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Burnside's Lemma

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SENIOR PROJECT

Burnside's Lemma

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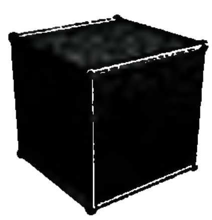
1 Introduction

Throughout my time at the University of Redlands, I have been drawn towards areas of math that are abstract. I enjoy knowing that an equation made up of completely abstract ideas can can be used to solve real world problems in a variety of subjects other than math.

This paper covers Burnside's Lemma including a proof and a variety of examples. It culminates with counting the number of unique Escher paintings that can be made. Also within this paper are discussion and proofs about both Pólia Enumeration and Sylow p-subgroups.

2 Example

Imagine that you wanted to paint the sides of a cube and you had three options of colors to use for each side. Initially one would think that, because there are six sides with three different options for colors there are $3^6 = 729$ ways to accomplish this task. This, however, is not the number of unique ways to accomplish this task. After noticing that a cube can be rotated and leave you with the same coloring of sides, you can see that the number of unique ways to color the cube is less than 729. This is where Burnside's Lemma comes in handy, because like George Pólya (1887-1985) said, "Mathematics consists of proving the most obvious thing in the least obvious way"(N. Rose).



3 History

In order to solve problems like the previous one and more complicated problems, a sophisticated approach is needed. In 1887, Georg Frobenius was able to provide the mathematical community with such an approach. His lemma, however, was not widely known until William Burnside proved it in 1911 in a book on group theory. Later on, in 1937, another mathematician, Pólya, used this lemma to solve combinatorial problems, which is now the main use of the lemma.

4 Permutations

In order to work towards the statement and proof of Burnside's Lemma, definitions for various concepts must first be discussed. These definitions will be used in the proofs that are to come. There is also a lemma and a theorem about permutations that will be utilized later in the paper.

Definition
A **bijection** is a function from a set A to a set B that is one-to-one and onto B. Another definition would be that the the function from A to B has an inverse.

Definition
A **permutation** of a set A is a function from A to A that is bijective. A **permutation group** of a set A is a set of permutations of A that forms a group under function composition.

Definition
The family of all permutations of a set X, denoted by S_X, is called the **symmetric group** on X. When $X = \{1, 2, ..., n\}$, S_X is usually denoted by S_n, and it is called the **symmetric group on n letters.**

The following Lemma, Theorem and proofs are adapted from *Advanced Modern Algebra* by Joseph J. Rotman.

Lemma 1.
If $\gamma, \alpha \in S_n$, then $\alpha\gamma\alpha^{-1}$ has the same cycle structure as γ. More specifically if the disjoint cycle decomposition of γ is

$$\gamma = \beta_1\beta_2...(i_1 i_2...)...\beta_t,$$

then $\alpha\gamma\alpha^{-1}$ is the permutation that is obtained from γ by applying α to the symbols in the cycles of γ.

Proof. Let β_i be a cycle and β_i' be the cycle when α is applied to each element in β_i. Let $\sigma = \beta_1'\beta_2'...(\alpha(i_1)\alpha(i_2)...)...\beta_t'$. If $\sigma(i) = i$, then it can be seen that σ fixes $\alpha(i)$. We also know that $\alpha\gamma\alpha^{-1}$ fixes $\alpha(i)$, because

$$\alpha\gamma\alpha^{-1}(\alpha(i)) = \alpha\gamma(i) = \alpha(i)$$

The last step is true, because we assumed that $\gamma = (i) \Rightarrow \sigma = (\alpha(i))$, which means that σ fixes $\alpha(i)$.

Now assume that γ does not fix i and instead moves i_1 to i_2. Thus $\gamma(i_1) = i_2$. Thus in the complete factorization of γ, there must be a cycle of the form $(i_1 i_2...)$. By the definition of σ, we know that one of the cycles is of the form $(kl...)$, where $\alpha(i_1) = k$ and $\alpha(i_2) = l$. Thus since $\gamma(i_1) = i_2$, $\sigma : k \to l$.

On the other hand, $\alpha\gamma\alpha^{-1} : k \to i_1 \to i_2 \to l$. Thus we now know that $\alpha\gamma\alpha^{-1}(k) = \sigma(k)$. Therefore, in general everything of the form $k = \alpha(i_1)$ are agreed upon by both σ and $\alpha\gamma\alpha^{-1}$. We know that α is surjective and thus every k is of the form $\alpha(i_1)$. Thus $\sigma = \alpha\gamma\alpha^{-1}$. $\square$

Theorem 1.
Permutations γ and σ in S_n have the same cycle structure if and only if there exists an $\alpha \in S_n$ such that $\sigma = \alpha\gamma\alpha^{-1}$.

Proof. $\Leftarrow$ Proved in the previous lemma.
$\Rightarrow$ Let the complete factorizations of γ and σ be as they are written below:

$$\begin{array}{ccccccccc} \gamma & = & \delta_1 & \delta_2 & ...(& i_1 & i_2 & ...) & ...\delta_t \\ \sigma & = & \nu_1 & \nu_2 & ...(& k & l & ...) & ...\nu_t \end{array}$$

Thus we will create a downward function $\alpha : \gamma \to \sigma$ (i.e. $\alpha(i_1) = k$ and $\alpha(i_2) = l$) and naming the function α. It can be seen that α is a permutation, because there was no repetition of elements when γ was factorized. Therefore by the previous lemma, $\sigma = \alpha\gamma\alpha^{-1}$. $\square$

5 Group Actions

Groups actions are a fundamental part of the study of algebra and will be used throughout this paper to help aid in definitions and proofs. The contents of this section can be attributed to *Abstract Algebra* by Dummit and Foote.

Definition
A **group action** of a group G on a set A is a map defined by $(g, a) \mapsto g \cdot a \in A$ for all $(g, a) \in G \times A$ that satisfies the following properties:

1. $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$, for all $g_1, g_2 \in G$ and $a \in A$, and
2. $e \cdot a = a$, for all $a \in A$, where e is the identity element of G.

Now a few examples will be presented in order to insure that the idea of a group action is understood and to work with using them inside a proof. From now on we will be using S instead of A to denote the set that is getting acted on.

Example 1.

Let G be the symmetric group S_4 acting on the set $S = \{1, 2, 3, 4\}$ by permuting its elements. In general, if $\sigma \in G = S_4$, then $\sigma \cdot i = \sigma(i)\ \forall i \in S = \{1, 2, 3, 4\}$. This defines a map from $G \times S$ to S. The the following example will illustrate the action of G on S:

- $(132)(12) \cdot 2 = 3$
- $(1234) \cdot 2 = 3$

- $(12)(34) \cdot 1 = 2$
- $(12)(3)(4) \cdot 3 = 3$
- $(1)(2)(3)(4) \cdot 3 = 3$

To prove this is a group action, properties 1 and 2 from the definition of a group action must be satisfied.

- Let $\sigma, \tau \in G$ and $i \in S$. Then

$$\begin{aligned} \sigma \cdot (\tau \cdot i) &= \sigma \cdot \tau(i) \\ &= \sigma(\tau(i)) \\ &= (\sigma \circ \tau)(i) \\ &= (\sigma \circ \tau) \cdot i \end{aligned}$$

- Let $i \in S$ and notice that (1) is the identity in the symmetric group. Then $(1) \cdot i = i \; \forall i \in S$.

Example 2.

Every group G acts on itself, $S = G$, by conjugation, which for each $g \in G$, is defined $\forall s \in S$ by $g \cdot s = gsg^{-1}$. This defines a map from $G \times G$ to G. Again, the two properties of a group action must be satisfied.

- Let $g, f, s \in G$. Then

$$\begin{aligned} g \cdot (f \cdot s) &= g \cdot (fsf^{-1}) \\ &= g(fsf^{-1})g^{-1} \\ &= (gf)s(f^{-1}g^{-1}) \\ &= (gf)s(gf)^{-1} \\ &= (gf) \cdot s \end{aligned}$$

- Since G is a group, it has an identity element e such that $ex = x = xe$ $\forall x \in G$. Thus using the group action, for any element $s \in G$, $e \cdot s = ese^{-1} = s$.

The definition of a group action leads to a few observations about the interactions between the group G acting on a set A that will be helpful in the following proofs and observations. In fact these observations can even be used as the definition of a group action.

Lemma

For each fixed $g \in G$ we get a map σ_g defined by

$$\sigma_g : A \to A.$$

$$\sigma_g(a) = g \cdot a.$$

Two important results that will be proven:

1. for each fixed $g \in G$, σ_g is a permutation of A, and
2. the map from G to S_A defined by $g \mapsto \sigma_g$ is a homomorphism.

Proof. (of 1) In order to see that σ_g is a permutation of A, it suffices to show that the map from A to A has a two-sided inverse. In fact, it will be shown that the map $\sigma_{g^{-1}}$ is the two sided inverse of σ_g. For all $a \in A$

$$\begin{aligned}(\sigma_{g^{-1}} \circ \sigma_g)(a) &= \sigma_{g^{-1}}(\sigma_g(a)) \\ &= g^{-1} \cdot (g \cdot a) \\ &= (g^{-1}g) \cdot a \\ &= 1 \cdot a \\ &= a\end{aligned}$$

This shows that $(\sigma_{g^{-1}} \circ \sigma_g)(a) = a$, $\forall a \in A$ and thus is the identity map from A to A. Similarly $(\sigma_g \circ \sigma_{g^{-1}})(a) = a$, $\forall a \in A$. Therefore $\sigma_{g^{-1}}$ is the two-sided inverse of σ_g and thus σ_g is a permutation of A. □

Proof. (of 2) Let $\theta : G \to S_A$ be the map defined by $\theta(g) = \sigma_g$. In order for θ to be a homomorphism, it needs to satisfy $\theta(g_1g_2) = \theta(g_1) \circ \theta(g_2)$ for all $g_1, g_2 \in G$. This is true if and only if $\theta(g_1g_2)(a)$ and $(\theta(g_1) \circ \theta(g_2))(a)$ are equal for every element $a \in A$. For all $a \in A$

$$\begin{aligned}\theta(g_1g_2)(a) &= \sigma_{g_1g_2}(a) \\ &= (g_1g_2) \cdot a \\ &= g_1 \cdot (g_2 \cdot a) \\ &= \sigma_{g_1}(\sigma_{g_2}(a)) \\ &= (\theta(g_1) \circ \theta(g_2))(a)\end{aligned}$$

This shows that $\theta(g_1g_2)(a) = \theta(g_1) \circ \theta(g_2)(a)$ for every element $a \in A$ and thus θ is a homomorphism. □

6 Orbit, Stabilizer, and Fixed Set

We will now look at a few other definitions that will be useful along with the properties of group actions. These definitions and properties were found in *Contemporary Abstract Algebra* by Gallian.

Definition

Let G be a group of permutations of a set S. For each $i \in S$, let the **orbit of i under G** be

$$orb_G(i) = \{g \cdot i | g \in G\}$$

Definition
Let G be a group of permutations of a set S. For each i in S, let the **stabilizer** of i in G be
$$stab_G(i) = \{g \in G | g \cdot i = i\}$$

Definition
Let G be a group of permutations of a set S. For each $g \in G$ the **fixed set of** g is
$$fix(g) = \{i \in S | g \cdot i = i\}$$

Each of these three definitions will be illustrated in the following example.

Example 3.

These concepts are easy to see in the following example. Let $S = \{1,2,3,4\}$ and $G = \{(1),(12),(34),(12)(34)\}$ with G acting as a permutation group on S. Then

$$\begin{array}{lll} orb_G(1) = \{1,2\} & stab_G(1) = \{(1),(34)\} & fix((1)) = \{1,2,3,4\} \\ orb_G(2) = \{1,2\} & stab_G(2) = \{(1),(34)\} & fix((12)) = \{3,4\} \\ orb_G(3) = \{3,4\} & stab_G(3) = \{(1),(12)\} & fix((34)) = \{1,2\} \\ orb_G(4) = \{3,4\} & stab_G(4) = \{(1),(12)\} & fix((12)(34)) = \phi \end{array}$$

Now that these terms have been defined, we will move on to prove a few theorems that will be useful when proving Burnside's Lemma. The first one will show that the $stab_G(s)$ is a subgroup of G.

Lemma 2.
If G acts on S, and $s \in S$, then $stab_G(s)$ is a subgroup of G.

Proof. In order to show that $stab_G(s)$ is a subgroup of G, it must be shown that it is a nonempty set that is itself a group.

Let $s \in S$. By the definition of a group and group action, there exists an identity element $e \in G$ such that $e \cdot s = s$. Therefore $e \in stab_G(s)$ and thus $stab_G(s)$ is nonempty.

Now we will check that $stab_G(s)$ is closed under inverses. Let $s \in S$ and $a \in stab_G(s)$.

$$\begin{aligned} & a \cdot s = s \\ \Rightarrow\ & a^{-1} \cdot (a \cdot s) = a^{-1} \cdot s \\ \Rightarrow\ & (a^{-1}a) \cdot s = a^{-1} \cdot s \qquad \text{Using property 1 from group action definition} \\ \Rightarrow\ & e \cdot s = a^{-1} \cdot s \\ \Rightarrow\ & s = a^{-1} \cdot s \end{aligned}$$

Thus $a^{-1} \in stab_G(s)$ and $stab_G(s)$ is closed under inverses.

We must now check that $stab_G(s)$ is closed under the group operation (product). Let $a, b \in stab_G(s)$ and $s \in S$. Then using the definition of the group action:

$$(ab) \cdot s = a \cdot (b \cdot s) = a \cdot s = s$$

Thus $ab \in stab_G(s)$ and $stab_G(s)$ is closed under the group operation (product) and $stab_G(s)$ is a subgroup of G.

□

The following is a well known theorem about the order of a group and will be useful while proving Burnside's Lemma.

Orbit Stabilizer Theorem
Let G be a finite group of permutations of a set S. Then for any i from S, $|G| = |orb_G(i)||stab_G(i)|$.

Proof. By Lagrange's Theorem, we know that for any $i \in S$

$$|G| = (\text{number of left cosets of } stab_G(i)) \cdot |stab_G(i)|$$

Thus it is sufficient to show a bijection between the set of left cosets of $stab_G(i)$ and $orb_G(i)$. In order to simplify the notation, we let $H = stab_G(i)$. Therefore define a function

$$\psi : \{\text{left cosets of } H\} \to orb_G(i)$$

by $\psi(aH) = a \cdot i$, for an element $a \in G$. Therefore ψ maps into $orb_G(i)$, because $a \cdot i \in orb_G(i)$. Thus we need to show that ψ is well defined, surjective, and injective.

In order to show that ψ is well defined, it must be shown that if $aH = bH$, then $\psi(aH) = a \cdot i = b \cdot i = \psi(bH)$. Since $aH = bH$, it is also true that $a \in bH$. This means that for some $h \in H$, $a = bh$. Therefore

$$a \cdot i = (bh) \cdot i = b \cdot (h \cdot i) = b \cdot i$$

which is what we wanted to show. Notice that the middle two expressions are equivalent by the definition of group action. Thus ϕ is well defined.

Next we want to prove that ψ is surjective. Let $y \in orb_G(i)$. Then by definition of orbit, $\exists g \in G$ such that $g \cdot i = y$. Then $\psi(gH) = g \cdot i = y$. Thus ψ is surjective.

Lastly, we want to prove that ψ is injective. Suppose that $\psi(aH) = \psi(bH)$. Then we know that $a \cdot i = b \cdot i$. Thus

$$\begin{aligned}
i &= e \cdot i && \text{(by group action definition)} \\
&= (a^{-1}a) \cdot i && \text{(by identity definition)} \\
&= a^{-1} \cdot (a \cdot i) && \text{(by group action definition)} \\
&= a^{-1} \cdot (b \cdot i) && \text{(by assumption)} \\
&= (a^{-1}b) \cdot i && \text{(by group action definition)}
\end{aligned}$$

Therefore $a^{-1}b \in H = stab_G(i)$. Thus $a^{-1}bH = H$ and $bH = aH$ which shows that ψ is injective.

We have shown that ψ is well defined and bijective as needed. Therefore, the number of left cosets of $H = stab_G(i)$ equals $|orb_G(i)|$. □

Definition
A family of subsets $\{A_i\}$ of a set X is called pairwise disjoint if $A_i \cap A_j = \phi$ for all $i \neq j$. A **partition** of a set X is a family of pairwise disjoint nonempty subsets whose union is all of X.

Theorem 2.
Let G be a group of permutations acting on a set S. The orbits of the elements in S constitute a partition of S.

Proof. Let $i, j, k \in S$. Then because the identity element sends i to i, $i \in orb_G(i)$. Thus each element $i \in S$ is in at least one orbit. Let $j \in orb_G(i) \cap orb_G(k)$. Thus $j \in orb_G(i)$ and $j \in orb_G(k)$. Then for some $\alpha, \beta \in G$, $j = \alpha(i)$ and $j = \beta(k)$. Therefore $\alpha(i) = \beta(k)$. Applying β^{-1} to each side will lead to $\beta^{-1}(\alpha(i)) = \beta^{-1}(\beta(k)$. Using the first property of group actions and the definition of an inverse, $(\beta^{-1}\alpha)(i) = k$. Let $x \in orb_G(k)$; then for some $\gamma \in G$, $x = \gamma(k) = (\gamma\beta^{-1}\alpha)(i)$. Thus $x \in orb_G(i)$. Therefore $orb_G(k) \subseteq orb_G(i)$. Similarly, $orb_G(i) \subseteq orb_G(k)$. Therefore, $orb_G(k) = orb_G(i)$. This means that if $k \in orb_G(i)$, then $orb_G(i) = orb_G(k)$. Thus all orbits must be disjoint or equal and all the elements in S must be in some orbit. Therefore the orbits of the elements in S constitute a partition of S. □

7 Burnside's Lemma

Burnside's Lemma
If G is a finite group of permutations acting on a set S, then the number of orbits of G on S is

$$\frac{1}{|G|}\sum_{g \in G} |fix(g)|.$$

This paper will include a variety of proofs of Burnside's Lemma. The first proof will be one fashioned after Gallian's proof in his book *Contemporary Abstract Algebra.*

Proof. In order to prove Burnside's Lemma, we shall count in two different ways the number of pairs, n, of the form (g, i), where $g \in G$, $i \in S$ and $g \cdot i = i$. The first way would be to look at $fix(g)$. For every g in G, the number of pairs

(g, i) where $g \cdot i = i$, would be $|fix(g)|$. Then in order to get the total number of pairs, one would need to sum $|fix(g)|$ for each $g \in G$ and thus

$$n = \sum_{g \in G} |fix(g)|$$

The next way to count the number of pairs would be to look at $stab_G(i)$. For each i in S, the number of pairs (g, i) where $g \cdot i = i$, would be $|stab_G(i)|$. Then in order to get the total number of pairs, one would need to sum $|stab_G(i)|$ for each $i \in S$ and thus

$$n = \sum_{i \in S} |stab_G(i)|$$

Thus we have two different expressions that allow us to count the number of pairs, n, of the form (g, i) where $g \cdot i = i$. Therefore, we are able to set them equal to each other:

$$n = \sum_{g \in G} |fix(g)| = \sum_{i \in S} |stab_G(i)| \tag{1}$$

Now we will focus on a single orbit of G. If s, t are in the same orbit of G, then because the orbits of the elements of S form a partition of S from the proof of Theorem 2,

$$orb_G(s) = orb_G(t)$$

Since the Orbit-Stabilizer Theorem says that $|G| = |orb_G(i)||stab_G(i)|$. Then it is also true that

$$\frac{|G|}{|orb_G(i)|} = |stab_G(i)|$$

Since both s and t have the same size for their orbit and the order of G is the same for both, then it follows that

$$|stab_G(s)| = |stab_G(t)|$$

For each orbit, we will sum the order of the stabilizer over the elements in the orbit

$$\begin{aligned} \sum_{t \in orb_G(s)} |stab_G(t)| &= |orb_G(s)||stab_G(s)| && \text{by summing over the orbits} \\ &= |G| && \text{by the Orbit Stabilizer Theorem} \end{aligned}$$

Now we will combine (1) about a single orbit.

$$\begin{aligned} n &= \sum_{g \in G} |fix(g)| = \sum_{i \in S} |stab_G(i)| \\ &= \sum_{orbits} \sum_{t \in orb_G(s)} |stab_G(t)| = |G| \cdot \text{(number of orbits)} \end{aligned}$$

From the second and the last part of this equation we have

$$\text{(number of orbits)} = \frac{1}{|G|}\sum_{g\in G}|fix(g)|$$

$\square$

Now that the elements of Burnside's Lemma are understood and one proof for Burnside's Lemma is complete, we will move on to show some examples of its use.

Example 4.

A jeweler would like to make necklaces containing six beads, each of which could be black or yellow. How many unique necklaces can they make?

In order to solve this problem using Burnside's Lemma, we will view each of the beads on the necklace as a vertex on a hexagon. Thus the symmetry group that we will be looking at is the dihedral group D6. Note that $S =$ the set of 2^6 colorings of the beads, when the beads are numbered $1-6$. There is this number, because each of the six beads has two options for the colors. Numbering the beads $1-6$ will allow us to keep track of which placement of a bead we are looking at on the necklace since there really is no clasp or start point on the type of necklace that we are discussing. Therefore the number of unique necklaces and the number of distinct colorings is equal to the number of orbits of $G = D6$ on S. This is because for example the necklace with beads in the odd positions colored yellow and beads in the even positions colored black is the same as the one with beads in the odd positions colored black and beads in the even positions colored yellow. This can be seen in Figure 1.

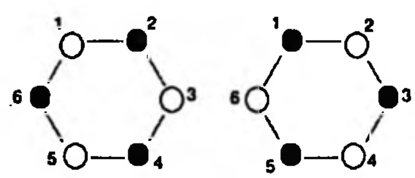


Figure 1: Equivalent Necklace Colorings

However when counting the number of elements in S, the two necklaces are different colorings, because we are looking at which color is assigned to each of the specific numbers.

In Figure 2 are the various rotations within D6. Each of these can be attained by rotating the vertices by a multiple of 60°, thus giving six options for rotations. In Figure 3 are the various reflections within D6. Each of the reflections is across the green line.

Thus the order of the group D6 would be 12. Now we will look at each element g of D6 and find the order of $fix(g)$.

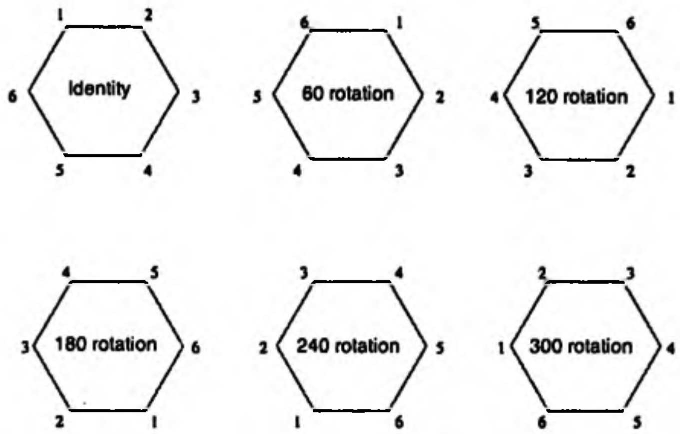


Figure 2: Rotations in D6

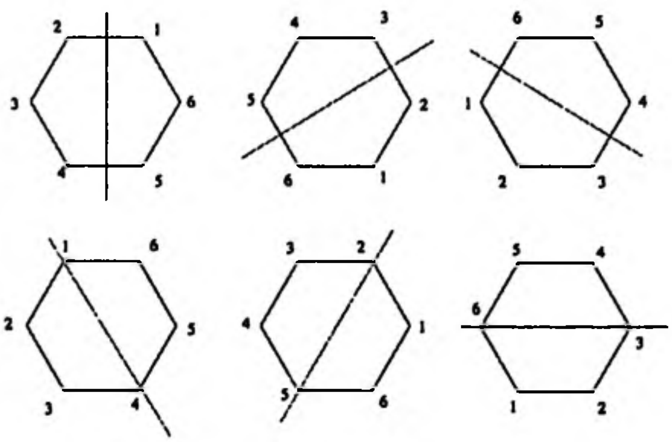


Figure 3: Reflections in D6

- If g is the identity element, then each one of the vertices is fixed under this action. Thus no matter what each vertex is colored, the entire coloring will be fixed. Therefore, $|fix(g)| = 2^6$.
- If g is the $60°$ rotation, then none of the vertices are fixed and there are no subsets of vertices that get sent to one another. Thus for an entire coloring to be fixed, all of the vertices must be the same color and since there are 2 colors, $|fix(g)| = 2^1$.
  - The same argument can be given if g is the $300°$ rotation.

- If g is the 120° rotation, then the vertices $1, 3$, and 5 rotate to one another and the vertices $2, 4$, and 6 rotate to one another. Thus as long as the vertices in the first set are colored the same and the vertices in the second set are colored the same, the coloring will be fixed. Therefore having two color choices for each set, $|fix(g)| = 2^2$.
  - The same argument can be given if g is the 240° rotation.
- If g is the 180° rotation, then the vertices 1 and 4 will be switched with one another, as will 2 and 5 with one another and 3 and 6 with one another. Thus as long as a vertex is the same color as the one that it is being switched with, the coloring will be fixed. Therefore having two color choices for each pairing, $|fix(g)| = 2^3$.
- If g is any one of the three reflections on the top of Figure 2, then it is a reflection across a line that is not through vertices. Thus no vertices get fixed and each vertex is switched with one other. Thus as long as a vertex is the same color as the one that it is being switched with, the coloring will be fixed. Therefore having two colors choices for each pairing will give each of these reflections, $|fix(g)| = 2^3$.
- If g is any one of the three reflections on the bottom of Figure 2, then it is a reflection across a line that is through two vertices. Thus two vertices get fixed, while the other four have a partner that they are switched with. Thus the color of the fixed vertices do not matter, but the other vertices must be the same color as the one it is being switched with. Therefore, having two color choices, for each of these reflections, $|fix(g)| = 2^4$.

A table of the previous calculations can be seen below.

Description of g	$\lvert fix(g)\rvert$
Identity	2^6
60° Rotation	2^1
120° Rotation	2^2
180° Rotation	2^3
240° Rotation	2^2
300° Rotation	2^1

Description of g	$\lvert fix(g)\rvert$
Reflection Through 1 and 4	2^4
Reflection Through 2 and 5	2^4
Reflection Through 3 and 6	2^4
Reflection Between 1 and 2	2^3
Reflection Between 2 and 3	2^3
Reflection Between 3 and 4	2^3

Therefore, using Burnside's Lemma, the number of unique colorings of the vertices and hence the number of unique necklaces that can be made is

$$\frac{1}{|G|}\sum_{g\in G}|fix(g)| = \frac{1}{12}[2^6+(2^1\times 2)+(2^2\times 2)+2^3+(2^3\times 3)+(2^4\times 3)] = \frac{1}{12}(156) = 13$$

These 13 options for unique necklaces can be seen in Figure 4.

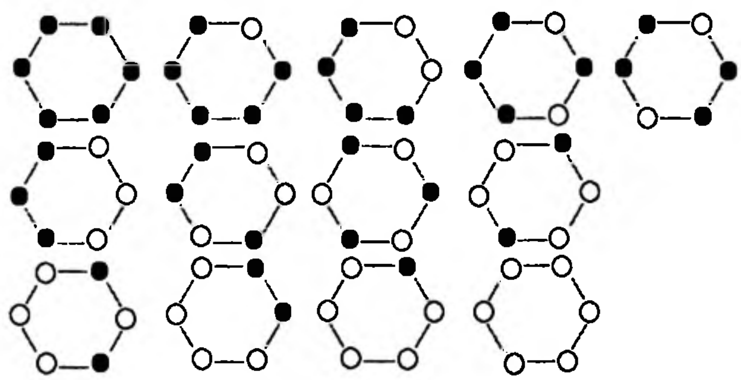


Figure 4: Unique necklaces with six beads

8 Another Way to Organize Calculations

The way that Burnside's Lemma is proved in *Contemporary Abstract Algebra* is straightforward; however it is not easy to visualize why it works. Thus the matrix organization in *Modern Algebra and Discrete Structures* may help clarify the proof and suggests another way to solve problems. Burnside's Lemma is restated below as a reminder.

Burnside's Lemma
If G is a finite group of permutations acting on a finite set S, then the number of orbits of G on S is

$$\frac{1}{|G|}\sum_{g\in G}|fix(g)|.$$

Suppose $G = \{g_1, g_2, ..., g_r\}$ and $S = \{i_1, i_2, ..., i_s\}$. Then the first step would be to create a matrix $A = [a_{kj}]$ of the size $r \times s$ where

$$a_{kj} = \begin{cases} 1, & \text{if } g_k \cdot i_j = i_j \\ 0, & \text{if } g_k \cdot i_j \neq i_j \end{cases}$$

The setup for such a matrix can be seen below.

	i_1	i_2	.. ..	i_s
g_1				
g_2				
$\vdots$				
$\vdots$				
g_r				

The next step is to count the number of 1's in the k^{th} row, which is equal to the number of elements $i \in S$ that are fixed when acted on by g_k and thus is $|fix(g_k)|$.

It is also helpful to note that the number of 1's in the j^{th} column will equal the number of elements $g \in G$ that fix i_j and thus is $|stab_G(i_j)|$.

Note that $\sum_{k=1}^{r} |fix(g_k)| = \sum_{j=1}^{s} |stab_G(i_j)|$. This is because both of these expressions are equal to the total number of 1's in the matrix, which equals n in our previous proof of Burnside's Lemma. We then have reached the point of equation (1) in that proof and can continue on from there exactly like we did before.

Example 5.

A student at a university has four keys on her key ring; two of which are for her apartment and two are for her office. The keys to the apartment are identical and the keys to the office are identical. How many unique orderings of the keys on the ring can there be?

To start off, notice that instead of looking at what color a vertex can be like in the previous examples, we will be looking at if a vertex is an apartment key or an office key. Thus we will be looking at a square with each vertex as a key as in Figure 5.

Figure 5: Picture of Key Arrangement

Now observe that that the group of symmetries for the arrangement of keys is made up of four rotations and four reflections. Thus the order of G is 8.

Below is a table like that introduced above and shows the calculations of Burnside's Lemma. Notice that the rows correspond to the elements of the symmetry group and thus the elements of G. The columns correspond to the arrangement of keys, where A stands for a key to their apartment and O stands for a key to their office.

	AA OO	AO OA	OA OA	AO AO	OA AO	OO AA
0°	1	1	1	1	1	1
90°	0	0	0	0	0	0
180°	0	1	0	0	1	0
270°	0	0	0	0	0	0
Horizontal Reflection	0	0	1	1	0	0
Vertical Reflection	1	0	0	0	0	1
Diagonal Reflection 1	0	1	0	0	1	0
Diagonal Reflection 2	0	1	0	0	1	0

By summing the elements in row g, we obtain $|fix(g)|$. Thus below is a table with $|fix(g)|$ for each of the g's.

g	$\|fix(g)\|$
0°	6
90°	0
180°	2
270°	0
Horizontal Reflection	2
Vertical Reflection	2
Diagonal Reflection 1	2
Diagonal Reflection 2	2

Thus the number of unique ways for the keys to be arranged on the ring is

$$\frac{1}{|G|}\sum_{g\in G}|fix(g)| = \frac{1}{8}[6+0+2+0+2+2+2+2] = \frac{1}{8}[16] = 2$$

This can easily be visualized, because either the two apartment keys are next to one another or they are separated by an office key.

9 Another Way to Count the Number of Distinct Colorings

Having done the last two examples, some patterns can be noticed in the calculation of $|fix(g)|$; thus a new formula for this value will now be introduced. The content in this section follows from *Modern Algebra and Discrete Structures* by R.F. Lax and from *Beyond Burnside's Lemma* by Lucas O. Wagner. To start a few variables must first be defined.

Notation

Let c be the number of color choices.

Notation

Let **x(g)** be the number of cycles within the permutation $g \in G$ when it is written in disjoint cycle form.

Examples of x(g)

- $g = (1234) \Rightarrow x(g) = 1$
- $g = (12)(34) \Rightarrow x(g) = 2$
- $g = (1)(2)(3)(4) \Rightarrow x(g) = 4$

Examples of x(g) in Example 4 about the necklace

- 120° rotation: $g = (135)(246) \Rightarrow x(g) = 2$
- Reflection through 1 and 4: $g = (1)(26)(35)(4) \Rightarrow x(g) = 4$
- Reflection between 2 and 3: $g = (23)(14)(56) \Rightarrow x(g) = 3$

Lemma 3.
If G is a finite group of permutations of n objects acting on a set S of colorings of the n objects in which the object can be any of c color choices and $g \in G$, then

$$|fix(g)| = c^{x(g)}$$

Proof. The original definition for $|fix(g)|$ was the number of colorings which are unchanged when acted on by g. As can be noticed in Example 4, a coloring remains the same when g acts on it, if all vertices within a vertex cycle have the same color. Thus there are c color choices for each of the vertex cycles. Therefore $\forall g \in G$, $|fix(g)| = c^{x(g)}$. $\square$

Now that there is a new way to calculate $|fix(g)|$, we will use it and Burnside's Lemma for another example.

Example 6.

Consider a checkerboard with four squares such that each square could be painted one of three colors (white, black, and gray). A picture of such a checkerboard is shown below in Figure 6. How many unique colorings of the board are there?

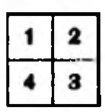


Figure 6: checkerboard with four squares

Just like in the previous example, we will treat each one of the corners as a vertex and thus the symmetry group that we will be looking at is the dihedral group D4 of the checkerboard. Note that S = the set of 4^3 colorings of the squares, when the corners are numbered $1-4$. Figure 7 shows the various elements of D4. There are four rotations, each of a degree that is a multiple of 90 and four reflections.

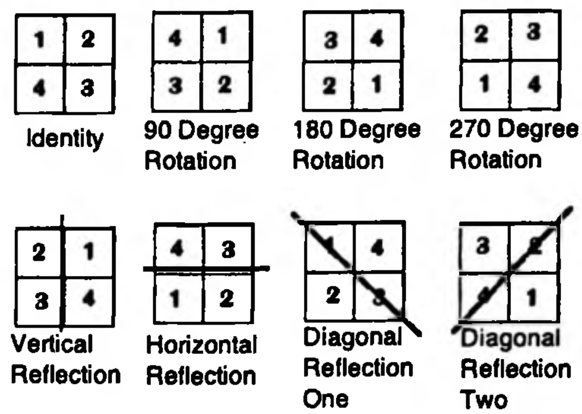


Figure 7: Rotations and Reflections of the Square

Now we must write each of the elements of D4 in cycle form in order to calculate $c^{x(g)}$. Below is a table with each of these calculations.

Description of g	Cycle form	$c^{x(g)}$
Identity	(1)(2)(3)(4)	3^4
90° Rotation	(1234)	3^1
180° Rotation	(13)(24)	3^2
270° Rotation	(1432)	3^1
Vertical Reflection	(12)(34)	3^2
Horizontal Reflection	(14)(23)	3^2
Diagonal Reflection One	(1)(24)(3)	3^3
Diagonal Reflection Two	(13)(2)(4)	3^3

Thus the number of unique colorings of a checkerboard with four squares and three options for colors is

$$\frac{1}{|G|}\sum_{g\in G} c^{x(g)} = \frac{1}{8}(3^4+3^1+3^2+3^1+3^2+3^2+3^3+3^3) = \frac{1}{8}(168) = 21$$

These 21 unique colorings can be seen below in Figure 8.

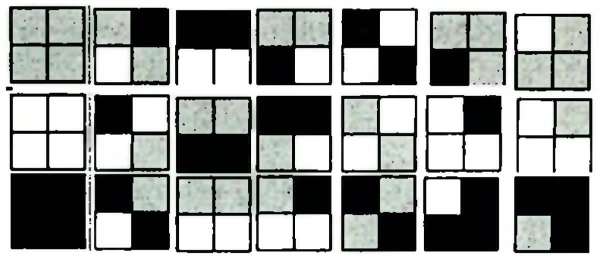


Figure 8: Unique Colorings of a Checkerboard

10 Pólya Enumeration

This section is written based on a chapter in *Modern Algebra and Discrete Structures*. The idea of Burnside's Lemma, especially the way of counting that was presented in Section 9, is very closely related to that of Pólya enumeration theory. Georg Pólya, who has been mentioned previously in this paper, was a Hungarian mathematician who developed this theory in order to count the number of different isomers of chemical compounds.

Definition
If $\rho \in S_n$ has $d_i(\rho)$ cycles of length i, for $i \in \{1, 2, ..., n\}$, in its representation as a product of disjoint cycles, then we let the polynomial associated to ρ be called the **cycle index of** ρ and be defined by

$$P_\rho(X_1, X_2, ..., X_n) = X_1^{d_1(\rho)} X_2^{d_2(\rho)} ... X_n^{d_n(\rho)}$$

The **cycle index of** a permutation group G is the polynomial

$$P_G(X_1, X_2, ..., X_n) = \frac{1}{|G|} \sum_{\rho \in G} P_\rho$$

In order to visualize this definition, we will apply it to D_4, which again is the dihedral group.

Example 7.

Remember that $D_4 = \{(1)(2)(3)(4), (1234), (13)(24), (1432), (12)(34), (13)(2)(4), (14)(23), (1)(24)(3)\}$ and acts on $S_4 = \{1, 2, 3, 4\}$. Therefore we can compute

$$P_{D_4} = (X_1^4 + 2X_1^2 X_2 + 3X_2^2 + 2X_4)/8$$

Notice that this is true, because (1)(2)(3)(4) has four cycles of length one (hence X_1^4), both (13)(2)(4) and (1)(24)(3) have two cycles of length one and one of length two (hence $2X_1^2X_2$), all three of (13)(24), (12)(34), and (14)(23) have two cycles of length two (hence $3X_2^2$), and finally both (1234) and (1432) have one cycle of length four (hence $2X_4$).

Notation

Let R be the set of elements that are being assigned to the elements in S.

In our examples so far, R would consist of the colors. In Example 6, R would have the elements white, black and grey.

Corollary 1.

The number of distinct orbits can be obtained by substituting the number,m, of elements in R into P for each X_i in the cycle index of G. Thus we have that

$$\text{(number of orbits)} = P_G(c, c, ..., c)$$

Proof. Notice that in Section 9, we denoted the number of elements in R by c. Note also that $x(g) = d_1(g) + d_2(g) + ... + d_n(g)$. Thus we have $P_g(c, c, ..., c) = c^{d_1(g)}c^{d_2(g)}...c^{d_n(g)} = c^{x(g)}$.

We have proved in Lemma 3 that $c^{x(g)}$ is equivalent to $|fix(g)|$. Therefore substituting into P_G, we have

$$P_G(c, c, ..., c) = \frac{1}{|G|}\sum_{g \in G}|fix(g)|$$

Based on Burnside's Lemma, one can see that $P_G(c, c, ..., c) =$ (the number of orbits). □

Definition

Suppose that $i \in S$. The **weight of** i, denoted by $W(i)$, is defined by

$$W(i) = r_1^{e_1(i)}r_2^{e_2(i)}...r_m^{e_m(i)}$$

where $r_s \in R$ for all $s \in \{1, 2, ..., m\}$ and $e_s(i)$ is the number of times the color r_s appears in the coloring of $A = \{1, 2, ..., n\}$, where A is the set on which the original permutation group G acts. In our previous examples, this would be the squares or the vertices.

We will now look at Example 6 and continue to number the squares of the checkerboard like in Figure 6. Denote a coloring by (V, X, Y, Z), where V is the color of square 1, X is the color of square 2, Y is the color of square 3, and Z is the color of square 4. The colors white, black, and gray will be denoted by the letters W, B, and G respectively. Therefore the weight of the coloring (G, B, G, W) is WBG^2 and the weight of the coloring (B, G, B, G) is B^2G^2.

Lemma 4.

Two elements in the same orbit of G on S have the same weight.

Proof. Suppose that s and t are in the same orbit. This is true iff $\exists g \in G$ such that $s = g \cdot t$. In this case, g permutes the elements of A so that $s = g \cdot t$ without changing any of the colors of the elements of A. Therefore $e_k(s) = e_k(t)$ for $k = 1, 2, ..., m$. Therefore $W(s) = W(t)$. □

Definition
Suppose that W_1, W_2,...,W_b are all of the possible weights of the elements of S and let p_z denote the number of orbits such that each element of that orbit has weight W_z, $z = 1, 2, ..., b$. Then

$$p_1W_1 + p_2W_2 + ... + p_bW_b$$

is called the **pattern inventory** of G on S.

Theorem 3. ***(Pólya-Redfield)***
Let $Y_i = r_1^z + r_2^z + ... + r_m^z$ for $r = 1, 2, ..., m$. Then

1. *The weights of elements of S that are fixed by $g \in G$ are the terms of $P_g(Y_1, Y_2, ..., Y_m)$.*
2. *The pattern inventory of G on S is given by*

$$P_G(Y_1, Y_2, ..., Y_m)$$

The previous theorem can be found in *Modern Algebra and Discrete Structures*, which states that the proof is tiresome and not very enlightening. Thus in this paper, we will see why it holds true in an example.

Example 8.

We will be using the same problem and notation from Example 4 to demonstrate these new concepts.

Description of g	Cyclic Decomposition of g	cycle index
Identity	(1)(2)(3)(4)(5)(6)	X_1^6
60° Rotation	(123456)	X_6
120° Rotation	(135)(246)	X_3^2
180° Rotation	(14)(25)(36)	X_2^3
240° Rotation	(153)(246)	X_3^2
300° Rotation	(165432)	X_6
Reflection Through 1 and 4	(1)(26)(35)(4)	$X_1^2X_2^2$
Reflection Through 2 and 5	(13)(2)(46)(5)	$X_1^2X_2^2$
Reflection Through 3 and 6	(15)(24)(3)(6)	$X_1^2X_2^2$
Reflection Between 1 and 2	(12)(36)(45)	X_2^3
Reflection Between 2 and 3	(14)(23)(56)	X_2^3
Reflection Between 3 and 4	(16)(25)(34)	X_2^3

Thus the cycle index of D_6 is

$$(X_1^6 + 3X_1^2X_2^2 + 4X_2^3 + 2X_3^2 + 2X_6)/12$$

Therefore the number of unique necklaces that can be made is

$$(2^6 + 3 \cdot 2^2 + 4 \cdot 2^3 + 2 \cdot 2^2 + 2 \cdot 2)/12 = 13$$

If we let Y denote a yellow bead and B denote a black bead, then the pattern inventory would be

$$\frac{1}{12}[(Y+B)^6+3(Y+B)^2(Y^2+B^2)^2+4(Y^2+B^2)^3+2(Y^3+B^3)^2+2(Y^6+B^6)]$$

$$=Y^6+Y^5B+3Y^4B^2+3Y^3B^3+3Y^2B^4+YB^5+B^6$$

Looking back at Figure 4, it is easy to see that these in fact are the various patterns that are possible for unique necklaces.

11 Escher's Patterns

Now that we have looked at a proof and some examples of the application of Burnside's Lemma, we will apply the lemma to a more complicated example, patterns in Escher's paintings. We will compute the number of unique patterns that Escher could make subject to certain restrictions. The work in this section is based off of work in *Escher's Combinational Patterns* by Dorris Schattschneider.

Escher's paintings, like the one in Figure 9, are made by repeating a certain image many times within the same painting, thus creating a pattern. Due to the repetition within the pattern, Burnside's Lemma can be used to count the number of unique patterns.

Figure 9: http://library.thinkquest.org/16661/index2.html

To start off with, rules of how the paintings can be created must be established. Escher would begin by taking a square and place some design inside of it. This square would be called a ***motif*** and would be created on a stamp. This motif would be repeated in order to create a 2x2 array that will be called the ***translation block***. Each of the four placements of the motif on the array can be a rotated or reflected version of the original motif, those being a different ***aspect***. Notice that if the motif is reflected, then another stamp must be created. The final step would be to take the translation block and place an exact version of it over and over again perpendicular to the edges of the plane that it is being placed on. This would then make a repeated pattern. This of course leads to the question; how many patterns can be made with the use of only one motif?

Like all other problems that require the use of Burnside's Lemma, the challenge is to make sure that patterns are not counted twice. We will consider the following cases, because they are the ones that Escher himself created.

1. If only one stamp is created and thus the translation block is only made up of rotations of the original motif.
2. Two of the images on the translation block are from one stamp and two are from the reflected stamp. This case itself also has a few different options within it.
   (a) The two that use the original stamp have the same aspect and the two from the reflected stamp have the same aspect.
   (b) The two that use the original stamp have different aspects and the two from the reflected stamp have different aspects.

In order to differentiate the various patterns that are produced, we must first establish a way of representing them. Thus we will represent the original motif by the letter A, the 90° rotation by the letter B, the 180° rotation by the letter C, and the 270° rotation by the letter D. Thus an example of a motif and its rotations would be Figure 10. These are then used to create the example translation block in Figure 11.

Figure 10: Schattschneider, Dorris, Escher's Combinational Patterns

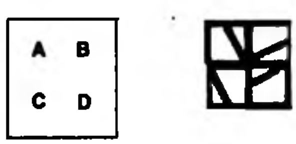


Figure 11: Example Translation block

Each translation block will receive a signature in order for us to refer to them more easily. This signature will be created by listing the letters starting in the top left corner and going clockwise. Therefore the signature for the translation block in Figure 11 would be ABDC.

For this example, S would be the set of patterns created by the motif. It is important to notice there are four distinct translation blocks and thus four signatures that represent the same pattern. Figure 12 is created by using the translation block from Figure 11. Notice that this pattern can be created using the signatures, ABDC, BACD, CDBA, and DCAB.

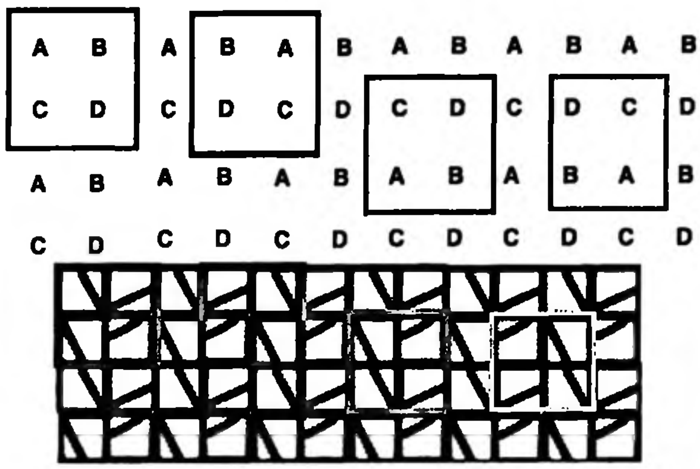


Figure 12: Pattern Created from Figure 11

11.1 Escher's Single-Stamp Pattern

Now we will look at Escher's single-stamp pattern and denote these repetitions of signatures within patterns as K_4. This would be the group of products of disjoint transpositions of the set $\{1, 2, 3, 4\}$. The elements of K_4 are listed in the table below along with the signature and color of box that corresponds to it from Figure 12. With these elements of K_4, we will let $A = 1$, $B = 2$, $C = 4$, and $D = 3$, therefore giving us a correspondence to the aspects in our example.

Element of K_4	Signature	Color of Box
$k_0 = e$	ABDC	Red
$k_1 = (12)(34)$	BACD	Purple
$k_2 = (14)(23)$	CDAB	Green
$k_3 = (13)(24)$	DCAB	Orange

Now we will define C_4 to be the group generated by r, where r rotates the entire translation block by 90° clockwise. Thus the rotations for r^2 and r^3 would be 180° and 270° respectively. In order to simplify notation later on, if X is an aspect of a motif, then X', X'' and X''' will represent the aspect that results from rotating the motif clockwise 90°, 180°, and 270° respectively. Thus again using the translation block from Figure 11, the effects of r^1, r^2, and r^3 can be seen in Figures 13-15 respectively, using both notation options. Notice that A' and B both represent a 90° rotation of the A motif. Both notations are used though, because B deals with the initial motif being rotated, whereas A' has to do with when the element of K_4 acts on the translation block. This is also true for the elements A'' and C and the elements A''' and D.

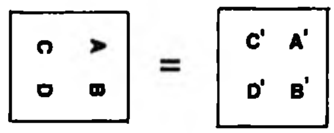


Figure 13: r^1 acting on the translation block from Figure 11

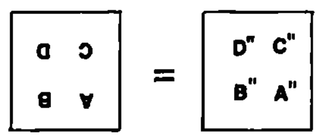


Figure 14: r^2 acting on the translation block from Figure 11

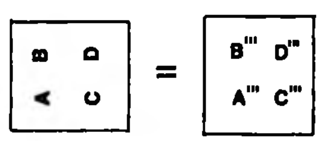


Figure 15: r^3 acting on the translation block from Figure 11

Now that we have defined both K_4 and C_4, we define H to be the group of elements generated by the products of elements in K_4 and C_4. This group H would act on the signatures and would produce equivalent signatures. This is because C_4 normalizes K_4 which will be shown later.

Before moving on a few terms must first be defined.

Definition
Let G be a group and N be a subgroup of G. Let g be an element in G with

its inverse denoted by g^{-1}. Then an element g is said to **normalize** N if $gNg^{-1} = N$. N is a **normal** subgroup if every element of G normalizes N. If A and B are subgroups of a group G, then B **normalizes** A if for every $b \in B$, $\{b^{-1}ab : a \in A\} = A$.

Now it will be shown that C_4 normalizes K_4.

Proof. Since $C_4 \leq S_4$ and $K_4 \leq S_4$, we want to show that every element $g \in C_4$ normalizes K_4 in S_4. We know that $C_4 = \{(1), (1234), (13)(24), (4321)\}$ and $K_4 = \{(1), (12)(34), (14)(23), (13)(24)\}$. Let $g \in C_4$ and $(\alpha\beta), (\gamma\delta) \in K_4$. Then $g(\alpha\beta)(\gamma\delta)g{-}1$ by Theorem 1 equals $(g(\alpha)g(\beta))(g(\gamma)g(\delta)) \in K_4$. Since this is true for every element of C_4, then we know that C_4 normalizes K_4. □

Also, by looking at the groups, we can see that $K_4 \cap C_4 = \{e\}$. Thus H is the semidirect product K_4C_4 and the order of H is 16. An element of H would be k_jr^i, where $k_j \in K_4$ and $r^i \in C_4$.

We will now go through all of the elements in H and determine which signatures are fixed by them. While doing these calculations, let P, Q, R, and S denote the four different aspects A, B, C and D. This will allow us at the end to multiply the number of fixed signatures by four since there are four different options for each aspect.

- $k_1(PQRS) = QPSR$
  Thus in order for the signature to be fixed we must have $P = Q$, $Q = P$, $R = S$ and $S = R$. Thus signatures that are fixed are of the form $PPQQ$.
- $k_2(PQRS) = SRQP$
  Thus in order for the signature to be fixed we must have $P = S$, $Q = R$, $R = Q$ and $S = P$. Thus signatures that are fixed are of the form $PQQP$.
- $k_3(PQRS) = RSPQ$
  Thus in order for the signature to be fixed we must have $P = R$, $Q = S$, $R = P$ and $S = Q$. Thus signatures that are fixed are of the form $PQPQ$.
- $r(PQRS) = S'P'Q'R'$
  Thus in order for the signature to be fixed we must have $P = S'$, $Q = P' = S''$, $R = Q' = S'''$ and $S = R' = S'''' = S$. Thus signatures that are fixed are of the form $PP'P''P'''$.
- $r^2(PQRS) = R''S''P''Q''$
  Thus in order for the signature to be fixed we must have $P = R''$, $Q = S''$, $R = P'' = R'''' = R$ and $S = Q'' = S'''' = S$. Thus signatures that are fixed are of the form $PQP''Q''$.
- $r^3(PQRS) = Q'''R'''S'''P'''$
  Thus in order for the signature to be fixed we must have $S = P'''$, $R = S''' = P'''''' = P''$, $Q = R''' = P''''' = P'$, and $P = Q''' = P'''' = P$. Thus signatures that are fixed are of the form $PP'P''P'''$.

- $k_1r(PQRS) = k_1(S'P'Q'R') = P'S'R'Q'$
  Thus in order for the signature to be fixed we must have $P = P'$, $Q = S'$, $R = R'$ and $S = Q' = S''$. This however is not possible, because $S \neq S''$. Thus there are no signatures that are fixed by k_1r.

- $k_1r^2(PQRS) = k_1(R''S''P''Q'') = S''R''Q''P''$
  Thus in order for the signature to be fixed we must have $P = S''$, $Q = R''$, $R = Q'' = R$ and $S = P'' = S$. Thus signatures that are fixed are of the form $PQQ''P''$.

- $k_1r^3(PQRS) = k_1(Q'''R'''S'''P''') = R'''Q'''P'''S'''$
  Thus in order for the signature to be fixed we must have $P = R'''$, $Q = Q'''$, $R = P'''$ and $S = S'''$. This however is not possible, because $S \neq S'''$. Thus there are no signatures that are fixed by k_1r^3.

- $k_2r(PQRS) = k_2(S'P'Q'R') = R'Q'P'S'$
  Thus in order for the signature to be fixed we must have $P = R'$, $Q = Q'$, $R = P'$ and $S = S'$. This however is not possible, because $S \neq S'$. Thus there are no signatures that are fixed by k_2r.

- $k_2r^2(PQRS) = k_1(R''S''P''Q'') = Q''P''S''R''$
  Thus in order for the signature to be fixed we must have $P = Q''$, $Q = P'' = Q$, $R = S''$ and $S = R'' = S$. Thus signatures that are fixed are of the form $PP''Q''Q$.

- $k_2r^3(PQRS) = k_1(Q'''R'''S'''P''') = P'''S'''R'''Q'''$
  Thus in order for the signature to be fixed we must have $P = P'''$, $Q = S'''$, $R = R'''$ and $S = Q'''$. This however is not possible, because $P \neq P'''$. Thus there are no signatures that are fixed by k_2r^3.

- $k_3r(PQRS) = k_1(S'P'Q'R') = Q'R'S'P'$
  Thus in order for the signature to be fixed we must have$S = P'$, $R = S' = P''$, $Q = R' = P'''$, and $P = Q' = P'''' = P$. Thus signatures that are fixed are of the form $PP'''P''P'$.

- $k_3r^2(PQRS) = k_1(R''S''P''Q'') = P''Q''R''S''$
  Thus in order for the signature to be fixed we must have $P = P''$, $Q = Q''$, $R = R''$ and $S = S''$. This however is not possible, because $S \neq S''$. Thus there are no signatures that are fixed by k_3r^2.

- $k_3r^3(PQRS) = k_1(Q'''R'''S'''P''') = S'''P'''Q'''R'''$
  Thus in order for the signature to be fixed we must have $P = S'''$, $Q = P''' = S''$, $R = Q''' = S'$ and $S = R''' = S$. Thus signatures that are fixed are of the form $P'''P''P'P$.

Now that we have gone through all of the elements of H, below is a table that lists each element and the number of signatures that it fixes. Notice that there are four choices for every unique letter in the signature.

Element of H	# of Fixed Signatures
e	256
r	4
r^2	16
r^3	4
k_1	16
k_2	16
k_3	16
k_3r	4
k_1r^2	16
k_2r^2	16
k_3r^3	4

Now we are able to use Burnside's Lemma. Thus the number of unique patterns that can be created using a single motif and the rules of Escher's first case is

$$\frac{1}{16}(256 + 6 \times 16 + 4 \times 4) = \frac{1}{16}(368) = 23$$

Figures 16 and 17 show an example of the 23 different patterns that can be produced using the motif in Figure 111.

11.2 Escher's Case Two (A)

Remember that the second case that Escher created is when two of the images on the translation block are from one stamp with the same aspect and two are from the reflected stamp with the same aspect.

Thus we must now change the groups that we are working with in order to account for the fact that two of the images are reflected. Thus, instead of C_4, we will be using D_4, which includes all four rotations and four reflections. We will, however, still be using the same K_4 from Escher's first case. Now define M to be the group of elements generated by the products of elements in K_4 and D_4. We must now show that D_4 normalizes K_4.

Proof. Since $D_4 \leq S_4$ and $K_4 \leq S_4$, we want to show that every element $g \in D_4$ normalizes K_4 in S_4. We know that $D_4 = \{(1), (1234), (13)(24), (4321), (12)(34), (14)(23), (1)(24)(3), (2)(13)(4)\}$ and $K_4 = \{(1), (12)(34), (14)(23), (13)(24)\}$. Let $g \in D_4$ and $(\alpha\beta)(\gamma, \delta) \in K_4$. Then $g(\alpha\beta)(\gamma\delta)g^{-1}$ by Theorem 1 equals $(g(\alpha)g(\beta))(g(\gamma)g(\delta)) \in K_4$. Since this is true for every element of D_4, then we know that D_4 normalizes K_4. □

The next step would be to go through each of the elements of this semidirect product to see which of the signatures are fixed. This however would entail looking at 32 different elements. The best way to approach this problem would be to create a computer program that would run through each of the signatures. This paper does not include the program or the way to solve it, because of the time limits.

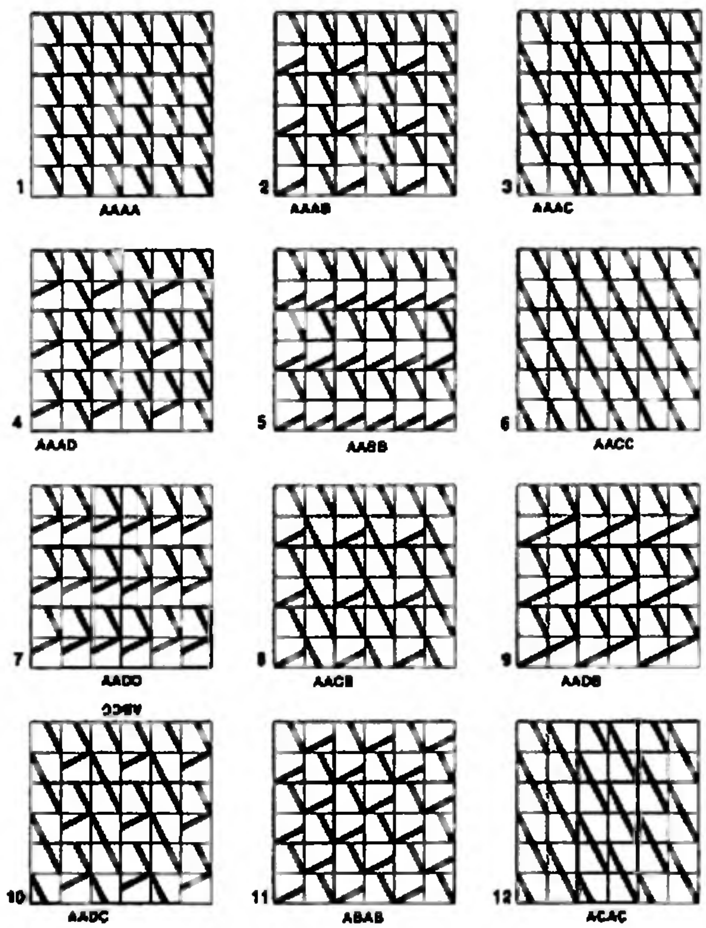


Figure 16: Schattschneider, Dorris, Escher's Combinational Patterns

11.3 Escher's Case Two (B)

This case that Escher created is when two of the images on the translation block are from one stamp with different aspects and two are from the reflected stamp with different aspects. With these restrictions, there are $6 \times (4 \times 4) \times (4 \times 4) = 1536$ different signatures that are possible. This is because there are 6 different ways to have two motifs from one stamp and two from the reflected stamp placed on a translation block. Then there are four choices for the aspect of the original motif and four choices for the aspect of the reflected motif.

Due to the number of signatures that are possible and the sizes of the groups that we would be working with, the author of *Eschers Combinational Patterns* states that the best way to approach this case would be to create a computer

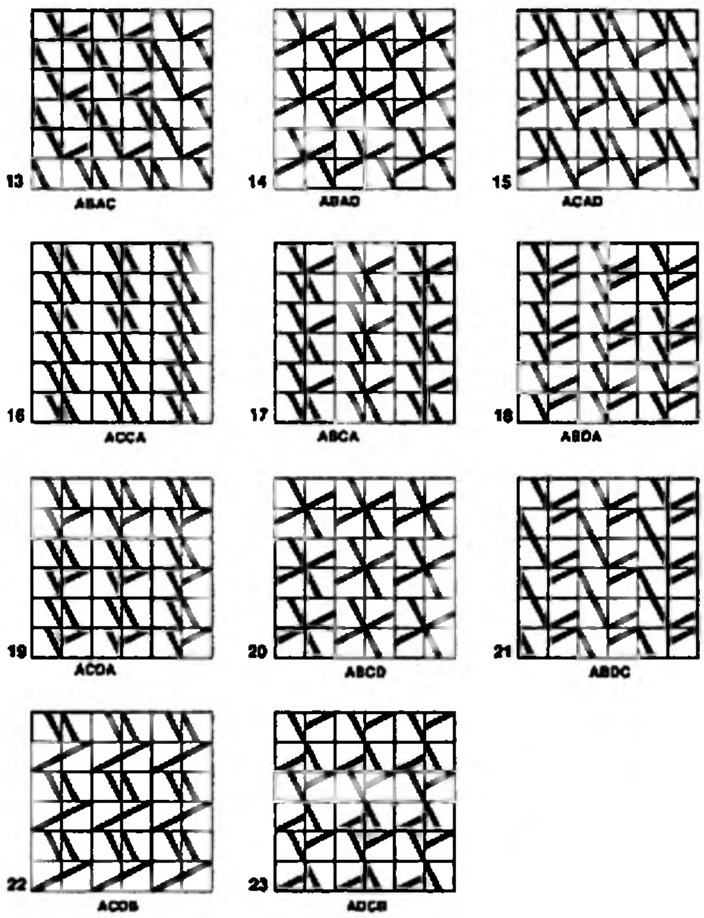


Figure 17: Schattschneider, Dorris, Escher's Combinational Patterns

program that "performs permutations on the signatures and sorts them into equivalence classes". Therefore this paper will not include the calculations for this case. However, it is interesting to note the number of patterns which is 49.

12 The Colorings of a Cube

Before concluding this paper, we will go back to the very first example that was given and solve it using Burnside's Lemma. This problem stated that you want to paint the sides of a cube and you had three options of colors to use for each side. With this example, we will be using the group S_6 acting on the set containing each of the faces of the cube numbered $1-6$. The tables below list each of the elements of S_6 written out in disjoint cycle form, along with the calculation of its fix using the method presented in section 9. The elements in the first table are rotations about an axis through the center of two opposite faces. The elements in the second table are rotations about an axis through two diagonally opposite vertices. The elements in the third table are rotations about an axis through the midpoints of two diagonally opposite edges.

Element of S_6	$\lvert fix(g)\rvert$
(1)(2)(3)(4)(5)(6)	3^6
(1)(3)(2546)	3^3
(1)(3)(24)(56)	3^4
(1)(3)(2645)	3^3
(2)(4)(1536)	3^3
(2)(4)(13)(56)	3^4
(2)(4)(1635)	3^3
(5)(6)(1234)	3^3
(5)(6)(13)(24)	3^4
(5)(6)(1432)	3^3

Element of S_6	$\lvert fix(g)\rvert$
(152)(364)	3^2
(125)(346)	3^2
(145)(263)	3^2
(154)(236)	3^2
(164)(235)	3^2
(146)(253)	3^2
(126)(345)	3^2
(162)(354)	3^2

Element of S_6	$\lvert fix(g)\rvert$
(15)(24)(36)	3^3
(14)(23)(56)	3^3
(16)(24)(35)	3^3
(12)(34)(56)	3^3
(13)(26)(45)	3^3
(13)(25)(46)	3^3

Now we are able to use Burnside's Lemma. Thus the number of unique colorings of a cube with three color options is

$$\frac{1}{24}\{(3^6) + (3^4 \times 3) + (3^3 \times 12) + (3^2 \times 8)\} = \frac{1}{24}(1368) = 57$$

Due to the fact that 57 pictures of a cube would take up a lot of room in this paper, the various unique colorings of a cube will not be depicted.

13 Conclusion and Future Research

This paper has included definitions and necessary proofs of lemmas and theorems that built up to Burnside's Lemma. These parts helped combine into the proof of Burnside's Lemma. After having proven it, a number of examples were solved using a few different ways to compute the numbers needed to plug into Burnside's Lemma.

This Lemma is used for counting the number of orbits of one group acting on another and thus has a lot of different applications in a variety of subjects. The one that interested me the most was the patterns in Escher's paintings, because it is the type of problem that most people do not believe can be solved using a mathematical equation.

I have always been very interested in Abstract Algebra and its usefulness in other subjects. I am fascinated by how such abstract thinking can be used to prove such complex and concrete problems. In writing this paper, I have been able to further my knowledge in this field of math that I love. Having to learn new ways to prove theorems and lemmas and improving my math writing skills has challenged me and I am really happy for that.

If I were to continue on with this project, I would spend the time creating the computer program to solve Escher's part B of the second case. This would challenge not only my understanding of the math behind Burnside's Lemma, but also the programming skills that I have learned through my computer science minor.

Throughout this journey of working on my project, I believe that there are two very important lessons that I will take away. The first one is to be careful about your citations, because you might end up getting credit for something someone else discovered. The other lesson is to brag about everything that you accomplish, because if Georg Frobenius had done so, my project may have been titled Frobenius' Lemma. Therefore I am not afraid to say that I am proud of this completed project!

14 References

Dummit, David Steven, and Richard M. Foote. *Abstract Algebra.* Upper Saddle River, NJ: Prentice Hall, 1999. Print.

Gallian, Joseph A. *Contemporary Abstract Algebra.* Boston, MA: Houghton Mifflin, 2006. Print.

Gilbert, William J. *Modern Algebra with Applications.* New York: Wiley, 1976. Print.

Lax, R.F. *Modern Algebra and Discrete Structures.* New York: HarperCollins Publishers Inc. 1991. Print.

Rotman, Joseph J. *Advanced Modern Algebra.* Upper Saddle River, NJ, 1996. Print.

Schattschneider, Doris, Escher's Combinational Patterns, *Electronic Journal of Combinatorics*, 4(2) (1997): R17.

Shifrin, Theodore. *Abstract Algebra: A Geometric Approach.* Englewood Cliffs, NJ: Prentice Hall, 1996. Print.

N. Rose, *Mathematical Maxims and Minims*, Raleigh NC: Rome Press Inc., 1988.

Wagner, Lucas O. *Beyond Burnside's Lemma*, June, 2008. http://www.rose-hulman.edu/mathjournal/archives/2008/vol9-n2/paper8/v9n2-8pd.pdf