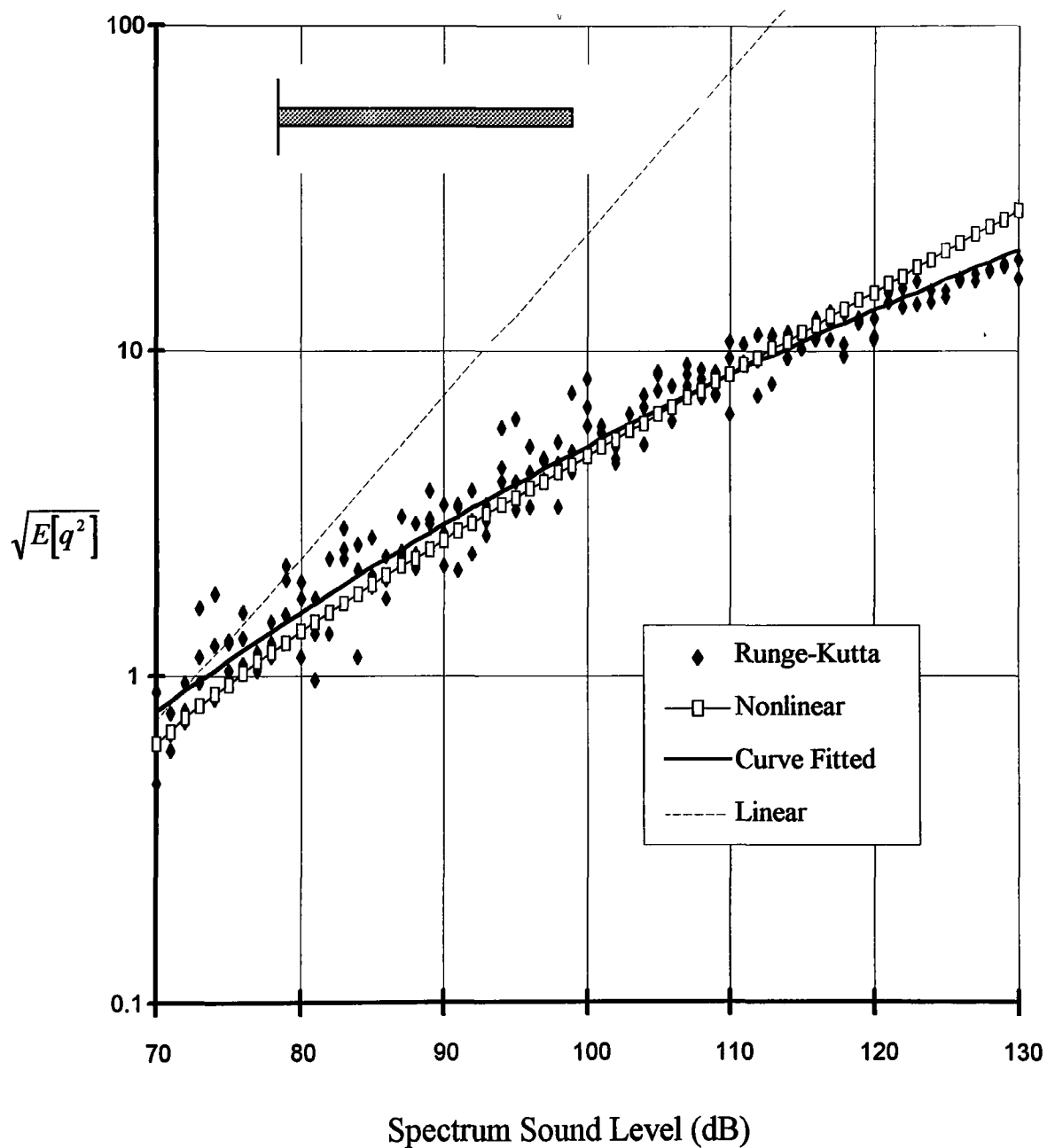


Figure 19. Comparison of $E[q^2]$ obtained by different methods for a 12x2x0.060 inch cantilever angle-ply beam.

5.3 Problem 3: White Noise Exciting a Cross-Ply Beam.

5.3.1 Statement of Problem.

This problem is identical to Problem 2, except the lamination lay-up sequence is that of a cross-ply. In other words, the only difference is

Lay-up sequence $[0^\circ/90^\circ/0^\circ/90^\circ]$

5.3.2 Solution.

The solution process, also, is identical to the one in Sections 5.1.2 and 5.2.2. The resulting properties turned out to be as follows.

Simply-supported beam:

Natural frequency, $\omega_0 = 184.0$ Hz

Coefficient $\alpha = -6004.4$

Coefficient $\beta = 9075.7$

Moment of inertia, $I = 0.000036$ in⁴

Area, $A = 0.12$ in²

Radius of Gyration, $R = 1.732 \times 10^{-2}$

Mass, $m = 1.614 \times 10^{-6}$

Clamped-clamped beam:

Natural frequency, $\omega_0 = 424.9$ Hz

Coefficient $\alpha = 0$

Coefficient $\beta = 48404$

Mass, $m = 4.841 \times 10^{-6}$

Cantilever beam:

Natural frequency, $\omega_0 = 35.7$ Hz

Coefficient $\alpha = 1.237 \times 10^{-12}$

Coefficient $\beta = 136054$

Mass, $m = 7.319 \times 10^{-7}$

These constants are used with Equation (4.67) to generate Figures 20, 21, and 22. Note that Equation (4.67) is plotted against results from linear analysis, Runge-Kutta analysis, and a curve, based on a best fit correlation of the Runge-Kutta analysis.

Random Response of a Simply-Supported Laminated Cross-Ply

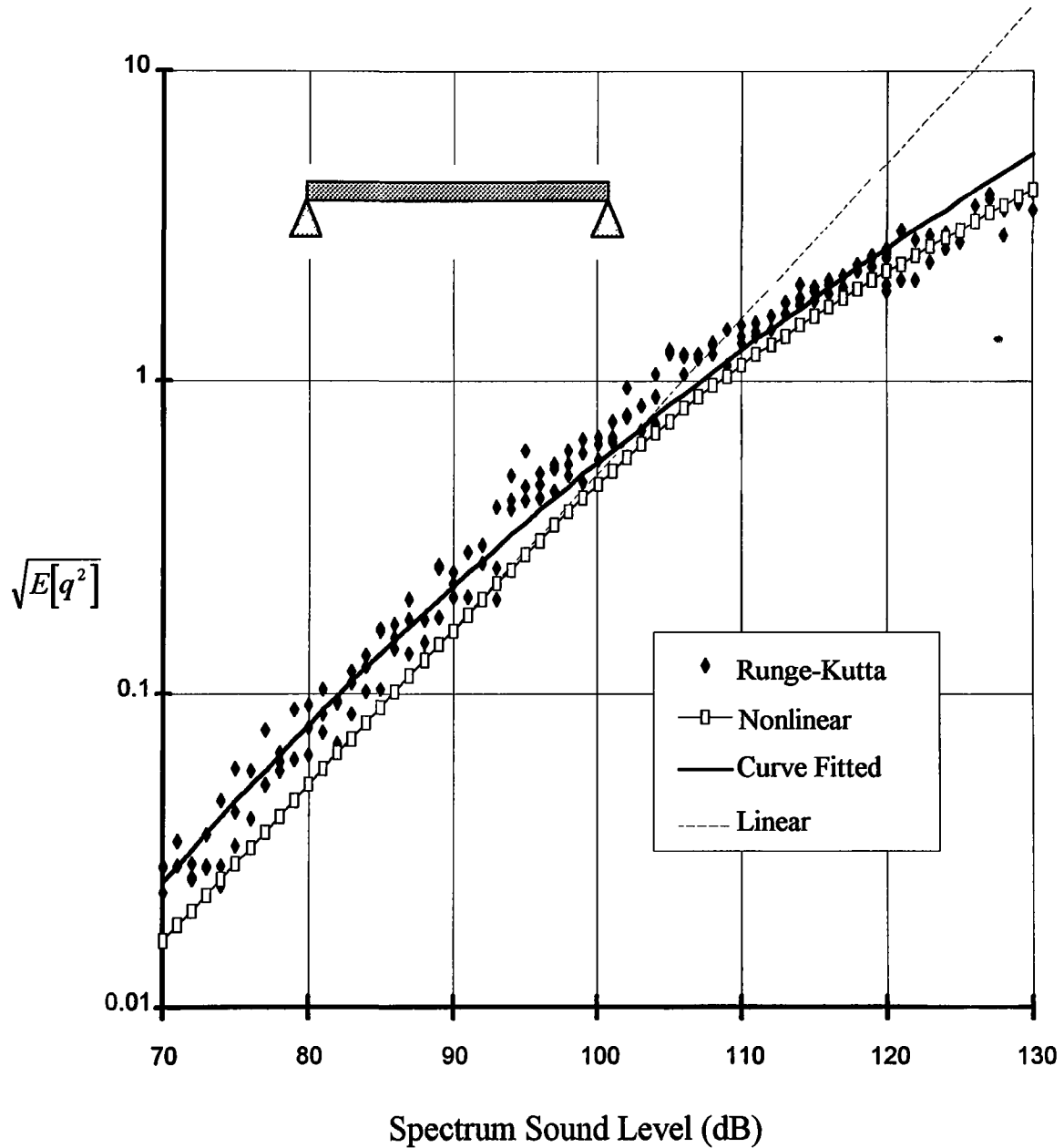


Figure 20. Comparison of $E[q^2]$ obtained by different methods for a 12x2x0.060 inch simply-supported cross-ply beam.

Random Response of a Clamped-Clamped Laminated Cross-Ply

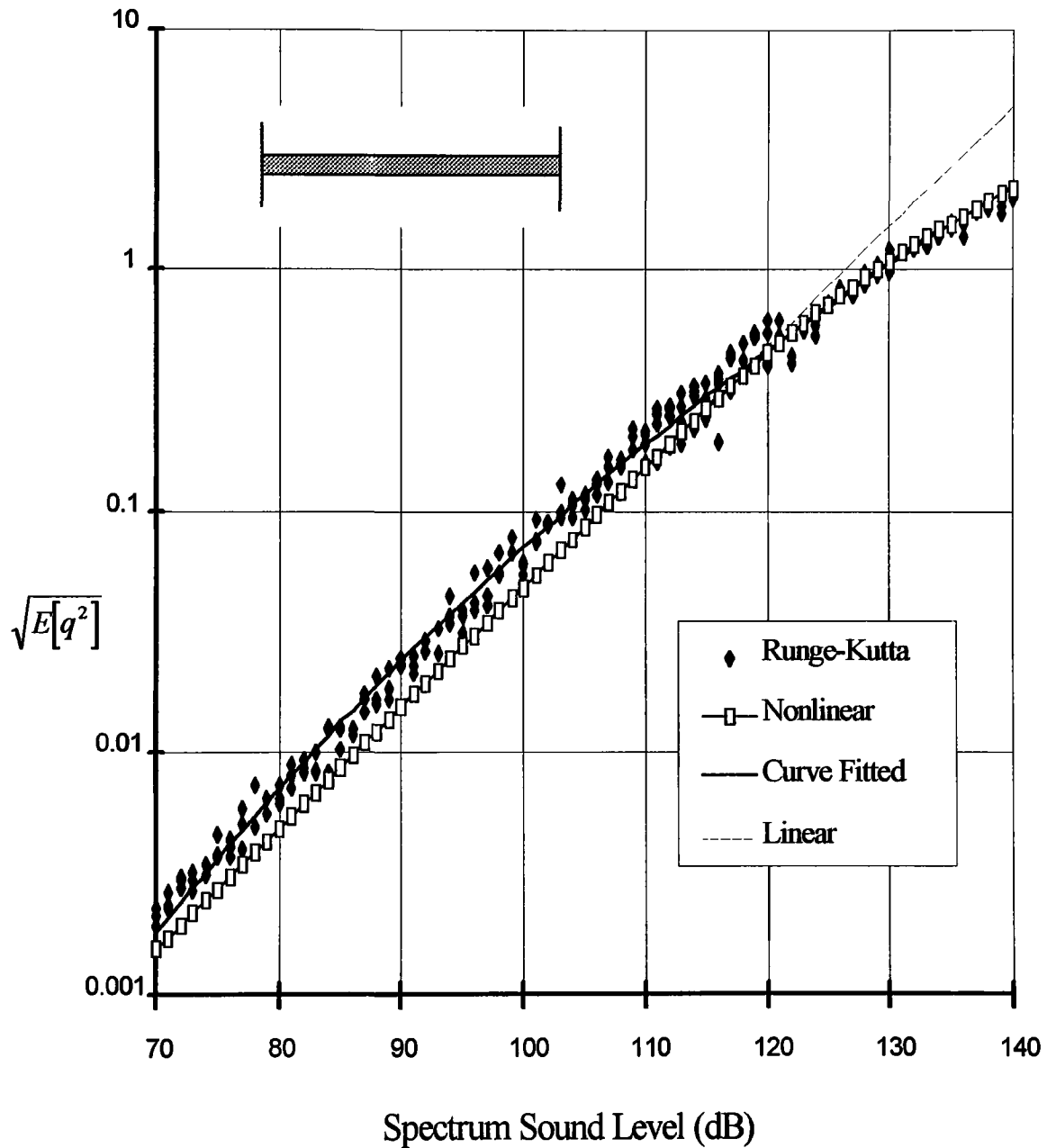


Figure 21. Comparison of $E[q^2]$ obtained by different methods for a 12x2x0.060 inch clamped-clamped cross-ply beam.

Random Response of a Cantilever Laminated Cross-Ply

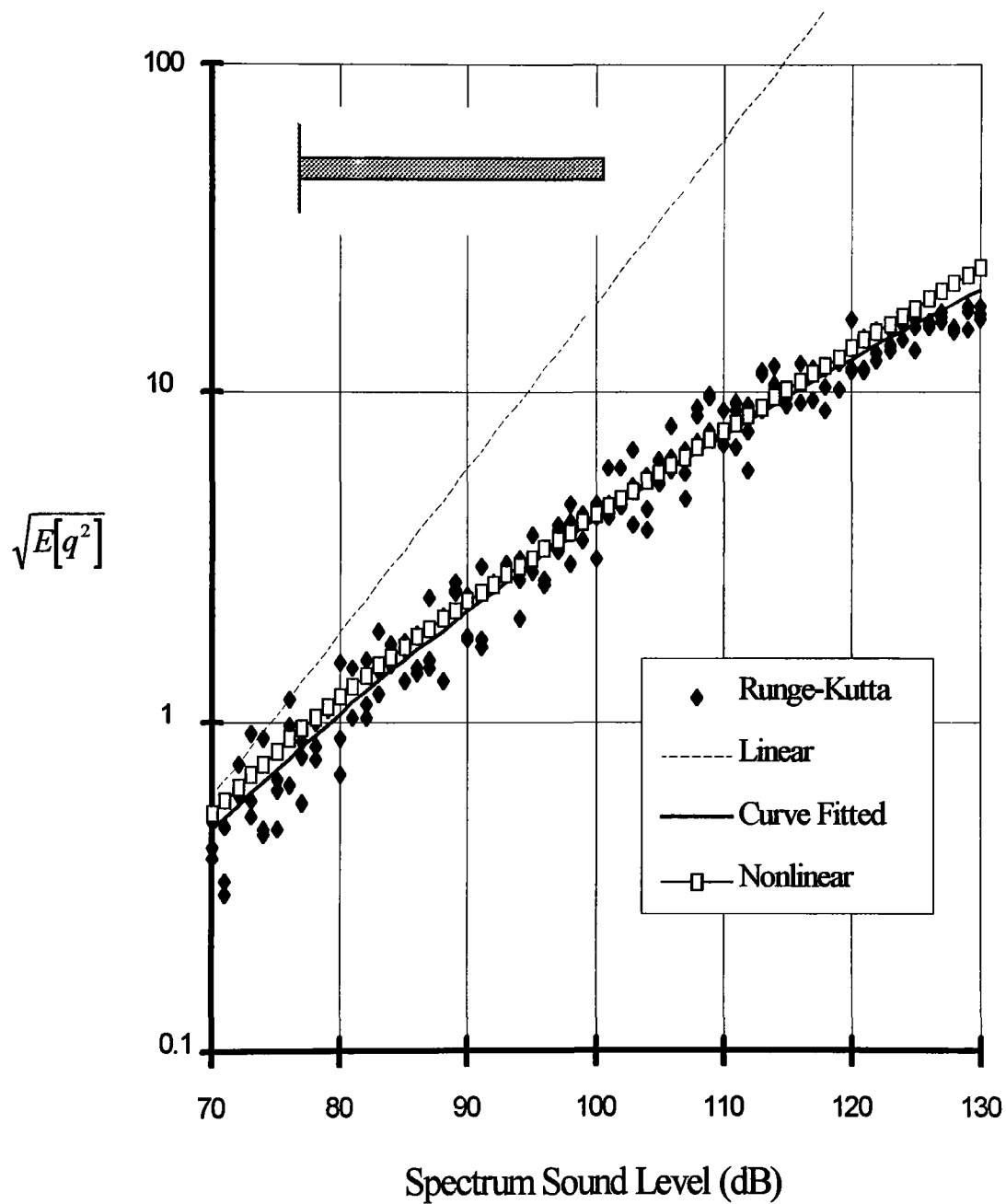


Figure 22. Comparison of $E[q^2]$ obtained by different methods for a 12x2x0.060 inch cantilever cross-ply beam.

5.4 Comparison of Numerical and Analytical Solutions.

It can be seen that the analytical solution of the Duffing equation (Equation (4.67)) does not contain the coefficient α . Thus, it is of importance to consider the effects that the coefficient α has on its solution. The application of the Runge-Kutta method enables such a comparison. Figure 23 shows two kinds of differences plotted versus SSL in dB. The first one is the average difference which is determined as follows. The average RMS at a given SSL obtained from the Runge-Kutta analysis is determined by writing

$$\left(RMS_{AVERAGE}\right)_{SSL} = \frac{1}{n_{SSL}} \sum_{i=1}^{n_{SSL}} \left(RMS_i\right)_{SSL} \quad (5.3)$$

where n_{SSL} is the number of datapoints retrieved from the Runge-Kutta analysis. The average difference at this SSL is therefore

$$\left(Diff_{AVERAGE}\right)_{SSL} = \left(RMS_{AVERAGE}\right)_{SSL} - \left(RMS_{ANALYTICAL}\right)_{SSL} \quad (5.4)$$

where $\left(RMS_{ANALYTICAL}\right)_{SSL}$ is the RMS obtained from Equation (4.67) at that particular SSL . The second difference is determined

by using a least-square curve-fit¹³ that goes through the entire data domain. This curve-fit difference is determined as follows

$$(Diff_{CURVE-FIT})_{SSL} = (RMS_{CURVE-FIT})_{SSL} - (RMS_{ANALYTICAL})_{SSL} \quad (5.5)$$

The effects of not having the coefficient α in Equation (4.67) are clearly significant from this analysis and are discussed in further detail in Chapter 6. •

¹³ The curve-fit equation is selected from a set of several equations; a polynomial, a hyperbolic, semilogarithmic, or logarithmic. The one that yielded the best correlation with the data domain from the Runge-Kutta analysis was selected. These equations appear in Appendix L.

Difference in Random Responses of a Simply-Supported Cross-Ply using Analytical and Numerical Methods

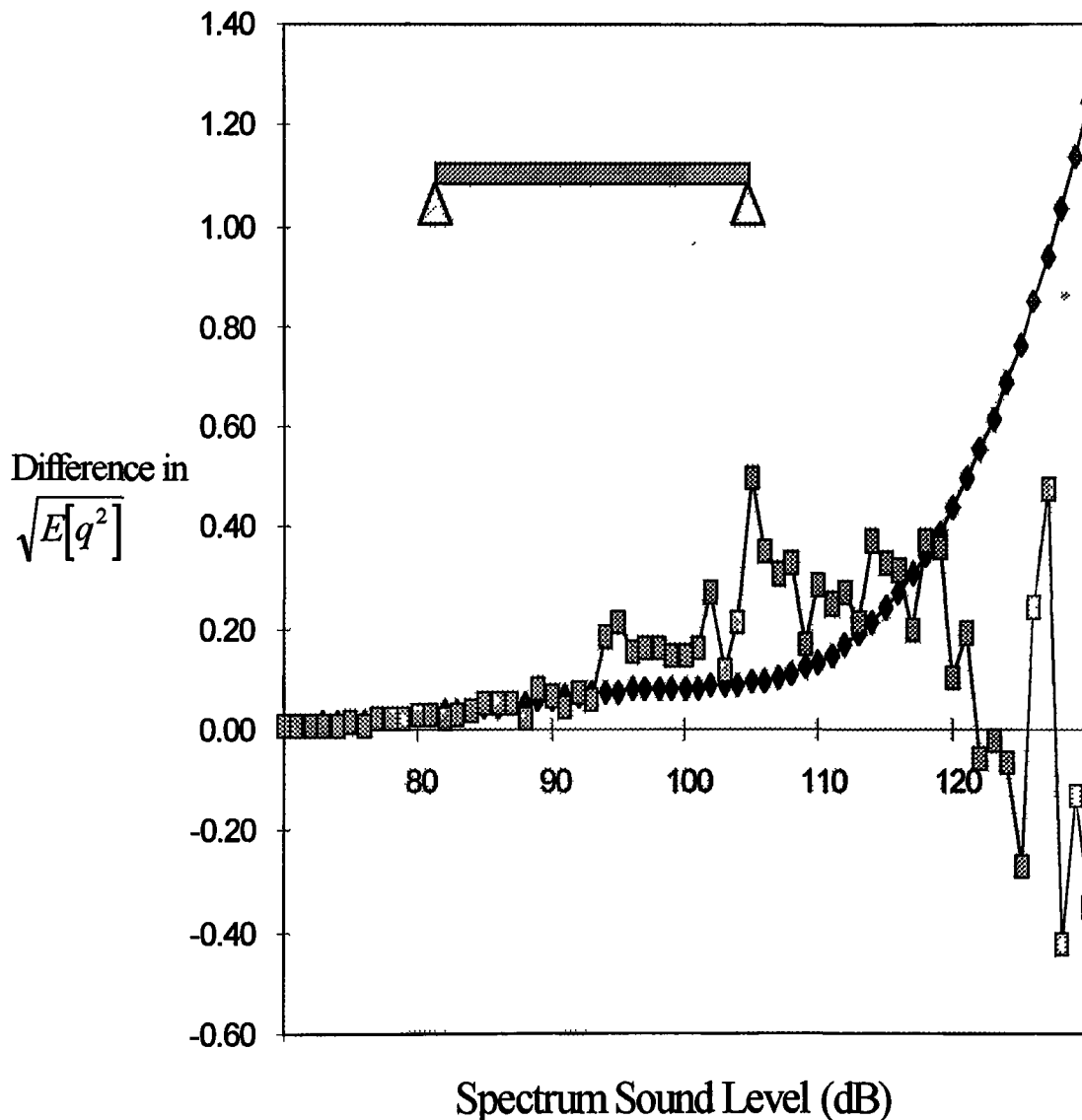


Figure 23. Difference between the random responses of a simply-supported 12x2x0.06 inch 4-layer cross-ply, using the analytical method (with $\alpha = 0$) and Runge-Kutta numerical scheme (with $\alpha \neq 0$).

5.5 Response Map for an Angle-Ply

Figure 24 shows a random response map for the simply-supported angle-ply of Section 5.2. It is generated by considering a range of angles, through which the lay-up sequence changes, on which a range of pressure spectral densities impinge. The advantage of the map is the immediate identification of the lay-up technique for the lowest response (or minimum fatigue).

Figure 25 shows a two dimensional representation of the response map at the point where $SSL = 100$ dB. It can also be thought of as a two dimensional plane cutting through the response map.

Response Map for a Simply-Supported Angle-Ply versus Lay-up Angles and *SSL*

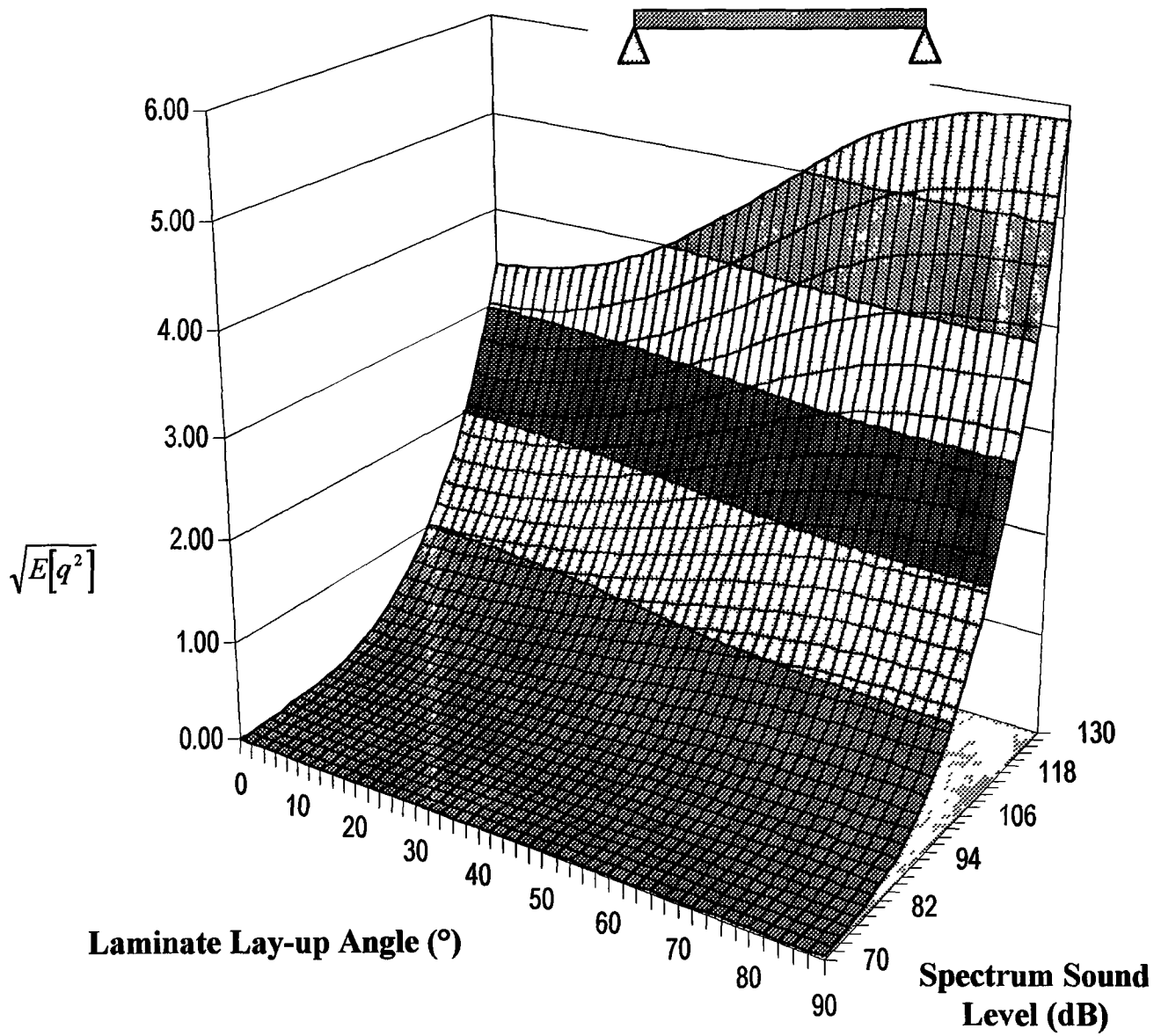


Figure 24. Random response map for a simply-supported 12x2x0.06 inch 4-layer angle-ply.

Random Response of a Simply- Supported Angle-Ply with respect to Angle Orientation at $SSL=100\text{dB}$

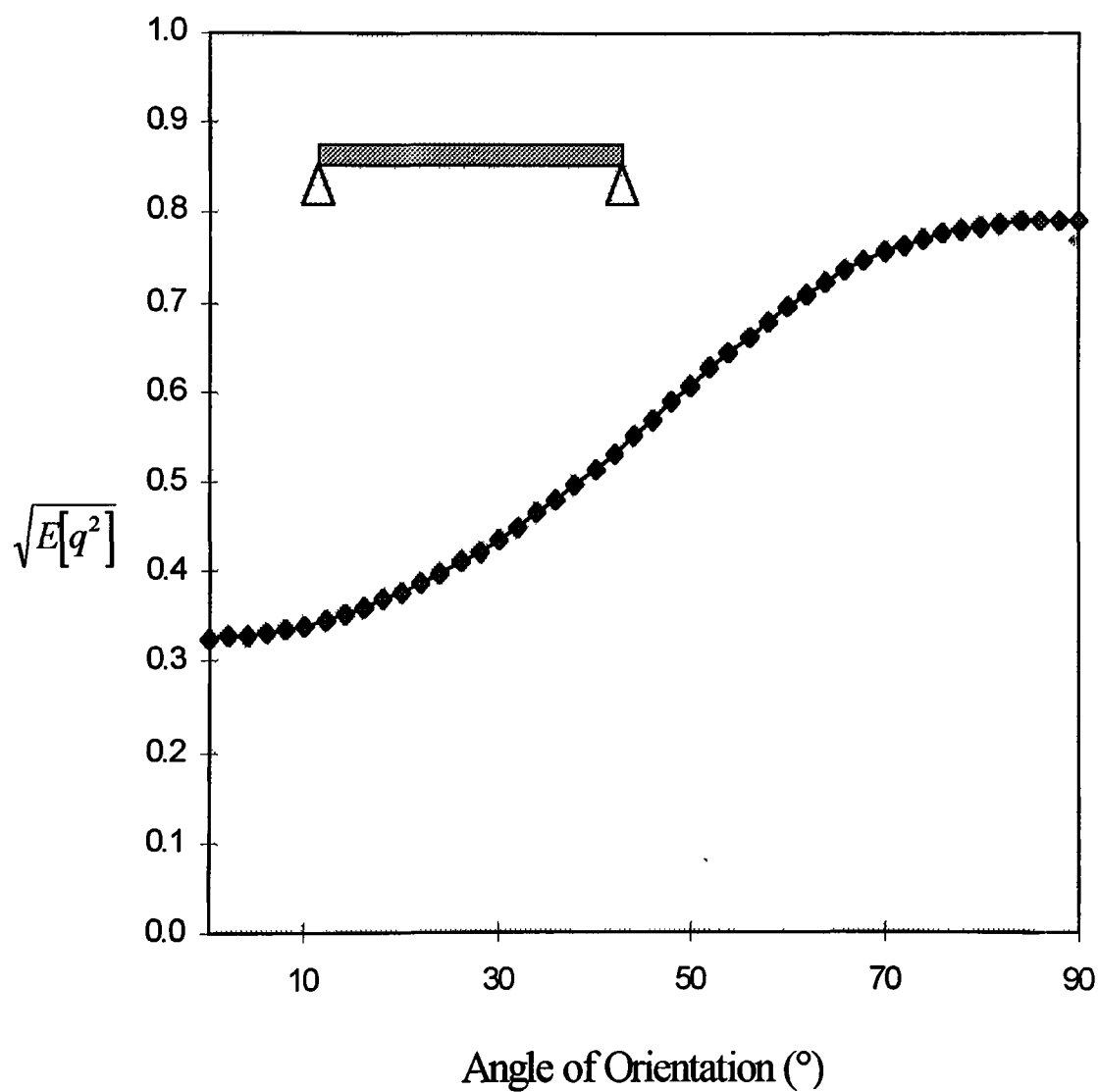


Figure 25. Random response behavior versus angle of orientation of laminate fibers for a simply-supported 12x2x0.06 inch 4-layer angle-ply.

CHAPTER 6

CONCLUSION

6.1 Concluding Remarks.

In this thesis it was shown how the theory of elasticity can be used in the analysis of randomly excited composite beams. The most important result is the achievement of a single generic equation yielding the mean-square deflection. This is a considerable simpler approach than Mei and Prasad's [44], but their method required a numerical iteration scheme, most practically implemented with a computer. Equation (4.67) in this thesis, on the other hand, accepts any kind of lamination lay-up, unsymmetrically or symmetrically laminated or even isotropic. Additionally, it was found that for an unsymmetrical angle-ply, an increase in angle of orientation results in an increase in random response (i.e. a smaller angle yields a stiffer beam - see Figure 21).

6.2 Accuracy and Limitations of the Results.

It must be kept in mind that the results of the nonlinear analysis of laminated beams are original, and consequently cannot be compared to existing research. However, previous research on isotropic materials (Seide [45] and Mei/Prasad [44]) is in good agreement with the results presented herein. That lends at least some support to the results for laminated beams.

One way of gaining further support is to compare two different approaches, as was implemented in the thesis. To make a reasonable comparison of the numerical solution of the Duffing equation to that of the analytical solution, one must make a reference to Figures 13 through 20. The similarity in trends and propinquity of values is very evident. Nevertheless, these are two different solutions, one incorporating a nonzero α (Runge-Kutta solution), the other neglecting the effects of α (analytical). The analytical solution, although presenting an interesting simplicity in use, suffers from the fact that a criteria including the coefficient α was not found. This results in a difference between the two solution techniques that implies that a correction function should be included with Equation (4.67). Thus, a more appropriate expression should be

$$E[q^2] = -\frac{\omega_0^2}{6\beta} + \sqrt{\left(\frac{\omega_0^2}{6\beta}\right)^2 + \frac{S}{3\beta\xi}} + C \quad (6.1)$$

where C is the correction function and would ideally be dependent upon both α and SSL . The dependency upon α could be implicit in that the coefficient α depends on the angle of orientation of the laminate. Thus, the correction could be formulated by considering the coordinate axes of the response map of Figure 21 and as such can be considered a three dimensional surface. To obtain the function is not a straight forward derivation, although quite possible. However, it requires a large number of data points from the Runge-Kutta analysis and a criteria that minimizes error, such as a least-squares analysis.

6.3 Recommendations.

An important area of future improvements is the search for a criteria that incorporates the coefficient α in Equation (4.67). The effects of a two mode solution is an area that should be studied as well, although, Prasad [44] showed limited effects of such a solution technique on the random response of an isotropic beam. Furthermore, thermal and hygrothermal effects call for some attention.

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Appendix A

Properties of Normal Processes in Random Analysis

In general, a normal (or a Gaussian) distribution is given with

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x-m)^2}{2\sigma^2}\right] \quad (\text{A.1})$$

where m is the mean of the distribution and s is the standard deviation of the distribution. If the mean is zero, i.e. $m = 0$, then equation (A.1) becomes

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{x^2}{2\sigma^2}\right] \quad (\text{A.2})$$

The general mathematical expectation of a value x to the power n is given by

$$E[x^n] = \int_{-\infty}^{\infty} x^n p(x) dx \quad (\text{A.3})$$

By inserting equation (A.2) into (A.3) for several values of n , and performing the integration, leads to the following relationships

$$E[x_1] = 0$$

$$E[x_1 x_2] = 0 \quad \text{if } x_1 \neq x_2$$

$$E[x_1 x_2 x_3] = 0$$

$$E[x_1 x_2 x_3 x_4] = E[x_1 x_2] E[x_3 x_4] + E[x_2 x_3] E[x_1 x_4] + E[x_1 x_3] E[x_2 x_4]$$

$$E[x_1 x_2 x_3 x_4 x_5] = 0$$

$$E[xx] = E[x^2] \neq 0$$

$$E[x^3] = \int_{-\infty}^{\infty} x^3 p(x) dx = 0$$

$$E[x^4] = \int_{-\infty}^{\infty} x^4 p(x) dx = 1 \cdot 3 \cdot (E[x_1^2])^2 = 3(E[x_1^2])^2$$

$$E[x^5] = \int_{-\infty}^{\infty} x^5 p(x) dx = 0$$

$$E[x^6] = \int_{-\infty}^{\infty} x^6 p(x) dx = 1 \cdot 3 \cdot 5 \cdot (E[x_1^2])^3 = 15(E[x_1^2])^3$$

$$E[x^7] = \int_{-\infty}^{\infty} x^7 p(x) dx = 0$$

$$E[x^8] = \int_{-\infty}^{\infty} x^8 p(x) dx = 1 \cdot 3 \cdot 5 \cdot 7 \cdot (E[x_1^2])^4 = 105(E[x_1^2])^4$$

$$E[x^9] = \int_{-\infty}^{\infty} x^9 p(x) dx = 0$$

$$E[x^{10}] = \int_{-\infty}^{\infty} x^{10} p(x) dx = 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot (E[x_1^2])^5 = 945(E[x_1^2])^5$$

In general

$$E[x^n] = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \int_{-\infty}^{\infty} x^n p(x) dx = 1 \cdot 3 \cdot 5 \cdot \dots \cdot (n-1) \cdot (E[x_1^2])^{n/2} & \text{if } n \text{ is even} \end{cases}$$

Appendix B

Derivation of the General Governing Equations of Motion for Unsymmetrically Laminated Plates

B.1 Equations of Motion in Terms of Stress Components

Consider the elemental cube of Figure B.1, which represents the k th layer of a laminate. A right-handed Cartesian coordinate system is used and is oriented such that the z -direction points downward. Then, normal and shear stresses in any of the three directions are analyzed as follows.

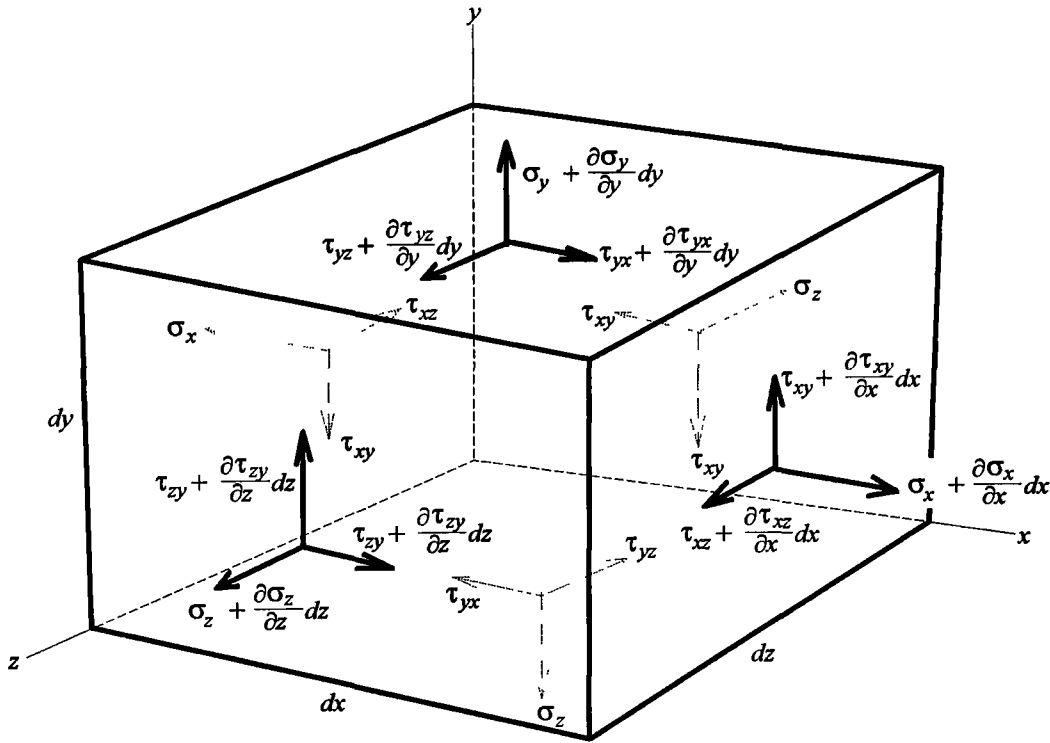


Figure B.1 The state of stress on a cube.

B.1.1 Stresses in the x -Direction:

According to Figure B.1, the stress components in the x -direction add up as follows

$$\rightarrow \sum F_x = ma_x = (\rho dV) \frac{\partial^2 u}{\partial t^2} = \rho dx dy dz \frac{\partial^2 u}{\partial t^2}$$

or

$$\left[-\sigma_x^k + \left(\sigma_x^k + \frac{\partial \sigma_x^k}{\partial x} dx \right) \right] dydz + \left[-\tau_{zx}^k + \left(\tau_{zx}^k + \frac{\partial \tau_{zx}^k}{\partial z} dz \right) \right] dxdy + \left[-\tau_{yx}^k + \left(\tau_{yx}^k + \frac{\partial \tau_{yx}^k}{\partial y} dy \right) \right] dxdz = \rho^k dxdydz \frac{\partial^2 u}{\partial t^2}$$

which reduces to

$$\frac{\partial \sigma_x^k}{\partial x} dxdydz + \frac{\partial \tau_{zx}^k}{\partial z} dxdydz + \frac{\partial \tau_{yx}^k}{\partial y} dxdydz = \rho^k dxdydz \frac{\partial^2 u}{\partial t^2} \quad (\text{B.1})$$

Then, dividing through Equation (B.1) by $dxdydz$ yields the equation of motion in terms of stresses in the x -direction

$$\frac{\partial \sigma_x^k}{\partial x} + \frac{\partial \tau_{yx}^k}{\partial y} + \frac{\partial \tau_{zx}^k}{\partial z} = \rho^k \frac{\partial^2 u}{\partial t^2} \quad (\text{B.2})$$

B.1.2 Stresses in the y -Direction:

Again, making a reference to Figure B.1, the stress components in the y -direction are added up similarly, yielding the equation of motion in terms of stresses in the y -direction

$$\frac{\partial \tau_{yx}^k}{\partial x} + \frac{\partial \sigma_y^k}{\partial y} + \frac{\partial \tau_{yz}^k}{\partial z} = \rho^k \frac{\partial^2 v}{\partial t^2} \quad (\text{B.3})$$

B.1.3 Stresses in the z -Direction:

The stresses in the z -direction is a little more involved. Figure B.2 depicts how only the stress components acting in the x -direction are considered. The resultant force acting in the z -direction due to σ_x alone is:

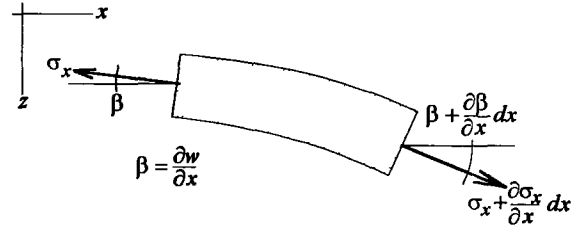


Figure B.2

$$\begin{aligned} (F_z)_{\sigma_x} &= \left[\sigma_x^k + \frac{\partial \sigma_x^k}{\partial x} dx \right] dydz \left[\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx \right] - \left(\sigma_x^k dydz \right) \frac{\partial w}{\partial x} \\ &= \left[\frac{\partial \sigma_x^k}{\partial x} \frac{\partial w}{\partial x} + \sigma_x^k \frac{\partial^2 w}{\partial x^2} + \frac{\partial \sigma_x^k}{\partial x} \frac{\partial^2 w}{\partial x^2} dx \right] dxdydz \end{aligned}$$

The term $\frac{\partial \sigma_x^k}{\partial x} \frac{\partial^2 w}{\partial x^2} dx$ can be neglected, since it is a higher order term. This leads to

$$(F_z)_{\sigma_x} = \frac{\partial}{\partial x} \left(\sigma_x^k \frac{\partial w}{\partial x} \right) dxdydz \quad (\text{B.4})$$

According to Figure B.3, the resultant force acting in the z -direction due to τ_{xz} alone is

$$\begin{aligned} (F_z)_{\tau_{xz}} &= \left[\tau_{xz}^k + \frac{\partial \tau_{xz}^k}{\partial z} dz \right] dx dy \left[\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x \partial z} dz \right] - \left(\tau_{xz}^k dx dy \right) \frac{\partial w}{\partial x} \\ &= \left[\frac{\partial \tau_{xz}^k}{\partial z} \frac{\partial w}{\partial x} + \tau_{xz}^k \frac{\partial^2 w}{\partial x \partial z} + \frac{\partial \tau_{xz}^k}{\partial z} \frac{\partial^2 w}{\partial x \partial z} dz \right] dx dy dz \end{aligned}$$

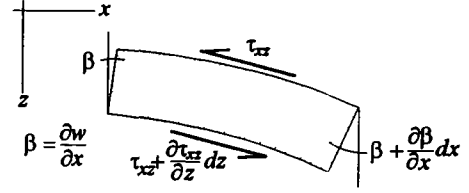


Figure B.3

Similarly, this leads to

$$(F_z)_{\tau_{xz}} = \frac{\partial}{\partial x} \left(\tau_{xz}^k \frac{\partial w}{\partial x} \right) dx dy dz \quad (\text{B.5})$$

The resultant force acting in the z -direction due to τ_{xy} alone is (see Figure B.4)

$$\begin{aligned} (F_z)_{\tau_{xy}} &= \left[\tau_{xy}^k + \frac{\partial \tau_{xy}^k}{\partial y} dy \right] dx dz \left[\frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial y \partial x} dy \right] - \left(\tau_{xy}^k dx dy \right) \frac{\partial w}{\partial x} \\ &= \left[\frac{\partial \tau_{xy}^k}{\partial y} \frac{\partial w}{\partial x} + \tau_{xy}^k \frac{\partial^2 w}{\partial y \partial x} + \frac{\partial \tau_{xy}^k}{\partial y} \frac{\partial^2 w}{\partial y \partial x} dy \right] dx dy dz \end{aligned}$$

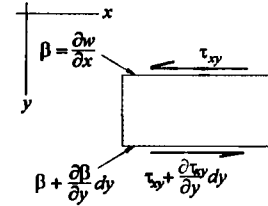


Figure B.4

which becomes

$$(F_z)_{\tau_{xy}} = \frac{\partial}{\partial y} \left(\tau_{xy}^k \frac{\partial w}{\partial x} \right) dx dy dz \quad (\text{B.6})$$

Considering stress components acting in the y -direction only. The resultant force acting in the z -direction due to σ_y alone is

$$(F_z)_{\sigma_y} = \frac{\partial}{\partial y} \left(\sigma_y^k \frac{\partial w}{\partial y} \right) dx dy dz \quad (\text{B.7})$$

$$(F_z)_{\tau_{yz}} = \frac{\partial}{\partial z} \left(\tau_{yz}^k \frac{\partial w}{\partial y} \right) dx dy dz \quad (\text{B.8})$$

$$(F_z)_{\tau_{xy}} = \frac{\partial}{\partial x} \left(\tau_{xy}^k \frac{\partial w}{\partial y} \right) dx dy dz \quad (\text{B.9})$$

Finally, consider stress components acting in the z -direction only. The resultant force acting in the z -direction due to σ_y , τ_{xz} , and τ_{yz} alone is

$$(F_z)_{\sigma_z, \tau_{xz}, \tau_{yz}} = \left(\frac{\partial \tau_{xz}^k}{\partial x} + \frac{\partial \tau_{yz}^k}{\partial y} + \frac{\partial \sigma_z^k}{\partial z} \right) dx dy dz \quad (\text{B.10})$$

The sum of all the forces must equal the mass times acceleration, or

$$\sum F_z = ma_z = m \frac{\partial^2 w}{\partial t^2}$$

Therefore, the addition of the results from equations (B.4) through (B.10) yields the *equation of motion in the z-direction*

$$\begin{aligned} \frac{\partial}{\partial x} \left(\tau_{xz}^k + \sigma_x^k \frac{\partial w}{\partial x} + \tau_{xy}^k \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y} \left(\tau_{yz}^k + \tau_{xy}^k \frac{\partial w}{\partial x} + \sigma_y^k \frac{\partial w}{\partial y} \right) + \\ \frac{\partial}{\partial z} \left(\sigma_y^k + \tau_{xz}^k \frac{\partial w}{\partial x} + \tau_{yz}^k \frac{\partial w}{\partial y} \right) = \rho^k \frac{\partial^2 w}{\partial t^2} \end{aligned} \quad (\text{B.11})$$

where u , v , w are displacements in the x , y , and z directions, respectively and $t = \text{time}$.

B.2 The Governing Equations of Motion

Equations (B.2), (B.3) and (B.11) are the equations of motions in terms of the stress components in the k^{th} layer lamina of a laminate. These equations can be expressed in terms of the resultant forces, transverse shear resultants, and resultant bending moments. Integrating equation (B.2) with respect to z over the thickness, h , of the laminate, yields

$$\int_{-h/2}^{h/2} \frac{\partial \sigma_x^k}{\partial x} dz + \int_{-h/2}^{h/2} \frac{\partial \tau_{xy}^k}{\partial y} dz + \int_{-h/2}^{h/2} \frac{\partial \tau_{xz}^k}{\partial z} dz = \int_{-h/2}^{h/2} \rho_0^k \frac{\partial^2 u}{\partial x^2} dz \quad (\text{B.12})$$

The definition of the resultant force matrix is given by

$$[N] = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} [\sigma]_k dz = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k dz \quad (\text{B.13})$$

Interchanging the order of differentiation and integration, and by using (B.13) yields a new expression of equation (B.12)

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = \rho \frac{\partial^2 u_0}{\partial t^2} \quad (\text{B.14})$$

where ρ stands for the average mass density of the laminate, and is defined by

$$\rho = \int_{-h/2}^{h/2} \rho_0^k dz \quad (\text{B.15})$$

Similarly, integrating equation (B.3) yields the following equation

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = \rho \frac{\partial^2 v_0}{\partial t^2} \quad (\text{B.16})$$

Finally, integrating equation (B.11) with respect to z yields

$$\begin{aligned}
N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + \\
\frac{\partial w}{\partial x} \left(\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} \right) + \frac{\partial w}{\partial y} \left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} \right) + P = \rho \frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{B.17}$$

where P is the distributed transverse loading, or in general, it is the sum of the external forces and is defined by

$$P = \sigma_z^k \Big|_{-h/2}^{h/2} \tag{B.18}$$

In most engineering applications of thin plates, the inplane load inertia effects $\frac{\partial^2 u_0}{\partial t^2}$, $\frac{\partial^2 v_0}{\partial t^2}$ can be neglected because the motion of the plates is predominantly transverse. Under this condition, equations (B.14), (B.16), and (B.17) become

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \tag{B.19}$$

$$\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \tag{B.20}$$

$$N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + P = \rho \frac{\partial^2 w}{\partial t^2} \tag{B.21}$$

By definition, the resultant moment can be written in a matrix form as follows

$$[M] = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} z dz \tag{B.22}$$

Then, multiply both sides of equation (B.2) by z and integrate the resulting equation with respect to z over the thickness of the plate, and use equation (B.22) to obtain

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} + \int_{-h/2}^{h/2} \frac{\partial \tau_{xz}^k}{\partial z} z dz = \int_{-h/2}^{h/2} \rho_0^k \frac{\partial^2 u}{\partial t^2} z dz \tag{B.23}$$

After some algebraic manipulations, equation (B.23) becomes

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \tag{B.24}$$

The integration on the right-hand side of equation (B.23) vanishes due to the fact that it leads to a rotary inertia term. Similar operations with equation (B.3) yields

$$\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \tag{B.25}$$

Differentiating equations (B.24) and (B.25) with respect to x and y respectively, yields

$$\frac{\partial Q_x}{\partial x} = \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_{xy}}{\partial x \partial y} \quad (\text{B.26})$$

$$\frac{\partial Q_y}{\partial y} = \frac{\partial^2 M_{xy}}{\partial x^2} + \frac{\partial^2 M_y}{\partial x \partial y} \quad (\text{B.27})$$

Finally, substituting these equations in equation (B.22) results in

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + P = \rho \frac{\partial^2 w}{\partial t^2} \quad (\text{B.28})$$

Equations (B.19), (B.20), and (B.28) represent the *governing the equations of motion* for laminated thin plates, and they have the same form of those of classical homogeneous, and isotropic plate theory. note that generally when these are applied to beams, the density, ρ , is in $\text{N sec}^2/\text{m}^4$ or $\text{lb}_f \text{sec}^2/\text{in}^4$, so equation (B.28) is rewritten as follows

$$\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2} + P = \rho h \frac{\partial^2 w}{\partial t^2} \quad (\text{B.29})$$

Appendix C

Derivation of the General Equation of Motion for an Unsymmetrically Laminated Beam

C.1 The Nonlinear Equation

The governing equations, which are given by (3.14), (3.15), and (3.16), yield the equations of motion for a beam. The first step in deriving the equation of motion is to determine selected terms of Equation (3.16) using Equations (3.14) and (3.15). This is done as follows

$$\begin{aligned} \text{From Equation (3.16)} \quad \frac{\partial N_x}{\partial x} = 0 \quad \Leftrightarrow \quad N_x = \text{Constant} \\ \text{From Equation (3.5)} \quad \frac{\partial N_x}{\partial x} = A_{11} \left[\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \right] + A_{16} \frac{\partial^2 v_0}{\partial x^2} - B_{11} \frac{\partial^3 w}{\partial x^3} \end{aligned} \quad (\text{C.1})$$

$$\text{From Equation (3.7)} \quad \frac{\partial N_{xy}}{\partial x} = A_{16} \left[\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \right] + A_{66} \frac{\partial^2 v_0}{\partial x^2} - B_{16} \frac{\partial^3 w}{\partial x^3} \quad (\text{C.2})$$

$$\text{From Equation (3.8)} \quad \frac{\partial M_x}{\partial x} = B_{11} \left[\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \right] + B_{16} \frac{\partial^2 v_0}{\partial x^2} - D_{11} \frac{\partial^3 w}{\partial x^3} \quad (\text{C.3})$$

$$\frac{\partial^2 M_x}{\partial x^2} = B_{11} \left[\frac{\partial^3 u_0}{\partial x^3} + \frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial x} + \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] + B_{16} \frac{\partial^3 v_0}{\partial x^3} - D_{11} \frac{\partial^4 w}{\partial x^4} \quad (\text{C.4})$$

The occurrence of the midplane displacements u_0 and v_0 in the above equations requires their determination. This is done in the following manner. Begin by rewriting Equations (3.14) and (3.15) using the computed values in Equations (C.1) and (C.2)

$$A_{11} \left[\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \right] + A_{16} \frac{\partial^2 v_0}{\partial x^2} - B_{11} \frac{\partial^3 w}{\partial x^3} = 0$$

$$A_{16} \left[\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \right] + A_{66} \frac{\partial^2 v_0}{\partial x^2} - B_{16} \frac{\partial^3 w}{\partial x^3} = 0$$

Rearranging the above equations yields

$$A_{11} \frac{\partial^2 u_0}{\partial x^2} + A_{16} \frac{\partial^2 v_0}{\partial x^2} = B_{11} \frac{\partial^3 w}{\partial x^3} - A_{11} \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \quad (\text{C.5})$$

$$A_{16} \frac{\partial^2 u_0}{\partial x^2} + A_{66} \frac{\partial^2 v_0}{\partial x^2} = B_{16} \frac{\partial^3 w}{\partial x^3} - A_{16} \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \quad (\text{C.6})$$

The above equations represent two linearly independent equations. Note that when solving two equations of the form

$$Ax + By = C$$

$$Dx + Ey = F$$

the solution is given by

$$x = \frac{BF - CE}{BD - AE} \quad \text{and} \quad y = \frac{CD - AF}{BD - AE}$$

When this analogy (i.e. x is analogous to $\frac{\partial^2 u_0}{\partial x^2}$ and B is analogous to A_{16} , etc) is used for equations (C.5) and (C.6), the result becomes

$$\frac{\partial^2 u_0}{\partial x^2} = \frac{A_{16} \left(B_{16} \frac{\partial^3 w}{\partial x^3} - A_{16} \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \right) - A_{66} \left(B_{11} \frac{\partial^3 w}{\partial x^3} - A_{11} \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \right)}{A_{16} A_{16} - A_{11} A_{66}} = \left(\frac{A_{16} B_{16} - A_{66} B_{11}}{A_{16}^2 - A_{11} A_{66}} \right) \frac{\partial^3 w}{\partial x^3} - \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x}$$

and

$$\frac{\partial^2 v_0}{\partial x^2} = \frac{A_{16} \left(B_{11} \frac{\partial^3 w}{\partial x^3} - A_{11} \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \right) - A_{11} \left(B_{16} \frac{\partial^3 w}{\partial x^3} - A_{16} \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \right)}{A_{16} A_{16} - A_{11} A_{66}} = \left(\frac{A_{16} B_{11} - A_{11} B_{16}}{A_{16}^2 - A_{11} A_{66}} \right) \frac{\partial^3 w}{\partial x^3}$$

Rewriting this in a more compact form

$$\frac{\partial^2 u_0}{\partial x^2} = K_1 \frac{\partial^3 w}{\partial x^3} - \frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \quad (\text{C.7})$$

$$\frac{\partial^2 v_0}{\partial x^2} = K_2 \frac{\partial^3 w}{\partial x^3} \quad (\text{C.8})$$

where

$$K_1 = \left(\frac{A_{16} B_{16} - A_{66} B_{11}}{A_{16}^2 - A_{11} A_{66}} \right) \quad (\text{C.9})$$

and

$$K_2 = \left(\frac{A_{16} B_{11} - A_{11} B_{16}}{A_{16}^2 - A_{11} A_{66}} \right) \quad (\text{C.10})$$

The stress resultant, N_x , is manipulated in a special way before it is used in Equation (3.16). This is done as follows. First integrate Equation (C.5) over the length, L , of the beam

$$\int_0^L N_x dx = A_{11} \int_0^L \frac{\partial u_0}{\partial x} dx + \frac{A_{11}}{2} \int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx + A_{16} \int_0^L \frac{\partial v_0}{\partial x} dx - B_{11} \int_0^L \frac{\partial^2 w}{\partial x^2} dx$$

which becomes

$$N_x = \frac{A_{11}}{L} [u_0(L) - u_0(0)] + \frac{A_{16}}{L} [v_0(L) - v_0(0)] + \frac{A_{11}}{2L} \int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx - \frac{B_{11}}{L} \int_0^L \frac{\partial^2 w}{\partial x^2} dx \quad (C.11)$$

The state of the midplane, which is represented by the terms $[u_0(L) - u_0(0)] = 0$ and $[v_0(L) - v_0(0)] = 0$, depends on the type of beam the equation of motion is being solved for. Therefore, the above equation will be left as this for now. Equations (C.7) and (C.8) must be differentiated before they can be inserted into Equation (C.4), which in turn, is used in Equation (3.16)

$$\begin{aligned} \frac{\partial^3 u_0}{\partial x^3} &= K_1 \frac{\partial^4 w}{\partial x^4} - \left(\frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial x} + \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right) \\ &= K_1 \frac{\partial^4 w}{\partial x^4} - \frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial x} - \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \end{aligned} \quad (C.12)$$

$$\frac{\partial^3 v_0}{\partial x^3} = K_2 \frac{\partial^4 w}{\partial x^4} \quad (C.13)$$

The equation of motion was given by

$$\frac{\partial^2 M_x}{\partial x^2} + N_x \frac{\partial^2 w}{\partial x^2} + P = \rho h \frac{\partial^2 w}{\partial t^2} \quad (3.16)$$

Using the previously derived expressions, the equation of motion can now be rewritten as follows

$$\begin{aligned} B_{11} \left[\frac{\partial^3 u_0}{\partial x^3} + \frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial x} + \left(\frac{\partial^2 w}{\partial x^2} \right)^2 \right] + B_{16} \frac{\partial^3 v_0}{\partial x^3} - D_{11} \frac{\partial^4 w}{\partial x^4} + \\ \left(\frac{A_{11}}{L} [u_0(L) - u_0(0)] + \frac{A_{16}}{L} [v_0(L) - v_0(0)] + \frac{A_{11}}{2L} \int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx - \frac{B_{11}}{L} \int_0^L \frac{\partial^2 w}{\partial x^2} dx \right) \frac{\partial^2 w}{\partial x^2} + P = \rho h \frac{\partial^2 w}{\partial t^2} \end{aligned}$$

Rearranging yields

$$\begin{aligned} B_{11} \frac{\partial^3 u_0}{\partial x^3} + B_{16} \frac{\partial^3 v_0}{\partial x^3} + B_{11} \frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial x} + B_{11} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 - D_{11} \frac{\partial^4 w}{\partial x^4} + \\ \left(\frac{A_{11}}{L} [u_0(L) - u_0(0)] + \frac{A_{16}}{L} [v_0(L) - v_0(0)] + \frac{A_{11}}{2L} \int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx - \frac{B_{11}}{L} \int_0^L \frac{\partial^2 w}{\partial x^2} dx \right) \frac{\partial^2 w}{\partial x^2} + P = \rho h \frac{\partial^2 w}{\partial t^2} \end{aligned}$$

Inserting equations (C.7) and (C.8) into the above results and expanding, results in

$$\begin{aligned}
& B_{11}K_1 \frac{\partial^4 w}{\partial x^4} - B_{11} \frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial x} - B_{11} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + B_{16}K_2 \frac{\partial^4 w}{\partial x^4} + B_{11} \frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial x} + B_{11} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 - D_{11} \frac{\partial^4 w}{\partial x^4} + \\
& \left(\frac{A_{11}}{L} [u_0(L) - u_0(0)] + \frac{A_{16}}{L} [v_0(L) - v_0(0)] + \frac{A_{11}}{2L} \int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx - \frac{B_{11}}{L} \int_0^L \frac{\partial^2 w}{\partial x^2} dx \right) \frac{\partial^2 w}{\partial x^2} + P = \rho h \frac{\partial^2 w}{\partial t^2}
\end{aligned}$$

which simplifies to the general equation of motion for any beam in terms of deflection

$$\begin{aligned}
& (B_{11}K_1 + B_{16}K_2 - D_{11}) \frac{\partial^4 w}{\partial x^4} + \\
& \left(\frac{A_{11}}{L} [u_0(L) - u_0(0)] + \frac{A_{16}}{L} [v_0(L) - v_0(0)] + \frac{A_{11}}{2L} \int_0^L \left(\frac{\partial w}{\partial x} \right)^2 dx - \frac{B_{11}}{L} \int_0^L \frac{\partial^2 w}{\partial x^2} dx \right) \frac{\partial^2 w}{\partial x^2} + P - \rho h \frac{\partial^2 w}{\partial t^2} = 0
\end{aligned} \tag{C.14}$$

C.2 The Linear Solution

Some steps leading to the linear equation of motion are presented here only for the purpose of comparison to the nonlinear case. As a start Equation (2.2) becomes

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{11} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ 0 \\ \frac{\partial v_0}{\partial x} \\ \frac{\partial^2 w}{\partial x^2} \\ 0 \\ 0 \end{Bmatrix} \tag{C.15}$$

The expressions analogous to Equations (C.7) and (C.8) are

$$\frac{\partial^2 u_0}{\partial x^2} = K_1 \frac{\partial^3 w}{\partial x^3} \tag{C.16}$$

$$\frac{\partial^2 v_0}{\partial x^2} = K_2 \frac{\partial^3 w}{\partial x^3} \tag{C.17}$$

where K_1 and K_2 are given also by Equations (C.9) and (C.10). After applying similar process as before, the linear equation of motion for a simply supported beam becomes

$$(B_{11}K_1 + B_{16}K_2 - D_{11}) \frac{\partial^4 w}{\partial x^4} + \left(\frac{A_{11}}{L} [u_0(L) - u_0(0)] + \frac{A_{16}}{L} [v_0(L) - v_0(0)] - \frac{B_{11}}{L} \int_0^L \frac{\partial^2 w}{\partial x^2} dx \right) \frac{\partial^2 w}{\partial x^2} + P = \rho h \frac{\partial^2 w}{\partial t^2} \tag{C.18}$$

Appendix D

A Unit Consistency Test

The Duffing equations for the simply-supported, clamped-clamped, and cantilever cases are given by Equations (4.27), (4.36), and (4.56), respectively. In order to verify the correctness of these equations, a unit test can be applied, which shows if the equations are correctly dimensionalized. Here, the simply-supported case is used for demonstration purposes. The Duffing equation for this case is given by

$$\ddot{q} + 2\xi\omega_0\dot{q} + \omega_0^2q + \alpha q^2 + \beta q^3 = \frac{F(t)}{m} \quad (4.27)$$

where the corresponding coefficients are given by

$$\omega_0^2 = (D_{11} - B_{11}K_1 - B_{16}K_2) \frac{\pi^4}{\rho h L^4} \quad (4.21)$$

$$\alpha = 2B_{11} \frac{\pi^3 R}{\rho h L^4} \quad (4.22)$$

$$\beta = A_{11} \frac{\pi^4 R^2}{4\rho h L^4} \quad (4.23)$$

$$F(t) = \int_0^L P(x,t) \sin \frac{\pi x}{L} dx \quad (4.24)$$

$$m = \frac{\rho h R L}{2} \quad (4.25)$$

The unit test is applied as follows

$$\omega_0^2 = (D_{11} - B_{11}K_1 - B_{16}K_2) \frac{\pi^4}{\rho h L^4} = \left(\text{kg} \cdot \text{m} / \text{s}^2 \right) \frac{1}{\left(\text{kg} / \text{m}^4 \right) (\text{m}) (\text{m}^4)} = \frac{1}{\text{s}^2}$$

$$\alpha = 2B_{11} \frac{\pi^3 R}{\rho h L^4} = \left(\text{kg} / \text{s}^2 \right) \frac{(\text{m})}{\left(\text{kg} / \text{m}^4 \right) (\text{m}) (\text{m}^4)} = \frac{1}{\text{s}^2}$$

$$\beta = A_{11} \frac{\pi^4 R^2}{4\rho h L^4} = \left(\text{kg} / \text{m} \cdot \text{s}^2 \right) \frac{(\text{m}^2)}{\left(\text{kg} / \text{m}^4 \right) (\text{m}) (\text{m}^4)} = \frac{1}{\text{s}^2}$$

$$\frac{f(t)}{m} = \frac{2}{\rho h R L} \int_0^L P(x,t) \sin \frac{\pi x}{L} dx = \frac{1}{\left(\text{kg} / \text{m}^4 \right) (\text{m}) (\text{m}) (\text{m})} \left(\frac{\text{kg}}{\text{m} \cdot \text{s}^2} \right) = \frac{1}{\text{s}^2}$$

Appendix E

Minimization of the Error of Linearization

As stated in Chapter 4, page 76, it is seen that the linearized frequency, Ω , introduces the nonlinear effects to the response. In order to minimize the error that arises from the difference between the linearized Equation (4.57) and the original Equation (4.27), the conditions for the least error must be determined. A necessary condition that includes the effects of both α and β and minimizes the error has not been determined yet. Generally, the error is minimized with respect to Ω^2 (note that minimizing the error with respect to $\omega_0^2 - \Omega^2$ yields the same condition), or

$$\frac{\partial}{\partial \Omega^2} E[err^2] = 0 \quad (E.1)$$

Inserting Equation (4.58) into (E.1) results in

$$\frac{\partial}{\partial \Omega^2} E[err^2] = \frac{\partial}{\partial \Omega^2} E\left[\left(\left(\omega_0^2 - \Omega^2\right)q + \alpha q^2 + \beta q^3\right)^2\right]$$

which expanded gives

$$\begin{aligned} \frac{\partial}{\partial \Omega^2} E[err^2] &= \frac{\partial}{\partial \Omega^2} E\left[q^2 \omega_0^4 - 2q^2 \omega_0^2 \Omega^2 + 2q^3 \omega_0^2 \alpha + 2q^4 \omega_0^2 \beta + q^2 \Omega^4 - 2q^3 \Omega^2 \alpha - 2q^4 \Omega^2 \beta + \alpha^2 q^4 + 2\alpha\beta q^5 + \beta^2 q^6\right] \\ &= \frac{\partial}{\partial \Omega^2} \left\{ E[q^2] \omega_0^4 - 2E[q^2] \omega_0^2 \Omega^2 + 2E[q^3] \omega_0^2 \alpha + 2E[q^4] \omega_0^2 \beta + E[q^2] \Omega^4 \right. \\ &\quad \left. - 2E[q^3] \Omega^2 \alpha - 2E[q^4] \Omega^2 \beta + \alpha^2 E[q^4] + 2\alpha\beta E[q^5] + \beta^2 E[q^6] \right\} \end{aligned}$$

which becomes

$$\begin{aligned} \frac{\partial}{\partial \Omega^2} E[err^2] &= -4E[q^2] \omega_0^2 \Omega + 4E[q^2] \Omega^3 - 4E[q^3] \Omega \alpha - 4E[q^4] \Omega \beta \\ &= -E[q^2] \omega_0^2 + E[q^2] \Omega^2 - E[q^3] \alpha - E[q^4] \beta \\ &= E[q^2] \omega_0^2 - E[q^2] \Omega^2 + E[q^3] \alpha + E[q^4] \beta = 0 \end{aligned}$$

Consequently, there is a relationship between the equivalent linear frequency and the mean-square displacement as follows, were the properties of a Gaussian distribution are employed (see appendix A)

$$\Omega^2 E[q^2] = \omega_0^2 E[q^2] + \beta \left\{ 3 \left(E[q^2] \right)^2 \right\}$$

or

$$\Omega^2 = \omega_0^2 + 3\beta E[q^2] \quad (E.2)$$

Consequently, replacing the pertinent terms in Equation (4.61) results in

$$E[q^2] \cong \frac{S}{\xi\Omega^2} = \frac{S}{\xi\{\omega_0^2 + 3\beta E[q^2]\}} \quad (\text{E.3})$$

Manipulating this yields the below quadratic equation

$$3\beta(E[q^2])^2 + \omega_0^2 E[q^2] - \frac{S}{\xi} = 0 \quad (\text{E.4})$$

whose solution is

$$E[q^2] = \frac{-\omega_0^2 \pm \sqrt{\omega_0^4 - 4(3\beta)\left(-\frac{S}{\xi}\right)}}{2(3\beta)} = -\frac{\omega_0^2}{6\beta} + \sqrt{\left(\frac{\omega_0^2}{6\beta}\right)^2 + \frac{12}{36\beta}\left(\frac{S}{\xi}\right)}$$

or

$$E[q^2] = -\frac{\omega_0^2}{6\beta} + \sqrt{\left(\frac{\omega_0^2}{6\beta}\right)^2 + \frac{S}{3\beta\xi}} \quad (\text{E.5})$$