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Hyperbolic Geometry: A Guide to Models and Motion

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Hyperbolic Geometry

A Guide to Models and Motions

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Chapter 1

Introduction

Twentieth century mathematics can be characterized by the study of functions. The most interesting functions are those that preserve the structure at hand. In many branches of mathematics transformations are looked at and classified. In geometry, transformations of a plane are very important. A transformation that distance is called an *isometry*. The classification of isometries is known in Euclidean geometry, and one will learn about each of them in any geometry course. Classification of isometries for the hyperbolic plane are not as well known, but they have been classified. In this paper we will discover hyperbolic geometry, investigate some of the models used to view hyperbolic geometry and discuss some of the isometries of the hyperbolic plane.

1.1 History of Geometry

In order to discuss a branch of branch of mathematics called "Hyperbolic Geometry," one must understand how the subject came about. The word "geometry" comes from the Greek word *geometrein*, where the first part *geo-* means "earth" and the second part *metrein* means "to measure." Geometry can be dated back to before 3000 B.C.E, and in ancient times was typically used for surveying land, construction, and astronomy; the geometry of this time was limited to properties that were physically observed.

Around 600 B.C.E the Greeks insisted that geometry be derived from studying rather than experimenting; this was the birth of axiomatic geometry. One of the most famous geometers, and considered the father of geometry, was Euclid. Euclid wrote a book entitled *The Elements*, which consisted of thirteen volumes that encompassed all of the known geometry. Euclid's Elements was not the first geometry text, but because it was so comprehensive it made all previous texts unnecessary. In the thirteen volumes of geometry that Euclid wrote, he included five postulates which could not be proven from the axioms. One of these postulates, the famous *Fifth Postulate of Parallels* (see appendix), had many mathematicians working on a proof using other postulates and axioms. Geometry with this famous postulate is called "Euclidean Geometry."

1.1.1 Euclidean Geometry

It is important to note the significance of finding a proof to the fifth postulate. Finding a proof would mean that one would not have to assume Euclid's fifth postulate to be true because it could be proved from the other axioms. This would make the list of postulates and axioms more concise. Many mathematicians tried to prove Euclid's famous "fifth postulate," but they were unsuccessful; some of these mathematicians were Girolamo Saccheri (1667-1733), Adrien-Marie Legendre (1752-1833), and Johann Heinrich Lambert (1728-1777), who made significant findings in many different branches of mathematics. Euclid tried to not include his fifth postulate, and proved the first twenty eight propositions without using it. It has been suggested that Euclid himself tried to find a proof to the fifth postulate before finally giving up and including it as one of his postulates. The geometry without the parallel postulate and any propositions proven using the parallel postulate is called "Absolute Geometry." This geometry provides a basis for "Non-Euclidean Geometry." If one removes the parallel postulate and replaces it with a different parallel postulate, non-Euclidean geometry is formed.

1.1.2 Hyperbolic Geometry

János Bolyai (1802-1860) published his discovery of non-Euclidean geometry in an appendix of a mathematical treatise by his father, Farkas Bolyai, entitled *Tentamen* in 1831. Farkas had a long-time friendship with Carl Fredrich Gauss from when they were both students. Farkas sent Gauss a copy of the *Tentamen* and Gauss sent a response to the Bolyai pair. Gauss claimed that he could not praise the work of János because "to praise it would amount to praising myself." [2] Gauss had come up with the same findings some thirty years earlier. He claimed that he did not feel that it was worthy enough to publish, and Gauss was afraid of what the public might think of his discoveries.

What neither János or Gauss knew was that Nikolai Ivanovich Lobachevsky (1792-1856) was the first to actually publish an account of non-Euclidean geometry, which he did in 1829. Lobachevsky wrote his paper in Russian which is most likely why Gauss and Bolyai did not know about it, since German and French were the predominant languages for mathematics papers during that time. Lobachevsky received many negative reviews, as Gauss feared with his work, but he was not discouraged and kept publishing his works. He published a treatise in German which he sent to Gauss, and later, with Gauss' influence, was elected an honorary member of the Gottingen Scientific Society. Finding hyperbolic geometry was not enough;

someone needed to prove that hyperbolic geometry was consistent. This ended up being shown with the help of models.

Many people set out to find models of this so called "non-Euclidean geometry." Eugenio Beltrami (1835- 1899) wrote a paper in 1868, *Saggio di Interpretazione della Geometria Non-Euclidea* ("Essay on an Interpretation of Non-Euclidean Geometry"), that included his findings for a model for the entire real hyperbolic plane as a disk in $\mathbb{R}^2$ [2]. Beltrami was the first to show that hyperbolic geometry was consistent, using different models that he worked with. Felix Klein (1849- 1925) presented Beltrami's disk model using projective geometry. His model, sometimes known as the Beltrami-Klein model, is much simpler than Beltrami's original model and has come to be known the Klein Disk Model. Klein proved that Euclidean geometry was consistent if and only if hyperbolic geometry was. This ended most debate about hyperbolic geometry being "fake." Two other models of hyperbolic geometry are attributed to Henri Poincaré (1854- 1912); he created another disk model as well as the upper half-plane model. His two models will be discussed after hyperbolic geometry is understood.

1.2 Absolute Geometry

Absolute geometry is based on the first four postulates (listed in Appendix). When Euclid wrote *The Elements* he proved twenty-eight propositions without using his fifth postulate, and those propositions can be considered the basis for absolute geometry. All of the theorems proved using absolute geometry are true in both Euclidean geometry and hyperbolic geometry.

Since hyperbolic geometry came out of a controversy over the fifth postulate, it is no surprise that there is a concept of parallelism in absolute geometry. It should be noted that the word "parallel" in the following theorem simply means non-intersecting. The way that parallelism is mostly thought about is that the lines are everywhere equidistant (like train tracks), but the reader should note that this is *not* the case.[1]

Theorem 1.2.1. *If two lines are cut by a transversal such that a pair of alternate interior angles are congruent, then the lines are parallel.*

Proof. Suppose lines l and m are cut by transversal t with a pair of alternate interior angles congruent. See figure 1.1. Let t cut l and m at points A and B, respectively. Assume that lines l and m intersect at point C. Let C' be the point on m such that B is between C and C' and $\overline{AC} \cong \overline{BC'}$. Consider $\triangle ABC$ and $\triangle BAC'$. We know $\overline{AC} \cong \overline{BC'}$ (by construction), $\overline{AB} \cong \overline{AB}$ (reflexive property), and $\angle ABC' \cong \angle BAC$ (by hypothesis), so by SAS $\triangle ABC \cong \triangle BAC'$. By CPCTC, $\angle BAC' \cong \angle ABC$. Well, $\angle ABC'$ and $\angle ABC$ are supplementary angles and since $\angle ABC' \cong \angle BAC$ then $\angle BAC'$

[1]Unless otherwise noted, all theorems and defintions are from [2]

and $\angle BAC$ are supplementary angles. This means that A lies between C and C', $C * A * C'$, so C' lies on l. This means that l and m intersect at two distinct points, which contradicts Postulate I. Therefore, l and m do not intersect, and are parallel. □

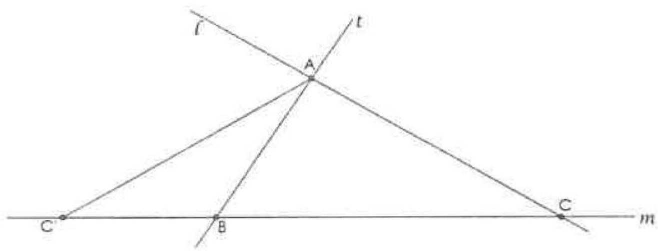

Figure 1.1: Parallelism in absolute geometry.

The type of parallel just discussed, where the lines are not necessarily everywhere equidistant, is quite different than the way most people think about it. As mentioned earlier, parallel here does not mean that the two lines are equidistant everywhere. When we get to the hyperbolic parallel postulate, one can see how assuming that the lines are equidistant is a problem. There are many things in absolute geometry that may seem strange to someone who has only seen Euclidean geometry. In Euclidean geometry the angle sum of a triangle is strictly 180, and it will later be shown that in hyperbolic geometry the angle sum of a triangle is strictly less than 180. We mentioned earlier that everything proven in absolute geometry is true in both Euclidean and hyperbolic geometry, so how does this work with the angle sum of the triangle? It turns out that in absolute geometry, the angle sum of a triangle is less than or equal to 180, which is true in both Euclidean and hyperbolic geometries.

1.3 Hyperbolic Geometry

Hyperbolic geometry is built from absolute geometry, which consists of Euclid's first four postulates, and includes the non-Euclidean parallel postulate.

Theorem 1.3.1 (Axiom P-2). *If l is any line and P any point not on l, there exists more than one line passing through P parallel to l.*

This statement will be better realized with the introduction to models. This is because figure 1.2 does not seem to make sense in the way we are used to looking at lines. It seems that the two lines that go through P will in fact intersect line l at some far away point.

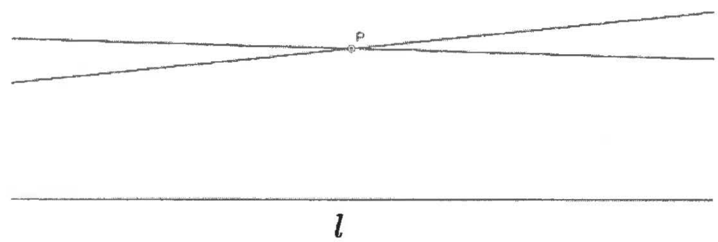

Figure 1.2: An example of multiple lines through P parallel to l.

Another interesting fact about parallelism in hyperbolic geometry is that there are many different types of parallel lines, one being *asymptotically parallel.*

Definition 1.3.1. *Given a line l and a point P not on l, let Q be the foot of the perpendicular from P to l.* A limiting parallel ray *to l emanating from P is a ray $\overrightarrow{PX}$ that does not intersect l and such that for every ray $\overrightarrow{PY}$ which is between $\overrightarrow{PQ}$ and $\overrightarrow{PX}$, $\overrightarrow{PY}$ intersects l.*

Lines l and m are said to be *asymptotically parallel* if m contains a limiting parallel ray to l.

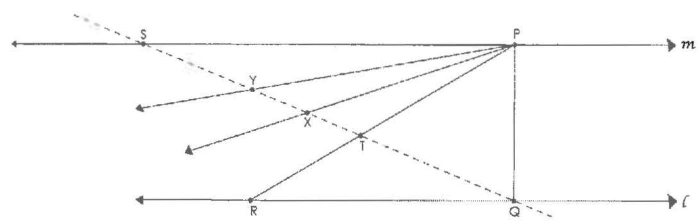

Figure 1.3: Limiting parallel ray PX; m and l are asymptotically parallel.

Seeing the difference in parallelism, one can understand that there must be other differences in the geometries as well. An important characteristic of hyperbolic geometry is that the sum of the measures of the angles of any triangle is less than 180. In order to show this, we will prove that all right triangles have an angle sum of less than 180, and a corollary to this theorem is the result that shows this to be true for every triangle.[2]

Theorem 1.3.2. *The sum of the measures of the angles of any right triangle is less than 180.*

Following the proof in [4],

[2]This next theorem comes from [4]

Proof. Let $\triangle ABC$ be a right triangle with right angle at B. Through A, draw the line $\overleftrightarrow{AP}$, which we will call m, perpendicular to $\overleftrightarrow{AB}$, which P on the C-side of line $\overleftrightarrow{AB}$. By theorem 1.2.1, $m \parallel \overleftrightarrow{BC}$. By the parallel postulate 1.3.1, there is another line n through A parallel to $\overleftrightarrow{BC}$, we may assume that one of its rays from A, $\overrightarrow{AQ}$, lies interior to $\angle BAP$. Now to finish the proof, we will break it up into sections.

1. From above, there exists a ray $\overrightarrow{AQ}$ parallel to line $\overleftrightarrow{BC}$ such that $\overrightarrow{AB} * \overrightarrow{AQ} * \overrightarrow{AP}$. Set $t = m\angle PAQ$.

2. Next, we want to locate a point in W so far out on ray $\overrightarrow{BC}$ that $m\angle AWB$ is arbitrarily small, and in this case, smaller than t. This is an argument Lengendre made that turns out to be true. Thus, there exists some W on $\overrightarrow{BC}$ such that $B{-}C{-}W$ and $m\angle AWB = m\angle W < t$.

3. Now if the order of the rays through A were $\overrightarrow{AB} - \overrightarrow{AQ} - \overrightarrow{AW}$, by the Crossbar Theorem ray $\overrightarrow{AQ}$ would meet segment $\overline{BW}$, which is a contradiction. Therefore, $\overrightarrow{AB} - \overrightarrow{AW} - \overrightarrow{AQ}$. By the first part, we know $\overrightarrow{AB} - \overrightarrow{AQ} - \overrightarrow{AP}$; putting them together we get, $\overrightarrow{AB} - \overrightarrow{AW} - \overrightarrow{AQ} - \overrightarrow{AP}$.

4. Now to estimate the angle sum of $\triangle ABW$:

$$\begin{aligned} m\triangle ABW &= 90 + m\angle W + m\angle BAW \\ &< 90 + t + m\angle BAQ \\ &= 90 + m\angle BAP \\ &= 180 \end{aligned}$$

5. By theorem A.9.2 on page 27, Angle Sum $\triangle ABC < 180$.

□

An obvious consequence to this theorem is that:

Theorem 1.3.3. *The sum of the measures of the angles of any triangle is less than 180.*

Out of any triangle, one can make two right triangles. By use of supplementary angles and the fact that the sum of the angles of a right triangle is less than 180, it can be shown that the original triangle has angle sum less than 180. This result can be extended further by saying that the sum of the angles of a quadrilateral is strictly less than 360.

Since the angle sum of a triangle in hyperbolic geometry is less than 180, we will introduce a new term to help describe a characteristic of the triangle. The *defect* of a triangle is the difference between 180 and the sum of the angles of the triangle. In Euclidean geometry the angle defect is zero for all triangles, which is most likely why the term is never used. For a triangle the defect of $\triangle ABC$ is:

$$D(\triangle ABC) = 180 - m\angle A - m\angle B - m\angle C$$

It can be generalized for convex polygons as well.

Definition 1.3.2. *The* defect *of the convex polygon* $P_1P_2P_3 \ldots P_n$ *is the number* $D(P_1P_2P_3 \ldots P_n) = 180(n-2) - m\angle P_1 - m\angle P_2 - m\angle P_3 - \cdots - m\angle P_n$.

The defect of any triangle in hyperbolic geometry will be a positive value. This can be seen easily for triangles since the angle sum of any triangle is strictly less than 180, then the defect (180 - the angle sum of the triangle) will be between 0 and 180, which is a positive value. Another interesting fact is that the defect of a polygon turns out to be additive. If a convex polygon and its interior is divided in any manner into convex subpolygons, then the sum of the defects of the subpolygons will equal the defect of the original polygon. We will look at an example to understand this idea.

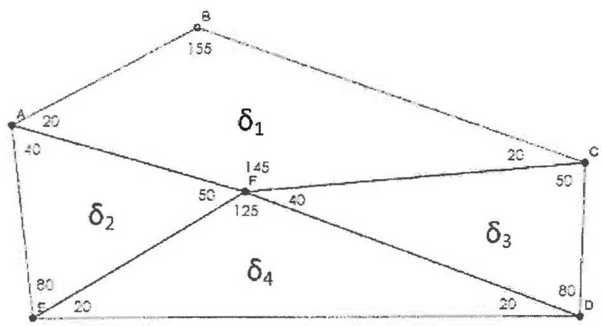

Figure 1.4: An example of the additivity of the defects.

Example 1.3.1. $\delta_1 = 360-$*(Angle sum of* $\diamond ABCF$*)* $= 360 - 340 = 20$

$\delta_2 = 180-$*(Angle sum of* $\triangle AEF$*)* $= 180 - 170 = 10$

$\delta_3 = 180-$*(Angle sum of* $\triangle CDF$*)* $= 180 - 170 = 10$

$\delta_4 = 180-$*(Angle sum of* $\triangle DEF$*)* $= 180 - 165 = 15$

$\delta_1 + \delta_2 + \delta_3 + \delta_4 = 55$
The angle sum of polygon $ABCDE = 60 + 155 + 70 + 100 + 100 = 485$. *Using the formula for the defect,* $\delta = 3 \cdot 180 - 485 = 540 - 485 = 55$. *As seen by this example,* $\delta = \delta_1 + \delta_2 + \delta_3 + \delta_4$.

In Euclidean geometry there are different criteria for showing that two triangles are congruent, such as SAS (Side-Angle-Side), ASA (Angle-Side-Angle), and SSS (Side-Side-Side), but AAA (Angle-Angle-Angle) was a similarity criteria, not a congruence criteria. In hyperbolic geometry all of the criteria hold true, and AAA is a *congruence* criterion!

Theorem 1.3.4. *Triangles that are similar are congruent*

Following a proof in [4],

Proof. Given two similar triangles ABC and A'B'C', assume that they are not congruent; that is that corresponding angles are congruent, but corresponding sides are not. In fact, no corresponding pair of sides may be congruent, or by ASA, the triangles would be congruent. This means that one triangle must have two sides that are greater in length than their counterparts in the other triangle. Suppose that $AB > A'B'$ and $AC > A'C'$. This means that we can find points D and E on sides $\overline{AB}$ and $\overline{AC}$ respectively such that $AD \cong A'B' and AE \cong A'C'$. By SAS, $\triangle ADE \cong \triangle A'B'C'$ and corresponding angles are congruent. Therefore $\angle ADE \cong \angle A'B'C' \cong \angle ABC$ and $\angle AED \cong \angle A'C'B' \cong \angle ACB$. This tells us that quadrilateral $\lozenge DECB$ has angle sum 360. This is a contradiction to the angle sum of a quadrilateral being less than 360, therefore $\triangle ABC \cong \triangle A'B'C'$. □

Now that we have looked at some interesting characteristics of hyperbolic geometry, we will look at how to view hyperbolic geometry using models. A *model* in geometry is an interpretation of the "undefined" terms. All of the terms that can be defined will be interpreted in the model as well, since they are defined in terms of the "undefined." For a model to be valid, all of the axioms must be true under the model, and in turn, all of the theorems, which are proved from the axioms, must hold true as well. There are many models in hyperbolic geometry, some of which will be discussed. There are three models that are most common, the Klein Disk Model, the Poincaré Disk Model, and the Poincaré Upper Half-Plane Model, which were mentioned earlier. Using the Beltrami-Klein Disk Model one can show that all models of hyperbolic geometry are isomorphic to that model; since Beltrami showed that this model was consistent in hyperbolic geometry, then each model that we are discussing is consistent in hyperbolic geometry. Since the Klein Disk Model will not be discussed in the paper, this isomorphism will not be shown. Each model of hyperbolic geometry has its own advantages and disadvantages. The Poincaré models will each be discussed more in depth in the following sections and both models are realized in the Euclidean plane, which is what most people can understand.

1.3.1 Poincaré Upper Half Plane

The Poincaré upper half plane is a model of hyperbolic geometry. This model resides in $\mathbb{R}^2$ and the points of this model are $\{(x, y)|x, y \in \mathbb{R}, y > 0\}$.

The lines in this model are vertical rays emanating from the x-axis (set of points satisfying $x = b$, where b constant, and $y > 0$) and Euclidean semi-circles whose diameter lies on the x-axis (the set of points satisfying $(x-c)^2 + y^2 = r^2$ and $y > 0$, where $b, c, r \in \mathbb{R}, r > 0$). Both lines are orthogonal to the x-axis, see figure 1.5. The set of *ideal points* are all points on the x-axis and the point at infinity. This means that each type of line contains two ideal points; these would be the points at the "end" of the semi-circle and for the vertical ray, the point on the x-axis and the point at infinity.

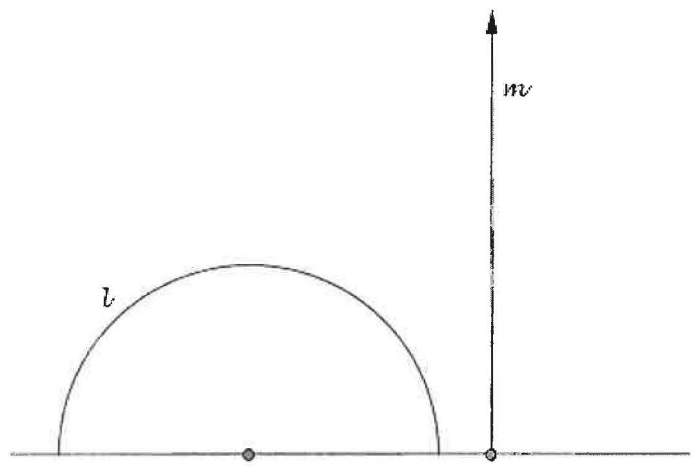

Figure 1.5: UHP examples of lines.

The measurement of the angles in UHP is fairly straightforward.The measure of angle $\angle ABC$ is defined as the measure of the angle formed by the Euclidean rays tangent to BA and BC at B in the same direction as hyperbolic-rays BA and BC; see figure 1.6.

Before we discuss the metric on the UHP, we need to define a cross ratio.

Definition 1.3.3. *Given four points in the plane, A, B, P, and Q, we define the* cross ratio *by:*

$$(AB, PQ) = \frac{AP \cdot BQ}{AQ \cdot BP}$$

The metric on the Upper Half-Plane is defined as follows (see figure 1.7): To find the hyperbolic distance between points A and B, let P and Q be ideal points on the line $\overleftrightarrow{AB}$. For the Euclidean semi-circle, we get:

$$h(AB) = |ln \frac{AP \cdot BQ}{AQ \cdot BP}| = |ln(AB, PQ)|$$

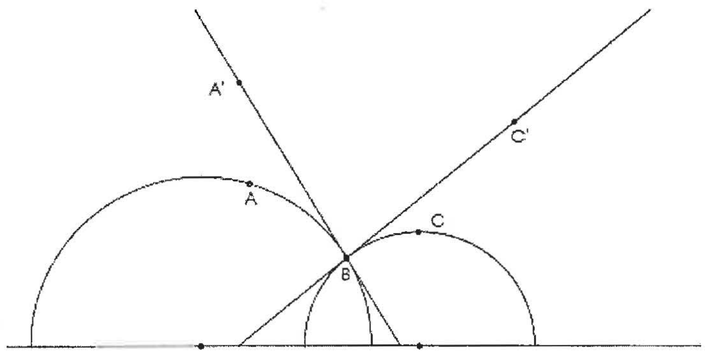

Figure 1.6: UHP examples of measurement of angles. $m\angle ABC = m\angle A'BC'$.

For the Euclidean vertical ray, we get:

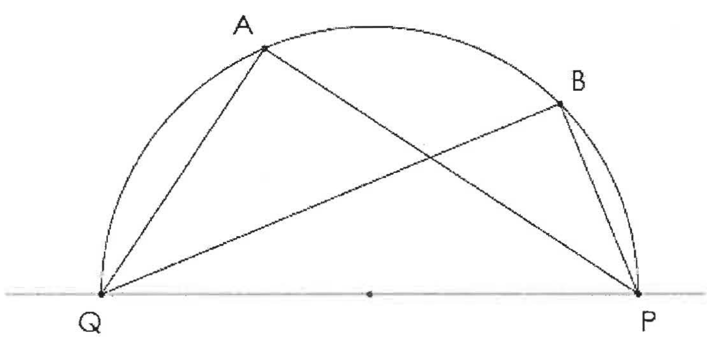

Figure 1.7: UHP example of hyperbolic length using cross ratio.

$$h(AB) = |ln\frac{AP}{BP}|$$

Another way to think about the vertical ray is that both of the ideal points do not lie on the x-axis; one ideal point lies on the x-axis and the other is the point at ∞. This can be derived from the Euclidean Semi-Circle formula by: $h(AB) = |ln\frac{AP \cdot BQ}{AQ \cdot BP}| = |ln\frac{AP}{BP} + ln\frac{BQ}{AQ}$ and as $Q \to \infty$, $\frac{BQ}{AQ} \to 1$ and $ln1 \to 0$, so we are left with the formula for the Euclidean Vertical Ray.

For the UHP model, there is another way to represent the hyperbolic distance using angle measures [5]. Let A, B, P and Q be defined as above, let C be the center of the semi-circle and let α denote the angle between

AC and CQ and let β denote the angle between BC and BQ; see figure 1.8. Using the law of cosines, we get the following result:

$$
\begin{aligned}
(AB, PQ) &= \frac{BQ \cdot AP}{BP \cdot AQ} \\
&= \frac{\sqrt{2r^2 - 2r^2\cos(\beta)} \cdot \sqrt{2r^2 - 2r^2\cos(\pi - \alpha)}}{\sqrt{2r^2 - 2r^2\cos(\pi - \beta)} \cdot \sqrt{2r^2 - 2r^2\cos(\alpha)}} \\
&= \sqrt{\frac{(1-\cos(\beta)) \cdot (1+\cos(\alpha))}{(1+\cos(\beta)) \cdot (1-\cos(\alpha))}}
\end{aligned}
$$

Multiplying by a form of 1, we get:

$$
\begin{aligned}
&= \sqrt{\frac{(1-\cos(\beta)) \cdot (1+\cos(\alpha))}{(1+\cos(\beta)) \cdot (1-\cos(\alpha))}} \cdot \sqrt{\frac{(1-\cos(\beta)) \cdot (1-\cos(\alpha))}{(1-\cos(\beta)) \cdot (1-\cos(\alpha))}} \\
&= \sqrt{\frac{(1-\cos(\beta))^2 \cdot \sin^2(\alpha)}{\sin^2(\beta) \cdot (1-\cos(\alpha))^2}} \\
&= \frac{(1-\cos(\beta)) \cdot \sin(\alpha)}{\sin(\beta) \cdot (1-\cos(\alpha))} \\
&= \frac{\csc(\beta) - \cot(\beta)}{\csc(\alpha) - \cot(\alpha)}
\end{aligned}
$$

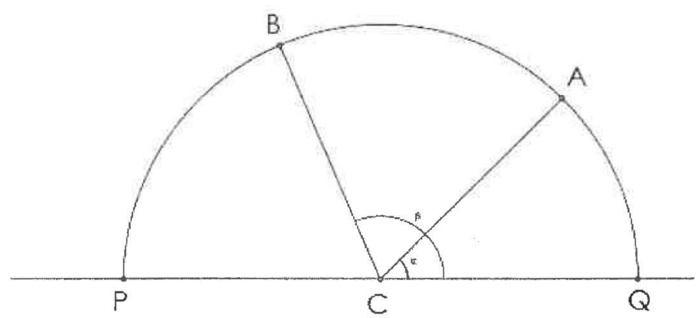

Figure 1.8: UHP example of hyperbolic length using angles.

Therefore, another way to define hyperbolic distance in UHP is:

Theorem 1.3.5. *Let γ be a circle with center $C(c, 0)$ and radius r. If A and B are points of γ such that the radii CB and CA make angles α and β ($\alpha < \beta$) respectively, with the positive x-axis, then the hyperbolic length of arc AB is equal to:*

$$|ln\frac{csc(\beta) - cot(\beta)}{csc(\alpha) - cot(\alpha)}|$$

By a simple substitution from the earlier definition we can get a similar definition for a vertical ray.

Theorem 1.3.6. *The hyperbolic length of the Euclidean line segment joining the points $B(a, y_1)$ and $A(a, y_2)$, $0 < y_1 \leq y_2$ is:*

$$ln\frac{y_2}{y_1}$$

When we started looking at hyperbolic geometry, we discussed the parallel postulate and how the models will make the statement more clear. Using the UHP model, we can see how the parallel postulate looks; see figure 1.9.

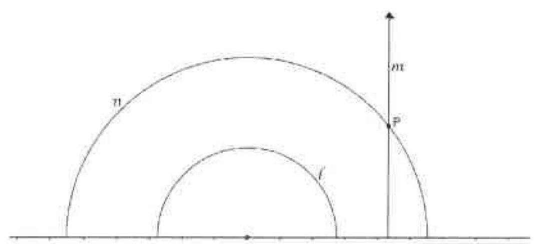

Figure 1.9: Lines m and n are parallel to l and go through P.

We can see that the lines that go through P do not intersect l, which seems more accurate than the original diagram that was shown. Now that we have seen what the Upper Half Plane is and how it works, we will move into the Disk model and do the same.

1.3.2 Poincarè Disk Model

The Poincaré Disk Model resides in the interior of the unit circle, γ, in the Euclidean Plane. The points of this model are the points lying interior to γ. The lines of this model are Euclidean circles orthogonal to γ or diameters of γ; see figure 1.10.

The PDM and the UHP model are very similar in how their metrics are defined; both use cross ratios in their definition.

The metric of the Poincaré Disk Model is defined as follows; where P, Q are the ideal points on the line $\overleftrightarrow{AB}$:

$$h(AB) = \frac{1}{2}|ln(\frac{AP \cdot BQ}{AQ \cdot BP})| = \frac{1}{2}|ln((AB, PQ))|$$

The angle measure of the angle formed at point A by lines l and m is defined as the angle formed by lines l' and m' at A where l' and m' are the Euclidean lines tangent to l and m, respectively, at A; see figure 1.11.

We have already seen how the parallel postulate is viewed in UHP, and for the PDM it can be seen in figure 1.12. It can be seen in the figure

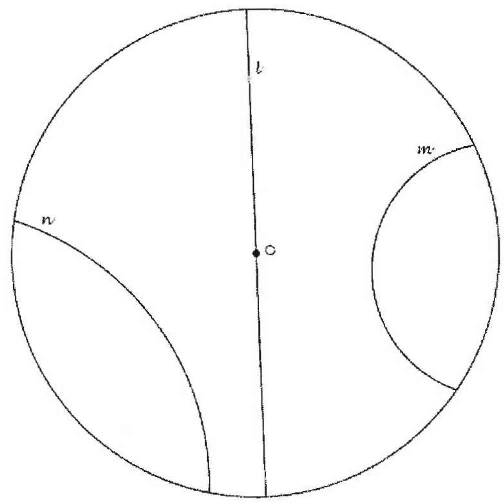

Figure 1.10: What lines look line in the PDM.

that the two lines that go through P do not intersect l. The reader should note that the lines look like they intersect at the "ends" of line l (the ideal points), but one must remember that those points are *not* in the model.

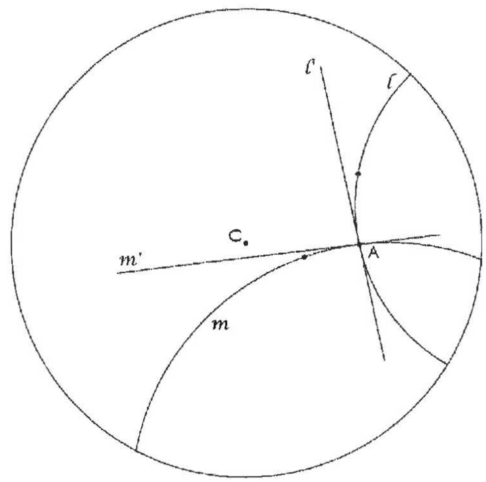

Figure 1.11: How to measure angles in PDM.

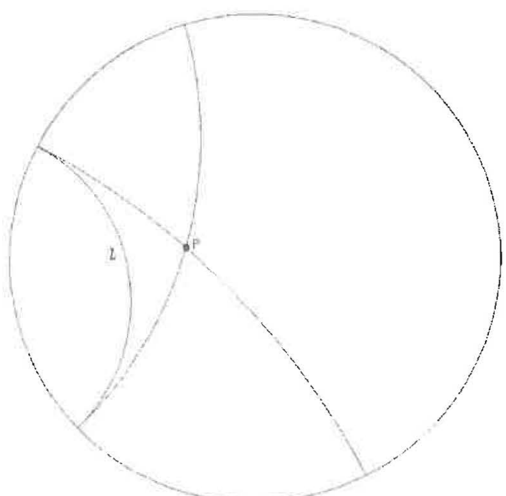

Figure 1.12: Two lines are parallel to l and go through P.

Chapter 2

Rigid Motions

We will begin our discussion on motions by first defining what a motion actually is.

Definition 2.0.1. *A transformation T of the entire plane onto itself is called a* motion *(also called* rigid motion*) or an* isometry *if length is invariant under T, i.e. if for every segment AB, $\overline{AB} = \overline{A'B'}$, or equivalently $AB \cong A'B'$.*

There is a well known result in Absolute geometry that will be very useful when describing isometries.

Theorem 2.0.2. *An isometry is angle-measure preserving (A), betweenness preserving (B), collinearity preserving (C), and distance preserving (D).*

By the above theorem, we know that isometries are distance preserving which is very important in theorems to come. It is well known that the isometries of the Euclidean plane are reflections, rotations, translations, and glide reflections, and these are discussed in every Euclidean geometry course. We will briefly discuss each of them in order to extend the results to hyperbolic geometry.

The reflection in a given line l is defined as follows:

$$\rho_l : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \rho_l(P) = P'$$

The points of l are fixed under ρ_l and l is the perpendicular bisector of PP'. It is obvious that a reflection is its own inverse, meaning $\rho_l(\rho_l(P)) = P$.

A translation through a given vector AB is defined as follows:

$$\tau_{AB} : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \tau_{AB}(P) = P'$$

Vector PP' is of the same length as vector AB and PP' is either collinear with or parallel to AB. The inverse of the translation in vector AB is the translation in vector BA, meaning $\tau_{AB}(\tau_{BA}(P)) = P$. There are no fixed points for a translation in a non-zero vector.

A rotation about a point C through angle α is defined as follows:

$$R_{C,\alpha} : \mathbb{R}^2 \to \mathbb{R}^2 \qquad R_{C,\alpha}(P) = P'$$

The segments CP and CP' are congruent and $\angle PCP' = \alpha$. The only fixed point is C, the center of rotation. The inverse of the rotation is the rotation about C by an angle $-\alpha$, meaning $R_{C,-\alpha}(R_{C,\alpha}(P)) = P$.

A glide reflection is a composition of a translation by a vector AB with a reflection in line AB, defined as:

$$G_{AB} : \mathbb{R}^2 \to \mathbb{R}^2 \qquad G_{AB}(P) = \rho_{AB} \circ \tau_{AB}(P) = P'$$

Note that the order of the composition does not matter. There are no fixed points in a glide reflection with a non-zero vector AB.

Reflections are the most fundamental type of motions. Using reflections, we will generate all other motions. It turns out that every motion can be written as a product of at most three reflections; a motion written as a product of three reflections can be categorized as either a reflection or a glide. We will denote a reflection by ρ_m across line m, its axis, and we will denote the image of a point A under ρ_m by A^m. A transformation that is not the identity and is equal to its own inverse is called an *involution*. It so happens that reflecting across m twice sends every point back where it came from, $\rho_m\rho_m = I$ or $\rho_m = (\rho_m)^{-1}$. Another important characteristic that is used to classify motions is fixed points. A *fixed point* of a transformation T is a point A such that $A' = A$. It is clear that the fixed points of a reflection ρ_m are the points lying on m. The following theorems are about fixed points and motions [2].

Theorem 2.0.3. *If a motion T fixes two points A, B, then it fixes every point on line $\overleftrightarrow{AB}$.*

Proof. Let T be a motion and let A and B be fixed under T. Then $A = A'$ and $B = B'$ which means $\overline{AB} = \overline{A'B'}$, since isometries preserve distance. Let C be a third point on $\overleftrightarrow{AB}$. Consider the case A*B*C (All other cases are similar). Since motions preserve betweenness, by 2.0.2, A*B*C' and since motions preserve distance, $\overline{AC} = \overline{AC'}$ and therefore, $C = C'$. □

Theorem 2.0.4. *If a motion fixes three noncollinear points, then it is the identity.*

Proof. If A, B, C are fixed noncollinear points, then by theorem 2.0.3 so is every point on the lines joining these three points, namely $\overleftrightarrow{AB}$, $\overleftrightarrow{AC}$, and $\overleftrightarrow{BC}$. If D is not on those three lines, choose any E between A and B. By Pasch's theorem (A.9.3 on page 27), line $\overleftrightarrow{DE}$ meets another side of $\triangle ABC$ in a point F. Since E and F are fixed, theorem 2.0.3 tells us that D is fixed. □

Theorem 2.0.5. *If a motion fixes two points A, B, and is not the identity, then it is the reflection across line $\overleftrightarrow{AB}$.*

Proof. Let T be a motion that fixes two points A and B, and let $T \neq I$. 2.0.3 ensures that every point on $\overleftrightarrow{AB}$ is fixed. Let C be any point not on $\overleftrightarrow{AB}$ and let F be the foot of the perpendicular from C to $\overleftrightarrow{AB}$. Since motions preserve angle measure, they preserve perpendicularity, so C' must lie on $\overleftrightarrow{CF}$. Theorem 2.0.4 ensures that $C' \neq C$ (since $T \neq I$), and since $\overline{CF} = \overline{C'F}$, C' is the reflection of C across $\overleftrightarrow{AB}$. □

When one thinks of motions they might think about a continuous movement of an object to its image. This next theorem seems to have that idea of motion.

Theorem 2.0.6. *$\triangle ABC \cong \triangle A'B'C'$ if and only if there is a motion sending A, B, C, respectively, onto A', B', C' and that motion is unique.*

Following the proof in [2],

Proof. Using theorem 2.0.4 we can show uniqueness. If T and T' had the same effect on A, B, C then $T^{-1}T'$ would fix these points; hence $T^{-1}T' = I$ and $T = T'$. Since motions preserve distance, then the motion maps $\triangle ABC$ onto a congruent triangle, by SSS. We will start with $\triangle ABC \cong \triangle A'B'C'$ and create the motion T. Assume $A \neq A'$ and let t be the perpendicular bisector of AA'. Then reflection across t sends A to A' and B, C to points B^t and C^t. If $B^t = B'$ and $C^t = C'$, we are done. Assume $B^t \neq B'$. Then $A'B' \cong AB \cong A'B^t$. Let u be the perpendicular bisector of $B'B^t$, so that ρ_u sends B^t to B'. This reflection fixes A', because if A', B^t, B' are collinear, A' is the midpoint of $B'B^t$ and lies on u, otherwise if they are not collinear, u is the perpendicular bisector of the base of isosceles triangle $\triangle B'AB^t$ and u passes through the vertex A'. Thus, $\rho_u \rho_l$ sends the pair (A, B) to the pair (A', B'). If it also sends C to C', we are done. Let C'' be its effect on C. Then $A'C' \cong AC \cong A"C''$ and $B'C' \cong BC \cong B'C''$ so that $\triangle A'B'C' \cong \triangle A'B'C''$. See figure 2.1. Since $\overline{A'C'} \cong \overline{A'C''}$, $\triangle C'AC''$ is an isosceles triangle. Thus, by theorem A.9.4 (on page 27), A' lies on the perpendicular bisector of $C'C''$ (this same argument can be made for B' lying on the perpendicular bisector of $C'C''$ as well). Thus $v = \overleftrightarrow{A'B'}$ will send C'' to C' and it will fix A' and B'. Thus $\rho_v \rho_u \rho_t$ is the motion we seek. □

Since an isometry preserves angle measure and distance, the isometry is completely determined by a triangle and its image under the isometry. What turns out to be a consequence to the above theorems is that,

Theorem 2.0.7. *Every motion is the product of at most three reflections.*

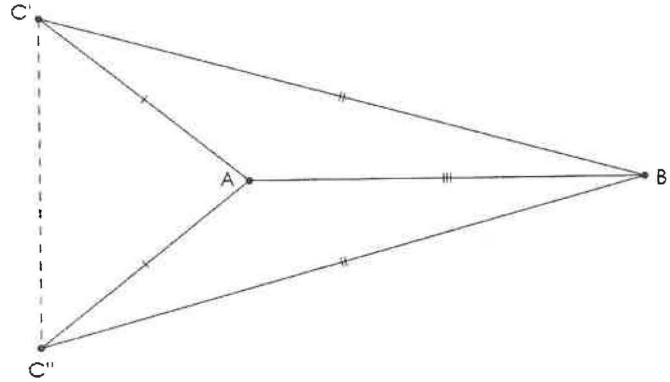

Figure 2.1: l is the perpendicular bisector or BB'.

As seen in the above proof, sending one triangle to its image takes up to three reflections, and thus, every motion is the composition of at most three reflections.

The identity motion is the product of zero reflections, a reflection is the product of one reflection and later we will see it is also the product of three reflections under certain circumstances. We will now move onto products of two reflections. Rotations are the product of two reflections, $T = \rho_l \rho_m$, where l and m meet at some point A.

Theorem 2.0.8. *A motion $T \neq I$ is a rotation if and only if T has exactly one fixed point.*

Proof. $\Rightarrow$ Suppose T has only one fixed point, A, and choose a point B, where $B \neq A$. See figure 2.2. Let l be the perpendicular bisector of BB'. Since $AB \cong AB'$, A lies on l by theorem A.9.4 (page 27). The motion $\rho_l T$ fixes both A and B. If $\rho_l T = I$, then $T = \rho_l$, which contradicts the hypothesis that T has only one fixed point. Hence if $m = \overleftrightarrow{AB}$, theorem 2.0.5 implies $\rho_l T = \rho_m$, so that $T = \rho_l \rho_m$ and T is a rotation about A. See figure 2.2. $\Leftarrow$ Given rotation $T = \rho_l \rho_m$ about A, assume that a point B, where $B \neq A$, is fixed. Then the reflection of B about l, B^l, is equal to the reflection of B about m, B^m. Joining $B^l = B^m$ to B gives a line perpendicular to both l and m, which is a contradiction. Therefore, T has only one fixed point. □

This next theorem will help us prove the theorem on three reflections.

Theorem 2.0.9. *If T is a rotation about A and m is any line through A, then there is a unique line l through A such that $T = \rho_l \rho_m$. If l is not perpendicular to m, then for any point $B \neq A$,*

$$(\angle BAB')^\circ = 2d^\circ$$

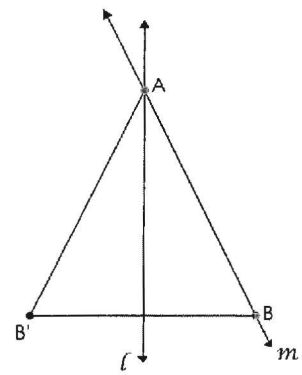

Figure 2.2: Showing that m is the line through A and B.

where d is the number of degrees in the acute angle made by intersecting lines l and m.

Proof. Let T be a rotation about A. By theorem 2.0.8, A is the only fixed point of T. Let B be a point such that $B \neq A$. Let l be the perpendicular bisector of BB'. Since $AB \cong AB'$, A lies on l by theorem A.9.4 (page 27). The motion $\rho_l T$ fixes both A and B. If $\rho_l T = I$ then $T = \rho_l$ which is a contradiction to T being a rotation since ρ_l fixes a line. If $m = \overleftrightarrow{AB}$, theorem 2.0.5 implies that $\rho_l T = \rho_m \Rightarrow T = \rho_l \rho_m$. To show the second part, we know $\triangle ACB \cong \triangle ACB'$ (where C is the intersection of l and BB'). This implies that $\angle CAB \cong \angle CAB'$. This means, $\angle BAB' = 2 \cdot \angle CAB = 2d$. □

Now we are able to prove the theorem on three reflections. The following are three types of *pencils of lines* which we will need to classify motions.

1. The pencil of all lines through a given point P.
2. The pencil of all lines perpendicular to a given line t.
3. The pencil of all lines through a given ideal point Σ (hyperbolic plane only).

This theorem leads to the complete classification of motions.

Theorem 2.0.10. *Let $T = \rho_l \rho_m \rho_n$.*

1. *If l, m, and n belong to a pencil, then T is a reflection in a unique line of that pencil.*
2. *If l, m, and n do no belong to a pencil, then T is a glide.*

Following a similar proof in [2], we are able to prove this theorem.

Proof. 1. Let l, m, and n belong to the pencil of all lines through a point A. Since A lies on all three lines, A is a fixed point of T, so T has at least one fixed point. Assume A is the only fixed point of T. Then T is a rotation, by theorem ?? about A. Since T is a rotation about A and n is any line through A, then there exists a unique line p through A such that $T = \rho_l \rho_m \rho_n = \rho_p \rho_n$. This means that $\rho_l \rho_m \rho_n \rho_n = \rho_l \rho_m = \rho_p$. But the product of two reflections cannot equal a reflection because the number of fixed points are different. Thus, A is not the only fixed point of T. If T fixes three noncollinear points, then T is the identity, but T cannot be the identity since T is the product of an odd number of reflections. So T must fix another point, B. Thus by 2.0.5, T is a reflection across $\overleftrightarrow{AB}$.

Let l, m, and n belong to the pencil of all lines perpendicular to a line t. Let K be a translation such that $K = \rho_l \rho_m$. Since $n \perp t$, then by theorem A.10.1, there exists a unique p such that $p \perp t$ and $K = \rho_p \rho_n$. Then $\rho_l \rho_m = \rho_p \rho_n$ and by simple arrangement, $\rho_l \rho_m \rho_n = \rho_l$. Since $T = \rho_l \rho_m \rho_n$, then $T = \rho_p$; therefore T is a reflection in a unique line of the pencil of all lines perpendicular to a line t.

Let l, m, and n belong to the pencil of all lines through ideal point Σ. Let K be the parallel displacement, $K = \rho_l \rho_m$. By theorem A.10.2, since n is a line through Σ, $\rho_l \rho_m = \rho_h \rho_n$, then $T = \rho_l \rho_m \rho_n = \rho_h$, so T is a reflection across a line in the pencil.

2. Assume the lines do not belong to a pencil. See figure 2.3. Choose any point A on l. Let m' be the line through A belonging to the pencil determined by m and n, since two lines determine a pencil. Then line n' exists such that

$$\rho_{m'} \rho_m \rho_n = \rho_{n'}$$

by part (1). Let B be the foot of the perpendicular k from A to n'. Since l, m', and k pass through A, line h exists such that

$$\rho_l \rho_{m'} \rho_k = \rho_h$$

by part (1). Then B does not lie on h (since l, m, and n do not belong to the same pencil), so by theorem A.10.3 on page 28, $\rho_h H_B$ is a glide along the perpendicular to h through B. But

$$\rho_h H_B = \rho_h(\rho_k \rho_{n'}) = \rho_l \rho_{m'} \rho_k \rho_k \rho_{m'} \rho_m \rho_n = \rho_l \rho_m \rho_n = T.$$

Therefore, T is a glide.

□

With this theorem one can classify any T that can be written as the product of three reflections. This is a very interesting result because it is true in both hyperbolic geometry and Euclidean geometry, although the definition of pencils would be different in both.

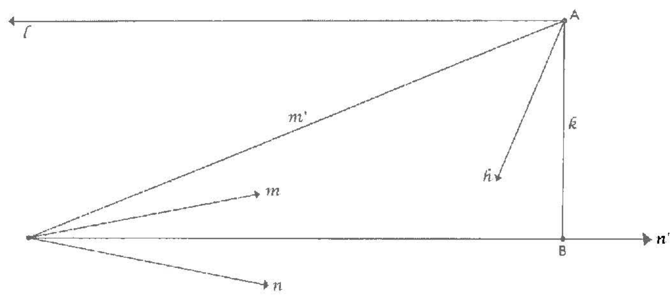

Figure 2.3: The proof of the theorem of three reflections.

2.1 Conclusion

The theorem on three reflections is a major result for isometries of the plane, but there are many more results that this paper does not cover. One can go deeper and discuss the isometries of the Poincaré models. This becomes more advanced and uses complex coordinates in order to fully understand. Reflections in the Upper Half Plane are easily seen and are included in the appendix, see page 28. By delving deeper into the models, one will be able to prove many theorems dealing with different types of parallel lines, one of which was mentioned (asymptotically parallel), as well as looking at the ideal points in each model. A major theme in geometry is looking at triangles and how motions affect the triangle (such as reversing the orientation), and one can look at the orientation-preserving isometries as well as asymptotic triangles (having one, two or three vertices at an ideal point) and how motions affect those triangles. Sadly, these topics mentioned were not in the scope of this paper, but it is strongly encouraged to keep learning about this “strange new world”.

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Appendix A

Axioms

A.1 Postulates

I For every point P and for every point Q, with $P \neq Q$, there exists a unique line that passes through P and Q.

II For every segment AB and for every segment CD there exists a unique point E on line $\overleftrightarrow{AB}$ such that B is between A and E and segment CD is congruent to segment BE.

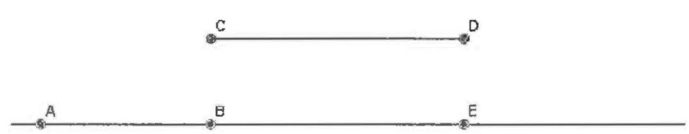

Figure A.1: $CD \cong BE$.

III For every point O and every point A, with $A \neq O$, there exists a circle with center O and radius OA.

IV All right angles are congruent to one another.

V If two lines in the same place are cut by a transversal so that the sum of the measures of a pair of interior angles on the same side of the transversal is less than 180, the lines will meet on that side of the transversal. This can be restated, see Parallel Postulate section of the appendix.

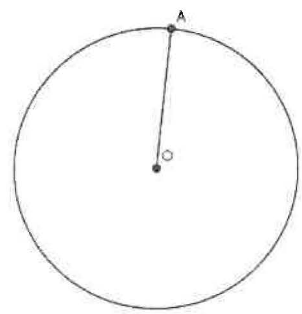

Figure A.2: Circle with center O and radius OA.

A.2 Incidence Axioms

Axiom I-1 Each two distinct points determine a line.

Axiom I-2 Three noncollinear points determine a plane.

Axiom I-3 If two points lie in a plane, then any line containing those two points lies in that plane.

Axiom I-4 If two distinct planes meet, their intersection is a line.

Axiom I-5 Space consists of at least four noncoplanar points, and contains three noncollinear points. Each plane is a set of point of which at least three are noncollinear, and each line is the set of at least two distinct points.

A.3 Metric Axioms

Axiom D-1 Each pair of points A and B is associated with a unique real number, called the *distance* from A to B, denoted AB.

Axiom D-2 For all points A and B, $AB \geq 0$, with equality only when $A = B$.

Axiom D-3 For all points A and B, $AB = BA$.

Axiom D-4 (Ruler Postulate) The points of each line l may be assigned to the entire set of real numbers x, $-\infty < x < \infty$, called *coordinates*, in such a manner that

1. each point on l is assigned to a unique coordinate
2. no two points are assigned to the same coordinate

3. any two points on l may be assigned the coordinates zero and a positive real number, respectively.
4. if points A and B on l have coordinates a and b, then $AB = |a-b|$.

A.4 Angle Axioms

Axiom A-1 Each angle $\angle ABC$ is associated with a unique real number between 0 and 180, called its *measure* and denoted $m\angle ABC$. No angle can have measure 0 or 180.

Axiom A-2 If D lies in the interior of $\angle ABC$, then $m\angle ABD + m\angle DBC = m\angle ABC$. Conversely, if $m\angle ABD + m\angle DBC = m\angle ABC$, then ray $\overrightarrow{BD}$ passes through an interior point of $\angle ABC$.

Axiom A-3 (Protractor Postulate) The set of rays $\overrightarrow{AX}$ lying on one side of a given line $\overleftrightarrow{AB}$, including ray $\overrightarrow{AB}$, may be assigned to the entire set of real numbers x, $0 \leq x < 180$, called *coordinates*, in such a manner that

1. each ray is assigned a unique coordinate
2. not two rays are assigned to the same coordinate
3. the coordinate of $\overrightarrow{AB}$ is 0
4. if rays $\overrightarrow{AC}$ and $\overrightarrow{AD}$ have coordinates c and d, then $m\angle CAD = |c-d|$.

Axiom A-4 A linear pair of angles is a supplementary pair.

A.5 Plane Separation Postulate

Definition A.5.1. *A set K in S is called* convex *provided it has the property that for all points $A \in K$ and $B \in K$, the segment joining A and B lies in K ($\bar{AB} \subseteq K$).*

Axiom H-1 Let l be any line lying in any plane P. The set of all points in P not on l consists of the union of two subsets H_1 and H_2 or P such that

1. H_1 and H_2 are convex sets
2. H_1 and H_2 have no points in common
3. If A lies in H_1 and B lies in H_2, the line l intersects the segment $\bar{AB}$.

A.6 Congruence Axioms

Axiom C-1 (SAS) Under the correspondence $ABC \leftrightarrow XYZ$, let two sides and the included angle of $\triangle ABC$ be congruent, respectively, to the corresponding two sides and the included angle of $\triangle XYZ$. Then $\triangle ABC \cong \triangle XYZ$.

A.7 Parallel Postulates

Hilbert's Euclidean Parallel Postulate For every line l and every point P not lying on l there is at most one line m through P such that m is parallel to l.

Euclid's Fifth Postulate If two lines are intersected by a transversal in such a way that the sum of the degree measures of the two interior angles on one side of the transversal is less than 180°, then the two lines meet on that side of the transversal.

Negation of Hilbert's Euclidean Parallel Postulate There exist a line l and a point P not on l such that at least two distinct lines parallel to l pass through P.

A.8 Equivalent Forms of Euclid's Fifth Postulate

- The area of a right triangle can be made arbitrarily large.
- The angle sum of all triangles is constant.
- The angle sum of a single triangle equals 180.
- Rectangle exist.
- A circle can be passed through any three noncollinear points.
- Given an interior point of an angle, a line (transversal) can be drawn through that point intersecting both sides of the angle.
- Two parallel lines are everywhere equidistant.
- The perpendicular distance from one of two parallel lines to the other is always bounded.

A.9 Theorems

Theorem A.9.1 (Crossbar Theorem). *If $\overrightarrow{AD}$ is between $\overrightarrow{AC}$ and $\overrightarrow{AB}$, then $\overrightarrow{AD}$ intersects segment BC.*

Theorem A.9.2. *If $\triangle ABC$ has angle sum less than 180 and D is any point on side $\bar{BC}$, then both $\triangle ABD$ and $\triangle ADC$ have angle sum less than 180.*

Theorem A.9.3 (Pasch's Theorem)**.** *If A, B, C are distinct noncollinear points and l is any line intersecting AB in a point between A and B, then l also intersects either AC or BC. If C does not lie on l, then l does not intersect both AC and BC.*

Theorem A.9.4. *If $\triangle BAB'$ is an isosceles triangle, A lies on the perpendicular bisector of $B\bar{B}'$*

Proof. Let $\triangle BAB'$ be an isosceles triangle with $A\bar{B} \cong A\bar{B}'$. Let D be the midpoint of $B\bar{B}'$. Construct the line segment AD. By definition of median, $A\bar{D}$ is a median of $\triangle BAB'$. We want to show that $A\bar{D}$ is also an altitude of $\triangle BAB'$. Since $A\bar{D}$ is a median, $B\bar{D} \cong D\bar{B}'$. Also, $A\bar{B} \cong A\bar{B}'$ by given. By the reflexive property, $A\bar{D} \cong A\bar{D}$. By SSS, $\triangle ABD \cong \triangle AB'D$. By CPCTC, $\angle ADB \cong \angle ADB'$. Thus, $\angle ADB$ and $\angle ADB'$ are supplementary, congruent angles. Hence $A\bar{D}$ and $B\bar{B}'$ are perpendicular, and $A\bar{D}$ is an altitude of $\triangle BAB'$. Since $A\bar{D}$ is a median and an altitude of $\triangle BAB'$, it lies on the perpendicular bisector of $B\bar{B}'$. □

Theorem A.9.5. *a The summit angles of a Saccheri quadrilateral are congruent to each other.*

b The line joining the midpoints of the summit and the base is perpendicular to both the summit and the base.

Theorem A.9.6. *Consider $\triangle ABC$ and its perpendicular bisectors. If perpendicular bisectors l and m are asymptotically parallel in the direction of ideal point Ω, then the third perpendicular bisector n is asymptotically parallel to l and m in the same direction Ω.*

Theorem A.9.7. *If a rotation has an ideal fixed point, then it is the identity.*

Theorem A.9.8. *The ends of m are the only ideal fixed points of the reflection ρ_m.*

A.10 Theorem on Three Reflections

In order to prove the theorem on three reflections, the following definitions and theorems are needed.

Theorem A.10.1. *If T is a translation along t and m is any line perpendicular to t, then there is a unique line $l \perp t$ such that $T = \rho_l \rho_m$.*

Proof. Let T be a translation along t and let m be a line perpendicular to t. Let m cut t at Q, and let l be the perpendicular bisector of QQ'. Then $\rho_l T$ fixes Q. Let P be any point on m, so that $\diamond PQQ'P'$ is a Saccheri quadrilateral. Since l is perpendicular to the base QQ' at its midpoint, l is also perpendicular to the summit PP' at its midpoint, by theorem A.9.5 (b). So P is the reflection of P' across l. Thus $\rho_l T$ fixes every point on m, and therefore $\rho_l T = \rho_m$, and $T = (\rho_l)(\rho_l)T = \rho_l \rho_m$. □

In order to discuss the next theorem needed, one must know what a *parallel displacement* is.

Definition A.10.1. *Let $T = \rho_l \rho_m$. If l and m are asymptotically parallel in the direction of an ideal point Ω, T is called a* parallel displacement *about Ω.*

Theorem A.10.2. *Given a parallel displacement $T = \rho_l \rho_m$, where l and m are asymptotically parallel in the direction of ideal point Σ. Then let k be any line through Σ and A any point on k; then Σ lies on the perpendicular bisector h of AA' and $T = \rho_h \rho_k$.*

Proof. Let k be any line through Σ and A be any point on k. Σ lies on two perpendicular bisectors l and m of $\triangle AA^m A'$, so by theorem A.9.6, Σ also lies on the third perpendicular bisector h. Then $\rho_h T$ fixes A and Σ. By theorem A.9.7, $\rho_h T$ cannot be a rotation about A. By theorem 2.0.5 (page 17), $\rho_h T$ must be a reflection, and by theorem A.9.8, it has to be the reflection across the line k joining A to Σ. □

In the next theorem the term H_A is used. H_A is a *half turn* about A, which means it is a rotation about A by 180 degrees.

Theorem A.10.3. *Given point B and line l, let t be the perpendicular to l through B. Then $\rho_l H_B$ is a glide along t if B does not lie on l and is ρ_t if B does lie on l.*

Proof. Let B lie on l. Then $H_B = \rho_l \rho_t$, and therefore $\rho_l H_B = (\rho_l)(\rho_l)(\rho_t) = \rho_t$. Let B not lie on l, let $m \neq l$ be the perpendicular to t through B. Then $T = \rho_l \rho_m \neq I$, since $l \neq m$. Also, $H_B = \rho_m \rho_t$, and therefore $\rho_l H_B = \rho_l \rho_m \rho_t = T\rho_t$. □

A.11 Reflections in UHP

Refections in the Upper-Half-Plane model turn out to be Euclidean reflections in the vertical lines or inversions in circles γ whose center is on the x-axis. It should be obvious to the reader that the reflection across a vertical line in the UHP model is an isometry, but a figure might help; see figure A.4. We shall now discuss inversion.

Definition A.11.1. *Let C be a circle of radius r, center O. For any point $P \neq O$, the* inverse *P' of P with respect to C is the unique point P' on ray $\overrightarrow{OP}$ such that $(\bar{OP})(\bar{OP'}) = r^2$.*

Some useful information about inversion is as follows:

Theorem A.11.1. *Properties of Circular Inversion (C is the circle of inversion and O is the center).*

1. *Points inside C map to points outside of C.*
2. *Points outside C map to points inside of C.*
3. *Each point on C is self-inverse (maps to itself).*
4. *Lines or circles map to lines or circles.*
5. *A line through O is invariant, but the individual points of that line are changed.*
6. *A circle through O maps to a line not through O, and the image line is perpendicular to the line that passes through O and the center of the given circle.*
7. *Cross Ratio is invariant under circular inversion.*
8. *A circular inversion is* **conformal**, *that is, it preserves curvilinear angle measure.*

We will start by looking at an example of circular inversion in the Euclidean plane.

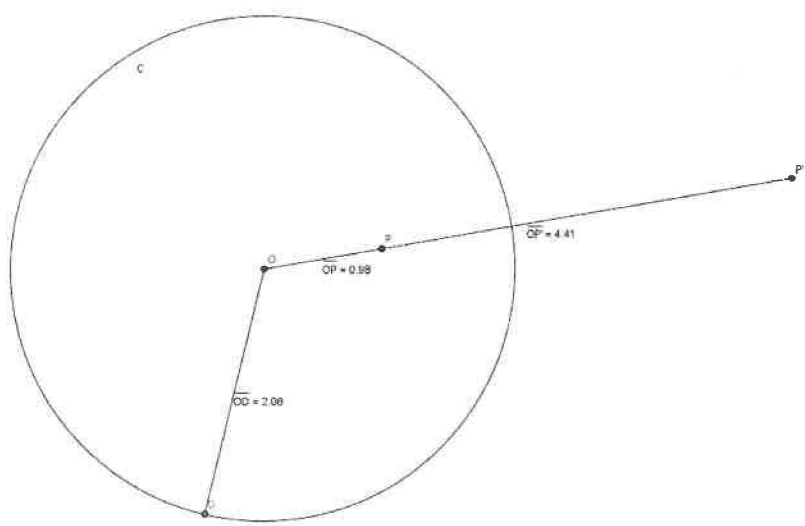

Figure A.3: An example of circular inversion.

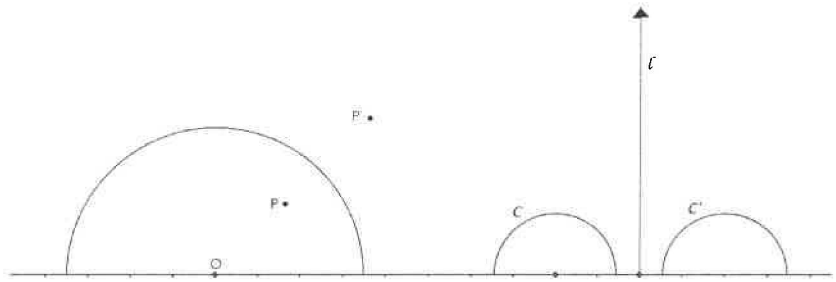

Figure A.4: An example of reflections in the upper half plane. Left is circular inversion and to the right is reflection across a vertical line.

In figure A.3, we can see how circular inversion works in $\mathbb{R}^2$. In the example, O is the center of the circle C with $r = 2.08$. Our point, P, lies inside of C. As seem in A.11.1, P' lies outside of C, with $OP = 0.98$ and $OP' = 4.41$. From definition A.11.1, we see that $(0.98) \cdot (4.41) = (2.08)^2 = 4.3264$.

Circular inversion does not map the center of inversion to a point, and therefore is not a transformation of the Euclidean plane $\mathbb{R}^2$, unless the point at infinity was included, $\mathbb{R}^2 \cup \{\infty\}$. Then the point at infinity would map to the center and vice versa. Thankfully, in the UHP the center of inversion is an ideal point, and thus is not in the model. Inversion of the UHP can also be seen in a figure; see figure A.4.