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Optimizing Attitude Control Systems

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Summary

This paper discusses a practical approach to attitude control system mechanization. Previous efforts reported in the literature have either resulted in systems too complex to mechanize or have not considered the problem in enough of its aspects to make the work meaningful. The classical control optimization techniques are briefly summarized and a critique of these methods is given. The solutions obtained with the classical techniques are either open loop, which is unsatisfactory from an attitude control standpoint, or they are closed loop. These closed loop solutions in general may require measuring all of the system variables, which may not be possible, or they may be far too complex to mechanize. In the proposed approach, called specific optimal control, sensor and actuator characteristics are given and the form of the controller is chosen. Controller parameters are then chosen so as to minimize some performance index. Three analytical methods being developed to perform this optimization are hill climbing, two point boundary value problem formulation, and differential approximation. Each of these methods is discussed. Numerical examples showing the application of these techniques are given in a reference.[9]

Introduction

In the design of a space vehicle, the need for attitude control arises. Solar cells mounted on solar panels must be pointed toward the sun. High gain antennas must be oriented in the direction of the Earth. From a temperature control standpoint it is desirable to have the Sun in a fixed position relative to the spacecraft. In order to perform trajectory corrections it is necessary to command the vehicle to an arbitrary orientation. Thus, attitude control is a necessary subsystem of a space vehicle.

In the past, the attitude control problem has been considered as three relatively independent tasks: (1) acquisition in which the spacecraft must be brought to the proper orientation from the rates occurring at separation or after a disturbance, (2) cruise control for the major portion of the mission, and (3) commanded turns for orienting the spacecraft prior to a trajectory correction. For a more efficient design, however, it is better to consider this as all one problem and to make design decisions accordingly. That is, a control system that will work for both acquisition and cruise conceptually seems more desirable than two separate, independent systems which would each perform just one function. In this paper an attempt is made to show how such mechanization considerations can be included in the analysis phase of attitude control system design. For example, a control system that is known to give satisfactory performance in the cruise mode of operation might be used to indicate constraints or possible solutions in an analysis of the acquisition problem.

Even though the dynamics for a single axis of a spacecraft are relatively simple, the design of an attitude control system for acquisition becomes rather complex. It is not sufficient to consider only a single axis for the acquisition problem because of interaxis dynamic, geometric, and possibly sensor coupling. This is particularly true if one is trying to optimize system performance.

In the discussion that follows we will primarily be concerned with techniques of optimizing attitude control systems for the acquisition requirement. Methods of simulating the space vehicle are mentioned in order to indicate some of the computational problems that arise. The classical optimization techniques are introduced and a manner in which they can be applied to the attitude control problem is proposed. Throughout the entire discussion, it should be noted that the object is to start with a system that either can be or has been built and then optimize it according to the chosen performance criteria.

The attitude control systems for the Mariner-Venus probe of 1962 and the Mariner-Mars probe of 1964 each operated in a dual mode manner. For small position and rate errors a passive technique of obtaining rate information for damping was used.[1] A block diagram of an attitude control system employing this technique, derived rate stabilization, is shown in Figure 1. The sun sensor provides the necessary position information. A switching amplifier, having a minimum on-time or holding characteristic for short input signals, drives the nozzle actuators. They operate in an on-off manner to torque the vehicle. The signal E_T to these actuators is proportional to the acceleration about the torqued axis; hence, E_D contains velocity information. This signal is very closely proportional to the velocity increment when the on-time of the switching amplifier is short compared to the time constant τ. When the switch is in position A the control system is thus passively stabilized.

When the position errors exceeded certain values, the gyros were turned on to measure the angular rates about the spacecraft body axes and the switch was moved from the A to the B position. The gyros stayed on until the position errors remained less than the threshold value. This mode of operation was used for all acquisitions and whenever a disturbance moved the spacecraft away from its references.

When gyros are used for acquisition the control system performance is close to optimum in a minimum fuel and time sense. For high rates about the body axes, there is first a period of rate reduction in which the nozzle actuators continously decrease the angular velocities. Then the velocity is low enough, the position sensors become effective and the system acquires the references with only a small amount of overshoot. Since a small increase in acquisition time over the minimum achievable is not significant in assessing control system performance, the increased complexity necessary to achieve theoretically optimal performance could not be justified.

This dual mode system involved a certain amount of logic and necessitated performing several switching functions. It was first necessary to establish whether acquisition had actually been achieved or if the spacecraft was just momentarily in the acquired position. Next the gyros had to be turned on and switched into the loop when the derived rate compensation was switched out. Since the gyros could not be turned on until after a disturbance had occurred, they were not available for use in all cases as soon as they were needed. In some cases, turning the gyros on caused larger transients than the original disturbance itself. Thus, it becomes desirable to have a single-mode, passive system that can perform both acquisition and cruise functions. Initial studies of three-axis, derived-rate acquisitions, however, indicate that the system performance is considerably inferior to that obtained from a system employing rate gyros. It therefore becomes desirable to optimize the passively compensated system to obtain the best possible performance.

Attitude control optimization efforts have been reported in the literature during the last several years. For the most part the optimization has been done for a single axis of the space vehicle[2,3]. Since there usually will be some cross coupling from one axis to another, this limited type of analysis is of only marginal interest in a practical case. Also, the results seem to present a rather complex mechanization problem[4]. In another reference[5] only the problem of angular velocity control with no position is discussed. While these references illustrate some of the difficulties one encounters in attempting to optimize an attitude control system, they do not give much assistance in the design of an actual system. The approach to be described in this paper will attempt to circumvent these

objections and to present a practical method of attitude control system design.

Development of the Mathematical Model

In the optimization problem the attitude of the space vehicle is represented by three Euler angles defined with respect to an inertial reference frame and by three angular velocities about a set of body axes. The initial conditions for acquisition are established when the spacecraft is separated from the boost vehicle. As an example, consider the following three equations which express the Euler angle rates corresponding to a roll-pitch-roll (ϕ, θ, Ψ) Euler sequence.

$$\dot{\phi} = \frac{\omega_x \sin \Psi + \omega_y \cos \Psi}{\sin \theta} \tag{1.1}$$

$$\dot{\theta} = \omega_x \cos \Psi - \omega_y \sin \Psi \tag{1.2}$$

$$\dot{\Psi} = \omega_z - \dot{\phi} \cos \theta \tag{1.3}$$

The initial conditions, which are ω_{x0}, ω_{y0}, and ω_{z0}, are determined at separation. These equations for the rigid body dynamics are the familiar Euler equations and are given in vector form by

$$I \, \dot{\bar{\omega}} = \bar{N} - \bar{\omega} \times I \, \bar{\omega} \tag{1.4}$$

where

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

$$\bar{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

and

$$\bar{N} = \begin{bmatrix} N_x \\ N_y \\ N_z \end{bmatrix}$$

The cross products of inertia are included because in general the control axes do not coincide with the principle axes. Equation (1.4) represents a set of three simultaneous equations. These equations together with the equation (1.1), (1.2) and (1.3) and the expression for the control law are then programmed for a solution on the digital computer.

Several problems arise in computer simulations of these equations which might be mentioned.[6] First of all, there is a large dynamic range which must be considered. The rates at separation are usually considered to be on the order of ten thousand degrees per hour while the rates for the limit cycle occurring at the end of acquisition are on the order of 1 degree per hour. Thus we have a ten thousand to one dynamic range in vehicle rate. This gives rise to accuracy problems in that to achieve sufficient accuracy in the 1 degree per hour range, the accuracy may have to be the same for the ten thousand degree per hour range. Thus the total accuracy of the problem during the initial part of the acquisition phase may have to be on the order of one part in a hundred thousand or even in a million. An associated problem area is the solution sensitivity to errors in computation. This problem must be investigated in order to determine just what errors in computation may be permissible without adversely effecting the resulting solutions. The third problem is that of time scaling. In general it is assumed that the acquisition will take approximately 15 minutes to half an hour. For a derived rate acquisition there are, however, phases of the acquisition in which time intervals on the order of 10 to 20 milliseconds will be important. Thus any computer program must detect these periods of short time importance and switch the time scaling of the problem in order to retain sufficient accuracy as well as to conserve computation time. An additional simulation problem results from the singularities in the equations for the Euler angle rates. Associated with each set of Euler angles are several spacecraft positions where the Euler angle rates approach infinity. In the vicinity of these points the computer must select another sequence of Euler angles whose associated rates are well behaved.

To illustrate this difficulty consider equations (1.1), (1.2) and (1.3). It can be seen that in the vicinity of

$$\theta = n\,\pi; \; n = 0, \, 1, \, 2, \, \ldots \ldots$$

equations (1.1) and (1.3) possess singularities.

Thus if the spacecraft body axes were aligned to the reference axes ($\emptyset$, θ and Ψ all zero) and the spacecraft was rotated an incremental angle $\Delta\alpha$ about its yaw axis, the three Euler angles would become

$$\emptyset = +\pi$$
$$\theta = +\Delta\alpha$$
$$\Psi = -\pi$$

from their starting values of zero, as shown in Figure 2. This rapid change of Euler angles is the result of attempting to use this Euler sequence in the vicinity of a point of singularity. If a roll-pitch-yaw ($\emptyset'$, θ', Ψ') sequence was chosen the corresponding Euler rate expressions would be

$$\dot{\emptyset}' = \frac{\omega_z \cos \Psi' - \omega_x \sin \Psi'}{\cos \theta} \tag{1.5}$$

$$\dot{\theta}' = \omega_z \sin \Psi' + \omega_x \cos \Psi' \tag{1.6}$$

$$\dot{\Psi}' = \omega_y - \dot{\emptyset}' \sin \theta' \tag{1.7}$$

where $\emptyset' =$ roll, $\Theta' =$ pitch, and $\Psi' =$ yaw. The $\Delta\alpha$ rotation about the yaw axis would correspond to a rotation from $\emptyset' = \Theta' = \Psi' = 0$ to

$$\emptyset' = 0$$
$$\Theta' = 0$$
$$\Psi' = \Delta\alpha$$

which indicates that this sequence is well behaved for the same spacecraft position. Thus we have a mathematical model which can be used in the optimization studies.

The Optimal Control Problem

In this section, the possibility of using optimal control theory for solving the attitude control problem will be examined. A brief introduction to the existing theory will be given and shortcomings of applying this theory to the attitude control problem will be pointed out. An extension of the theory which incorporates the physical and economic constraints will be developed.

A Typical Optimal Control Problem[7,8,9]

To enable a quick presentation of the existing optimization techniques pertinent to the attitude control problem, a fairly general optimal control problem will be posed next. A special case of the general problem is the attitude control problem.

The plant to be controlled is described by a system of ordinary differential equations of the form

$$\dot{x} = F(t, x, u) \tag{2.1}$$

where x, F, and u are column vectors of the form

$$x = \text{col} (x_1,\ldots\ldots, x_n)$$
$$F = \text{col} (F_1,\ldots\ldots, F_n) \tag{2.2}$$
$$u = \text{col} (u_1,\ldots\ldots, u_m)$$

In equation (2.1), it is assumed that F possesses at least piecewise continuous second partial derivatives with respect to all arguments.

In the attitude control problem eq. (2.1) corresponds to equations (1.1), (1.2) (1.3) and (1.4).

The vector x is defined as the state of the system and the vector u is defined as the control that can be exerted on the plant to change its state.

In all practical problems, the vector u is required to be in a closed set in the m dimensional Euclidean space R^m at each instant of time. Symbolically this is expressed as

$$u(t) \in U \tag{2.3}$$

where, for our discussion, we will assume that U is independent of x and t.

In the attitude control problem, the set U is defined from the physical requirement that each component of the torque exerted by its gas jets is limited in magnitude. In this case U is a closed bounded set in three dimensional Euclideon space defined by $|u_i| \leq k$, $i = 1, 2, 3$.

It is assumed that the system is in an initial state

$$x(0) = C \qquad (2.4)$$

The plant might be in the initial state given by (2.4) due to some prior disturbance.

The optimal control problem consists of exerting control on the plant over a suitable non-zero time interval $[0,T]$ in such a manner that a performance index of the type

$$I_1(u) = \varphi\left[x(T)\right] + \int_0^T f_1(t,x,u)dt \qquad (2.5)$$

is minimized.

In equation (2.5), φ and f are suitable scalar valued functions which possess at least sectionally continuous second partial derivatives with respect to all their arguments.

The terminal time T may be explicitly determined beforehand or implicitly specified.

There is no loss in generality in considering a performance index of the type

$$I(u) = \int_0^T f(t, x, u)dt \qquad (2.6)$$

The terminal state x(T) of the plant may be free or fixed depending on the particular plant to be controlled.

In the attitude control acquisition problem, one would require that the terminal state should correspond to a desired state which in general will require that the positions correspond to suitable orientation of the space vehicle and that the angular velocities are zero. A meaningful performance index will be to minimize the amount of fuel used in the acquisition. In general, the time available to accomplish the acquisition is fixed beforehand.

The minimum fuel type performance index will correspond to

$$f(t,x,u) = u^2 \qquad (2.7)$$

in equation (2.6)

The two basic methods for solving the optimal control problem yield different types of solutions. The first method yields the so called "control function" or "open loop" solution and the second method yields the "control law" or "closed loop or feedback" solution. Intuitively, one would feel that a closed loop solution is more desirable than the open loop solution.

The two methods will now be briefly sketched out.

<u>The Pontriagin's Maximum Principle</u>[8,10]: The Pontriagin's Maximum Principle
consists of forming an auxiliary function called the Hamiltonian which is intimately
related to the dynamical equations and performance index of the optimal control
problem. Before stating the maximum principle, it is convenient to define some
terms.

Assume that an optimal control u(t) exists in the interval $[0, t]$ such that
the performance index is indeed minimized with this control. Denote this <u>optimal
control</u> by u*(t). It is of course necessary to assume u(t)∈U for each value of t.
Let the associated trajectory obtained by solving equation (2.1) with initial
condition (2.4) be denoted by x*(t). The trajectory x*(t) is called the <u>optimal
trajectory</u>. Specifically, x*(t) satisfies

$$\dot{x}*(t) = F(t, x*, u*) \tag{3.1}$$

with $$x*(0) = C \tag{3.2}$$

The assumptions that u*(t), $0 \leq t \leq T$, is optimal implies from (2.6)

$$I(u*) \leq I(u) \tag{3.3}$$

for any control u(t).

The Pontriagin's Maximum Principle states that

$$H(t, x*, u*, \lambda*) \leq H(t, x*, u, \lambda*) \tag{3.4}$$

at each instant of time t, $0 \leq t \leq T$.

In equation (3.4), the scalar valued Hamiltonian H is defined as

$$H(t, x, u, \lambda*) = f(t, x, u) + \langle \lambda*, F(t, x, u) \rangle \tag{3.5}$$

where $\langle , \rangle$ denotes the Euclidean inner product.

In equation (3.5), $\lambda*$ is a column vector of the form

$$\lambda* = \text{col} (\lambda_1^*, \ldots \lambda_n^*) \tag{3.6}$$

which satisfied the system of equations

$$\dot{\lambda}* = - \frac{\partial H(t, x*, u*, \lambda*)}{x*} \tag{3.7}$$

where $$\frac{\partial H}{\partial x} = \text{col} (\frac{\partial H}{\partial x_1}, \ldots \frac{\partial H}{\partial x_n}) \tag{3.8}$$

From (3.5), it is evident that

$$\dot{x}* = \frac{\partial H(t, x*, u*, \lambda*)}{\partial \lambda*} \tag{3.9}$$

The minimization implied by the inequality (3.4) will yield, in principle,
the optimal control in the form

$$u* = k(t, x*, \lambda*) \tag{3.10}$$

Now define

$$H*(t, x*, \overset{*}{\lambda}) = H(t, x*, k(t, x*, \lambda*), \lambda*) \tag{3.11}$$

Rewriting equations (3.7) and (3.9) using (3.11) leads to the so called "canonic equations" which are satisfied along the optimal trajectory in the form

$$\dot{\lambda}* = - \frac{\partial H*(t, x*, \lambda*)}{\partial \lambda*} \tag{3.12}$$

and

$$\dot{x}* = \frac{\partial H*(t, x*, \lambda*)}{\partial x*} \tag{3.13}$$

Equations (3.12) and (3.13) are a set of 2n first order differential equations. To obtain a solution to these equations requires 2n boundary conditions. Notice that n initial conditions are specified by equation (3.2). To obtain additional conditions, one uses the transversality conditions.[8] The transversality conditions in the case when terminal time T is fixed and terminal state x(T) is free lead to

$$\lambda*(T) = 0 \tag{3.14}$$

If equations (3.12) and (3.13) are now solved with the boundary conditions (3.2) and (3.14), one obtains x*(t) and λ*(t) which determine the "open loop" control u*(t), $0 \leq t \leq T$, from equation (3.10). It is evident that obtaining the open loop solution for an optimal control problem leads to solving a two-point boundary value problem (TPBVP). Obtaining numerical solutions of TPBVP is far from trivial. A computational algorithm for solving certain classes of TPBVP is furnished by the method of quasilinearization[11, 12].

Bellman's Dynamic Programming Method[13,14,15]: It was pointed out that the use of the Pontriagin's Maximum principle leads to the formidable problem of solving a TPBVP. Moreover, the solution of the TPBVP yields an optimal control and trajectory for a specific value of the initial state of the plant. To obtain an optimal control and trajectory for a differential state will involve solving the TPBVP all over again. This problem can be circumvented in general by using the dynamic programming approach which yields the optimal control and trajectory for any arbitrary initial states and starting times. It is this fact which leads to a feedback solution.

Dynamic Programming is concerned with multi-stage decision processes. The key to dynamic programming is the "method of invariant imbedding" and the "principle of optimality".

The underlying idea for invariant imbedding is the following. Faced with the problem of determining certain properties of one particular process, one may attempt the analysis by considering that one process in isolation. However, it is often profitable and simpler to consider a whole family of processes of which the original process is a member, and try to interconnect the properties of neighbouring processes. Many structural properties of the given process can be determined using this technique.

The original optimal control problem is concerned with a process which starts at time zero in a fixed initial state C. Consider now the process with the same dynamical equations as the original process which starts at an arbitrary time $t = \tau$ in an arbitrary state $x(\tau) = z$ where optimization involves minimizing

$$\int_{\tau}^{T} g(t, x, u(dt \tag{4.1}$$

where $0 \leq \tau \leq T$ and $-\infty < z < \infty$

The original process has now been inbedded in a more general family of processes. When $\tau = 0$ and $z = C$ in the general family, the original process is singled out.

Now define a value function $J(z, \tau)$ as

$$J(z, \tau) = \underset{\substack{u(t)\epsilon U \\ \tau \leq t \leq T}}{\text{Min}} \int_{\tau}^{T} g(t, x, u)dt \tag{4.2}$$

for a process governed by equation (2.1) which at time τ is in state $x(\tau) = z$

Rewriting (4.2) as

$$J(z, \tau) = \underset{\substack{u(t)\epsilon U \\ \tau \leq t \leq T}}{\text{Min}} \left[\int_{\tau}^{\tau+\Delta} g(t, x, u) \, dt + \int_{\tau+\Delta}^{T} g(t, x, u) \, dt \right] \tag{4.3}$$

involving the principle of optimality, and taking limits when $\Delta \to 0$ yields

$$\frac{\partial J}{\partial \tau} + \underset{u(\tau)\epsilon\Omega}{\text{Min}} \left[g(\tau, z, u(\tau)) + < F(\tau, z, u(\tau)), \frac{\partial J}{\partial z} > \right] = 0 \tag{4.4}$$

Replacing τ by t the current time and z by x* the current state on an optimal trajectory in equation (4.4) yields

$$\frac{\partial J}{\partial t} + \underset{u(t)\epsilon\Omega}{\text{Min}} \left[g(t, x^*, u(t)) + < F(t, x^*, u(t)), \frac{\partial J}{\partial x^*} > \right] = 0 \tag{4.5}$$

From equations (3.5) and (4.5), it is evident that

$$\frac{\partial J}{\partial t} + \underset{u(t)\epsilon\Omega}{\text{Min}} \ H(t, x^*, u, \frac{\partial J}{\partial x^*}) = 0 \tag{4.6}$$

and $\lambda^* = \dfrac{\partial J}{\partial x^*}$ \hfill (4.7)

For a fixed terminal time, free terminal state problem it is clear from equation (4.2) that the boundary condition to be used in solving the nonclassical partial differential equation (4.6) is

$$J(z, T) = 0 \qquad (4.8)$$

for any finite z.

Equation (4.6) is the functional equation of dynamic programming (Bellman's equation) which when solved with the boundary condition (4.8) will explicitly yield $J(x^*, t)$. This in turn will yield from equations (3.10) and (4.7) the optimal control in the form

$$u^*(t) = k(t, x^*, \frac{\partial J}{\partial x^*} (x^*, t))$$

$$= \varphi(t, x^*) \qquad (4.9)$$

Equation (4.9) expresses the optimal control as a function of the current time t and the current state x^* which is indeed a feedback solution.

Critique of Methods of Optimal Control Theory

Open Loop Solution: The open loop solution in the first place is very undesirable for the attitude control problem. The control function is evaluated using specific initial conditions and has to be recomputed if the initial conditions are different. It is practically impossible to determine the state of the space vehicle when the acquisition mode is initiated. Moreover, even if the initial state is known, one is faced with the formidable problem of having to solve a TPBVP to determine the control function.

Closed Loop Solution: The closed loop solution is more appealing than the open loop solution. However it involves solving a nonlinear partial differential equation which is far from trivial. Even if one can obtain the control law explicitly, there are some serious practical problems in instrumenting the controller with physical hardware. In the first place, the optimal feedback controller is nonlinear in general. In the second place it requires knowledge of all the states for manipulation. In practice, this is a fiction because it is very difficult to measure all the states either due to inaccessibility of some of the states or unavailability of suitable transducers. Hence it is often uneconomical to build an optimal controller. In general one is forced to use some type of "sub-optimal" controller which is more economical. One such type of controller is the so called specific optimal controller. The specific optimal control (SOC) problem[16] is outlined next.

It should be pointed out here that it is sometimes possible to estimate the states of the system based on partial measurement. Often one can use the estimated values of the states in place of the actual values as inputs to the controller[17,18].

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The specific optimal control problem starts with the reasonable practical assumption that one knows beforehand the states which are accessible for manipulation by the controller as well as permissible forms of control laws. In other words the sensors and actuators are specified and the form of the controller can be decided upon before the system is optimized. It is assumed that the only unknowns in the controller are a finite set of parameters. The permissible forms of control laws are generally dictated by economic considerations. These somewhat vague statements will be made precise below.

The specific optimal control problem is the same as the optimal control problem formulated in a previous section with the additional condition that the form of the controller is restricted to be of the form

$$u = h(y, b) \tag{5.1}$$

where h, y and b are column vectors of the form

$$
\begin{aligned}
h &= \text{col } (h_1, \ldots h_m) \\
y &= \text{col } (y_1, \ldots y_p) \\
b &= \text{col } (b_1, \ldots b_k)
\end{aligned}
\tag{5.2}
$$

The vector y represents the variables accessible for manipulation and is of the form

$$y = \alpha(t, x) \tag{5.3}$$

where α is a known p dimensional vector function of its arguments.

In equation (5.1) it is assumed that h is a known vector function of its arguments and b represents an unknown set of k parameters.

The specific optimal control problem consists of determining the values of the parameter set b such that the index of performance is minimized when the controller is constrained to have the form given by equation (5.1)

The specific optimal control problem can now be reformulated as follows: From equations (2.1), (5.1) and (5.3), the modified dynamical equations are:

$$
\begin{aligned}
\dot{x} &= F\left[t, x, h\left\{\alpha(t, x), b\right\}\right] \\
&= G(t, x, b)
\end{aligned}
\tag{5.4}
$$

The system is in an initial state, from equation (2.4)

$$x(0) = C \tag{5.5}$$

From equation (2.6), the modified performance index to be minimized is

$$
\begin{aligned}
I\left[h\left\{\alpha(t, x), b\right\}\right] &= V(b) \\
&= \int_0^T f\left[t, x, h\left\{\alpha(t, x), b\right\}\right] dt
\end{aligned}
$$

i.e. $\quad V(b) = \int_0^T g(t, x, b)dt \tag{5.6}$

The vector valued function $G(t, x, b)$ in equation (5.4) and the scalar valued function $g(t, x, b)$ in equation (5.6) are defined in an obvious manner.

The specific optimal control problem can now be stated as follows: Given the dynamical equations in the form of equation (5.4) where the initial state is given by (5.5), determine the value of the constant vector b to minimize a performance index of the form given by equation (5.6). In this manner mechanization constraints can be included in the controlder design.

Methods of Solving the Specific Optimal Control Problem

Basically, three methods are available for solving the SOC problem. They may be generally classified as:

(a) Hill-Climbing methods
(b) TPBVP methods
(c) Differential approximation methods

All these methods are computational methods which are equally applicable to both linear and nonlinear systems. The first two methods yield exact solutions whereas the third method yields an approximate solution which is often justifiable on the basis of ease in computation. A brief description of the three methods follows:

Hill Climbing Methods[19,20]: This method views the SOC problem as that of determining the minimum of a k-dimensional hill which is described by $V(b)$ of equation (5.5). $V(b)$ is a function of a k-dimensional vector b or equivalently a function of k variables. Any choice for the vector b will yield a value for $V(b)$ which is obtained by numerically solving the differential equation (5.4) with initial condition (5.5) and then evaluating the integral in the right hand side of equation (5.6)

Any version of the hill climbing method is an iterative method involving a suitable algorithm for changing the value of the constant vector b during each iteration in such a manner that the value of $V(b)$ is smaller than its value in the previous iteration. The simplest version which is certainly not the most efficient consists of changing one component of b at a time in a suitable manner to assure a decrease in the value of $V(b)$. More sophisticated versions of Hill Climbing methods involve changing the value of b by a suitable amount at each iteration based on the value of the gradient of $V(b)$ at the trial value of b.

TPBVP Methods[16]: This method views the SOC problem as a variational problem. The Euler-Lagrange equations and the transverslity conditions of this variational problem lead to a TPBVP. Specifically, one adjoins to equation (5.4) the first vector differential equation.

$$\dot{b} = 0 \qquad (6.1)$$

which just re-iterates that the parameter vector is a constant. Thus equation (6.1) does not lead to any loss in generality.

The SOC problem is now equivalent to minimizing (5.5) subject to the differential constraints (5.4) and (6.1)

The Euler-Lagrange differential equations for this variational problem can be immediately written in terms of the scalar valued Lagrangian $L(t, x, \dot{x}, b, \dot{b}, \lambda, \mu)$ as

$$\frac{d}{dt} L_{\dot{x}} = L_x$$

$$\frac{d}{dt} L_{\dot{b}} = L_b \tag{6.2}$$

$$L_\lambda = 0$$

$$L_\mu = 0$$

where the Lagrangian is defined by

$$L(t, x, \dot{x}, b, \lambda, \mu) = g(t, x, b) + \langle \lambda, G(t, x, b) - \dot{x} \rangle - \langle \mu, \dot{b} \rangle \tag{6.3}$$

The vectors λ and μ are the "Lagrange Multiplier" vectors with components

$$\lambda = \text{col} (\lambda_1, \ldots, \lambda_n) \tag{6.4}$$

$$\mu = \text{col} (\mu_1, \ldots, \mu_k)$$

In equation (6.2), L_x is a vector with components

$$L_x = \text{col} \left(\frac{\partial L}{\partial x_1}, \ldots, \frac{\partial L}{\partial x_n} \right) \tag{6.5}$$

The vectors $L_{\dot{x}}, L_{\dot{b}}, L_b, L_\lambda$ and L_μ are similarly defined.

Equation (6.2) represents a system of $2n + 2k$ first order ordinary differential equations. The boundary conditions imposed on this set of equations is obtained from the transversality condition as

$$x(0) = C$$
$$\mu(0) = 0$$
$$\mu(T) = 0 \tag{6.6}$$
$$\lambda(T) = 0$$

The solution of equation (6.2) with boundary conditions (6.6) will yield $b(0)$ which is the optimal value of the parameters in the specific controller. One method of solving the TPBVP is the method of quasilinearization.

The Differential Approximation Method.[16]

The differential approximation method furnishes an approximate solution to the SOC problem. The control parameter vector b is picked in such a manner that the trajectory of the system with a specific controller is "closest" to the trajectory of the optimal control system in which no constraints are placed on the form of the controller. The term closest will be made precise later.

The differential approximation method requires a knowledge of the open loop solution to the optimal control problem with no constraint in the form of the controller. This implies that the TPBVP arising from the use of the Pontriagin's Maximum principle has been solved. In practice, it is always necessary to solve this problem to determine the absolute minimum that the performance index can take if no controller constraints are imposed so that one has a standard or basis for

comparing a specific optimal controller of a given form. If the SOC performance index is considerably larger than the optimal performance index, one may be forced to resort to a more complicated specific controller. Hence even if one uses the first two methods proposed here to solve the SOC problem, it is often necessary to obtain the open loop optimal control junction and the corresponding optimal trajectory for the optimal control problem with no controller constraints to obtain the optimal performance index to judge the "goodness" of the specific optimal controller of the given form.

Hence assume that the optimal trajectory for the optimal control problem has been determined using Pontriagin's maximum principle and the transversality conditions, the numerical solution for the TPBVP having been obtained by any method, for example, the quasi-linearization method. As usual this trajectory is denoted by $x*(t)$.

Denote the trajectory corresponding to the system with a specific controller by $x_s(t, b)$ which indicates the explicit dependence of the "specific trajectory" on the parameter vector b. It is evident from equation (5.4) that if there exists a value of the parameter vector b such that

$$\dot{x}*(t) = G(t, x*(t), b) \tag{6.7}$$

then the specific trajectory corresponding to this value of b will be exactly equal to the optimal trajectory $x*(t)$ and the specific performance index will be equal to the optimal performance index. This will then yield the best specific controller among all possible specific controllers. Unfortunately, it will not be possible to satisfy the identity (6.7) in general.

However, if one defines an "error" e(t, b) by

$$e(t, b) = \dot{x}^*(t) - G(t, x^*(t), b) \tag{6.8}$$

it seems reasonable to attempt to pick b such that a suitable "length" of the error is minimized. This vague concept can be made precise by requiring that b be picked such that

$$\int_0^T \left\| e(t, b) \right\|_\varphi^2 \, dt \tag{6.9}$$

is minimized. In equation (6.9), $\left\| e(t, b) \right\|_\varphi$ is a suitable quasi-norm in an n-dimensional vector space. In particular

$$\left\| e(t, b) \right\|_\varphi^2 = \langle e(t, b), \varphi e(t, b) \rangle \tag{6.10}$$

where φ is a suitable positive semi-definite matrix.

Using equation (6.8), minimization of expression (6.9) implies

$$\frac{\partial}{\partial b_i} \int_0^T \left\| \dot{x}^*(t) - G(t, x^*(t), b) \right\|_\varphi^2 \, dt = 0 \tag{6.11}$$

$$i = 1, 2, \ldots k$$

Equation (6.11) in general leads to k algebraic equations involving the k unknown components of b. This can be solved numerically to yield an approximate solution to the SOC problem.

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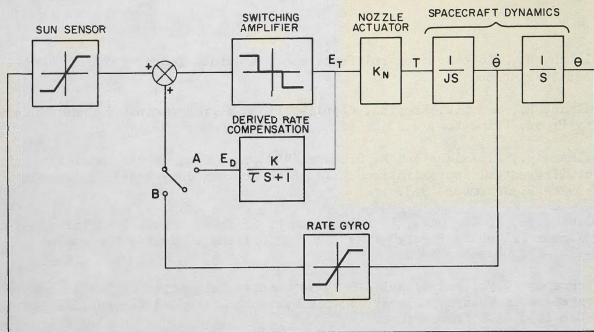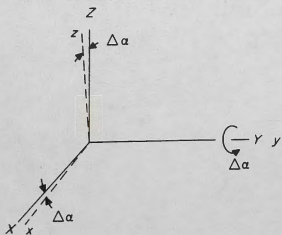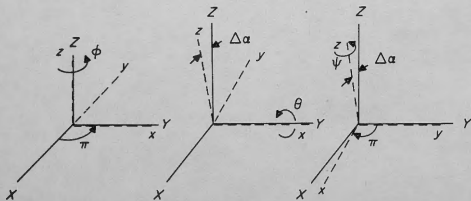
Figure 1 - Attitude Control System Block Diagram

SPACECRAFT AFTER A $\Delta\alpha$ YAW TURN

Figure 2 - Coordinate System