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A METHODOLOGY FOR THE RANK ORDERING OF ALTERNATIVES USING A BAYESIAN DECISION MODEL, WITH APPLICATION TO THE SPACE PROGRAM

By

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Summary

The primary objective of this paper is to provide a reasonably general and essentially unified approach to those problems involving "value" judgments and "subjective" decision making, without regard to excessive rigor. The principle areas and methods of attack developed are:

(1) The selection of a "value" measure which emphasizes the fact that the criterion of optimum performance is quite arbitrary, its merits reflecting only the constraints on the problem and the objectives sought,

(2) The utilization of statistical decision theory as a basis for the solution and subsequent evaluation of a class of problems in which a priori "value" judgments must be assigned by an individual or committee under uncertainty, and

(3) The application of the methodology to those areas in which the relative uncertainty level of a decision need be assessed in terms of a cost or penalty incurred in reaching the conclusion. A particularly important application is the selection of alternatives (i.e., projects by corporate executives) and the subsequent sensitivity analysis of the decision.

Selection of A Value Measure

Utility in The Classical Sense

The notion of utility, as defined in the classical sense, refers to the capacity of a certain amount of money or goods for satisfying the needs of individuals. Other terms used for this same concept are "moral gain" (Laplace)[1] and subjective value". In general, the following law holds:

"Law of Diminishing Marginal Utility" - If a certain gain (a certain amount of goods or money) is added to an initial fortune f_o then the utility of this gain is the smaller, the higher f_o.

This is, of course, an empirical law concerning the reactions of human beings, hence a law of psychology; but it is of primary importance for the application of value theory in determining practical decisions. This law was first pronounced by Daniel Bernoulli[2] and is well known in economics. As an illustration, consider the following hypothetical individual designated as "X".

The aim of "X" in all his actions is the satisfaction of his needs and the avoidance of suffering, which we may regard as negative satisfaction. Gains in money or goods are appreciated as means of obtaining satisfaction. Thus, what counts is their utility. Therefore, "X's" decisions must be guided by the principle of maximizing the utility of his gains rather than the gains themselves. Since, however, he cannot foresee future events, gains, and utilities with certainty, but only with probability, he must apply the maximizing principle to the estimate of utility rather than to the unknown utility itself. This, however, presupposes that certain problems are solved which involve serious difficulties. First, utility must be measurable, i.e., quantified, and further, a law must be known defining, as well as determining the utility of gains.

The first problem is to find a method of measuring the (positive or negative) utility of a gain (or a loss as a negative gain) for a certain person at a certain time. The (positive or negative) gain may consist in the acquisition (or loss) of money, goods, or other advantages. In other words, a quantitative value must be found for the otherwise inexact concept of utility, which is perhaps not quantitative, but merely comparative. Thus, the basic problem consists in measuring the utility of money. If this is possible, then it might be possible to measure the utility of other goods and advantages (or disadvantages) by establishing utility equivalences between them and amounts of money. This seems possible at least for those goods which can be exchanged, bought, and sold. But it might not be impossible even for the so-called imponderables; for example, a disease or the recovery from it, the positive or negative prestige gained by composing a good or a bad symphony, the gaining or losing of the love of a woman. However, it may be possible, at least "theoretically", to determine the utility of events of this kind for "X" by determining his preferential reactions. Even if neither "X" nor the medical authorities accessible to him know how to cure a certain disease, nevertheless, he can imagine a fairy confronting him with the alternative of either curing the disease or giving him a certain amount of money. Although the situation is imaginary, "X" can ask himself what he would prefer, and his answer measures his actual valuation. There are amounts of money which he will value less than the cure, and perhaps others which he will value more. There will also be intermediate amounts with respect to which he has no clear preference either way, and which will thus represent a money equivalent for the utility of the advantage or disadvantage in question.

It must be admitted that there are some serious problems involved in this assumption of the possibility of measuring the utility of all advantages and disadvantages for a given person at a given time on the basis of one common, one-dimensional scale. But something like this assumption is usually taken as a basis of an analysis of what is called "rational behavior" in many parts of social science, especially in economics and ethics, and it is indeed hard to see how such an analysis could be made without this assumption. For the present purpose, one need not enter into a critical examination of these assumptions. That belongs to the task of the methodology of the fields mentioned. What is presupposed here is the more general logical assumptions underlying an analysis of rational behavior, expressed in terms of a value measure.

Utility in The von Neumann-Morgenstern Sense

John von Neumann and Oskar Morgenstern[3] discuss the problem of a "quantitative" concept of utility and construct an axiomatic system for it. Against those economists who propose to use the concept of utility merely in a comparative form (e.g., in the method of indifference curves introducted by Pareto), they advance the following argument. Let us assume that the system of preferences of the person "X" is complete not only with respect to alternative events, which when chosen, occur with certainty but also with respect to uncertain events with given numerical probabilities. This means that "X" is able to say, for example, which of the following two alternative events he prefers or whether they are equally desirable to him: 1) he receives $1.00 in cash, or 2) he receives a lottery ticket which represents a chance of obtaining $100.00 with the probability 0.01. The authors show that this complete system of the preferences of "X" determines a quantitative concept of utility for "X" in all its essential features, leaving open only the choice of a zero point and a unit of the utility scale. The resulting numerical utility is "that thing for which the calculus of mathematical expectations is legitimate."

Many investigations by economists concerning decisions made by a person "X" (including the discussion of utility by von Neumann and Morgenstern just mentioned) are restricted to cases in which "X" knows the values of probability for certain events, especially for anticipated consequences of possible actions. The term "probability" is understood in these investigations in the sense of relative frequency.

Davidson-Siegel-Suppes Measure of Utility

Recent efforts during the last decade in the development of a "value" measure have received their principal stimulation from the theoretical work of von Neumann and Morgenstern, coupled with the experimentalism of E. A. Singer, Jr.[4] One of the most sophisticated procedures is that developed by Davidson, Siegel, and Suppes[5]. In a study designed to measure the utility of money in the sense of an interval scale, an event was constructed, which for most people had a subjective probability of one-half. The event was produced by means of dice containing nonsense syllables, and was utilized in a gambling situation in which subjects were competing for each other's "dollar" fortune.

The essential device that defined operationally how the individual's choices determined an ordered metric scaling was a one-person game in which the subject chose between two alternatives, each of which was a probability combination of two outcomes. Obviously, in this situation it is assumed that the subject makes a choice to maximize his expected utility. Furthermore, both the von Neumann and Davidson procedures assume that no value is placed upon the act of gambling itself. However, evidence to the contrary has been found by Royden, Suppes, and Walsh[6], who then added this consideration to their model. In trying the "enriched" model on a group of sailors and college students, it led to better predictions for the former, but worse for the latter.

Churchman-Ackoff[7] Approximate Measure of Value

In many cases the decision maker is required to express a preference between paired sets of outcomes, which are not assumed to be mutually exclusive and exhaustive. For example, consider a situation in which an individual desires black coffee and sugar. If obtaining coffee and sugar are treated as outcomes, they are not exclusive. Furthermore, the Churchman-Ackoff approximate measure of value is not based on the concept of a "standard gamble", and hence makes no assumptions concerning "subjective" probability, or the maximization of an expected value. As pointed out by these authors the technique resembles a procedure developed in chemistry for estimating values of a property of each of a set of objects, where only comparative evaluations are possible. The underlying assumptions are given in a paper by Churchman and Ackoff[7], and an illustrative example presented involving four outcomes. This procedure is repeated here for convenience:

"(1) Have the subject rank the four outcomes in order of importance and then assign numbers to each which reflect his relative evaluation of them. Let O_1, O_2, O_3, and O_4 represent these outcomes, ordered from the most to the least important.

(2) Determine which is preferred, O_1, or the combination of O_2, O_3, and O_4. If the combination is preferred, then

(2a) Determine which is preferred, O_1, or combination of O_2 and O_3.

(3) Determine which is preferred, O_2 or the combination O_3 and O_4.

(4) Determine whether or not the numbers assigned in step (1) are consistent with the preferences expressed in steps (2) and (3). If they are, the procedure is completed; if not, confront the subject with the inconsistency and have him modify either the number of the preferences until they are consistent."

The Expected Value Measure[8]

Numerous authors[5,7] have expressed the view that the "expected value measure" requires less complex judgments than those necessitated by either the von Neumann or the Davidson procedure. Ackoff[8] points out that this measure of value shares with von Neumann's theory, "applicability to both quantitative and qualitative outcomes, or to a combination of them." The methodology for obtaining this measure may be summarized as follows[8]:

Given a set of mutually exclusive and exhaustive responses, i.e., outcomes $R_1, R_2 \ldots R_n$, assume that one can obtain from a subject some judgments which yield a set of probabilities $P_1, P_2, \ldots P_n$ such that the choices $P_1 R_1, P_2 R_2, \ldots P_n R_n$ are equally preferred. Assume further that one maximizes the "expected value". Then the result is,

$$P_1 V_1 = P_2 V_2 = \ldots P_n V_n \qquad (1)$$

where V_1 is the relative value of R_1, and V_2 is the relative value of R_2, ... etc.

Let $\sum_j V_j = K$ where K is some arbitrary constant. For simplicity, let K = 1.

Then from equation (1), it follows that,

$$V_2 = \frac{P_1}{P_2} V_1$$

$$V_3 = \frac{P_1}{P_3} V_1$$

$$\vdots$$

$$V_j = \frac{P_K}{P_j} V_K$$

$$\text{where } V_1 = \left[\frac{1}{1 + \frac{P_1}{P_2} + \frac{P_1}{P_3} + \ldots \frac{P_1}{P_n}}\right] \qquad (2)$$

A Posteriori Probability and The Bayesian Model Rationale

The a posteriori approach to the problems of decision making under uncertainty, in conjunction with the Bayesian Model constitute a means of assessing the requisite risks associated with subjective "value" judgments by introducing a logical structure consisting of:

(1) the given states of nature, considered as random, i.e., the a priori probabilities,

(2) the results of an "experiment" in order to compute the conditional probabilities of the states of nature, i.e., the a posteriori probabilities, and

(3) a loss matrix for determining the weighted average of risks corresponding to the minimum Bayes strategy.

For example, by utilizing the "expected value measure" scaled to unity (i.e., K = 1) as a criterion for the rank ordering of alternatives, a set of a priori "utility probabilities" may be computed, which express the relative numerical preferences of the respective alternatives. The "experiment" associated with the Bayesian Model and the subsequent computation of the conditional probabilities is identified with the observer's uncertainty in ranking alternative one as one, alternative two as two, etc., as well as the uncertainty in mis-ranking an alternative, i.e., alternative one as two, one as three, etc. The loss matrix indicates the costs or penalties incurred in mis-ranking an alternative, and furnishes the necessary structure for generating the weighted average risks corresponding to the minimum Bayes strategy. Several types of loss matrices are considered, i.e., linear, quadratic, and cubic. Thus, in ranking alternative one as one there is zero loss (all diagonal terms are zero), while in the linear case a penalty of one unit is incurred in ranking alternative one as two (similarly, alternative two as one, i.e., the loss matrix is symmetric), a penalty of two units is incurred in ranking one as three, etc. For the quadratic and cubic cases the losses will be one and four, and one and eight units, respectively.

The analytical expressions which relate the a priori and conditional probabilities for computing the minimum Bayes strategy for both the continuous and discrete case is given in Appendix I and III, respectively.

Application to The Space Program and Results

The Computer Program

Equations (1), (2), (3), and (4) given in Appendix III have been programmed for the IBM 1130 computer, with the capability of a 20 by 20 matrix for $f(Z|\theta)$ and $R(d;\theta)$, the "experiment" and "loss" matrices respectively. A listing of this program is available upon request. In addition, the "value" function given by equation (2) in the text may be used to compute the vector of a priori "utility" probabilities, if this option is desired. Provision is also made for the insertion of an arbitrary set of "utility" probabilities. The program has various input-output format options which include a plotting subroutine, provisions for stepping the "experiment" matrix, as well as intermediate printouts of each component of the Bayes strategy.

A plot of the results of a sample problem is presented in Figure 1 for linear, quadratic or square law, and cubic loss matrices with perturbations in the a priori probabilities. The weighted average or risk corresponding to the minimum Bayes strategy is plotted as a function of the "pivot" element in the "observer" or "experiment" matrix, i.e., $f(Z|\theta)$, with the remaining entries for any state of nature θ_1, θ_2, etc., being equally divided. For example, in the 3x3 matrix illustrated, the uncertainty in ranking alternative one as one with probability .95 is equal to the uncertainty of ranking alternative two as two, and alternative three as three, i.e., .95, while the uncertainty associated with the ranking of alternative one as two is .05/2 = .025, which is equal to the uncertainty in ranking alternative one as three, etc. It should be observed that an option also exists in the program for inputting an arbitrary "experiment" matrix.

The non-admissible solution boundary region in Figure 1 corresponds to the condition where each entry in the matrix is the reciprocal of the rank of the matrix. Under these circumstances, $f(Z|\Theta)$ is purely random, which constitutes the lower bound of the decision-observation space.

Results Obtained

The specific problem at hand was to generate a set of priorities or rank orderings for a given number of candidate experiments to be considered for inclusion on a particular space flight in accordance with a stated flight profile or objective, and then assess the alternative decisions made in terms of a cost or penalty in the uncertainty level of the respective rankings obtained.

The approach adopted was to form a committee in order to achieve "concensus" concerning the "value" measure to be used, as well as subsequent determinations of the rank orderings themselves. In accordance with the requirements and rationale for a simple, quantitative, and expeditious technique, the "expected value" measure was adopted as the criterion of performance. It should be observed that concensus was achieved among the fifteen committee members (all engineering line personnel) after extensive and exhaustive individual assessments.

Once the criterion standard was selected, the process of ranking the candidate experiments for a given mission was initiated. The documentation of this procedure is contained in a series of Martin Company "daily reports" emanating from the committee during the month of February, 1966, and indeed, furnishes interesting reading. As a by product of the procedure, each member of the committee kept a daily log indicating perturbations in his individual preferences from day to day.

Utilizing the results of this committee, the priorities were transformed onto a value scale (from zero to unity) by equation (2) of the text. These computations were then used as the basis for the a priori "utility" probabilities of the true states of nature, i.e., $q(W)$, in the discrete Bayesian decision model formulation. (See Appendix III).

The "experiment" matrix, $f(Z|\Theta)$, indicated the committee's uncertainty level associated in ranking experiment number one as one, experiment two as two, etc. Alternatively, the probability of misranking an experiment was split equally among the remaining candidates.

As before, several loss matrices were proposed, including a linear and square law loss matrix. Utilizing the computer program developed, the weighted average of the risk corresponding to the minimum Bayes strategy was computed, and plotted as a function of the uncertainty in the committee rankings. Some typical results are shown in Figure 2 for the committee rankings of eighteen candidate experiments to be included on a given space mission.

For the square law loss matrix at the 0.50 uncertainty level a relative risk is shown corresponding to approximately 4.3 units as much as the linear case. As the uncertainty increases, the effect is exaggerated. However, at the 0.90 and 0.95 levels of uncertainty this condition is considerably less pronounced.

Appendix II illustrates a portion of the computer results obtained. The resultant committee concensus, i.e., the input to the program, is designated by the term "value coefficients", with each succeeding experiment ranked relative to the first experiment, which is arbitrarily given a value of unity. Thus, experiment one is 3.3334 times as important as experiments two, three and four, 6.6667 times as important as experiment five, 8.3334 times as important as experiment six, etc., until finally experiment one is 500 times as important as experiment number eighteen. Based on these rankings the a priori probabilities are computed using equations (1) and (2) of the text.

The loss matrix shown is an 18x18 square law matrix, folded at multiples of seven for purposes of printout. Some sample computations are shown, where the "pivot" probability element is in steps of 0.01.

Conclusions

As experience was gained by the committee in performing the rank orderings, and as immediate computer feedback became available illustrating the sensitivity of the committee rankings as a function of the decision uncertainty level for a given cost or penalty matrix, there was an inherent desire for individuals to utilize the computer program developed to compare their own rankings (day to day) with that of the committee. This interest enhanced the overall performance of the committee, and produced a high level of interaction between committee members. As a result, acceptability of the model and the associated methodology was improved as the work progressed.

Several additional potential areas were identified as being applicable, in terms of evolving improved decisions by utilizing the concepts developed in the model. At present, the methodology is being employed to further investigate the dynamics of decision making by concensus, as well as to describe a datum for the role of middle management in corporate level operations. For example, consider the project manager who must continually evaluate the relative success potential of competing projects and assess the risks involved in terms of funds and manpower expenditures. By running several alternative rank orderings and various loss matrices, the sensitivity of these parameters as a function of the decision uncertainty level can be explored, and hopefully, further insight gained in evolving a better decision.

The model itself represents a descriptive analytical technique to explore alternatives, rather than an optimization procedure. In this respect it should be observed that the usefulness and utility of the model is in terms of its flexibility for investigating perturbations, and thus provide appropriate inputs which serve as a basis for value decisions by humans. Ultimately, it will be the human who will accept or reject an alternative, and responsibility for the consequences rests with him, not the computer.

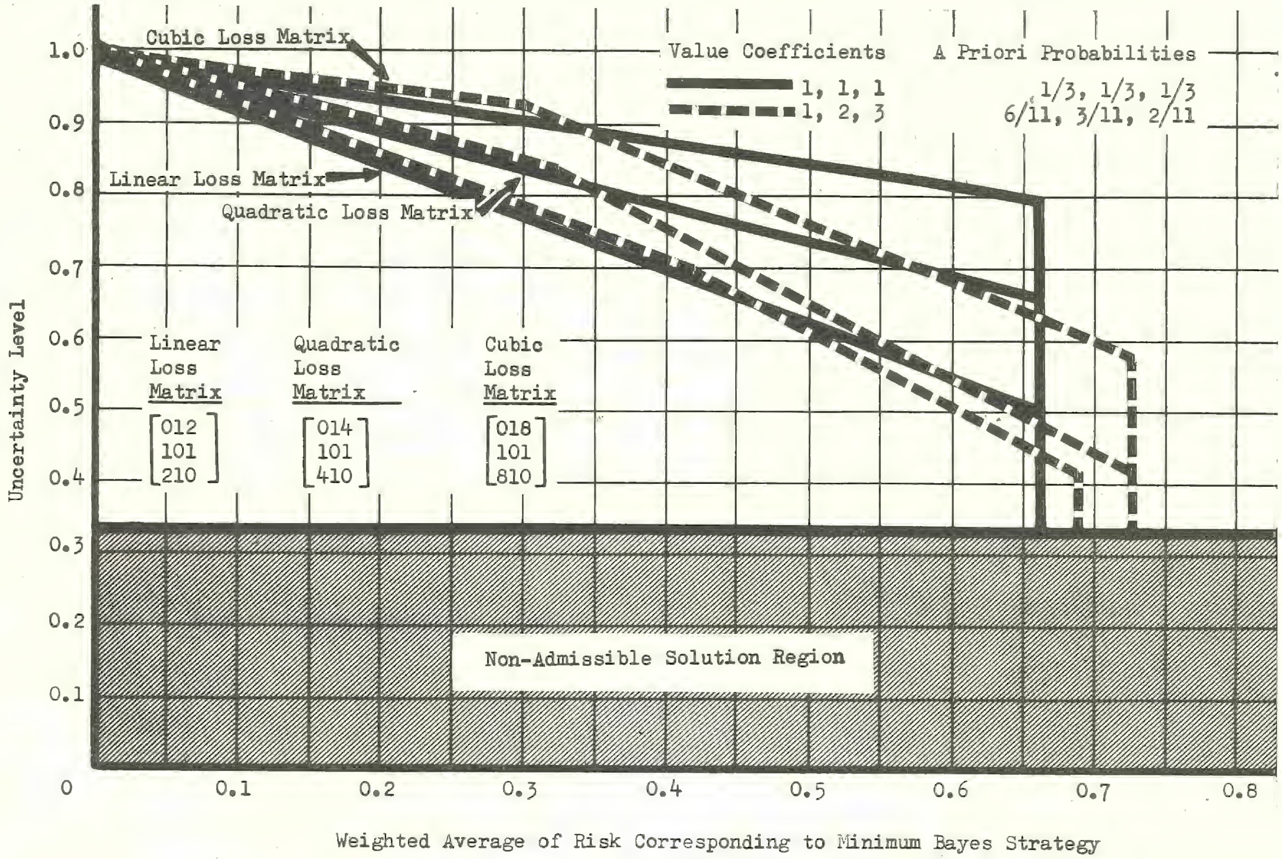

Figure 1. Illustrative Example

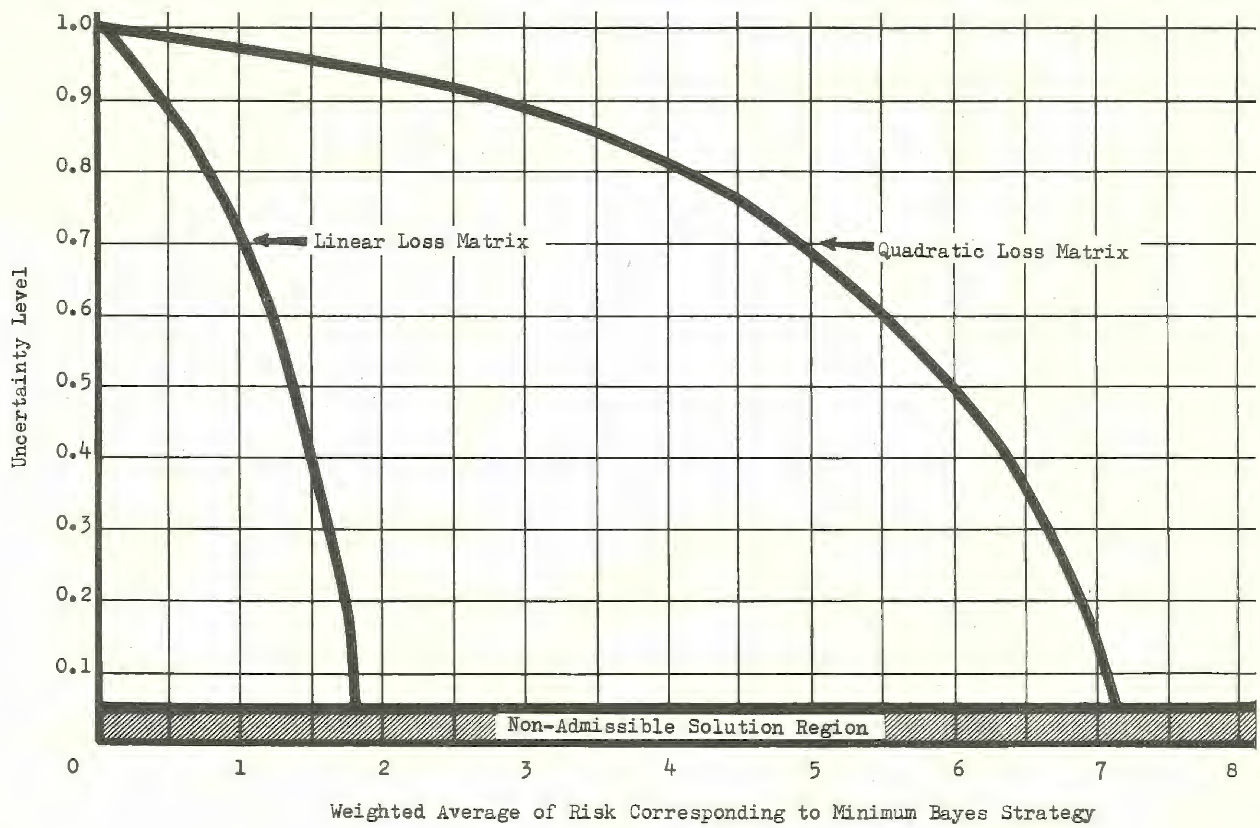

Figure 2. Application to Space Program

Appendix I

The Bayesian Decision Model For The Continuous Case[9]

Let S be an n-dimensional sample space, and let W_1 and W_2 be a partition of the sample space such that if a sample point

$$s = (Z_1, \dots, Z_n), \text{ taken from } f(Z,\theta) \tag{1}$$

falls in W_1 action a_1 is taken, and if s falls in W_2 action a_2 is taken. The action probabilities are defined as,

$$P(s \in W_1 | \theta), \quad P(s \in W_2 | \theta)$$

where $P(s \in W_i | \theta)$ is the probability that s falls in W_i (the probability that action a_i is taken)

when the true state of nature is θ.

A strategy is defined as the function "d" which assigns an action of A to each possible sample, where A, in this case is restricted to

$$A = \left\{ a : a = a_1 \text{ or } a_2 \right\} \tag{2}$$

The action which is taken is

$$a = d\ (Z_1, Z_2, \dots, Z_n) \tag{3}$$

The "loss" associated with action "a" and the state of nature θ is given by $L\ (a;\theta)$. The risk (expected loss) corresponding to strategy "d" is given by:

$$R(d;\theta) = \iint \dots \int_s \left\{ L \left[d(Z_1, Z_2, \dots, Z_n);\ \theta \right]. \right. \tag{4}$$

$$\left. f(z_1;\theta) \dots f(Z_n;\theta)\ dZ_1 \dots dZ_n \right\}$$

$$\therefore R(d;\theta) = L(a_1;\theta)\ P(s \in W_1 | \theta) +$$

$$L(a_2;\theta)\ P(s \in W_2 | \theta) \tag{5}$$

Consider W as a random variable, which has a probability distribution, so that

$$r(Z, W) = p(Z|W)\ q(W) = S(W|Z)\ t(Z) \tag{6}$$

$q(W)$ is the a priori distribution

$S(W|Z)$ is the a posteriori distribution.

From equation (5), the risk may be simply expressed as the expected value of the loss. Using a compact notation,

$$R(d;\theta) = R(W,d) = E_z \left\{ L \left[W,\ d\ (Z) \right] \right\} \tag{7}$$

A strategy "d" is a Bayes strategy corresponding to a minimization of the expected risk. Thus, determine "d" such that

$$\min_d \left\{ E_W \left[R(W,a) \right] \right\} = \min_d \left\{ \int_W R(W,d)\ q(W)\ dW \right\} \tag{8}$$

But

$$R(W,d) = E_Z \left\{ L \left[W,d(Z) \right] \right\} = \int_Z L \left[W,d(Z) \right]\ p(Z|W)\ dZ\ . \tag{9}$$

Substituting equation (9) into equation (8)

$$\min_d \left\{ E_W \left[R(W,d) \right] \right\}$$

$$= \min_d \left\{ \int_W \int_Z L \left[W,d(Z) \right]\ p(Z|W)\ q(W)\ dZdW \right\} \tag{10}$$

$$= \min_d \left\{ \int_W \int_Z L \left[W,d(Z) \right]\ S(W|Z)\ t(Z)\ dZdW \right\} \tag{11}$$

$$= \min_d \left\{ \int_Z t(Z) \left[\int_W L(W,d(Z)\ S(W|Z)\ dW \right] dZ \right\} \tag{12}$$

But $t(Z)$ is a probability distribution which is non-negative. Hence, if it is desired to minimize equation (12) over "d", only the function in the brackets need be considered. Thus,

$$\min_d \left\{ E_W \left[R(W,d) \right] \right\} = \min_d \left\{ \int_W L[W,d(Z)]\ S(W|Z) dW \right\} \tag{13}$$

or

$$\min_d \left\{ E_W \left[R(W,d) \right] \right\} = \min_d \left\{ \int_W L[W,d(Z)]\ p(Z|W) q(W) dW \right\} \tag{14}$$

For the particular case, in which the structure of the loss function is assumed to be of the form,

$$L \left[W,d(Z) \right] = (d - W)^2 \tag{15}$$

i.e., quadratic or square law loss function, equation (14) becomes

$$\min_d \left\{ \int_W (d-W)^2\ p(Z|W)\ q(W)\ dW \right\} \tag{16}$$

Differentiating equation (16) with respect to "d", and setting the result equal to zero in order to obtain the Bayes strategy yields,

$$\frac{\int_W W\ p(Z|W)\ q(W)\ dW}{\int_W p(Z|W)\ q(W)\ dW} = \frac{t(Z) \int_W W \cdot S(W|Z) dW}{\int_W r(Z,W)\ dW} \tag{17}$$

but $\int_W r(Z,W)\ dW = t(Z)$

Hence,

$$d = \int_W W\ S(W|Z)\ dW = E_W\ (W|Z) \tag{18}$$

which is the well known result obtained by Wald[10].

Appendix II

Sample Computer Output

VALUE COEFFICIENTS

1.0000	3.3334	3.3334	3.3334	6.6667	8.3334	33.3334
33.3334	33.3334	33.3334	35.7143	71.4284	166.6667	166.6667
200.0000	250.0000	333.3334	500.0000			

A PRIORI PROBABILITIES

0.42409	0.12722	0.12722	0.12722	0.06361	0.05089	0.01272	0.01272	0.01272	0.01272
0.01187	0.00593	0.00254	0.00254	0.00212	0.00169	0.00127	0.00084		

LOSS MATRIX

0.0000	1.0000	4.0000	9.0000	16.0000	25.0000	36.0000
49.0000	64.0000	81.0000	100.0000	121.0000	144.0000	169.0000
196.0000	225.0000	256.0000	289.0000			
1.0000	0.0000	1.0000	4.0000	9.0000	16.0000	25.0000
36.0000	49.0000	64.0000	81.0000	100.0000	121.0000	144.0000
169.0000	196.0000	225.0000	256.0000			
4.0000	1.0000	0.0000	1.0000	4.0000	9.0000	16.0000
25.0000	36.0000	49.0000	64.0000	81.0000	100.0000	121.0000
144.0000	169.0000	196.0000	225.0000			
9.0000	4.0000	1.0000	0.0000	1.0000	4.0000	9.0000
16.0000	25.0000	36.0000	49.0000	64.0000	81.0000	100.0000
121.0000	144.0000	169.0000	196.0000			
16.0000	9.0000	4.0000	1.0000	0.0000	1.0000	4.0000
9.0000	16.0000	25.0000	36.0000	49.0000	64.0000	81.0000
100.0000	121.0000	144.0000	169.0000			
25.0000	16.0000	9.0000	4.0000	1.0000	0.0000	1.0000
4.0000	9.0000	16.0000	25.0000	36.0000	49.0000	64.0000
81.0000	100.0000	121.0000	144.0000			
36.0000	25.0000	16.0000	9.0000	4.0000	1.0000	0.0000
1.0000	4.0000	9.0000	16.0000	25.0000	36.0000	49.0000
64.0000	81.0000	100.0000	121.0000			
49.0000	36.0000	25.0000	16.0000	9.0000	4.0000	1.0000
0.0000	1.0000	4.0000	9.0000	16.0000	25.0000	36.0000
49.0000	64.0000	81.0000	100.0000			
64.0000	49.0000	36.0000	25.0000	16.0000	9.0000	4.0000
1.0000	0.0000	1.0000	4.0000	9.0000	16.0000	25.0000
36.0000	49.0000	64.0000	81.0000			
81.0000	64.0000	49.0000	36.0000	25.0000	16.0000	9.0000
4.0000	1.0000	0.0000	1.0000	4.0000	9.0000	16.0000
25.0000	36.0000	49.0000	64.0000			
100.0000	81.0000	64.0000	49.0000	36.0000	25.0000	16.0000
9.0000	4.0000	1.0000	0.0000	1.0000	4.0000	9.0000
16.0000	25.0000	36.0000	49.0000			
121.0000	100.0000	81.0000	64.0000	49.0000	36.0000	25.0000
16.0000	9.0000	4.0000	1.0000	0.0000	1.0000	4.0000
9.0000	16.0000	25.0000	36.0000			
144.0000	121.0000	100.0000	81.0000	64.0000	49.0000	36.0000
25.0000	16.0000	9.0000	4.0000	1.0000	0.0000	1.0000
4.0000	9.0000	16.0000	25.0000			
169.0000	144.0000	121.0000	100.0000	81.0000	64.0000	49.0000
36.0000	25.0000	16.0000	9.0000	4.0000	1.0000	0.0000
1.0000	4.0000	9.0000	16.0000			

196.0000	169.0000	144.0000	121.0000	100.0000	81.0000	64.0000
49.0000	36.0000	25.0000	16.0000	9.0000	4.0000	1.0000
0.0000	1.0000	4.0000	9.0000			
225.0000	196.0000	169.0000	144.0000	121.0000	100.0000	81.0000
64.0000	49.0000	36.0000	25.0000	16.0000	9.0000	4.0000
1.0000	0.0000	1.0000	4.0000			
256.0000	225.0000	196.0000	169.0000	144.0000	121.0000	100.0000
81.0000	64.0000	49.0000	36.0000	25.0000	16.0000	9.0000
4.0000	1.0000	0.0000	1.0000			
289.0000	256.0000	225.0000	196.0000	169.0000	144.0000	121.0000
100.0000	81.0000	64.0000	49.0000	36.0000	25.0000	16.0000
9.0000	4.0000	1.0000	0.0000			

PIVOT PROBABILITY ELEMENT = 1.0000002
WEIGHT.AV.OF RISK CORRESP.TO MIN.BAYES STRAT. = 0.0000

PIVOT PROBABILITY ELEMENT = 0.9900000
WEIGHT.AV.OF RISK CORRESP.TO MIN.BAYES STRAT. = 0.6453

PIVOT PROBABILITY ELEMENT = 0.9799999
WEIGHT.AV.OF RISK CORRESP.TO MIN.BAYES STRAT. = 1.1005

PIVOT PROBABILITY ELEMENT = 0.9699997
WEIGHT.AV.OF RISK CORRESP.TO MIN.BAYES STRAT. = 1.4477

PIVOT PROBABILITY ELEMENT = 0.9599996
WEIGHT.AV.OF RISK CORRESP.TO MIN.BAYES STRAT. = 1.7410

PIVOT PROBABILITY ELEMENT = 0.9499995
WEIGHT.AV.OF RISK CORRESP.TO MIN.BAYES STRAT. = 1.9953

PIVOT PROBABILITY ELEMENT = 0.9399994
WEIGHT.AV.OF RISK CORRESP.TO MIN.BAYES STRAT. = 2.2241

PIVOT PROBABILITY ELEMENT = 0.9299993
WEIGHT.AV.OF RISK CORRESP.TO MIN.BAYES STRAT. = 2.4262

PIVOT PROBABILITY ELEMENT = 0.9199992
WEIGHT.AV.OF RISK CORRESP.TO MIN.BAYES STRAT. = 2.6156

Appendix III

Computation of Bayes Strategies For The Discrete Case

Although equations (17) and (18) give the Bayes strategies for the continuous case, the following tabular formulation is applicable when the a priori probabilities, loss function, and observations are discrete variables.

$$R(d;\theta) = \begin{array}{c} \\ \theta_1 \\ \theta_2 \\ \cdot \\ \cdot \\ \cdot \\ \theta_n \end{array} \begin{array}{c} \begin{array}{ccc} d_1 & d_2 \cdots\cdots\cdots & d_n \end{array} \\ \left[\begin{array}{ccc} r(d_1;\theta_1) & r(d_2;\theta_1) \cdots & r(d_n;\theta_1) \\ r(d_1;\theta_2) & & \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ r(d_1;\theta_n) & \cdots\cdots\cdots\cdots & r(d_n;\theta_n) \end{array}\right] \end{array}$$

$$B(\bar{W},d) \quad B(\bar{W},d_1) \cdots\cdots\cdots\cdots B(\bar{W},d_n)$$

$$q(W) = \begin{array}{c} \bar{W} \\ \left[\begin{array}{c} W_1 \\ W_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ W_n \end{array}\right] \end{array}$$

$$f(Z)|\theta) = \begin{array}{c} \\ \theta_1 \\ \\ \theta_2 \\ \theta_n \end{array} \begin{array}{c} \begin{array}{cccc} Z_1 & Z_2 & \cdots & Z_n \end{array} \\ \left[\begin{array}{ccc} f(Z_1|\theta) & \cdots & f(Z_n|\theta_1) \\ \cdot & & \\ \cdot & & \\ f(Z_1|\theta_n) & \cdots & f(Z_n|\theta_n) \end{array}\right] \end{array}$$

The Bayes strategy is the action which minimizes

$$B(\bar{W}) = \min_{d} \left\{ B(\bar{W},d_1),\ b(\bar{W},d_2),\ \ldots\ B(\bar{W},d_n) \right\} \qquad (1)$$

where $B(\bar{W})$ denotes the minimum value of the average expected loss. More explicitly,

$$B(\bar{W}) = \min_{\substack{d \\ 0 \angle i \angle n}} \left\{ B(\bar{W},d_i) \right\} = \min_{d} \left\{ \sum_{i=1}^{n} W_i r(d_i;\theta) \right\} \qquad (2)$$

$$= \min_{d} \left\{ \frac{1}{f(Z)} \sum_{i=1}^{n} W_i\ F(Z|\theta_i)\ r(d_i;\theta) \right\} \qquad (3)$$

where

$$f(Z) = W_1\ f(Z|\theta_1) + W_2\ f(Z|\theta_2) + \ldots\ W_n f(Z|\theta_n) \qquad (4)$$

The weighted average of the risks corresponding to the minimum Bayes strategy is simply the product of each $B(\bar{W})$ multiplied by the corresponding $f(Z)$.

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