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Black-Scholes and Monetary Black Holes

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Introduction

Mathematics has been a very dynamic force on Wall Street for over thirty five years. Until the introduction of computing to support complex theoretical models, Wall Street used basic algebra to determine future values of bonds and bills as well as good old fashioned intuition trying to "guess" the movements of the stock market. Beginning in 1973, Wall Street went hi-tech. Investment banking firms hired physicists, computer scientists, and mathematicians like the Yankees are hiring future hall of famers. There is an ever increasing need for employees with strong mathematical backgrounds and a keen interest in discovering new methods of evaluation for the ever changing world of finance. The boom in the industry of mathematical finance all started in 1973 with the publication and implementation of an article written by Fischer Black and Myron Scholes called "The Pricing of Options and Corporate Liabilities,"[1] but better known as the Black-Scholes formula.

The Black-Scholes formula is fundamental to modeling carried out in the financial world. Black-Scholes presents investment bankers a method of evaluating a stock in order to know how much to charge for a premium when dealing with options; a way of selling and/or buying securities at a predetermined time in the future. Understanding how Black-Scholes and options theory work relies on understanding probability, statistics, and Brownian motion. These tools make it possible for Black-Scholes to do its dirty work.

Public and private companies give out stock options to their employees annually. These options have the potential to cost the company millions or billions if the stock increases enough. The question is: How do companies account for these costs? Recent changes in the tax laws require public and private companies to determine the future value of the option(s) given. To account for the bonuses, public companies will use Black-Scholes or a method derived from Black-Scholes. The problem arises with private companies. Because they aren't publicly traded they can't use Black-Scholes directly. This paper will present a method for the valuation of a stock for a private company

Chapter 2

Language of Options

The Black-Scholes formula sets the value of an option. A good question to ask right now is: What is an option and how is it valued? Black-Scholes uses these terms as well as a number of others that probably are not familiar to the typical reader. The purpose of this section is to acquaint the reader with all of the necessary language of financial instruments and markets in order to grasp what Black-Scholes tells us.

2.1 Definitions

The broadest type of financial instrument is a **security.** A security is some form of financial asset whether it be stock, bonds, commodities, *etc.* **Equity**, simply put, is ownership. For instance, owning 100 shares of company XYZ means that you have equity in XYZ. For a publicly traded company, stock is equity. Stock is easily traded and therefore is highly liquid; in other words you can buy or sell stock that is publicly traded at any time. The **spot price**, S_t, of a stock is the current market value of the stock. For example, the spot price of company XYZ above could be \$50 per share today.

Black-Scholes was originally applied to options. Hence it is important to thoroughly understand what options are and how options work. An **option** is the right to buy (in the case of a **call** option) or sell (in the case of a **put** option) an equity in a certain company.[1] If one decides to act on their option then one **exercises** their right to purchase or sell equity. The **strike price** of an option is the price agreed upon by the buyer and seller of the option (or the market if the option is publicly traded) that will be paid if the option is exercised. It is important to understand the difference between the spot price and strike price. The spot price is the current market value of the equity whereas the strike price is a predetermined price that is agreed to be paid in the future for the equity if exercised.

[1]The options market originated in the financial industry as an alternative method to investing in order to entice more activity from investors.

The agreements on stock options typically last for 3, 6, or 12 months. These periods are referred to as **time to maturity.** At the time of maturity the buyer (in the case of a call option) must decide whether or not to exercise the option. Recall that the spot price is denoted by S_t. For the remainder of the paper the strike price will be denoted as K, the day on which the agreement begins will be denoted by t, and the expiration date will be denoted by T. For a call option, a stock is considered in-the-money if $(S_T - K) > 0$, denoted $(S_T - K)^+$ and out-of-the-money if $(S_T - K) \leq 0$, denoted $(S_T - K)^-$. The opposite is true for a put option. Investors want to enter this market because it minimizes their investment. If a stock finishes out-of-the-money then an investor does not exercise his/her right to buy the stock and only loses the cost of the call option.

Example 2.1.1. *Katie has entered into a call option with her trader. The spot price at time 0 is \$100 and the strike price is \$104. At time t the price of the stock is \$110. Since (S - K) is greater than zero or* $(\$110 - \$104) = \$6 > 0$, *Katie will exercise her option and buy the stock at \$104 per share. Since the stock is worth \$110 Katie will make \$6 per share and finish in-the-money.*

Example 2.1.2. *Allison has entered into a put option with her trader. The spot price at time 0 is \$70 and the strike price is \$65. At time t the price of the stock is \$67. Since* $(K - S)$ *is less than zero 0, or* $(\$65 - \$67) = -\$2 < 0$, *Allison will not exercise her option to sell the stock. In this case the option finished out-of-the-money and therefore was not exercised.*

In the above examples there were premiums involved with the transactions. A **premium** is the amount charged to the investor above the current spot price. In example 2.1.1 the premium was \$4 dollars per share, and the premium was \$5 per share in example 2.1.2. We are not so much concerned with premium as we are with the fair value. The premium is agreed upon and then the writer of the option needs an additional charge to make the bet fair. The **fair value** can be thought of as the 'production cost' of the option. It is the value at which the option can be sold, say, by a trader writing the deal, so that neither a profit is made nor a loss incurred in the transaction. The options premium, the amount for which the option is sold, is a different matter [3]. This can be seen through the following example.

Example 2.1.3. *Say we have a card game using only aces through tens of a regular playing deck. We draw one card from this deck. We win \$1 if the card drawn is odd and win nothing if an even card is drawn. How much should the house charge to play this game? Since half of the cards in the modified deck are odd then the house can expect to win half of the time and lose the other half. Thus a "fair value" to enter the game would be \$0.50. If the house were interested in turning a profit then the house could charge a "premium" of \$0.02 per game. In this instance each game would cost us \$0.52 to play with the house.*

2.2 Types of Options and Hedging

There are two different types of options relating to the time at which one can exercise an option, an **American** option and a **European** option. Entering an American option gives the holder the right to exercise the option at any time between the initial transaction and the maturity date. Whereas entering a European option, the holder can exercise the option only at the time of maturity. Originally, *Black-Scholes was used with European options only.* This meant that for Black-Scholes to work one must have been working with an option which could only be exercised at the time of maturity.

Calls and puts offer different types of positions, a long and short position respectively. A **long position** is a position in which a security is owned whereas a **short position** is a position in which a security is owed. In example 2.1.1 Katie exhibited a long position at T because she physically bought the stock and had equity in the company. In example 2.1.2 Allison could have held a short position if she had chosen to exercise her option. Her position would have been short because she would have sold stock that she didn't have. Hence, she would have been short the stock. This short of stock could be easily rectified by simply buying the stock from a trader.

When entering a market a broker would like to make 'bets' completely risk free. This means that no matter what happens-the stock goes up or down-the broker losses no money. This risk free idea can be achieved by hedging. **Hedging** simply refers to playing both sides in order to minimize risk, thus we hedge our risk yielding a riskless portfolio. More on hedging will be discussed later in this paper.

2.3 Interest Rates

We now shift our focus to interest rates and present value analysis. "The **Interest rate** is the cost to borrowers of obtaining money and the return (or yield) on money to lenders. **Present value** is the value today of funds to be received or paid on a future date" [2].

Let the **principal** (the amount borrowed) be P_0 and the annual interest rate at which you borrow the principal be r. Then the amount to be repaid one year from now is

$$P_0 + r \cdot P_0 = P_0 \, (1 + r) \, . \tag{2.3.1}$$

In this case the principal is compounded annually until the principal plus the interest is paid off. What if our debt is compounded quarterly? After 3 months we owe

$$P_0 \cdot \left(1 + \frac{r}{4}\right) .$$

The next three months we must make another payment on $P_0 \cdot (1 + \frac{r}{4})$ yielding

$$\left[P_0 \cdot \left(1 + \frac{r}{4}\right)\right] \cdot \left(1 + \frac{r}{4}\right) .$$

This pattern continues until a full year is elapsed, yielding

$$\left[\left[\left[P_0 \cdot \left(1+\frac{r}{4}\right)\right]\left(1+\frac{r}{4}\right)\right]\left(1+\frac{r}{4}\right)\right]\left(1+\frac{r}{4}\right) = P_0 \cdot \left(1+\frac{r}{4}\right)^4.$$

Leading to the general equation

$$A = P \cdot \left(1+\frac{r}{n}\right)^{n \cdot t}. \tag{2.3.2}$$

where A is the total amount owed on the principal P, when the interest rate is r, the number of times compounded in a year is n, and the number of years contracted is t. Looking at equation 2.3.2, a good question to ask would be: What happens if we let $n \to \infty$? In other words, what if we have continuous compounding? The answer to this question leads us to

$$\begin{aligned} \lim_{n\to\infty} P \cdot \left(1+\frac{r}{n}\right)^{n \cdot t} &= P \cdot \lim_{n\to\infty} \left(1+\frac{r}{n}\right)^{n \cdot t} \\ &= P \cdot \left(\lim_{n\to\infty} \left(1+\frac{r}{n}\right)^{n}\right)^{t} \end{aligned}$$

We know that

$$\lim_{n\to\infty} \left(1+\frac{r}{n}\right)^{n} = e^r$$

and substitution yields the final result

$$\lim_{n\to\infty} P \cdot \left(1+\frac{r}{n}\right)^{n \cdot t} = P \cdot e^{rt}. \tag{2.3.3}$$

What happens if we are on the other end; that is what if someone wants to borrow money from us? If someone promises to pay us some principal, P, plus some interest at a rate, r, in one year what is the value of that loan today? This question leads us to the idea of present value analysis.

2.4 Present Value Analysis

A **bond** is a debt instrument used by a company or a government to raise cash. A bond differs from stock insofar as a bond is a loan to a institution and the buyer of the bond holds no equity in that institution just a promise from that institution to pay the principal of the bond back plus interest. If the nominal interest rate, r, is compounded yearly, then the present value of the sequence of payments $x_i \, (i = 1, 2, 3, \ldots)$, where x_i is the payment due at the end of year i, is

$$\sum_{i=0}^{n} \left[(1+r)^{-i} \cdot x_i\right]. \tag{2.4.4}$$

Intuitively this makes sense because we are simply rearranging the terms in equation 2.3.2 and solving for P. Here $A = \sum_{i=0}^{N} x_i$ because each payment could be different. Since we are compounding yearly, $n = 1$, and i is the current payment number for N total payments. Now that present value is easily calculated, we can use this tool to find an important issue in options theory, the rate of return or yield.

Chapter 3

Mathematical Concepts

We now shift the focus of this paper to the necessary mathematical background needed in order to understand Black-Scholes. Probability theory and statistics play a vital role in Black-Scholes because we must model changes in stock prices and use historical data to try to determine future values of stocks. Applied mathematics provides the weaponry that will enable us to "predict" future prices of stocks with accuracy.

3.1 Basic Probability

Starting with the most basic idea in probability, a **random event** is an event which has an outcome that is undetermined before the event is observed. For instance, flipping a coin is considered a random event because either a head or a tail will come up and we don't know which one. For a fair coin we do know that the chance of a head landing face up is $\frac{1}{2}$. Another example could be made with a regular deck of cards. We know that we can draw a red colored card from a regular deck of cards with the same likelihood as flipping a coin, which is $\frac{1}{2}$.

What is meant by a fifty-fifty chance? The probability of some event happening must fall in a range of values. This range "weighs" the chance of some event happening. The range is $0 \leq p \leq 1$, where p is the probability of an event happening. In the examples above a fifty-fifty chance refers to a head falling face up with probability of $p = .5$. An example of a non-random event is "guessing" what the next color on a traffic light will be if you are in the United States and the current color of the light is red. This is a non-random event because we know that a green light will be next. Thus a non-random event is one in which the probability of an event occurring is 1 or 0. The relation of random events and probabilities to Black-Scholes is that a stock price increases or decreases with a certain probability. This event is called a random event because we do not know which outcome will occur.

In relation to the coin flipping game and card game above, the idea of independence will now be discussed. Two events, A and B, are said to be **independent** if the probability that A occurs given that B occurred is the probability that A occurs. In the previous example flipping a coin represents an independent event insofar as the probability that the second flip lands on a head is 0.5; in other words the result of the first flip contains no information about the second flip. In the card game above each turn is not independent, meaning the outcome of the second turn depends on the outcome of the first turn, and the outcome of the third turn depends upon the outcome of the first two turns and so forth. These events are dependent because if a red card is drawn on the first turn then the probability of drawing a red card on the second turn decreases while the probability of drawing a black card increases. The idea of independence is important in the study of Black-Scholes because the day to day stock prices of an asset (the outcome) is considered to be a sequence of independent events, that is, the previous day's trades have no affect on the current day's price.

Expected value is a weighted average of the possible values of X [where X is some event], where the weight given at a value is equal to the probability that X assumes that value. A more formal definition for the expected value of an event X, is

$$E[X] = \begin{cases} \sum_{i=1}^{n} x_i \cdot P(X = x_i) & \text{discrete} \\ \int_{-\infty}^{\infty} f_X(x)\,dx & \text{continuous} \end{cases} \tag{3.1.1}$$

An example will make this clearer.

Example 3.1.1. *Dan is playing a game with a fair six sided die. He wants to know what the expected value of the die is if he rolls it an infinite number of times. The probability of rolling any one number is* $\frac{1}{6}$ *since there are* 6 *sides and the die is fair. Thus if we take each value,* $x_i = 1, 2, 3, 4, 5, 6$, *and multiply them by their probabilities,* $\frac{1}{6}$, *and then sum the values we get*

$$\begin{aligned} E[X] &= \sum_{i=1}^{6} x_i \cdot \left(\frac{1}{6}\right) \\ &= \left(\frac{1}{6}\right) \cdot \sum_{i=1}^{6} x_i \\ &= 3.5 \end{aligned} \tag{3.1.2}$$

Therefore the value for which Dan is looking is 3.5.

An important note is that before Dan ever rolls the die the expected value is 3.5 which seems counter intuitive because 3.5 is not a feasible outcome. But with expected value we are calculating an average with certain weights and the weights in the case of a die are all the same, $\frac{1}{6}$, so it makes sense that the expected value is not an integer.

3.2 Mean and Standard Deviation

When we have a set of data we are often interested in how close to the **mean** the values fall. That is, how close is each individual event from the average of all events that were observed? This question leads us to variance. The sample **variance** of X, denoted by $Var(X)$, is defined by

$$Var(X) = \frac{\sum_{i=1}^{n} (x_i - \overline{X})}{n-1} \tag{3.2.3}$$

Where $\overline{x} = \frac{x_i}{n}$ is the sample mean.

Variance from a population is found by using the equation

$$Var(X) = \left[(X - E[X])^2\right]. \tag{3.2.4}$$

So the variance measures the average square of the distance between X and its expected value. What does it mean for a value to be something squared away from the mean? This leads us to standard deviation. The **standard deviation** is simply the square root of the variance. We now have understandable results in terms that we can interpret

Example 3.2.1. *Consider 11 stock prices, 40.70, 40.89, 40.41, 37.24, 39.68, 38.22, 49.49, 45.64, 43.23, 45.68, and 41.97 dollars, we can calculate the sample variance and sample standard deviation. First calculate the sample mean,* $\overline{x} = 42.1045$ *dollars. Now using equation 3.2.3 the variance is 13.2064 dollars squared. The square root of this gives us the standard deviation which is 3.634 dollars. This tells us that, on average, each stock price is about 3.634 dollars away from the mean of 42.1045 dollars.*

3.3 Distributions

Rolling dice and flipping cards are considered discrete random variables because there are a finite or countably infinite number of outcomes. For instance, a die can only land on the numbers one through six. Whereas a **continuous random variables** can take on any value within some interval. An example of a continuous random variable would be the height of randomly selected people. These "events" can be any number in the interval 0 inches to 8 feet. Every continuous random variable X has a **probability density function** [pdf], f_X, associated with it. The pdf determines the probabilities associated with X in the following manner. For and numbers $a < b$, the area under f_X between a and b is equal to the probability that X assumes a value between a and b. That is,

$$P\{a \leq X \leq b\} = \int_a^b f_X(x)\,dx$$

The most commonly used type of continuous random variable is the **normal random variable.** The pdf of a normal random variable is given by the equation

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}e^{-(x-\mu)^2/2\sigma^2}, -\infty < x < \infty. \tag{3.3.5}$$

This normal density function gives us the famous bell-shaped curve, where μ is the center of the curve and σ determines the flatness or skinniness of the curve. It can be shown that $\mu = E[X]$, and $\sigma^2 = Var(X)$. The proof will be left out as it is not needed in this paper. Refer to [5] for a detailed proof.

A normal density function with mean 0 and variance 1 is called a **standard normal random variable.** Let Z be a standard normal random variable. The function $\Phi(z) = P\{Z \leq z\}$, is called the standard normal distribution function. So $\Phi(z)$ gives us the probability that Z is less than or equal to z and this is equal to the area under the standard normal density function given in equation 3.3.5 with $\mu = 0$ and $\sigma = 1$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}, -\infty < z < \infty. \tag{3.3.6}$$

We can easily find values for negative z since the normal density curve is symmetric and the total area under the curve is one. We simply use the identity,

$$\Phi(-z) = 1 - \Phi(z) \tag{3.3.7}$$

An important property of normal random variables is that if X is a normal random variable then so is $aX + b$, when a and b are real constants. We can now transform any normal random variable into a standard normal random variable. By using the formula

$$Z = \frac{X - \mu}{\sigma} \tag{3.3.8}$$

Here, Z has expected value 0 and variance 1, therefore for $Z \sim N(\mu, \sigma^2)$ is a standard normal random variable. This allows us to easily compute probabilities of any normal random variable. Some values for $\Phi(z) = P\{Z \leq z\}$ can be found in Ross 2002.

Example 3.3.1. *Scores for a typical round of golf are approximately normally distributed with a mean of 71.67 strokes and standard deviation of 5.4 strokes. Calculate the probability that a player shoots 70 or under but higher than 66.*

Solution. *Let X be the score of a randomly chosen golfer. Then, since $P\{Z \leq b\} = P\{Z \leq a\} + P\{a < Z \leq b\}$, we see that $P\{a < Z \leq b\} =$*

$\Phi(b) - \Phi(a)$.

$$\begin{aligned}
P\{X \leq 70\} &= P\left\{\frac{X - 71.67}{5.4} \leq \frac{70 - 71.67}{5.4}\right\} \\
&= P\{Z \leq -.309259\} \\
&= 1 - \Phi(.309259) \\
&= 1 - .621 \\
&= .379 \\
P\{X < 66\} &= P\left\{\frac{X - 71.67}{5.4} < \frac{66 - 71.67}{5.4}\right\} \\
&= P\{Z < -1.05\} \\
&= 1 - \Phi(1.05) \\
&= 1 - .8531 \\
&= .1469 \\
P\{66 < X \leq 70\} &= .379 - .1469 \\
&= .2321
\end{aligned}$$

So the probability that a randomly selected golfer shoots 70 or under but higher than 66 is 0.2321, in other words s/he has a 23.21 percent chance of shooting in that range.

3.4 Lognormal Random Variables

The random variable Y is said to be a lognormal random variable with parameters μ and σ if $\log(Y)$ is a normal random variable with mean μ and variance σ^2. That is, Y is lognormal if it can be expressed as $Y = e^X$, where X is a normal random variable. The mean and variance of a lognormal random variable are as follows:

$$\begin{aligned}
E[Y] &= e^{\mu+\sigma^2/2} \\
Var(Y) &= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)
\end{aligned}$$

3.5 The Central Limit Theorem

One of the important theoretical results in probability theory is the central limit theorem. The reason for its importance is that it tells us that a sum of a large number of independent random variables, all having the same probability distribution, will itself be approximately a normal random variable.

Theorem 3.5.1. Central Limit Theorem *Let* $X_i \overset{iid}{\sim} F_X$ *with* $|E(x_i)| < \infty$ *and* $\sigma_X^2 < \infty$. *Then for large* n, $S_n = \sum_{i=1}^n x_i$ *will approximately be a normal* *random variable with expected value $n\mu$ and variance $n\sigma^2$. As a result, for any x we have*

$$P\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq x\right\} = \Phi(x), \tag{3.5.9}$$

with the approximation becoming exact as n tends toward infinity.

With the central limit theorem we can take a large number of stock prices and, assuming each stock price is independent, the sum of the prices becomes more and more normal. With these new tools at hand the paper now shifts into the final fundamental idea, Brownian motion, before the introduction of the Black-Scholes formula.

Chapter 4

Brownian Motion

The Brownian motion process is named after an English botanist Robert Brown. He first described the unusual motion of particles immersed in a liquid or gas. In 1905 Einstein showed mathematically that Brownian motion could be explained by describing the movement of the immersed particle as the result of continuous collisions with other particles in the medium.

Definition 4.0.1. *The collection of prices $S(y)$ of a security at a time y from the present, such that $0 \leq y < \infty$, is said to follow a* Brownian motion *with drift parameter μ and variance parameter σ^2 if, for all nonnegative values of y and t, the random variable*

$$S(t+y) - S(y) \tag{4.0.1}$$

is independent of all prices up to time y and, in addition, is a normal random variable with mean μt and variance $t\sigma^2$.

Brownian motion of a security has the property that a future price depends on the present and all past prices only through the present price. In other words the present price represents all we know about the security. An important aspect to remember is that Brownian motion depends upon the *difference* in prices.

A few problems arise when trying to use Brownian motion to model stock prices. With the assumption that the stock price is a normal random variable comes the possibility that the price of the stock could become negative which cannot happen in the real world. Secondly, Brownian motion uses the difference of the two security values. At first this seems like a good idea, but upon closer inspection the difference between security prices is not the best method. Instead a ratio of prices gives us a more accurate account. An example will make this second problem clearer.

Example 4.0.1. *Elizabeth bought two stocks 6 months ago. One hundred shares of FLBW were purchased at \$10 per share and one hundred shares of LHK were purchased at \$100 per share. Elizabeth wants to sell her shares* *in both companies. FLBW is currently selling at \$25 per share and LHK is selling at \$200 per share. Comparing the return on the stocks yields two different results. With FLBW, Elizabeth made \$15 per share but with LHK, Elizabeth made \$100 per share. Taking the ratios of the stock prices yields* $100 \cdot \frac{(25-10)}{10} = 150\%$ *return on FLBW and* $100 \cdot \frac{(200-100)}{100} = 100\%$ *return on LHK. Hence, even though LHK's stock value is higher than FLBW's, FLBW has a higher percent return.*

What do these two different results mean? Which value do we really care about? Brownian motion suggests that we use the non-ratio return. This doesn't make much sense because a security can have a lower value but still yield a higher rate of return as in the above example. A ratio of stock prices gives us a much better idea of profitability of a certain investment. Theories based on percentage changes rather than absolute changes are called geometric. Conversely, theories based on absolute changes are called arithmetic. Looking at percentage changes leads us into a discussion of geometric Brownian motion.

4.1 Geometric Brownian Motion

Now let us look at the same problem (trying to determine the price of some security over time) but instead of using the difference of prices we use a ratio.

Definition 4.1.1. *Let the present time be time* 0, *and let* $S(y)$ *denote the price of the security at time* y *from the present. We say that the collection of prices* $S(y)$, $0 \leq y < \infty$, *follows a geometric Brownian motion with drift parameter* μ *and volatility parameter* σ *if, for all nonnegative values of* y *and* t, *the random variable*

$$\frac{S(t+y)}{S(y)} \tag{4.1.2}$$

is independent of all prices up to time y*; and if, in addition*

$$\ln\left[\frac{S(t+y)}{S(y)}\right] \tag{4.1.3}$$

is a normal random variable with mean μt *and variance* $t\sigma^2$.

This tells us that a series of prices will exhibit geometric Brownian motion if the ratio of the price at time t in the future to the present price will, independent of the history of prices of a certain security, have a log-normal probability distribution with mean μt and variance $t\sigma^2$. Again, it is an important note that the history of the stock's price is only important to compute the drift (mean) and the volatility (standard deviation) and does not directly predict the value of future prices. Before we move on we must first understand what drift and volatility mean. Drift is the mean. This lets us know the average ratio of the price of the security and gives us the spot price at time t. If we did not include drift then every stock price would be assumed to start at 0 since the prices follow a Brownian motion which uses a standard normal distribution. Volatility is a little bit more complex. The higher the volatility of a stock the more the price will fluctuate in the short and long run. These observations lead us to equate volatility with riskiness. The higher the volatility the less precisely we can determine a stock's price and the riskier the asset becomes.

4.2 Computing Volatility

Many people view the computation of volatility as the single most important aspect of the Black-Scholes formula. The computation is a five step process. In effect we are computing the standard deviation of the short-term returns. This can be done as such.

1. Fix a standard time period Δt (one day, one month, etc.) and express it in terms of years. For example, if we are using weekly closing prices, then Δt is equal to one week. This is 1/52 expressed in years.

2. Collect the stock price data for each time period. For example collect the weekly closing prices for one year.

3. Compute the return from the beginning to the end of each period. If the closing price on day i is denoted S_i and the closing price at day $i+1$ is denoted S_{i+1}, then the one week return is given by,

$$r_i = \ln\left(\frac{S_{i+1}}{S_i}\right). \tag{4.2.4}$$

4. Compute the sample average value of the sample returns.

$$\begin{aligned} \bar{r} &= \frac{1}{N+1} \cdot (r_0 + \ldots + r_N) \\ &= \frac{\sum_{i=0}^{N} r_i}{N+1} \end{aligned} \tag{4.2.5}$$

5. Compute the annual standard deviation using the formula

$$\begin{aligned} \sigma &= \frac{1}{\sqrt{\Delta t}} \sqrt{\left(\frac{1}{N} \cdot (r_o - \bar{r})^2 + \ldots + (r_N - \bar{r})^2\right)} \\ &= \frac{1}{\sqrt{\Delta t}} \sqrt{\frac{\sum_{i=0}^{N} (r_i - \bar{r})^2}{N}} \end{aligned} \tag{4.2.6}$$

A few notes about the computation of volatility before we move on. In finding the standard deviation we multiply by $\frac{1}{\sqrt{\Delta t}}$ because the standard unit of time in options is one year. Thus we must express every value used in terms of one year. Also notice that $\frac{1}{N}$ is used instead of $\frac{1}{N+1}$. Logic would tell us that dividing by $N+1$ would be the correct computation because there are $N+1$ terms. Statistics has shown that using $N+1$ yields a biased sample standard deviation. We use N to make the sample standard deviation unbiased.

4.3 Justification of Geometric Brownian Motion

Now that we know the difference between Brownian motion and geometric Brownian motion we need to justify that we can use geometric Brownian motion as a model for the movement of stock prices. Let Δ be a small increment of time and suppose that, every Δ time units, the price of a stock goes up by the factor u with probability p or down by the factor d with the probability of 1-p, where

$$u = e^{\sigma\sqrt{\Delta}}, \; d = e^{-\sigma\sqrt{\Delta}}$$
$$p = \frac{1}{2}\left(1 + \frac{\mu}{\sigma}\sqrt{\Delta}\right). \tag{4.3.7}$$

Taking Δ smaller and smaller so that the price changes occur more and more frequently, the collection of prices follow a geometric Brownian motion. The verification is as follows:

Let Y_i equal 1 if the price of the stock goes up at time $i\Delta$, and let it be 0 if the price of the stock goes down. The number of times the stock price goes up in the first n time increments is $\sum_{i=1}^{n} Y_i$, and the number of times it goes down is $n - \sum_{i=1}^{n} Y_i$. Now, $S(n\Delta)$, its price at the end of this time, can be expressed as

$$S(n\Delta) = S(0)u^{\sum_{i=1}^{n} Y_i} d^{n-\sum_{i=1}^{n} Y_i}$$

or

$$S(n\Delta) = d^n S(0)\left(\frac{u}{d}\right)^{\sum_{i=1}^{n} Y_i}$$

If we now let $n = t/\Delta$, then the preceding equation can be expressed as

$$\frac{S(t)}{S(0)} = d^{t/\Delta}\left(\frac{u}{d}\right)^{\sum_{i=1}^{t/\Delta} Y_i}$$

We now take the ln of both sides

$$\begin{aligned} \ln\left(\frac{S(t)}{S(0)}\right) &= \frac{t}{\Delta}\ln(d) + \ln\left(\frac{u}{d}\right)\sum_{i=1}^{t/\Delta} Y_i \\ &= \frac{-t\sigma}{\sqrt{\Delta}} + 2\sigma\sqrt{\Delta}\sum_{i=1}^{t/\Delta} Y_i \end{aligned} \tag{4.3.8}$$

The equality from equation 4.3.8 comes from the definitions of u and d. We take the ln of both of the equalities, u and d, and substitute into 4.3.8. As Δ gets smaller and smaller the number of observations gets larger and larger. By the central limit theorem $\sum_{i=1}^{t/\Delta} Y_i$ tends in distribution to a normal random variable and thus equation 4.3.8 tends in distribution to a normal random variable. This means that $\ln(S(t)/S(0))$ becomes a normal random variable. The mean is computed as follows:

$$
\begin{aligned}
E\left[\ln\left(\frac{S(t)}{S(0)}\right)\right] &= \frac{-t\sigma}{\sqrt{\Delta}} + 2\sigma\sqrt{\Delta}\sum_{i=1}^{t/\Delta} E\left[Y_i\right] \\
&= \frac{-t\sigma}{\sqrt{\Delta}} + 2\sigma\sqrt{\Delta}\frac{t}{\Delta}p \\
&= \frac{-t\sigma}{\sqrt{\Delta}} + \frac{t\sigma}{\sqrt{\Delta}}\left(1+\frac{\mu}{\sigma}\sqrt{\Delta}\right) \\
&= \frac{-t\sigma}{\sqrt{\Delta}} + \frac{t\sigma}{\sqrt{\Delta}} + \frac{t\sigma\mu\sqrt{\Delta}}{\sigma\sqrt{\Delta}} \\
&= \mu t \qquad (4.3.9)
\end{aligned}
$$

$\sum_{i=1}^{t/\Delta} E\left[Y_i\right] = \frac{t}{\Delta}p$ because the expected value of $Y \ \forall \ i$ is 1 and the probability of that happening is p. When we sum from i to t/Δ we get (t/Δ) p's. The next substitution comes from using the definition of p, equation 4.3.7. Now the variance is as follows:

$$
\begin{aligned}
Var\left(\ln\left(\frac{S(t)}{S(0)}\right)\right) &= 4\sigma^2\Delta\sum_{i=1}^{t/\Delta} Var(Y_i) \\
&= 4\sigma^2 tp(1-p) \\
&= \sigma^2 t \qquad (4.3.10)
\end{aligned}
$$

Note that $\sum_{i=1}^{t/\Delta} Var(Y_i) = \left(\frac{t}{\Delta}\right)p(1-p)$ because our Y_i's are Bernoulli random variables with parameter p and we are summing $\frac{t}{\Delta}$ of the Y_i's. For small Δ, p is approximately 1/2. Therefore we can use geometric Brownian motion as a model for the price movements of a security. We end this section with an example.

Example 4.3.1. *We will compute the probability that a call option on a stock following the geometric Brownian motion model will expire "in the money." Suppose C is a European call option on S, a stock, struck at K, with time $T-t$ to expiration. Suppose also that the instantaneous expected rate of return on S is r and the volatility is σ. Computing the probability that C expires in the money is equivalent to computing the probability that $S_T \geq K$, where S_T is the value of the stock at expiration. Now we need to know the return (mean) on the stock from time t to time T. This is given by $(T-t)(r-\sigma^2/2)$. We know this to be true because we are assuming that the* *stock follows a geometric Brownian motion with* $\mu = (r - \sigma^2/2)$ *and time* $= (T - t)$. *The return on* S *from time* t *to time* T *is given by* $\ln(S_T/S_t)$, *where* S_t *is the spot price of* S *at time* t *and* S_T *is the spot price of* S *at time* T. *Since* S_T *is a random variable (it represents an unknown value in the future),* $ln(S_T/S_t)$ *is a random variable because of the central limit theorem. Now, using what we know about standard normal random variables we can compute the probability that a stock finishes "in-the-money",*

$$
\begin{aligned}
&P\{S_T \geq K\} \\
&= P\left\{\frac{\ln(\frac{S_T}{S_t}) - (T-t)(r - \frac{\sigma^2}{2})}{\sigma\sqrt{T-t}} \geq \frac{\ln(\frac{K}{S_t}) - (T-t)(r - \frac{\sigma^2}{2})}{\sigma\sqrt{T-t}}\right\} \\
&= P\left\{\frac{\ln(\frac{S_t}{S_T}) + (T-t)(r - \frac{\sigma^2}{2})}{\sigma\sqrt{T-t}} \leq \frac{\ln(\frac{S_t}{K}) + (T-t)(r - \frac{\sigma^2}{2})}{\sigma\sqrt{T-t}}\right\}
\end{aligned}
$$

Thus the probability that our stock finishes "in-the-money" is,

$$
\Phi\left(\frac{\ln(S_t/K) + (T-t)(r - \sigma^2/2)}{\sigma\sqrt{T-t}}\right) \tag{4.3.11}
$$

Chapter 5

The Black-Scholes Formula

Before the formula is actually given we must first understand a few ideas. Black-Scholes allows us to find the value of a call option without arbitrage. **Arbitrage** is taking advantage of a disparity in markets. For instance, if the market for wheat is $5 per pound in Iowa and only $3 per pound in Nebraska then an eager profiteer could buy tons of wheat in Nebraska and then sell it in Iowa. This person would be taking advantage of the disparity in price and therefore using arbitrage to their advantage. Ideally we would like to create a no-arbitrage market. Black-Scholes allows us to accomplish this arbitrage free market. We do this by setting up a hedge. We buy shares of the stock that we are selling and we complement that with a short sale of bonds. Theoretically this allows us to sell a call option without risk. For a more detailed survey to hedging see Chris Davis(2005).

5.1 Risk-Neutrality Derivation

Consider a call option with strike price K and expiration at time t, the risk-free rate of interest is r, compounded continuously, then the price of the security follows a geometric Brownian motion model with drift μ and volatility σ. Let $S(y)$ denote the price of the security at time y, where $0 < y < t$. Assuming the price of the stock follows a geometric Brownian motion with volatility parameter σ and drift parameter μ. We now need to decide how often we will collect the price data. That is, we need to decide if we want to collect the data every week, day, or hour, *etc*. Let number of periods be denoted as n. Since our model is based on t we need to adjust n so that we have it in units that we like. In a time span t we have n sub units. At every $\frac{t}{n}$ the price of the stock moves up u units with probability p, or down

d units with probability $1-p$, where,

$$u = e^{\sigma\sqrt{t/n}} \quad (5.1.1)$$
$$d = e^{-\sigma\sqrt{t/n}} \quad (5.1.2)$$
$$p = \frac{1}{2}\left(1+\frac{\mu}{\sigma}\sqrt{t/n}\right). \quad (5.1.3)$$

Since the probability of going up is p and down is $1-p$ we have an n-stage binomial model. Let

$$X_i = \begin{cases} 1 & \text{if} \quad S(it/n) = uS((i-1)t/n)\text{an up movement} \\ 0 & \text{if} \quad S(it/n) = dS((i-1)t/n)\text{a down movement} \end{cases}$$

Then we want our bets to be fair. A fair bet means that there is no arbitrage taking place. In other words, no one person, the buyer or the seller, is gaining an advantage. We can assume that each $X_i, \ldots, X_n$ is independent, and we want

$$p = P\{X_i = 1\}.$$

At this point we need to figure out a value for p. If we want our option to be arbitrage free then we want the expected gain on a bet to be zero. First we must determine the probability that the stock finishes "in-the-money", that is if $X_i = 1 \ \forall \ i$. Let,

$$\alpha = P\{X_1 = x_1, \ldots, X_{i-1} = x_{i-1}\}$$

denote the probability that the stock is purchased. Then

$$p = P\{X_i = 1 | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}\}$$

denotes the probability that a purchased stock increases for the next interval. We then have an expected profit/loss of,

$$\alpha[p(1+r)^{-1}uS(i-1) + (1-p)(1+r)^{-1}dS(i-1) - S(i-1)]. \quad (5.1.4)$$

Equation 5.1.4 can be summarized as follows:

$$p(1+r)^{-1}uS(i-1)$$

is the product of the probability, p, that a stock increases, the present value, $(1+r)^{-1}$, of the stock, the amount the stock will increase, u, and the previous value of the stock $S(i-1)$.

$$(1-p)(1+r)^{-1}dS(i-1)$$

is the product of the probability, $(1-p)$, that a stock decreases, the present value, $(1+r)^{-1}$, of the stock, the amount the stock will decrease, d, and the previous value of the stock $S(i-1)$. $S(i-1)$ is the spot price of the stock at time $i-1$. For the expected gain to be zero we need,

$$p(1+r)^{-1}uS(i-1)+(1-p)(1+r)^{-1}dS(i-1)-S(i-1)=0.$$

Solving for p we see that,

$$\begin{aligned}
0 &= p(1+r)^{-1}u+(1-p)(1+r)^{-1}d-1 \\
1 &= \frac{pu+(1-p)d}{(1+r)} \\
1+r &= pu+(1-p)d \\
1+r &= pu+d-pd \\
1+r-d &= p(u-d) \\
p &= \frac{1+r-d}{u-d}. \qquad (5.1.5)
\end{aligned}$$

If the probability of an increase is p then any bet on buying a stock will have zero expected gain. Basically, we have taken a bet with possibly high arbitrage and turned it into a fair, no-arbitrage bet.

We now need to adjust 5.1.5 so that it is in terms of our compounding period, t/n. Using,

$$p=\frac{1+rt/n-d}{u-d}$$

where rt/n represents the rate charged per compounding. We may substitute for u and d using equations 5.1.1 and 5.1.2 yielding,

$$p=\frac{1+rt/n-e^{-\sigma\sqrt{t/n}}}{e^{\sigma\sqrt{t/n}}-e^{-\sigma\sqrt{t/n}}} \qquad (5.1.6)$$

Using Taylor series expansion for

$$\begin{aligned}
e^{-\sigma\sqrt{t/n}} &\approx 1-\sigma\sqrt{t/n}+\sigma^2\,(t/2n) \\
e^{\sigma\sqrt{t/n}} &\approx 1+\sigma\sqrt{t/n}+\sigma^2\,(t/2n)
\end{aligned}$$

substitution yields

$$\begin{aligned}
p &= \frac{\sigma\sqrt{t/n}-\sigma^2 t/2n+rt/n}{2\sigma\sqrt{t/n}} \\
&= \frac{1}{2}+\frac{r\sqrt{t/n}}{2\sigma}-\frac{\sigma\sqrt{t/n}}{4} \\
&= \frac{1}{2}\left(1+\frac{r-\sigma^2/2}{\sigma}\sqrt{t/n}\right). \qquad (5.1.7)
\end{aligned}$$

Letting $n\to\infty$, equation 5.1.7 converges to geometric Brownian motion with drift parameter $r-\sigma^2/2$ and volatility parameter σ. What this tells us is that the only probability law that makes bets fair is given in equation 5.1.7. In other words we know that if we use this probability, then there will be no arbitrage and if we do not then the resulting output will be subject to arbitrage.

5.2 Black-Scholes Formula

We are now ready to present the risk-neutral geometric Brownian motion model with drift of $(r - \sigma^2/2)t$ and volatility $\sigma^2 t$ which is better know as the Black-Scholes formula. We have,

$$\begin{aligned} C &= e^{-rt}E[(S(t) - K)^+] \\ &= e^{-rt}E[S(0)e^{-W} - K^+] \end{aligned} \tag{5.2.8}$$

where W is a normal random variable with mean $(r - \sigma^2/2)t$ and variance $\sigma^2 t$. The reason why we use e^{-W} in equation 5.2.8 is so that we can use $S(0)$ instead of $S(t)$. Essentiality, we are using the value of the stock price at time zero, the price when the trade was struck, and then calculating the value of $S(t)$ for all t throughout the duration of the option. We now modify equation 5.2.8 into a form which we can easily use and understand.

Let

$$C(S, t, K, \sigma, r) = E[e^{-rt}(S(t) - K)^+]$$

be the risk-neutral cost of a call option at time t with price $S(t)$, time to maturity t, strike price K, risk-free rate of interest r. Then C follows a geometric Brownian motion model with volatility parameter σ. As shown above in section 5.1 the risk-neutral probability of $S(t)$ can be expressed as

$$S(t) = S(0) \cdot e^{(r-\sigma^2/2)t+\sigma\sqrt{t}Z} \tag{5.2.9}$$

where Z is a standard normal random variable.

Let I be the indicator random variable for the event that the option finishes in the money.

$$I = \begin{cases} 1 & \text{if} \quad S(t) > K \\ 0 & \text{if} \quad else \end{cases} \tag{5.2.10}$$

Lemma 5.2.1. *Using equation 5.2.9 to rewrite equation 5.2.10 as*

$$I = \begin{cases} 1 & if \quad Z > \sigma\sqrt{t} - \omega \\ 0 & if \quad else \end{cases}$$

where

$$\omega = \frac{rt + \sigma^2 t/2 - \ln(K/S)}{\sigma\sqrt{t}} \tag{5.2.11}$$

Proof

$$\begin{aligned} S(t) > K &\Leftrightarrow e^{(r-\sigma^2/2)t+\sigma\sqrt{t}Z} > K/S \\ &\Leftrightarrow (r-\sigma^2/2)t + \sigma\sqrt{t}Z > \ln(K/S) \\ &\Leftrightarrow \sigma\sqrt{t}Z > \ln(K/S) - (r-\sigma^2/2)t \\ &\Leftrightarrow Z > \frac{\ln(K/s) - (r-\sigma^2/2)t}{\sigma\sqrt{t}} \\ &\Leftrightarrow Z > \sigma\sqrt{t} - \omega. \end{aligned} \tag{5.2.12}$$

The following Lemma provides the expected value of I.

Lemma 5.2.2.

$$E[I] = P\{S(t) > K\} = \Phi(\omega - \sigma\sqrt{t}), \tag{5.2.13}$$

where Φ is the standard normal cumulative distribution function.

Proof *It follows from its definition that*

$$\begin{aligned} E[I] &= 1 \cdot P\{S(t) > K\} + 0 \cdot P(S(t) \leq K) \\ &= P(S(t) > K) \end{aligned}$$

By Lemma 5.2.1 this maybe written as

$$\begin{aligned} E[I] &= P\{Z > \sigma\sqrt{t} - \omega\} \\ &= P\{Z < \omega - \sigma\sqrt{t}\} \\ &= \Phi(\omega - \sigma\sqrt{t}) \end{aligned} \tag{5.2.14}$$

The following Lemma is used in the proof of theorem 5.2.1

Lemma 5.2.3.

$$e^{-rt}E[IS(t)] = S(t)\Phi(\omega). \tag{5.2.15}$$

Proof *We let $c = \sigma\sqrt{t} - \omega$.*

$$E[IS(t)] = \int\limits_{S(t)>K} 1 \cdot S(t)e^{(r-\sigma^2/2)t+\sigma\sqrt{t}x}\frac{1}{2\pi}e^{-x^2/2}\,dx + \int\limits_{S(t)\leq K} 0 \cdot S(t)f_Z(x)\,dx$$

Since $S(t) > K \Leftrightarrow z < \sigma\sqrt{t} - \omega = c$

$$\begin{aligned}
E[IS(t)] &= \int_c^\infty S(t)e^{(r-\sigma^2/2)t+\sigma\sqrt{t}x}\frac{1}{\sqrt{2\pi}}e^{-x^2/2}\,dx \\
&= \frac{1}{\sqrt{2\pi}}S(t)e^{(r-\sigma^2/2)t}\int_c^\infty e^{-(x^2-\sigma\sqrt{t}x/2)}\,dx \\
&= \frac{1}{\sqrt{2\pi}}S(t)e^{(r-\sigma^2/2)t}\int_c^\infty e^{-(x^2-2\sigma\sqrt{t}x+\sigma^2 t)/2+\sigma^2 t/2}\,dx \\
&= \frac{1}{\sqrt{2\pi}}S(t)e^{(r-\sigma^2/2)t}e^{\sigma^2 t/2}\int_c^\infty e^{-(x-\sigma\sqrt{t})^2/2}\,dx
\end{aligned}$$

Letting $y = x - \sigma\sqrt{t} \Rightarrow dy = dx$

$$E[IS(t)] = S(t)e^{rt}\frac{1}{\sqrt{2\pi}}\int_{-\omega}^\infty e^{-y^2/2}\,dy$$

By equation 3.1.1 we get

$$E[IS(t)] = S(t)e^{rt}P\{Z > -\omega\}$$

By symmetry

$$\begin{aligned}
E[IS(t)] &= S(t)e^{rt}P\{Z < \omega\} \\
&= S(t)e^{rt}\Phi(\omega).
\end{aligned}$$

And by dividing by e^{rt} *we have*

$$e^{-rt}E[IS(t)] = S(t)\Phi(\omega). \tag{5.2.16}$$

Lemma 5.2.1 through 5.2.3 lead to the Black-Scholes formula.

Theorem 5.2.1. *For spot price,* $S(t)$*, time,* t*, strike price,* K*, volatility parameter,* σ*, and risk-free rate of interest,* r*, the risk-neutral cost of a call option,* C*, is*

$$C(s,t,K,\sigma,r) = s\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}).$$

Proof

$$\begin{aligned}
C(s,t,K,\sigma,r) &= e^{-rt}E[(S(t)-K)^+] \\
&= e^{-rt}E[I(S(t)-K)] \text{ since } S(t)-K > 0 \Rightarrow S(t) > K \\
&= e^{-rt}E[IS(t)] - Ke^{-rt}E[I] \\
&= e^{-rt}S(t)e^{rt}\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}) \text{ by Lemma 5.2.1 and 5.2.3} \\
&= S(t)\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}). \qquad (5.2.17)
\end{aligned}$$

The Black-Scholes formula allows us to calculate the fair value of a call option with strike price K, spot price S, time to expiration t, risk-free rate of interest r, and assuming geometric Brownian motion parameter σ. The value of the call option, C, gives the value that needs to be charged to a call option given a certain strike price so that the option is a fair bet.

Example 5.2.1. *KT wants to enter a call option with her trader Jim. They agree on the strike price of* \$55 *for security XYZ currently selling at* \$50. *Jim wants to know what the price of the call option should be, given that the time to maturity is half a year, the risk-free rate of interest is 0.045 and the risk-free rate of interest is 0.045 and the volatility parameter is 0.15. We have,*

$$\begin{aligned}\omega &= \frac{0.045 \cdot 0.5 + 0.15^2 \cdot 0.5/2 - log(55/50)}{.15\sqrt{.5}} \\ &= -0.633428\end{aligned}$$

$$\begin{aligned}C &= 50\Phi(-.633428) - 55e^{-0.045 \cdot 0.5}\Phi(-0.739494) \\ &= 50(0.2643) - 55(0.977751)(0.2327) \\ &= 13.215 - 12.5137 \\ &= 0.7013 \qquad (5.2.18)\end{aligned}$$

What this tells us is that Jim must charge a fee of 70.13 cents on each share purchased if he wants to make the bet fair.

Chapter 6

Balance Sheet Valuations

Giving options to employees is a common event in the corporate world. At the end of each year many companies give out options as bonuses for an employee's hard work. Many times these bonuses are given in large quantity to high ranking officers of a company. Recent financial accounting changes require public and private companies to write off these bonuses as expenses on their balance for the year in which the options are offered. "Recently, in response to such developments as the public reaction to financial reporting scandals, which were connected in the public mind, rightly or not, with highly compensated executives' lucrative stock-based compensation programs, and the IASB's announced intent to mandate use of the FAS 123-like fair value method, a number of large publicly held corporations have embraced fair value reporting of options."[4] The Black-Scholes formula is the method of choice for the expensing of options under FAS-123. This has created some excitement and worry among Wall-Street and public and private companies alike.

6.1 Public vs. Private

We now revisit the discussion of volatility and its importance. Referring to section 4.2, the computation of volatility requires the collection of stock prices over a certain time period. For instance we could collect the stock prices for a company at the closing of the market for each day. This would give us 250 observations in one year (there are only 250 trading days in one year). We then compute the log return ratios of the observations and find the standard deviation. For a publicly traded company, like Microsoft, this is easily done because Microsoft stock is traded instantaneously and by the millions of shares each day. The numerous trades per day essentially allows us to collect as many data points as desired. This ease of data collection is not the case for a private company.

A private company is almost completely the opposite of a public company in relation to the ease of transaction of stock. What this means is that a collection of observed stock prices is hard to come by. In a private company, transaction of stock occurs when a stock holder agrees to sell their shares to an investor. This might sound similar to a public company but in a private company the buyer and seller must seek each other and there are no secondary markets in which trades occur. This limits the number of transactions and thus the number of observations we can collect for the computation of volatility. For example, Sooner Southwest Bankshares (SSW), a private bank holding company, has only 17 usable observations between January 1996 and December 2004. This provides a tremendous potential error for a volatility estimation. For this reason the Generally Accepted Accounting Principles manual (an accounting manual which defines accounting rules) states, "For nonpublicly traded companies, essentially the same approach [referring to the approach for public companies] is to be used, except that the measure of volatility, which would be difficult to assess, or even meaningless, in the absence of an active market for the shares, is dispensed with."[4] The author of this paper believes there is a way of using volatility and the Black-Scholes formula to expense the cost of an option for a private company.

6.2 Theory of Private Valuation

A step by step method will be presented with an example that follows. Valuation of privately held shares proceeds as follows:

1. Match the private company with several publicly traded companies that are similar industry type and asset size.
2. Collect end of the year stock prices for the publicly traded companies.
3. Compute the returns of the publicly traded companies using $\ln\left(\frac{S_{i+1}}{S_i}\right)$ as the value for the return.
4. Compare the returns between the public and private companies.
5. Use the volatility of the public company that most closely resembles the private company.

The first step is fairly simple. We want several companies that are similar in industry type to the private company which is being valued. If we are valuing an option on a private company that makes food products, it is not a good idea to use a public company that makes computer chips for a comparison. The markets for food and computer chips are so different that the volatilities will not match well. An increase in wheat might increase the expense of the food product producer and in return decrease the amount of profit made by the company which would affect the value of the private company and the value of the stock. This increase in wheat prices would have little or no effect on the price of the stock or the volatility of a computer chip maker. Another important key in matching private and public companies is asset size. If the asset size difference is too great, then changes in the market might affect the larger company while, at the same time, have no affect on the smaller company or visa versa.

Once similar companies are selected, we need to collect the end of the year stock prices for each of the public companies. The number of data collected should be similar to the number of data available for the private company. This means that if the private company has 12 end of the year prices (the private company has been in existence for 12 years) then the picked public companies should have, at minimum, a relatively close number of end of the year prices (this means the public company has been in existence for roughly the same number of years as the private company, at minimum). If the public company does not have enough prices (they have not been around long enough) then matching the public to private company becomes riskier, and we want to eliminate as much error as possible. The end of the year prices for the private company come from the ESOP stock appraised values. The ESOP value is given by an independent company that uses asset size and profitability to value the stock price of a company.

Step three requires us to compute the log return ratios of each company using $\ln\left(\frac{S_{i+1}}{S_i}\right)$ as the formula. This is simple and straight forward.

Once the log return ratios are computed, we have several different methods at our disposal to find the best 'match' for the private company. We can first look at the direct returns, comparing one public company at a time to the private company. This method is extremely subjective and caries many chances for error. We can use a "pairs plot" which places the returns for each year of the private company as an x value on a graph and the returns for each year of the public company as a y value. For example if private company XYZ had a return of 0.48 in 2000 and public company ABC had a return of 0.55 in 2000 then the point (0.48,0.55) would be plotted on a graph. The coordinates for each year of a private company and public company are graphed. Ideally the plotted points would fall on a line. A correlation of one would signify perfect linearity or a 'perfect' match. This method reduces some of the subjectiveness but does not reduce all of the error in our match. To reduce all of the subjectiveness of the pairs plot we can calculate the correlation coefficient. The correlation coefficient is a measure of the degree of linearity between X and Y, in our case the degree of linearity between the log ratios of prices between the public and private company. With the correlation values we simply find the public company with correlation value closest to one in absolute value.

We now pick the public company which most closely resembles the private company in relation to industry type, asset size, and return ratios. From this point we use the method discussed in section 4.2 to compute the volatility of the public company and use that value for the value of volatility of the private company. The remaining variables in the Black-Scholes formula pose no problems for private companies.

6.3 Application of Private Valuation

This section will follow the method described in section 6.2 with an actual private company. The company to be used is Sooner Southwest Bankshares (SSW) located in Tulsa, Oklahoma. SSW is a bank holding company which means they own banks but are not a bank themselves. In December of 2004 SSW asset size was $220,088,000. SSW gives options to their employees as bonuses. According to the new accounting rules, SSW must expense these bonuses on its income statement for the life of the option. [1] The problem arises when trying to expense the options; how is the volatility computed?

According to the first step we must find similar public companies to SSW. The public companies selected are BNCCorp Inc. (BNCC), Landmark Bankcorp Inc. (LARK), Ohio Legacy Corp. (OLCB), Pelican Financial Inc. (PFI), Southern Connecticut Bancorp Inc. (SSE), and Team Financial Inc. (TFIN). These companies have asset sizes of $673,710,000, $442,091,000, $195,052,000, $198,816,000, $81,695,000, and $664,083,000 respectively. Now that we have some companies we must collect their end of the year stock prices. This is straight forward and just takes time.[2]

The yearly returns of the given companies are presented in figure 6.1. The statistical package S-PLUS was used for this and the code for the program can be found in the Appendix A on page 33. The first aspect of

	BNCC	LARK	OLCB	PFI	SSE	TFIN	SSW
2003-04	-0.09	0.13	0.32	-0.59	NA	0.04	0.07
2002-03	0.95	0.21	NA	0.99	NA	0.23	0.11
2001-02	-0.04	0.23	NA	-0.16	NA	0.20	0.22
2000-01	0.20	0.24	NA	1.21	NA	0.23	0.07
1999-00	0.12	-0.16	NA	-0.70	NA	-0.23	0.11
1998-99	-0.61	-0.06	NA	NA	NA	NA	0.18
1997-98	-0.42	-0.02	NA	NA	NA	NA	0.48
1996-97	0.27	0.34	NA	NA	NA	NA	0.30

Figure 6.1: Yearly log returns, $\ln\left(\frac{S_{i+1}}{S_i}\right)$

table 6.1 that one should notice is the lack of data on OLCB and SSE. These companies are fairly new and probably do not have enough data to compute a comparable volatility for SSW. Also notice that PFI and TFIN

[1] All financial information for SSW was provided by SSW.

[2] All financial information for the public companies was found at http://finance.yahoo.com.

are missing 3 return years. While not entirely bad this might cause a little uncertainty in the volatility value. Since SSW has a remarkable record of never having a negative return year when comparing BNCC and LARK to SSW we might consider LARK a good match because they only have 3 negative return years whereas BNCC has 4 negative return years.

Another method should be used to double check the validity of LARK as a good match. Using a pairs plot allows us to graphically compare the log return ratios of the company. An r will be added to the symbols to denote the use of log return ratios. As described above, we plot the values of SSW along the horizontal axis (of the SSWr column) and the values of the public companies along the vertical axis (along each particular row). The data yields the plot figure 6.2. We are interested in the column of SSWr.

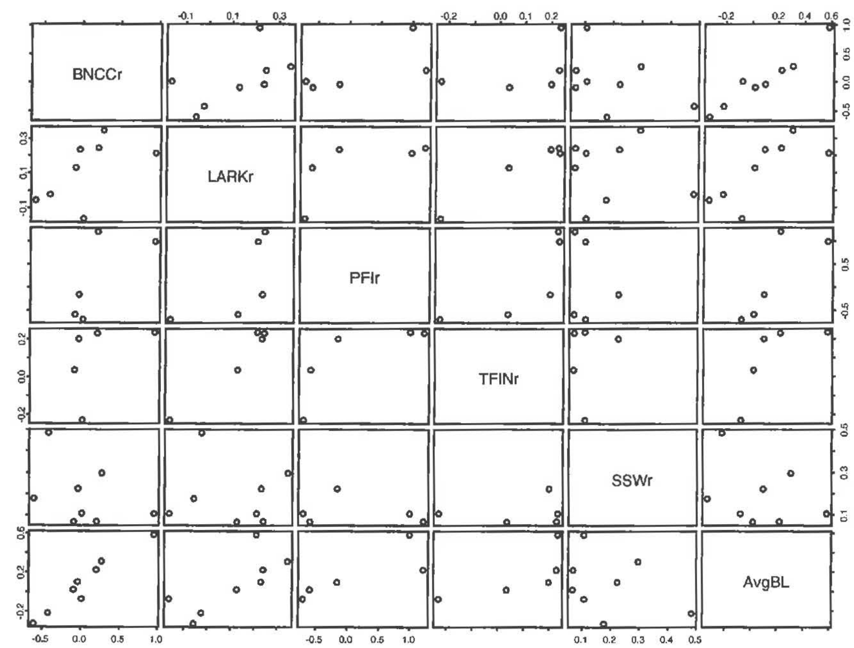

Figure 6.2: Pairs plot of log returns, $\ln\left(\frac{S_{i+1}}{S_i}\right)$

The box where SSWr and BNCCr intersect shows somewhat of a negative linear relationship. The box where SSWr and LARKr intersect shows a positive linear relationship but there is one point at the far right side of the box that could cause problems with our measurements. OLCBr and SSEr were not plotted because they did not have enough values to include in a chart. We mentioned early the lack of values for PFI and TFIN and this shows in the weak linear relationship for both of these companies with SSW.

	BNCCr	LARKr	PFIr	TFINr	SSWr	AvgBL
BNCCr	1.00	0.57	0.64	0.42	-0.38	0.96
LARKr	0.57	1.00	0.64	0.94	-0.06	0.75
PFIr	0.64	0.64	1.00	0.77	-0.09	0.70
TFINr	0.42	0.94	0.77	1.00	0.08	0.62
SSWr	-0.38	-0.06	-0.09	0.08	1.00	-0.33
AvgBL	0.96	0.75	0.70	0.62	-0.33	1.00

Figure 6.3: The Pearson correlations of the log returns.

An AvgBL is also included in this plot. This is the average (mean) of the returns from BNCC and LARK. The average appears to be the strongest linear relationship of all companies. From the pairs plot we might consider the average volatilities of BNCC and LARK for SSW.

One last measure of association we can use to determine the proper public company for SSW is correlation. Taking the returns of the companies and correlating each one with SSW yields the data in table 6.3. Again, using the SSWr column we notice that BNCCr is the closest company to the correlation value of one in absolute value. A correlation value of one means that the two companies compared are perfectly matched. As shown SSWr is perfectly correlated with SSWr which makes sense. From all of the different methods of evaluation, BNCC appears to be the best match for SSW.

The volatility for BNCC is calculated. The closing prices of each day for as many years as possible are used as the data points. The volatility is computed in S-PLUS and the code is given in the Appendix A on page 33. The volatility of BNCC is 0.030902. Now that we have σ for SSW all we need are r, $S(t_0)$, K, and t. We can find r from the Wall-Street Journal and on April 8th of 2005 it was 4.47%. We let $S(t_0) = 500$, $K = 500$, and $t = 3$. The values for $S(t_0)$, K, and t come from an actual stock option bonus given by SSW. We can find ω by using equation 5.2.11.

$$\begin{aligned}\omega &= \frac{0.0447 \cdot 3 + 0.030902^2 \cdot 3/2 - \ln(500/500)}{0.030902\sqrt{3}} \\ &= 2.53219\end{aligned}$$

Now that we have ω we can calculate the cost to SSW of the option given to the employee.

$$\begin{aligned}C &= S(t_0)\Phi(\omega) - Ke^{-rt}\Phi(\omega - \sigma\sqrt{t}) \\ &= 500\Phi(2.53219) - 500e^{-0.0447\cdot 3}\Phi(2.47867) \\ &= 497.15 - 434.278 \\ &= 62.872\end{aligned}$$

SSW wound expense this bonus on their income statement as \$62.87 per share over the life of the option.

6.4 Potential Problems

There are many problems that can arise through this method. They are:

1. We could have a bad match between the public and private company.

2. We could have a bad value for volatility.

3. There could be sudden unexplained market shifts.

In our example in section 6.3 the correlation value was 0.38 in absolute value. This weak correlation tells us that the use of BNCC to model SSW, while a decent match, is not as good as we would like. The securities and exchange commission may want to provide guidance for private companies regarding the expensing of options.

Even if we have a good correlation value the volatility value could still be a little off. Estimating the future volatility of a public company is difficult. Matching of volatilities creates an even greater chance for error. A good resolution to this problem would be to use the volatility of the comparable public company and calculate the current value of the private company from three years ago. For example we know SSW is worth \$535 per share today. We could look back three years and take the value of SSW at that time, \$360 per share, and use that price and the volatility given to compute the price today. If that price is somewhat close to \$535 we know we have a good volatility.

$$\begin{aligned} C &= 360\Phi(2.53219) - 360e^{-0.0447\cdot 3}\Phi(2.47867) \\ &= 357.948 - 312.68 \\ &= 45.268 \end{aligned}$$

This tells us that three years ago we would have valued the same call option at \$45.268. This is very wrong, it should be close to \$175.00. But the inaccuracy is expected due to our poor correlation. At this point we would want to search for a different public company to match with SSW.

Finally, the method given for the valuation of private companies and the Black-Scholes formula do not account for sudden, dramatic shifts in markets. If the risk-free rate of interest drops or increases dramatically then these models fail miserably. An example of the effects of sudden market changes on financial models is the story of Long-Term Capital Management (LTCP). LTCP was a trillion dollar hedge fund that collapsed with the default of the Russian bond market as well as turmoil in Southeast Asia. More on this subject can be found in Lowenstein. With many instances like LTCM it is safe to say that the element of unmeasurable events causes all of the models to fail and markets to crash. This "problem" with unforseen events of nature makes modeling the stock market with perfection impossible.

Chapter 7

Concluding Remarks

The Black-Scholes formula is a powerful tool used in financial mathematics. Before the formula was around, brokers had to guess the amount they would charge to customers for each particular option. Brokers can now enter simple information into a computer and have a result that is an accepted cost of an option.

The Black-Scholes formula has become the benchmark for options pricing. Public and private companies alike will use the formula to estimate the cost of options given as bonuses to company employees. The application given in chapter 6 uses the formula to value the option, but focuses on the estimation of volatility.

The formula is used as the fundamental foundation in mathematical finance like the limit is the foundation in calculus. Similar to calculus, financial mathematics has moved beyond the foundational aspects into more advanced methods. Examples would be binomial trees, general and multiple stock models, American options, and options on stocks where the company pays a dividend just to name a few.

Through the Black-Scholes formula, Myron Sholes and the late Fisher Black started a whole new branch of mathematics. They were able to take what they knew about probability and modeled the expected gain from a yet to be completed options contract that was, at the time, thought to be based upon a completely random process. From the formula, other mathematicians, physicists, and computer scientists were able to model more advanced financial processes. Most importantly what Fisher and Black did was show that in some way many things that were thought to be totally random can be modeled using probability and statistics.

Bibliography

[1] F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–59, 1973.

[2] Maureen Burton and Ray Lombra. *The Financial System and The Economy: Principles of Money and Banking.* Thomson South-Western, Mason, Ohio, 2003.

[3] Neil A. Chriss. *Black-Scholes And Beyond: Option Pricing Models.* McGraw-Hill, New York, 1997.

[4] Black Ervin L. Nach Ralph Epstein, Barry J. and Patrick R. Delaney. *GAAP 2005: Interpretation and Application of Benerally Accepted Accounting Principles.* John Wiley and Sons, Inc., DeKalb, Illinois, 2005.

[5] Sheldon M. Ross. *An Elementary Introduction to Mathematical Finance.* Cambridge University Press, Cambridge, 2003.

Appendix A

Programming code for S-PLUS

The computation of returns:

```
function (x = BNCCquotes) {
    n <- dim(x)[1]
    x[3:n, "Open.Log"] <- log(x[2:(n - 1), "Open"]/x[3:n, "Open"])
    x[3:n, "Close.Log"] <- log(x[2:(n - 1), "Close"]/x[3:n,
        "Close"])
    x
}
```

The computation of volatility:

```
function (x = BNCCquotes) {
    x <- stocklog(x)
    x.close <- sqrt(var(x[, "Close.Log"], na.method = "omit"))
    x.open <- sqrt(var(x[, "Open.Log"], na.method = "omit"))
    x <- c(x.open, x.close)
    names(x) <- c("Open", "Close")
    x
}
```