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"UNIVERSAL LAW FOR THE TRANSITION FROM CHAOS TO PERIODICITY IN NONLINEAR PHYSICAL SYSTEMS"

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Universal Law for the Transition from Chaos to Periodicity in Nonlinear Physical Systems

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Abstract

This paper investigates the Chaos phenomena in nonlinear physical systems described by differential equations. A prototypic system is the Duffing oscillator, described by the nonlinear second order differential equation which presents a mathematical model of the motion performed by a plane pendulum under a periodic external force. By using numerical and phase space analysis, the transition from periodic to chaotic behavior (and vice versa) is analyzed. By changing the damping parameter k, the transition to chaos through the bifurcations of limit cycles is demonstrated. Numerical results show that after four successful bifurcations, the 16-cycle unexpectedly exchanges with a stable 3-cycle, which further bifurcates to a 6- and 12-cycle, until the chaotic strange attractor is reached. Further decrease of the damping parameter provides the transition from chaos to odd periodic limit cycles. The stable 9-, 7- and 5-periodic limit cycles successfully lead the motion, the latter bifurcates to 10-cycle, and further to 15-cycle, which again leads to a chaotic strange attractor. Finally, for small values of the damping parameter, a stable 1-limit cycle emerges from the chaos.

1 Introduction

This paper investigates the Chaos phenomena in nonlinear physical systems described by differential equations. A prototypic equation is Duffing's oscillator, a driven, damped, and anharmonic oscillator described by the following second order differential equation:

$$\ddot{x}+k\dot{x}-x+x^3 = f(t), t > 0, \quad (1.1)$$

where

$$f(t) = a\cos(t).$$

The equation (1.1) presents a mathematical model for the physical problem of the oscillations of the plane pendulum ([5]). The pendulum consists of a heavy small diameter ball suspended on a rigid and very light rod (Fig.1).

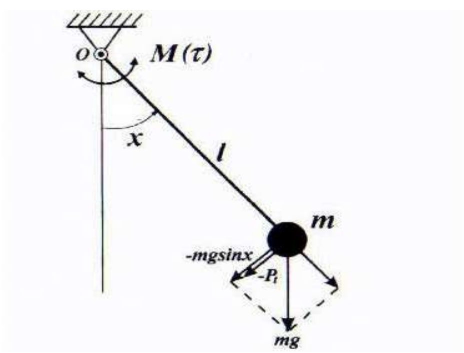

Figure 1. Forced pendulum ([5]).

The rod can rotate around the horizontal axis. The position of the ball is determined by a single time-dependent coordinate, for instance, the angular displacement denoted as $x(t)$. Accordingly, the first derivative $\dot{x}$, denotes the velocity of the pendulum, while the second derivative $\ddot{x}$, denotes the acceleration of the pendulum. The motion of the ball is ruled by gravitation, damping forces (friction, etc.), and the external periodic force. The Duffing equation (1.1) can be derived by a straightforward application of Newton's second law. Duffing's equation is often regarded as a precise approximation of numerous technical devices ([5, 6]). Consider an initial-value problem for the equation (1.1) under the condition

$$x(0) = x_o, \ \dot{x}(0) = y_o, \qquad (1.2)$$

where x_o is an initial position, and y_o is an initial velocity of the pendulum. By introducing a new function $y = \dot{x}$, the problem (1.1),(1.2) may be replaced with the following initial value problem for the system of two equations:

$$\dot{x} = y, \ x(0) = x_o, \qquad (1.3)$$

$$\dot{y} = -ky + x - x^3 + f(t),$$
$$y(0) = y_o. \qquad (1.4)$$

The function $f(t) = a\cos(t)$ portrays an external force with period $T = 2\pi$. For any fixed value of the damping parameter k, initial position x_o, and initial velocity y_o, the solution $(x(t), y(t))$ of the system (1.3),(1.3) is described as a continuous curve in the phase plane

$(x, \dot{x})$. $T = 2\pi$ presents natural time step, and the intermittent solution with period $2\pi N$, N being an integer, describes a closed curve on the phase plane of period $2\pi N$, that is to say, $x(t+2\pi N) = x(t)$, $y(t+2\pi N) = y(t)$. This solution is called an N-cycle. The main question addressed is to understand how the asymptotic behavior of $(x(t), y(t))$ as $t \to \infty$, depends on the damping parameter k. If for a certain range of initial values, the solution of the Duffing's system settles down on an $N - cycle$, then the latter is called a period N attractor, or N-limit cycle. We will use term stable N-cycle.

It is well-known that for large values of k, there is a period 1 attractor, or 1-limit cycle. When k decreases, this 1-limit cycle may become repelling or unstable, and accede to other m-limit cycles. However, an attractor is not necessarily a periodic and regular closed curve on the phase plane. In the time frame of the 1960s, Japanese mathematician Ueda made a remarkable discovery of, so called, **strange attractors** ([2]). He discovered that, for certain values of the damping parameter k, all solutions asymptotically converge to a strange non-periodic, irregular solution which makes random-like oscillations. Ueda's "strange attractor" portrayed the solution unpredictable in time, and extremely sensitive to initial values. In view of these properties, it is called a chaotic. A Poincare map of Ueda's strange attractor is presented in Figure 20 of the Appendix. This discovery opened a new chapter in the research of dynamical systems with continuous time.

2 Hypothesis and Procedures

The main hypothesis of this paper is that there is a universal transition route from periodicity to chaos and vice versa, from chaos to periodicity. After a chaotic regime, all odd periodic cycles are distributed in decreasing order. We aim to find whether there exist universal constants, which qualitatively describe this universal transition route. Finding this universal law and its related constants will be a great advantage in the prediction of the behavior of a chaotic systems described by nonlinear differential equations.

Our procedure consists of the

following algorithm:

Step 1. Find how the asymptotic behavior of the trajectory of the system (1.3), (1.4) depends on the initial point (x_o, y_o) for large values of the damping parameter k. This step will present a classification of the possible limit cycles for different ranges of the initial values.

Step 2. Choose an initial value as a typical point from each range to investigate the dependence of limit cycles on the damping parameter.

Step 3. Provided that the bifurcation of the limit cycles is observed, for an arbitrary positive integer n, signify by k_n the value of the damping parameter when the $2^{n-1} \cdot T$-periodic limit cycle bifurcates to a $2^n \cdot T$-periodic limit cycle (remember that $T = 2\pi$).

Step 4. Having three successive values k_n, k_{n+1}, k_{n+2}, calculate the convergence rate of the damping parameter as

$$\delta_n = \frac{k_n - k_{n+1}}{k_{n+1} - k_{n+2}}. \qquad (2.1)$$

It is expected that δ_n will be near the universal constant 4.6629... (see [1]) for large n

Step 5. Consider the formula (2.1), where n is replaced with $n+1$, and by substituting δ_n for δ_{n+1}, predict the value of k_{n+3} as

$$k_{n+3} = \frac{(1+\delta_n)k_{n+2} - k_{n+1}}{\delta_n} \qquad (2.2)$$

Step 6. If accuracy is achieved, go to step 7. Otherwise replace n with n+1, and go to step 4.

Step 7. Having the value k_∞, when the limit cycle becomes chaotic, decrease the value k slowly, and observe whether the transition from a chaotic strange attractor to another periodic limit cycle occurs.

Step 8. By changing the damping parameter near the chaotic regime, search for the limit cycles with an odd period $(2n+1)T$ and signify the related value of the damping parameter as λ_n.

Step 9. Attempt to find the order of the odd limit cycles, and modify steps 4-6 to find the convergence rate of the parameter values λ_n to possible universal constants.

3 Description of Results

We demonstrate the results when the external force is chosen as $f(t) = 0.3 \cos t$. It should be noted that the choice of the constant as specifically 0.3 has no qualitative influence on the described results. For all large values of the damping parameter k, the solution of the Duffing's system (1.3) converges to a stable 1-cycle. There are two symmetric 1-cycles, each being attractive for a particular range of initial values. In Figure 2, a stable 1-cycle is presented with the initial value chosen as (1,0), while the damping parameter $k = 0.43$. While decreasing the damping parameter, the 1-cycle becomes unstable and repelling. It eventually bifurcates to a 2-cycle (Figure 3). This bifurcation is repeated by further decreasing k, and a successive bifurcation to the 4-, 8- and 16-cycles is observed (Figures 4-6). However, the bifurcation doesn't continue *ad infinitum.* By slowly decreasing k, we observe that the 16-cycle unexpectedly transforms into a stable 3-cycle (see Figure 7). Moreover, by further decreasing k, we find that the 3- and 16-cycles repeatedly exchange each other.

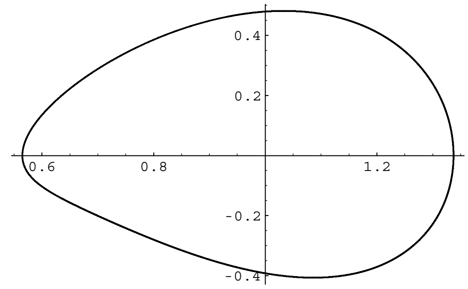

Figure 2. Stable 1-cycle, $(x_o, y_o) = (1, 0)$, $k = 0.43$.

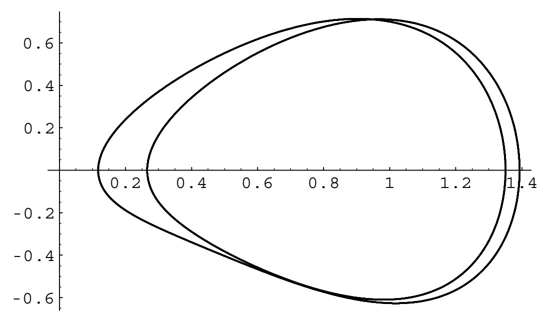

Figure 3. Stable 2-cycle, $(x_o, y_o) = (1, 0)$, $k = 0.425$.

A similar route is calculated for the different ranges of the initial values. In Figures 21-26 of the Appendix, a similar bifurcations route between the 1- and 16-cycles with successive exchange with the 3-cycle is presented with the initial value being chosen as (-1,0). We calculated the approximate convergence rate of the damping parameter by using the formula (2.1) of Step 4 of the algorithm from §2, according to four successive bifurcations between 2- and 16-cycles. For both different ranges of initial values the convergence rate comes close to Feigenbaum's universal constant $\delta = 4.6692...$ (see Table 1 and Table 2 in Appendix). One could expect that, as in descrete models with a quadratic maximium, this bifurcation continues ad infinitum. However, it is not.

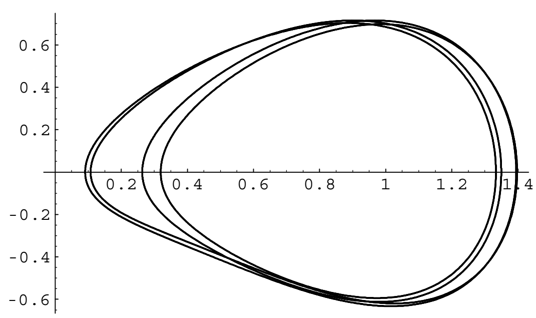

Figure 4. Stable 4-cycle, $(x_o, y_o) = (1, 0)$, $k = 0.418$.

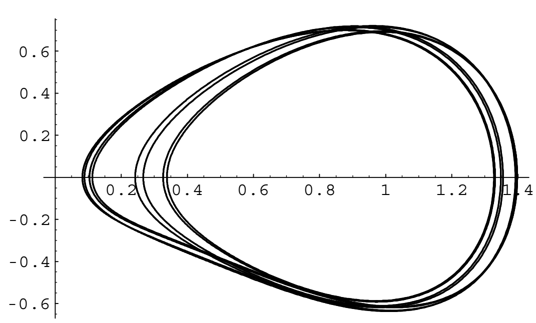

Figure 5. Stable 8-cycle, $(x_o, y_o) = (1, 0)$, $k = 0.416$.

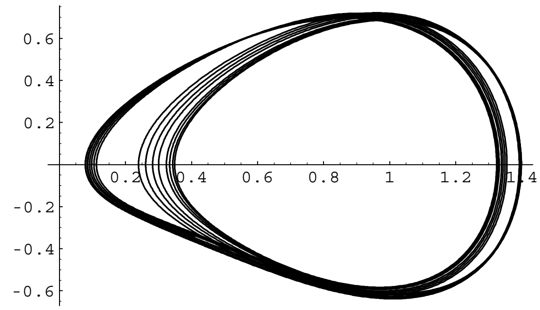

Figure 6. Stable 16-cycle, $(x_o, y_o) = (1, 0)$, $k = 0.414898$.

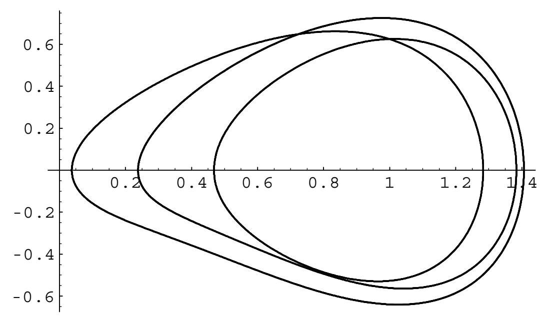

Figure 7. Stable 3-cycle, $(x_o, y_o) = (1, 0)$, $k = 0.41459176$.

Astoundingly, after the 16- cycle becomes repelling, a stable 3-cycle emerges (See Figure 7 and Figure 26 of Appendix). Moreover, by further reducing k, the 3- and 16- cycle exchange with each other. By decreasing the damping parameter further, the stable 3-cycle becomes repelling, and bifurcates to a 6- cycle (See Figure 8). Further reduction of the damping parameter leads to a 12- cycle (See Figure 9). It is expected that in this range of the damping parameter, periodic $3 \cdot 2^n$-cycles follow each other until a chaotic strange attractor appears (See Figure 10). It should be pointed out that up to this point, the dynamics of limit cycles in two different ranges of initial values evolve independetly from each other. However, as one see from Figure 10, the chaotic strange attractor expands out of half plane, and accordingly, two "symmetric" strange attractors correspond with each other. As a result, the trajectory starts jumping randomly between those overlapping attractors. This creates a unique strange attractor, which is a unification of two overlapping ones (Figure 11).

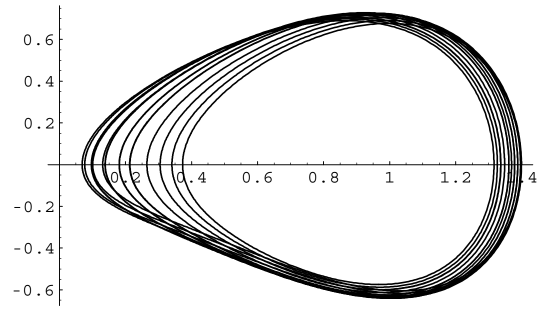

Figure 9. Stable 12-cycle, $(x_o, y_o) = (1, 0)$, $k = 0.41275$.

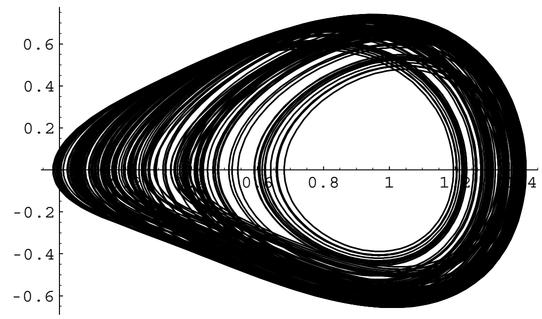

Figure 10. Chaotic attractor, $(x_o, y_o) = (1, 0)$, $k = 0.405$.

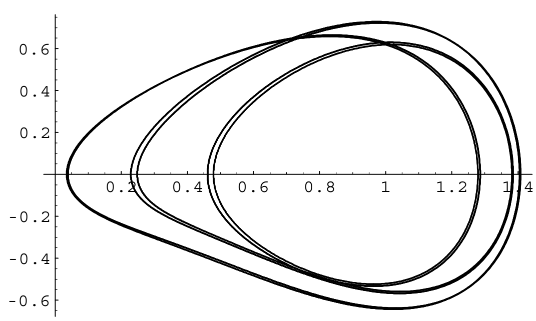

Figure 8. Stable 6-cycle, $(x_o, y_o) = (1, 0)$, $k = 0.41336$.

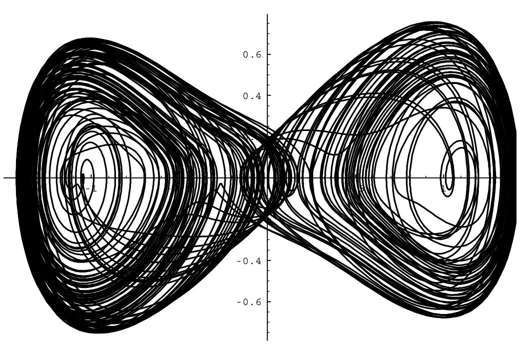

Figure 11. Overlapping chaotic attractor, $(x_o, y_o) = (1, 0)$, $k = 0.38$.

By further reducing the damping parameter, a chaotic strange attractor again leads to a periodic limit cycle. We observe the transition from Chaos to odd periodic limit cycles. In Figure 12, a stable 9- cycle is presented.

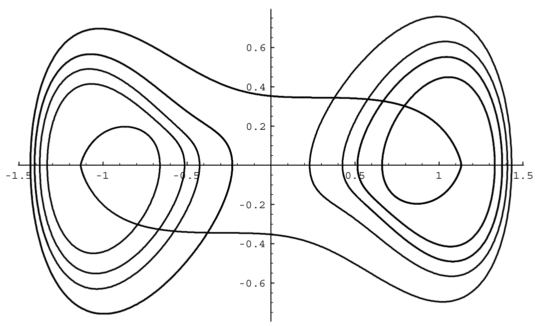

Figure 12. Stable 9 Cycle, $(x_o, y_o) = (1, 0)$, $k = 0.35$.

Further decrease of k, provides a transition from the 9- to a 7- cycle. The interesting transition mechanism is demonstrated in Figure 13. By loosing two loops, the stable 9-cycle smoothly transforms into a stable 7-cycle (Figure 14).

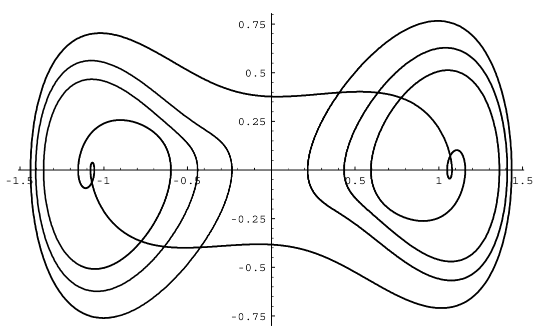

Figure 13. Transition from 9- to 7-cycle, $(x_o, y_o) = (1, 0)$, $k = 0.34$.

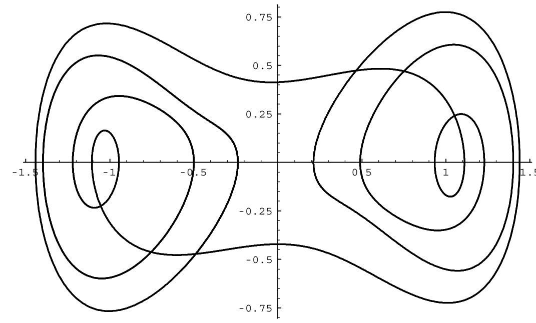

Figure 14. Stable 7 Cycle, $(x_o, y_o) = (1, 0)$, $k = 0.325$.

In a similar way, stable 7-cycle transforms to stable 5-cycle (Figure 15).

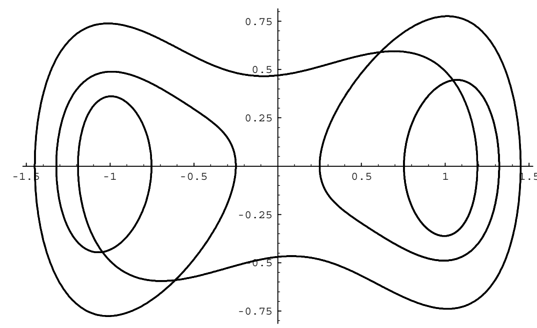

Figure 15. Stable 5 Cycle, $(x_o, y_o) = (1, 0)$, $k = 0.299$.

It was expected that the stable 5-cycle would be transformed into a stable 3-cycle by completing a series of all odd cyles. Instead, the 5- cycle unexpectedly bifurcates to a stable 10-cycle (Figure 16), which further tansforms to a stable 15-cycle (Figure 17). Further decrease of k, leads to a chaotic strange attractor (Figure 18).

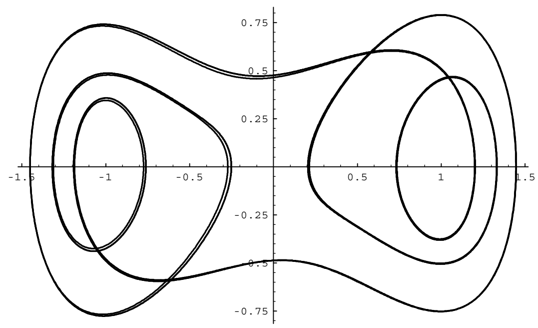

Figure 16. Stable 10 Cycle, $(x_o, y_o) = (1, 0)$, $k = 0.296$.

Finally, for small values of the damping parameter, a stable 1-limit cycle emerges again from Chaos(Figure 19).

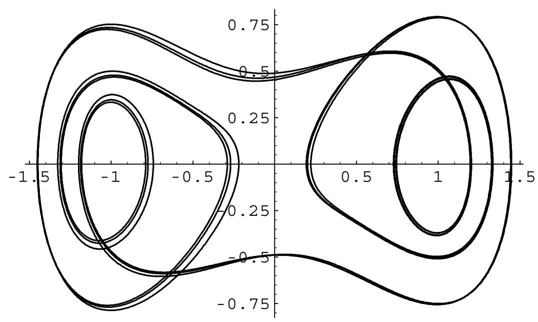

Figure 17. Stable 15 Cycle, $(x_o, y_o) = (1, 0)$, $k = 0.295$.

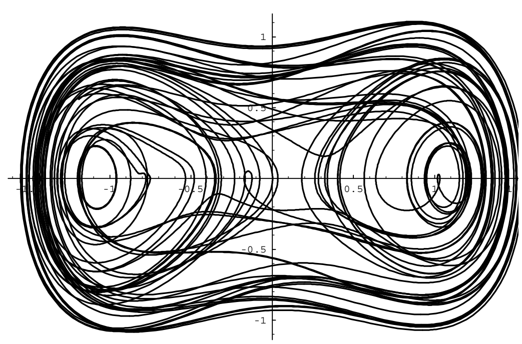

Figure 18. Chaos, $(x_o, y_o) = (1, 0)$, $k = 0.293..$

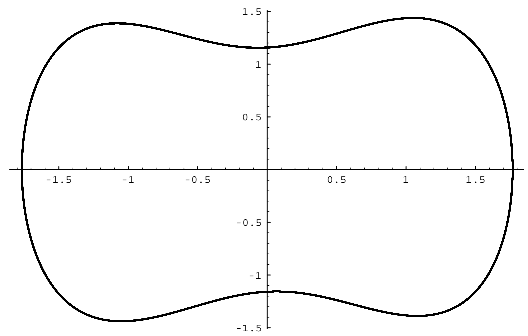

Figure 19. Stable 1 Cycle, $(x_o, y_o) = (1, 0)$, $k = 0.13..$

4 Appendix

Figure 20. Ueda's strange attractor.

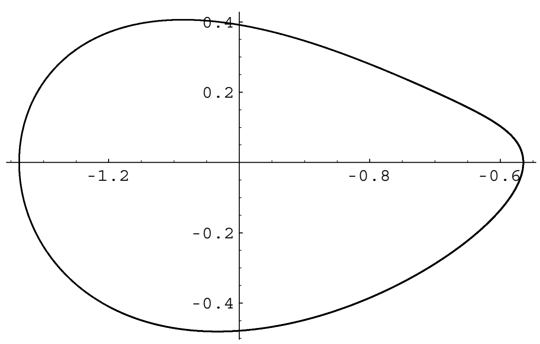

Figure 21. Stable 1-cycle, $(x_o, y_o) = (-1, 0)$, $k = 0.43$.

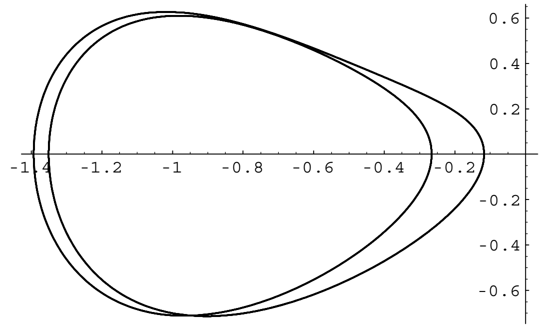

Figure 22. Stable 2-cycle, $(x_o, y_o) = (-1, 0)$, $k = 0.425$.

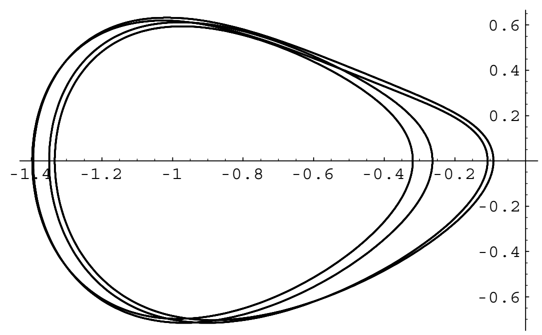

Figure 23. Stable 4-cycle, $(x_o, y_o) = (-1, 0)$, $k = 0.418$.

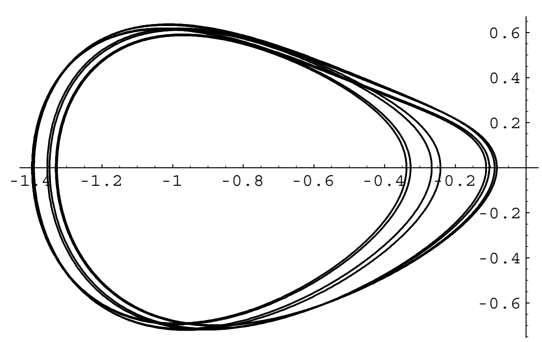

Figure 24. Stable 8-cycle, $(x_o, y_o) = (-1, 0)$, $k = 0.416$.

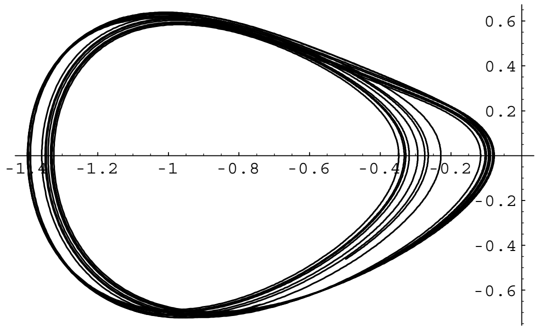

Figure 25. Stable 16-cycle, $(x_o, y_o) = (-1, 0)$, $k = 0.414896$.

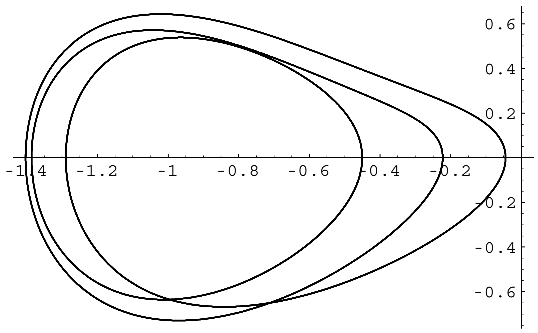

Figure 26. Stable 3-cycle, $(x_o, y_o) = (-1, 0)$, $k = 0.414891$.

Table 1

Transition	Damping Parameter +	Damping Parameter -	δ_n +	δ_n -
1-2 Cycle	0.42917655184	0.426613453057	—	—
2-4 Cycle	0.41965	0.4164662	—	—
4-8 Cycle	0.41645	0.416472	2.97705	2.98262
8-16 Cycle	0.415776	0.415785	4.74777	4.64338
16-3 Cycle	0.459654	0.459647	—	—

Table 2

Attractor	Damping Parameter
3- cycle	0.41459176
6-cycle	0.41336
12-cycle	0.41275
...Chaos...	0.41
9-cycle	0.35
7-cycle	0.325
5-cycle	0.297
10-cycle	0.298
15-cycle	0.295
...Chaos	0.293
1-cycle	0.1388

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