Spring 5-2017

Ode to Applied Physics: The Intellectual Pathway of Differential Equations in Mathematics and Physics Courses: Existing Curriculum and Effective Instructional Strategies

Brandon L. Clark
University of Maine

ODE TO APPLIED PHYSICS: THE INTELLECTUAL PATHWAY OF DIFFERENTIAL EQUATIONS IN MATHEMATICS AND PHYSICS COURSES: EXISTING CURRICULUM AND EFFECTIVE INSTRUCTIONAL STRATEGIES

by

Brandon L. Clark

A Thesis Submitted in Partial Fulfillment
of the Requirements for a Degree with Honors
(Mathematics and Physics)

The Honors College
University of Maine
May 2017

Advisory Committee:

John R. Thompson, Professor of Physics, Advisor
Thomas Bellsky, Assistant Professor of Mathematics
Natasha Speer, Associate Professor of Mathematics Education
Michael C. Wittmann, Professor of Physics
Edith Elwood, Adjunct Assistant Professor in Honors (Sociology)

Abstract

The purpose of this thesis is to develop a relationship between mathematics and physics through differential equations. Beginning with first-order ordinary differential equations, I develop a pathway describing how knowledge of differential equations expands through mathematics and physics disciplines. To accomplish this I interviewed mathematics and physics faculty, inquiring about their utilization of differential equations in their courses or research. Following the interviews I build upon my current knowledge of differential equations in order to reach the varying upper-division differential equation concepts taught in higher-level mathematics and physics courses (*e.g.,* partial differential equations, Bessel equation, Laplace transforms) as gathered from interview responses. The idea is to present a connectedness between the simplest form of the differential equation to the more complicated material in order to further understanding in both mathematics and physics. The main goal is to ensure that physics students aren't afraid of the mathematics, and that mathematics students aren't without purpose when solving a differential equation. Findings from research in undergraduate mathematics education and physics education research show that students in physics and mathematics courses struggle with differential equation topics and their applications. I present a virtual map of the various concepts in differential equations. The purpose of this map is to provide a connectedness between complex forms of differential equations to simpler ones in order to improve student understanding and elevate an instructor's ability to incorporate learning of differential equations in the classroom.

Contents

List of Figures

Chapter 1

Introduction

As an undergraduate double major in Mathematics and Physics, I've had an educational experience that has inspired the writing of this thesis. While it may be expected that the mathematics and physics departments would bear a strong connection, in my experience there was a noticeable disconnect. Specifically, I found the disconnect centered around the implementation of differential equations in both math and physics courses. In my differential equation courses, I learned how to solve a variety of differential equations with little to no context. In my physics courses, it was the opposite. Differential equations in physics are presented primarily through context with little to no mathematical formalism.

Be mindful that this is not an education research thesis. I present background literature in research in undergraduate mathematics education and physics education research in order to demonstrate content connections – and lack thereof – documented in prior studies on student learning. Although I use education research as supporting information, it is only a basis on which I intend to build in order to demonstrate the implementation of differential equations in mathematics and physics classrooms and how students and instructors utilize the different mathematical and physical tools at their disposal.

The purpose of this thesis is to explore the interconnection between physics and mathematics through differential equations, based primarily on my educational experiences but also on information gained from the education research literature and interviews of instructors of differential equations courses and relevant physics courses. The centerpiece of this thesis is a constructed map I have built to show intellectual links between mathematical theory and concepts in physics. The map provides a visual relating the various types of differential equations and the solution methods typically used to solve those differential equations. Additionally, there are branches of the map connecting physical applications typically taught in mathematics and physics courses to the different types of differential equations. To build this map I utilized my own personal class notes as well as instructor interviews to explore how content in differential equations is linked across mathematics and physics curricula. In the instructor interviews I asked questions related to the types of differential equations instructors taught in their courses, the solution methods to solving those differential equations, as well as any physical applications which correspond to the various different differential equations. Putting all the information together produced a detailed visual map highlighting the connections between various mathematics and physics topics through differential equations. The map is built as a tool for instructors and students alike to demonstrate how earlier mathematical ideas help build up to more complex mathematical and physical content. I want to emphasize the importance of what I refer to as experiential learning. I define experiential learning as utilizing previously learned content, either from life or the classroom, to understand and further develop skills through more complicated material. In mathematics, I consider experiential learning concepts to be content that (not necessarily math-focused) students have seen before and can utilize when solving more complex mathematical problems. The physics aspect provides an added element to experiential learning, contributing physical applications to aid in understanding beyond the raw mathematics. This is not a one-way street, where physics only helps make sense of the mathematics; in many instances the mathematics can clarify details in physics.

Overall, the visual map I've built has its place on a larger scale. This map focuses on a specific part of a global image which demonstrates the interrelated structure of differential equations. The best self-visualization I have is a family tree. A family tree comprised of the span differential equations cover with the ordinary differential equation set analogous to the oldest generation (the top of the tree). From the ordinary differential equation one can construct a pathway to more complex content in differential equations. This thesis puts a spotlight on a few select pathways from the ordinary differential equation to various applications in undergraduate physics courses. In order to present these ideas with the reader in mind, I supply a step-by-step sequence of differential equation topics that I hope demonstrate the correlation of distinct ideas taught in mathematics and physics courses in order to aid in understanding.

Chapter 2

Background Content

2.1 The Differential Equation

A differential equation (DE) expresses a relationship between a function and its derivative; this typically represents a relationship between a quantity and the rate of change of that quantity. For differential equations, the goal is no longer about algebraically solving for a number, but instead solving for functional solutions primarily using ideas from Calculus. The solution to a differential equation is a function that satisfies the relationship between the derivatives and the function described by the differential equations. There are a few categories in which to describe a differential equation, and commonly different categories present different solution types. The first category is the *order* of a differential equation which is determined by the highest order derivative term in the differential equation. The following are examples of first- and second-order differential equations:

$$\text{First Order DE}: \ \frac{dy}{dt} + a(t)y = b(t) \tag{2.1}$$

$$\text{Second Order DE}: \ -\frac{\hbar^2}{2m}\frac{d^2\varphi_E(x)}{dx^2} + \frac{1}{2}m\omega^2 x^2 \varphi_E(x) = E\varphi_E(x). \tag{2.2}$$

Equation 2.1 is a first order differential equation because the first derivative term $\frac{dy}{dt}$ is the highest order derivative term in the equation. The second-order differential equation (2.2) is the energy eigenvalue differential equation for the quantum harmonic oscillator which is discussed in detail later in the paper. It's a second order differential equation because the second-derivative term $\frac{d^2\varphi_E(x)}{dx^2}$ is the highest order derivative term in the equation.

The next category is whether a differential equation is *homogeneous* or *non-homogeneous*. The following equations show the difference between homogeneous and non-homogeneous differential equations:

$$\text{Non-Homogeneous First Order DE}: \frac{dy}{dt} + a(t)y = b(t), \;\; b(t) \neq 0 \tag{2.3}$$

$$\text{Homogeneous First Order DE}: \frac{dy}{dt} + a(t)y = 0, \;\; b(t) = 0. \tag{2.4}$$

Equation 2.3 is a non-homogeneous differential equation because the terms cannot be re-arranged such that the right hand side (in this case $(b(t)$ is equal to zero. On the other hand, 2.4 is a homogeneous differential equation because the terms can be rearranged such that the right hand side, $b(t)$, is zero. Whether an equation is homogeneous or non-homogeneous leads to different solution types for differential equations. The difference in homogeneity can be utilized when describing various physical systems as well. If the function and rates of change for a particular system have a functional dependence on the right hand side of the differential equation, it will be non-homogeneous.

The third category which describes differential equations is the notion of whether one is *linear* or *non-linear*. A differential equation is linear if the variables and derivatives are multiplied by constants or variables independent of the solution function. An example of a non-linear differential equation is

$$\frac{dy}{dt} = y^2 = y \cdot y$$

where the variable y is clearly not independent of the solution y itself. Non-linear differential equations can be challenging to solve. Typically it requires that the differential equation is linearized. This allows us to utilize linear solution techniques, which we focus on primarily in this paper, in order to solve the more complex differential equation. It is common to have systems of linear equations. Later on we discuss transforming between second-order differential equations and systems of first-order linear differential equations as a technique for solving second-order differential equations. The aspect of linear is a useful tool in context of differential equations in simplifying a more complex differential equation or system.

The last category used to differentiate differential equations is whether or not a differential equation is Ordinary or Partial. An ordinary differential equation (ODE) is a differential equation where the solution is an unknown function of one independent variable. A partial differential equation (PDE) is a differential equation that contains partial derivatives, as opposed to ordinary, where the solution is an unknown function of multiple independent variables. Below is an example of a partial differential equation:

$$u_t = u_{xx} + u_{yy}, \tag{2.5}$$

where $u(x, y, t)$ is a a function of three independent variables. Equation 2.5 is known as the heat equation (in two dimensions). Here the notation u_t and u_{xx} represent partial first and second derivatives where $u_t = \frac{\partial u}{\partial t}$ and $u_{xx} = \frac{\partial^2 u}{\partial x^2}$.

Knowing these classifications is the first step of many in relating the ideas and implementations of differential equations. The classifications produce a variety of different concepts to explore in the realm of differential equations. The rest of the paper breaks down the specific differential equations taught in a general undergraduate sequence building up to a interconnected visual map between the mathematics of differential equations and the experiential contextualization in physics. First we explore the literature that provides a foreground in student learning of differential equations in mathematics and physics to highlight some of the key implementations of differential equations found in earlier educational research studies that I use as a base to building my map.

Chapter 3

Background Literature

Despite the centrality of differential equations and their solution methods in undergraduate mathematics, science, and engineering curricula, there have been fewer than 24 empirical studies published in top journals in the past 12 years related to the teaching and learning of differential equations.[1] One reason for this may be the focus in research in undergraduate mathematics education (RUME) and physics education research (PER) on introductory courses at the undergraduate level. The following sections discuss research on common student difficulties and strategies for reinventing solutions in differential equations in each of these disciplines. This highlights how students utilize prior knowledge in mathematics or physics to solve more complex mathematical and physical ideas.

3.1 Student Difficulties: RUME

Student difficulties in undergraduate differential equations education is an expanding research topic(*e.g.*, [1-7]). A consistent finding in the research literature in undergraduate mathematics education is that students face epistemological challenges across multiple facets of solving differential equations. Additionally with respect to multiple representations of a solution (analytical, numerical, or graphical), students tend to privilege algebraic approaches to graphical, despite being in classes which emphasize graphical and qualitative analysis[1]. This is an example where experiential knowledge, such as algebraic approaches, may be favored by students, while students may avoid or have trouble understanding what they may consider more complicated methods (*e.g.,* graphical approaches).

Breaking away from experiential, solving for a function rather than a number may be a new idea for students in differential equations[2]. It's a challenge in mathematics education to create learning environments in which students generate, refine, and extend their intuitive and informal ways of reasoning to more sophisticated and formal ways of reasoning[3]. While mathematics and physics curricula are structured sequentially such that advanced classes and topics stem from previous courses/concepts, making the logical connections can be difficult.

In differential equation courses, solution methods can be taught using new or foreign concepts, preventing students from utilizing their intuition to help comprehend a solution. For instance, research states that most, if not all, differential equations textbooks solve systems of linear differential equations using techniques from linear algebra. Students are typically taught to find eigenvalues and corresponding eigenvectors, and from there they form an analytic solution. This solution strategy often stems from the characteristic equation. Mathematical ideas like eigenvalues, eigenvectors, and the characteristic equation tend to be poorly understood by students[3]. We will see later an example where students simplify solving a system of linear first order differential equations demonstrating knowledge of straight line graphing as opposed to typical eigenvalue/eigenvector solution approach. In this particular example students developed a strategy using their experiential knowledge of slopes and ratios as a substitute for the standard linear algebra approach.

Another major challenge comes from the fact that universities are now accepting a much larger and more diverse group of students[4]. This adds an additional layer of complexity to the expectation that students will exploit prior knowledge in order to enhance their understanding of differential equations. Consequently, the educational issues facing universities have changed, introducing new pedagogical challenges. One response to these challenges is to develop new curricular and instructional approaches based on contemporary theories of learning and instructional design. One such innovative approach, referred to as the Inquiry-Oriented Differential Equations (IO-DE) project, is establishing that graphical and numerical approaches should not be taught as ends in and of themselves, but rather should emerge as tools for students as they solve challenging problems [5]. I plan to show that physical context and mathematical formalism can share a similar functionality for students. In their respective disciplines, mathematics could view physical contextualization as a guide to enhance student understanding, and likewise, physics courses could avoid using mathematics purely as a means to a desired result. Physics and mathematics build off of one another and may be considered mutually exclusive. When we do not contextualize math or physics, students find that getting an answer is sufficient and that they're not expected to understand why the result makes sense. This is a consequence of rule-based explanations or the "because the professor said so" cliché[2].

A differential equation is a relationship between some quantity and that same quantity's rate of change[6]. These quantities are primarily expressed as functions in mathematics and physics. In a study by Kuster[6] one resource students accessed was what he called functional dependence, which provided support for relating specific values in differential equations, as well as determining which equations potentially matched a vector field based on which variables the value of the derivative term $\frac{dy}{dt}$ depended on. Kuster defines the term "resource" as a small set of small-scale knowledge elements that have a productive role during the process of problem solving. In Kuster's analysis of two students, he noticed that while both of the students utilized many of the same resources, their application of them in the individual tasks was different more times than not. Similarly the interpretations of the differential equations and their components within tasks were often different[6]. This supports that students approach content in mathematics differently, and that different solution methods and multiple representations of differential equations may increase overall student comprehension.

Students don't just have difficulties with differential equations in mathematics courses; researchers in physics education are exploring challenges students face with differential equations in physics as well. A few common findings are discussed in the next section.

3.2 Student Difficulties: PER

Research into student difficulties at the upper division is a growing area of physics education research (PER) (*e.g.*, [8-14]). Students in upper-division courses are asked to manipulate increasingly sophisticated mathematical tools as they tackle more advanced physics content[10].

In one study, researchers explored student difficulties with the mathematical procedure of separation of variables, which is common to upper division physics and is a common tool used when solving first order differential equations [9]. The equation students were asked to separate is:

$$mv\frac{dv}{dx} = mg - bv^2,$$

based off Newton's second law (which we will discuss in detail later). Wittmann and Black argue that there are multiple procedural resources that can be brought into problem solving, and these resources are used in different combinations by different students. In this study, the particular resources are algebra based, including the operations of multiplication, division, addition, subtraction, and grouping[9]. It is not uncommon to see that students have multiple approaches to solving a given problem, and many students appear more comfortable with their own personalized strategy in approaching it. In fact, students generally have a multitude of ways to correctly solving a problem, and it's important to recognize these different solution pathways, specifically recognizing their values and shortcomings[9]. In physics, there rarely is one specific way in which a solution to a problem can be found. Therefore, there may be more than one solution method to solving differential equations in a physics context. This goes back to allowing students to manipulate algebra in a way that makes sense to them. It may be that one pathway to a solution is more effective than another, but knowing what to suggest to a student in a given moment requires an understanding of the variety of student thinking [9].

There are obviously many factors which influence student thinking including their educational background. The variety of approaches in algebraic strategies utilized in separating an equation may stem from student differences in educational background. To account for the diversity in techniques, instructors should be prepared to share different approaches to solving problems in order to accommodate for student needs. This will be a key point in my project as I attempt to shine a light on different mathematical and physical solution pathways in order to meet the knowledge based needs of a broad range of students with varying educational backgrounds. Let's consider using mathematical actions as a kind of thought [9]. Every mathematical action may be a tool for students. For example, students are not just dividing, but the action of division serves a thoughtful purpose. As educators, we need to provide thought through mathematical action (as a tool) in order to improve student understanding. This thought through mathematical action is highly prevalent in a physics differential equations context. Many of the differential equation solutions in physics we will find lend themselves to the properties of the mathematical tool students use. There may be more than one tool capable of finding the solution, and as educators we must help students utilize these mathematical tools in order to make sense of this physics. Otherwise, students may become lost in the mathematics, and in turn, unable to provide any physics understanding. This becomes clear in the next study.

One common technique in solving partial differential equations in a physics context is separation of variables (SOV). Here they use the term SOV to refer to the technique of guessing a general solution with a functional form that allows the partial differential equation to be separated into several ordinary differential equations and then solving these ordinary differential equations individually with appropriate boundary conditions. This technique is not to be confused with the strategy, also conventionally referred to as separation of variables, used to solve separable ordinary differential equations by isolating terms with the function on one side of the equals sign and the independent variable on the other side and integrating both(discussed in the last study)[10]. In an undergraduate physics curriculum partial differential equations appear in numerous contexts including waves on a string, thermodynamics, and the Schrödinger equation. In one study students had difficulties with construction of the model (mapping between the physics and mathematics of a problem), specifically when they would inappropriately eliminate mathematical terms, incorrectly set up or fail to utilize the nonzero boundary condition, or set up an integral (i.e. Fourier's trick) incorrectly. The majority of these issues arose from incorrectly establishing an expression to match the nonzero boundary condition[10].

In the same study, when executing the mathematical formalism students typically would write down solutions from memory or an equation sheet. About twenty-percent of the students relying on memorization/regurgitation provided a general solution inconsistent with the ODE they were solving. Common mistakes included an incorrect function form based on the sign of the separation constant as well as misusing the separation constant in the general solution[10]. This begins to demonstrate that memorization/regurgitation of general solutions alone can be an inefficient method for determining specific solutions for differential equations. In order for students to correctly express the mathematics, they must additionally consider the context of the specific situation. Additionally, in the study when student's were asked to determine nonzero constants, the researchers found that more mathematical errors occurred applying Fourier's trick as opposed with term matching. This is most likely due to the mathematical rigor of a Fourier transform being an inherently more demanding mathematical strategy[10]. This supports that there may be alternate strategies to determining solutions (or specific aspects to solutions) which are mathematically favorable to students. The concept of a Fourier trick may be unfamiliar and/or complicated for students as opposed to the more algebraic approach with term matching.

3.3 Simplifying the Problem

An idea from engaged model construct theory is that students engaged in mathematical activity can reinvent formal mathematics starting with experientially real situations[1]. For experientially real situations consider physical and natural applications that students would be familiar with. Grounding mathematics in familiarity may provide students with the means to develop their own mathematical strategies to apply on current and future problems. Realistic Mathematics Education (RME) focuses specifically on engaging students in the reinvention of mathematical ideas in differential equations[3]. It may be possible to reinvent ideas using previous, possibly simpler solution methods and mathematical concepts. Typical in students' mathematical work is treating mathematical terms as physical objects, offering an interplay of metaphor and bodily motion, which are significant elements of doing mathematics[1]. Students naturally work toward making mathematics, including the pure symbolism, a physical construct which they can manipulate in order to enhance their mathematical comprehension. In differential equations, the derivatives represent physical rates of change, the solution models particular behavior, and solution methods for differential equations, like *separation of variables*, can have analogous physical attributes.

Currently typical instruction at the undergraduate level tends to not encourage students to create their own strategies, but recent educators have been exploring approaches that invite learners to build their own ideas and ways of presenting these ideas[3]. To see how students utilize experiential knowledge to develop their own solution methods let's discuss a study focused on linear systems of differential equations. Linear systems of differential equations typically arise in physical and natural sciences as a way to describe two or more simultaneous rates of change. These systems are formed in order to analyze solutions to higher-order differential equations by reduction of order. The typical method for solving linear systems of differential equations involves ideas from linear algebra. Students are typically taught to find eigenvalues and corresponding eigenvectors and then form an analytic solution, which often stems from the characteristic equation. These mathematical ideas tend to not be well understood by students. These concepts are discussed in more detail later in Section 5.2.15.1.

In this study [3] a unique approach called "eigenvector first approach" or "slope first approach" is developed as a substitute for the linear algebra eigenvalue method. Focusing on eigenvectors first extends students' strong mathematical and intuitive understanding of slope[3]. Any preexisting knowledge of slope is the experiential aspect of forming a new solution. It's assumed that students' comprehension of slopes is better than their understanding of eigen-based concepts from linear algebra. Using ideas from slope, students created an innovative analytic solution method that combined graphic and analytic representations[1]. The new solution method is referred to as the straight line solution (SLS) method which was designed as a simplification to the common eigenvalue method described above. SLSs are significant mathematical ideas because they serve the basic building blocks for all other solutions of linear and non-linear systems of differential equations[3]. To see my derivation of the straight line solution method see Section 5.2.15.2.

If a teacher wants students to reinvent important mathematical ideas, it is the responsibility of the teacher to foster in students the kind of curiosity and mathematical goals that have the potential to lead to the intended reinvention[3]. I interviewed course instructors to determine whether or not they are implementing multiple strategies for solution methods for students. Instructors have a variety of tools, but what if instructors are having students drive a nail with a wrench, as opposed to a hammer? Then students are attempting solutions with complex, less understood, methods. The goal is to determine methods from which students gain the most understanding. As we know, student learning is grounded in experientially "real" situations, which leads to the development of formal mathematics. For systems of differential equations, the "real" includes the slopes of vectors[3]. For example, it is common for students to have worked with slope-intercept form of linear expressions and graphing linear relationships in Algebra focused courses earlier in their mathematics education. The straight line solution method as fore-mentioned provides a graphical solution to supplement the analytic solution known as a phase diagram, or phase portrait. The phase portrait is the collection of solution graphs contained in the phase plane. This graphical representation for solutions emerges as a new mathematical reality for students.[3] This further supports the centrality of interplay between numerical, graphical, and analytic representations in students' mathematical work.[1]

Chapter 4

Methods

The goal of this thesis was to show how various differential equations content in mathematics and physics curricula are interrelated in my experience and analysis. Subsequently, I explored the interconnectedness of ideas and solution strategies implemented in a typical sequence of mathematics and physics courses. This information helped to build a map, visually demonstrating the intellectual connections between different topics in differential equations. In order to gather this information I went back though my class notes for courses from Calculus II up to a senior level Quantum & Atomic physics course. Along with the class notes, I went back to the textbooks associated with each course to further acquire data to include in the final mapping of ideas. To prevent the thesis being entirely biased by my course notes and textbooks, I vetted research literature in student learning of differential equations to gain further insight on the typical differential equations material covered in mathematics and physics courses. The literature did in fact provide ideas not provided by my classwork or instructor interviews.

Interviewing instructors was the third method by which I gathered information for building my map. Asking a series of eight questions to eight faculty across mathematics and physics disciplines, I gathered information on what exactly instructors implement in their classrooms. The instructors primarily helped in establishing a timeline of information flow typical for their coursers, as well as mathematics and physics curricula in general. In terms of the visual map, the faculty interviews helped connect the dots between the various facets of content between types of differential equations, their solution methods, and the physical applications attributed to them. This chapter breaks down the different methods going into more detail about their specific role in constructing the map.

4.1 Class Notes

The primary source material for this thesis is my own reflection on the education I've received as an undergraduate mathematics and physics double major. I've reviewed the notes and material from the courses I've taken as a guide to construct the intellectual progression from the ordinary differential equation to higher level concepts in mathematics and physics. The courses I've taken relevant to this project are Calculus II, Differential Equations, Classical Mechanics, Electricity and Magnetism, Physical Electronics, and Quantum & Atomic Physics. Many examples found throughout this project are taken directly from the notes I took for these courses. On top of the notes I've taken as a student I include any information garnered from the textbooks associated with each course. Additional examples are taken from variations of problems that I previously solved as a student of the course, either from the instructor or the textbooks. Any additional information from the textbooks is an inherent supplement of the education I received, and will provide its own unique perspective on the differential equation topics covered in each class. While many instructors teach "by the book," there were a few occasions where professors would deviate from the textbook, and the textbook would be an extra guide for my own growth and understanding. At times the textbook provides details the professor didn't cover and there are times when the professor would clarify vague aspects of the textbook, or go above and beyond what the textbook offers in terms of content. For instance, not all textbooks provide the same solution method for particular differential equations. Reasons akin to these are why I chose to include both my notes and textbook material as tools for my project on building a map of differential equation related ideas. A full list of textbooks will be listed in the References (*e.g.*, [22]-

[25]). My personal class notes are biased to my unique experience in the particular courses I took. To reduce biased information from the project I looked at research literature and conducted faculty interviews with instructors from the math and physics departments to broaden the scope of ideas and content for this thesis.

4.2 Literature

A portion of my research comes from research in the fields of mathematics and physics education. The literature provided different solution methods that I hadn't discovered through my own experiences and that I believe are invaluable to my overall project. The literature provided information that was not biased to my own experience or the course/faculty routine at the universities where I conducted the interviews. I chose to include aspects of the vetted literature as an attempt to explore a broader scope of learning in mathematics and physics education centered around differential equations. I am aware that my individual education did not provide me with all there is to know about differential equations, so I have taken examples from a few articles in order to enhance my interconnected mapping of concepts surrounding differential equations.

4.3 Interviews

In order to gather more of a perspective on the course progression at universities in the northeast and where/how differential equations are implemented in math and physics curricula, I interviewed eight faculty from both mathematics and physics departments. I asked each faculty member the same series of eight questions seen here:

1. What types of differential equations do you typically use in your courses? [First Order, Second Order, PDE, Higher Order, Homogeneous, Inhomogeneous, Linear, Nonlinear, etc.]

2. What applications are represented by these differential equations in your courses? Mathematical or physical?

3. What solution methods do you use for these differential equations? Have you ever considered more than one?

4. To what extent do you invoke initial and boundary conditions in applications of differential equations? Why or why not?

5. To what extent do you present DEs from a physical perspective? What contexts do you use, and for which DEs?

6. To what extent do you find that context, or the use of physical context in general, useful or helpful for the students? Is it more helpful at the time, or for future topics (either in this course or others)?

7. Have you ever found in your experience of more complex differential equations that it helps to rely on a previous, possibly simpler, solution or concept? If so, when?

8. How do you connect the simpler solution/concept to the present one?

The interviews were scheduled for an hour in a private one-on-one setting; they were either videotaped or notes were taken as part of an open discussion. With many faculty I would follow up with them after the interviews to either clarify ideas or expand on specific concepts that I believed imperative to my research. While building my project I would watch the videos and/or review any written artifacts to construct the progression of differential equations for each class. The interview data is intertwined with the material from my notes and the literature findings in order to paint the best picture for the reader.

Chapter 5

Mathematics Courses

This chapter works through two courses, Calculus II and Differential Equations, where differential equations are implemented in a typical undergraduate mathematics sequence. These two courses were selected based on the instructors interviewed and their courses taught. This section is written in a sequence that demonstrates the interconnectedness of mathematical concepts through differential equations. A majority of students see a differential equation for the first time in the second semester of the Calculus sequence, utilizing introductory solution methods like direct integration and separation of variables. Later in a Differential Equations course, students learn more complex strategies for solving multiple types of differential equations and discuss the mathematical theory that governs the solutions to differential equations. Each course will have examples of physical applications that apply to the different types of differential equations seen in each course. The motivation or lack thereof with physical relevance in the following mathematics sections reflects the instructional approach based on the instructor's thoughts for each course and the nature of my class notes. While I argue that experiential motivation is important for understanding, I present these ideas as typically taught based on the perspective of the instructor interviews and my own course work to not misrepresent how students may be seeing the material for the first time. At the end of each class section there will be a subsection focused on thoughts from the instructors as gathered from the interviews.

5.1 Calculus II

A majority of students see a differential equation for the first time in the second semester of the Calculus sequence. The typical solution methods in Calculus II are direct integration and separation of variables, which both rely on various methods of integration familiar to Calculus II students. Instructors use population and logistic models to provide physical contextualization to the mathematical rigor.

5.1.1 First Order Differential Equations

Generally students in Calculus II work with first-order differential equations, typically linear, but not always. In preserving traditional notation from calculus, students generally start working with first-order differential equations of the form

$$\frac{dy}{dx} = g(x) \cdot y, \tag{5.1}$$

where $g(x)$ is some function of x. From here students use concepts of anti-differentiation through integration as defined by the fundamental theorem of calculus in order to determine the solution to the differential equation. Let's quickly work through a solution to equation 5.1. First divide both sides of the equation by y.

$$\frac{dy}{dx} \cdot \frac{1}{y} = g(x).$$

Using the reverse chain rule, the left side becomes

$$\frac{d}{dx}(\ln|y|) = g(x).$$

By integrating both sides with respect to x

$$\int \frac{d}{dx}(\ln|y|)dx = \int g(x)dx,$$

and then applying the fundamental theorem of calculus, we get

$$\ln|y| = \int g(x)dx + C.$$

Exponentiating both sides of the equation yields

$$|y| = e^{\int g(x)dx+C} = e^C e^{\int g(x)dx},$$

where e^C is rewritten as a new constant C such that

$$|y| = Ce^{\int g(x)dx}.$$

The right side of the equation is always positive due to the exponential terms, therefore

$$y = Ce^{\int g(x)dx}. \tag{5.2}$$

This is the general solution to (5.1). Solving this differential equation required knowledge of anti-differentiation through integration and the fundamental theorem of calculus (FTC). For students in Calculus II, these concepts are previously explored mathematical ideas from Calculus I, which they may draw from in order to make sense of the solution process. Now that the mathematical solution to the differential equation is known, how can a physical context be applied to enhance experiential learning by providing relevance?

5.1.2 The Population and Logistics Models

A common instructional context employed in Calculus II is a model for population dynamics, for which the growth rate of population over time is given by the following equation:

$$\frac{dP}{dt} = r(t, P) \cdot P, \tag{5.3}$$

where $r(t, P)$ is the growth (or decay) function, which determines whether or not a population grows or declines. One might consider $r(t, P) =$ birth rate $-$ death rate. Hence, if $r(t, P) > 0$, there is population growth and if $r(t, P) < 0$, population is declining (decaying). This birth/death rate contextualization for the rate function $r(t, P)$ does not account for immigration/migration factors in population dynamics.

Let's consider a simplistic model with $r(t, P) = a$ where a is a rate constant. Then the population rate of change model (5.3) can be written as the linear differential equation

$$\frac{dP}{dt} = aP. \tag{5.4}$$

The solution method for this differential equation is similar to the solution method for the differential equation (5.1), where the solution for equation (5.4) is:

$$P(t) = Ce^{\int a dt}. \tag{5.5}$$

For this specific case where a is constant under integration, the solution is

$$P(t) = Ce^{at}.$$

One can now solve for population behavior with $r(t, P)$ defined as a constant. In the solution there is still this unknown constant term C. Is there anyway to solve for C? Does it offer any significance in terms of our solution? The answer to both of these questions is yes. So far our solutions to these differential equations have involved invoking indefinite integration. What if we apply some initial condition such that the we could attribute values to the bounds of our integral? Problems unto which we introduce an initial condition are called initial value problems (IVP). Introducing an initial condition will provide a more explicit solution to the differential equations (5.1) and (5.5), removing the ambiguity of the constant term C.

Mathematically, this is how a typical solution works in detail. Using equation (5.4), we start by dividing both sides by P:

$$\frac{dP}{dt} \cdot \frac{1}{P} = a.$$

which by reverse chain rule on the left side yields:

$$\frac{d}{dt}(\ln |P|) = a.$$

This time, when we integrate both sides of the equation, we want to consider an initial condition for the population at a time t_0, $P(t_0)$, and label that initial population P_0. Integrating both sides now as a definite integral from t_0 to a later time t our expression becomes the following integral, where the time terms have been assigned a new variable s as a notational preference to avoid the variables of the integrand matching the bounds of integration:

$$\int_{t_0}^{t} \frac{d}{ds}(\ln |P(t)|) ds = \int_{t_0}^{t} a ds.$$

Applying the FTC yields and knowing that a is constant gives:

$$\ln|P(t)| - \ln|P(t_0)| = a(t - t_0).$$

Simplifying the left side using the difference of logs rule yields:

$$\ln\left|\frac{P(t)}{P(t_0)}\right| = a(t - t_0).$$

Taking the exponential of both sides results in:

$$\left|\frac{P(t)}{P(t_0)}\right| = e^{a(t-t_0)},$$

and for our final solution we get:

$$P(t) = P_0 e^{a(t-t_0)}, \tag{5.6}$$

where $P(t_0) = P_0$. Another way to solve for C is that once we determine the general solution, we can evaluate the solution at the initial condition. Let's for concreteness say our initial population at time $t = 0$ is $P(t = 0) = 1000$. Evaluating the general solution

$$P(t) = Ce^{at}$$

at the initial condition gives

$$P(0) = Ce^{a \cdot 0} = 1000.$$

Therefore the constant $C = 1000$. The exact solution then is

$$P(t) = 1000e^{at}.$$

Equation (5.6) is the general solution for an initial value population model with a constant growth (or decay) rate. Does this particular model truly reflect how population behaves in reality? P_0 since a population value cannot be negative. An interesting result with this solution is that the population can be modeled in forward and backwards time. For this solution, if $a > 0$, then the population over time $P(t)$ will exponentially increase as time progresses. In reality, there are environmental factors and competition logistics that prevent populations from getting infinitely large. In short, every population has a carrying capacity. How can one incorporate carrying capacity into the solution? For a more advanced model we can add a competition term $-bP^2$ to the right side of equation (5.4). This added competition term for varying values of b might cause a quadratic decay, depending on the population size at an instant in time. As the population size approaches or is well above the carrying capacity (which we will show to be $\frac{a}{b}$) the competition term will dominate and the model will demonstrate a quadratic decay.

The new model,

$$\frac{dP}{dt} = aP - bP^2, \tag{5.7}$$

is known as the Logistic Law of Growth or the Logistic differential equation with $a >> b > 0$. Here the rate function $r(t, P) = a - bP$, and is no longer constant (but depends on the time and thus population). Therefore, our new differential equation is non-linear. For small population values, since the decay term is much smaller than the growth term, there will still be mainly exponential growth. On the other hand, for large P, the competition term $-bP^2$ is no longer negligible, and thus exponential population growth slows down or even reverses.

Once equation (5.7) is solved analytically, the carrying capacity can be determined. The first step in the solution is to divide by everything on the right and multiply by dt in order to separate variables (discussed in the next section) such that:

$$\frac{dP}{aP - bP^2} = dt.$$

Next we integrate both sides where we define the initial condition $P(t_0) \equiv P_0$ once again.

The definite integrals become:

$$\int_{P_0}^{P} \frac{dr}{ar - br^2} = \int_{t_0}^{t} ds,$$

where I have included a variable change in both integrals as to not confuse the bounds of the integral with the functions over which we are integrating. This is a common technique in integration, mainly used as a notational convenience. To solve the integral on the left we have to use another topic from Calculus II, partial fractions. Rewriting the integral on the left expressed in partial fractions we get:

$$\frac{1}{a}\int_{P_0}^{P}\left(\frac{1}{r}+\frac{b}{a-br}\right)dr=\int_{t_0}^{t}ds.$$

By the second part of the FTC, the right side is equivalent to $t-t_0$, the elapsed time. Integrating the left side via the first part of the FTC yields

$$\frac{1}{a}\int_{P_0}^{P}\left(\frac{1}{r}+\frac{b}{a-br}\right)dr=\left(\frac{1}{a}\ln|r|+\frac{b}{a}\cdot\frac{1}{-b}\ln|a-br|\right)\Bigg|_{P_0}^{P};$$

evaluating this expression at the bounds gives

$$\left(\frac{1}{a}\ln|r|+\frac{b}{a}\cdot\frac{1}{-b}\ln|a-br|\right)\Bigg|_{P_0}^{P}=\frac{1}{a}\Big(\ln|P|-\ln|a-bP|-\ln|P_0|+\ln|a-bP_0|\Big).$$

Using the rules for addition and difference of logs, our expression becomes:

$$\frac{1}{a}\Big(\ln|P|-\ln|a-bP|-\ln|P_0|+\ln|a-bP_0|\Big)=\frac{1}{a}\ln\left|\frac{P(a-bP_0)}{P_0(a-bP)}\right|$$

Rejoining the two sides of the integral equation gives:

$$\frac{1}{a}\ln\left|\frac{P(a-bP_0)}{P_0(a-bP)}\right|=t-t_0.$$

Multiplying both sides of the equation by a and then exponentiating each side, we get,

$$\left|\frac{P(a-bP_0)}{P_0(a-bP)}\right|=e^{a(t-t_0)},$$

and, through some meticulous algebraic manipulation, we come to our solution:

$$P(t)=\frac{P_0a}{P_0b+(a-bP_0)e^{-a(t-t_0)}}.$$

What does this solution even tell us? The relevant question is, what happens at large values of time – do we finally get the behavior we would expect for a more realistic population model? Taking the limit of our solution as time goes to infinity yields

$$\lim_{t\to\infty}P(t)=\lim_{t\to\infty}\frac{P_0a}{P_0b+(a-bP_0)e^{-a(t-t_0)}}\to\frac{aP_0}{bP_0}=\frac{a}{b}.$$

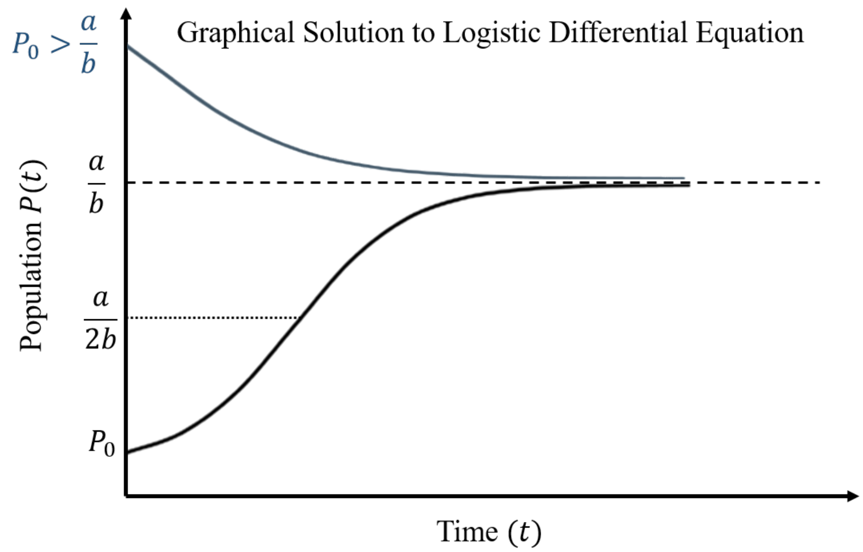

Figure 5.1: General Graphical Solution to The Logistic Growth Model

The limit of the population value at large time, since the exponential term decays to zero, is represented by the asymptote in Figure 5.1. This limit is known as the carrying capacity. Thus the value for the carrying capacity using our model of population dynamics is $\frac{a}{b}$. The carrying capacity is independent of P_0, therefore whether $P(t)$ increases with time, $0 < P_0 < \frac{a}{b}$, or $P(t)$ decreases with time if $P_0 > \frac{a}{b}$, the population still limits to the same carrying capacity.

The next feature of our differential equation (5.7) to examine is the derivative of the equation, which tells us about the change in population growth rate for differing values of population size. Specifically, it allows us to determine how the population growth rate behaves with respect to the relative closeness of the population to the carrying capacity. Taking the derivative of differential equation (5.7) with respect to time we find that

$$\begin{aligned}
\frac{d^2P}{dt^2} &= a\frac{dP}{dt} - 2bP\frac{dP}{dt} \\
&= (a-2bP)\frac{dP}{dt} \quad \text{substituting in } \frac{dP}{dt} = aP - bP^2 \\
&= (a-2bP)(aP-bP^2) \\
\frac{d^2P}{dt^2} &= (a-2bP)(a-bP)P.
\end{aligned}$$

This results in the following three inequalities:

$$\frac{d^2P}{dt^2} > 0 \text{ if } P < \frac{a}{2b};$$
$$\frac{d^2P}{dt^2} < 0 \text{ if } \frac{a}{2b} < P < \frac{a}{b};$$
$$\frac{d^2P}{dt^2} > 0 \text{ if } P > \frac{a}{b}.$$

Therefore our solution $P(t)$ is concave up for population values below half the carrying capacity as well as above the carrying capacity, and concave down for values between the $\frac{a}{2b}$ and $\frac{a}{b}$. These results also agree with our expectations for a logistic curve.

Now students have access to the analytical and graphical solutions for a logistic growth population model. It is important that students are aware of both representations. Having the graphical solution, derived using Calculus I tactics, allows for students to predict population behavior for different values of P_0, a, and b. Again, this is an idealized model of population dynamics: we truly expect populations to fluctuate about the carrying capacity, as opposed to gradually approaching the capacity as time gets large as seen in Figure 5.1. We can account for other population growth/decay factors mathematically using more complex population models such as systems of differential equations describing multiple populations which coexist, additional dynamic terms in the differential equation, etc. The solutions for these particular models are outside the scope of Calculus II and many require numerical approximation and cannot be confined to a single analytic solution.

5.1.3 Separation of Variables

In Calculus 2, the conventional separation of variables (SOV) solution method is used when the function on the right side is not just in terms of the dependent variable (varied by a constant) but depends on the independent variable as well. Consider the equation

$$\frac{dy}{dt} = \frac{g(t)}{f(y)} \tag{5.8}$$

where $f(y)$ is a continuous nonzero function of y (dependent variable) and $g(t)$ is a continuous function of t (independent variable). This form of differential equation could be linear or non-linear. The Logistic Growth population model is an example of a non-linear differential equation for which we can solve using separation of variables. As the method suggests, we want to separate the functions that depend on different variables. To do this, first multiply both sides of Equation (5.8) by $f(y)$ such that,

$$f(y)\frac{dy}{dt} = g(t).$$

If we express $f(y)$ as a derivative then we can let,

$$F(y) = \int f(y)dy$$

be any antiderivative of f. This allows us to say, using the chain rule,

$$f(y)\frac{dy}{dt} = \frac{d}{dt}[F(y)]$$

and

$$\frac{d}{dt}[F(y)] = g(t).$$

Taking the indefinite integral of both sides with respect to t gives,

$$\int \frac{d}{dt}F\big(y(t)\big)dt = \int g(t)dt.$$

Invoking the FTC leads to our general solution:

$$F\big(y(t)\big) = \int g(t)dt + C. \tag{5.9}$$

Equation (5.9) is the general solution of a differential equation with separable variables, *i.e.,* it provides a family of functions that satisfy the differential equation (12). What happens when we have an initial value for this function? The adjustment requires taking a definite integral instead of an indefinite integral. Looking back at equation (5.8), we now start our solution with an initial value $y(t_0) = y_0$, where if:

$$\frac{dy}{dt} = \frac{g(t)}{f(y)},$$

then we still have

$$\frac{d}{dt}[F(y)] = g(t).$$

Now we want to take a definite integral with lower bound t_0 and upper bound t such that

$$\int_{t_0}^{t} \frac{d}{dr} F\big(y(r)\big) dr = \int_{t_0}^{t} g(s) ds,$$

where once again we change variables in the integrands as a notational convenience. By the second part of the FTC this results in

$$F\big(y(t)\big) - F\big(y(t_0)\big) = \int_{t_0}^{t} g(s) ds, \tag{5.10}$$

which is the same as

$$\int_{y_0}^{y} f(r) dr = \int_{t_0}^{t} g(s) ds. \tag{5.11}$$

Both equations (5.10) and (5.11) represent the solution to the initial value problem for the separable equation (5.8).

The two solutions for separation of variables (general and initial value problem) go through the mathematical rigor highlighting the formalism of indefinite integration and the fundamental theorem of calculus through definite integration. However, it is not uncommon for mathematicians and physicists alike to take a "shortcut" when it comes to separable differential equations, avoiding the explicit calculus-based routine derived above. I refer to this supplemental strategy as a shortcut, when in fact it invokes a new conceptualization entirely. This new technique treats the derivative term $\frac{dy}{dt}$ of the differential equation as a ratio of infinitesimally small quantities dy and dt which can be manipulated algebraically. Starting with equation (5.8), I will quickly demonstrate the algebraic technique here, first multiplying both sides of the equation by $f(y)$:

$$f(y)\frac{dy}{dt} = g(t).$$

Treating dy and dt as very small quantities of y and t, I multiply both sides of the equation by dt, which gives

$$f(y)dy = g(t)dt.$$

Now the differentials dy and dt are on the side corresponding to the function dependent on the same variable. From here, an indefinite or definite integral, which correlate to initial value problems(IVP), can be taken on each side, leading to the same general solutions seen in equations (5.9) and (5.11), respectively:

$$F\big(y(t)\big)dt = \int g(t)dt + C \text{ and } \int_{y_0}^{y} f(r)dr = \int_{t_0}^{t} g(s)ds.$$

5.1.4 The Gompertz Equation

Let's go through a specific example using this shortcut exploring a special case of the logistics function known as the Gompertz equation. The Gompertz equation models tumor growth, market impact in finance, and populations in confined spaces. The Gompertz equation is unique in that it describes behavior where growth is slowest at the beginning and end of a time period. I choose this particular example because it is a unique application of differential equations in Calculus II brought up in an instructor interview. Mathematically, the Gompertz equation is defined as

$$\frac{dy}{dt} = ry\ln\left(\frac{K}{y}\right). \tag{5.12}$$

The Gompertz equation is a first-order differential equation where, r and K are positive constants and y is a positive function. It may not be clear quite yet, but this equation is separable. To start, divide everything by K so that:

$$\frac{d}{dt}\left[\frac{y}{K}\right] = \frac{ry}{K}\ln\left(\frac{K}{y}\right) = -\frac{ry}{K}\ln\left(\frac{y}{K}\right).$$

To make this equation more noticeably separable, substitute in $z = \frac{y}{K}$ ($z > 0$), rewriting the previous step as:

$$\frac{dz}{dt} = -rz\ln(z).$$

Applying the algebraic separation of variables technique, dividing both sides by $z\ln(z)$ and multiplying each side by dt, the resulting expression is:

$$\frac{dz}{z\ln(z)} = -rdt.$$

Taking an indefinite integral on either side leads to:

$$\ln|\ln(z)| = -rt + C.$$

Exponentiating both sides once simplifies to:

$$|\ln(z)| = e^{-rt+C} = e^C e^{-rt} = Ce^{-rt},$$

and then exponentiating again yields:

$$z = e^{\left(Ce^{-rt}\right)} = (e^C)^{e^{-rt}}.$$

Substituting $z = \frac{y}{K}$ back into the equation and solving for y gives our general solution to equation (5.12):

$$y(t) = K(e^C)^{e^{-rt}}. \tag{5.13}$$

To make this less ambiguous, let's include an initial condition $y(t_0) = y_0$ in order to solve for the constant C. Plugging in $t = t_0$ into equation (5.13) we get

$$y(t_0) = K(e^C)^{e^{(-rt_0)}} = y_0.$$

Solving for constant C gives

$$C = \ln\left(\frac{y_0}{K}\right)e^{rt_0}$$

and the solution to the initial value problem becomes:

$$y(t) = K\left(e^{\ln(\frac{y_0}{K})e^{rt_0}}\right)^{e^{-rt}} = K\left[e^{\ln(\frac{y_0}{K})}\right]^{e^{-r(t-t_0)}} \tag{5.14}$$

for any initial condition y_0. Looking back at equation (5.13) let's determine the behavior of the solution as time gets infinitely large by taking the limit as follows:

$$\lim_{t\to\infty} K(e^C)^{e^{-rt}} = K.$$

K is the equilibrium, similar to the carrying capacity of our population dynamics model discussed earlier, such that at large values of time t the solution will approach the asymptote $y(t) = K$.

5.1.5 Calculus II: Instructors' Thoughts

In this section I discuss the responses to the faculty interview questions and highlight supporting ideas for how the information was organized for Calculus II. Note that for Calculus

II I only interviewed one mathematics faculty member. In the interview, when asked ”what types of differential equations do you implement in your courses?” the instructor mentioned that “for the differential equations in Calculus II, derivatives are taken with respect to x or t in order to preserve traditional Calculus notation.” The instructor also implied that most differential equation content in Calculus II was restricted to first-order, linear, homogeneous ordinary differential equations. Later on, the instructor mentioned that they sometimes introduced students to second-order and non-homogeneous first-order differential equations. We will discuss these particular concepts in the next section of the thesis through the scope of a core differential equations course, with respect to my Calculus II experience not being as in depth with differential equations.

When asked what solution method to differential equations are seen in Calculus II, the instructor responded that “the solution methods in Calculus II focus on taking the antiderivative of the derivative, by the fundamental theorem of calculus,” which students typically see in a Calculus I course. In terms of applications of differential equations in Calculus II, the instructor said they use “population and logistic models” to provide physical contextualization to students in hopes to better student understanding. While the instructor suggests that they may not focus on the applications, the instructor agrees that it is important for students to be aware of them, as “many students will continue their education as engineers and scientists, where applications become more central to their learning experiences.” The population dynamics and logistics growth model make the students consider what is reasonable, and take into account that the physical relevance plays a role in the solution for mathematics.

5.2 Differential Equations

A differential equations course opens with concepts including first-order, linear, homogeneous differential equations, separation of variables, and populations models. All of these concepts were discussed in detail in the previous section. Depending on the curriculum, these topics may or may not be review for students.

5.2.1 The Integrating Factor

While the integrating factor is sometimes introduced in a second semester Calculus sequence, I first saw the integrating factor in a differential equations course, and that's why I've included it in the differential equations section. The integrating factor comes in handy when attempting to solve non-homogeneous, linear, first-order differential equations. Let's take a look at the following non-homogeneous linear differential equation:

$$\frac{dy}{dt} + a(t)y = b(t). \tag{5.15}$$

If $a(t) \neq b(t) \neq 0$ (non-homogeneous) and $a(t)$ and $b(t)$ are strictly functions of t (linear) then the differential equation (5.15) cannot be solved using direct integration or separation of variables. With no other tools currently in our solution method tool box, let's derive a new method in order to solve the non-homogeneous differential equation (5.15). To start, multiplying equation (5.15) by $\mu(t)$ gives

$$\mu(t)\frac{dy}{dt} + \mu(t)a(t)y = \mu(t)b(t). \tag{5.16}$$

The reason behind this first step becomes more apparent soon. We treat the left side of the equation as the derivative $\frac{d}{dt}(\mu(t)y)$. Differentiating using the power rule we get

$$\frac{d}{dt}(\mu(t)y) = \mu(t)\frac{dy}{dt} + \frac{d\mu(t)}{dt}y. \tag{5.17}$$

We need to determine a $\mu(t)$ such that

$$\frac{d\mu(t)}{dt} = \mu(t)a(t). \tag{5.18}$$

We choose this particular $\mu(t)$ to satisfy the two previous equations (5.16) and (5.17). Now, in terms of $\mu(t)$, (5.18) is an ordinary, linear, homogeneous differential equation. Thus the solution (akin to the solution of equation (5.1)) is

$$\mu(t) = e^{\int a(t)dt}. \tag{5.19}$$

Here, $\mu(t)$ is the integrating factor and it is mathematically represented by equation (5.19). So,

$$\mu(t)\frac{dy}{dt} + \mu(t)a(t)y = \frac{d}{dt}(\mu(t)y) = \mu(t)b(t) \tag{5.20}$$

and indefinitely integrating both sides of the equation with respect to t yields:

$$\int \frac{d}{dt}(\mu(t)y)dt = \int \mu(t)b(t)dt,$$

and furthermore by direct integration:

$$\mu(t)y = \int \mu(t)b(t)dt + C.$$

The general solution to equation (5.15) is then:

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)b(t)dt + C\right], \tag{5.21}$$

where the integrating factor is:

$$\mu(t) = e^{\int a(t)dt}.$$

We can construct an initial value problem for non-homogeneous differential equations as well by letting $y(t_0) = y_0$. Using equation (5.20) and taking the definite integral of the last two expressions with respect to t yields:

$$\int_{t_0}^{t} \frac{d}{dr}(\mu(t)y)dr = \int_{t_0}^{t} \mu(s)b(s)ds,$$

and by the second part of the fundamental theorem of calculus:

$$\mu(t)y(t) - \mu(t_0)y(t_0) = \int_{t_0}^{t} \mu(s)b(s)ds.$$

After two steps of algebra, to isolate $y(t)$, the integrating factor initial value problem solution is

$$y(t) = \frac{1}{\mu(t)}\left[\mu(t_0)y(t_0) + \int_{t_0}^{t} \mu(s)b(s)ds\right]. \tag{5.22}$$

In comparing equations (5.21) and (5.22), providing an initial condition presents a value for the constant C, where in this case $C = \mu(t_0)y(t_0)$.

The integrating factor approach works based off of ideas from anti-derivatives in calculus. We picked the integrating factor to match up with terms in the differential equation in equation (5.16) such that the reverse product rule for anti-differentiation holds. The integrating factor differential equation in equation (5.18) was solvable using earlier techniques such as direct integration or separation of variables. Once we had solved for the integrating factor, the final solution $y(t)$ was determined using integration techniques from calculus. The integrating factor is an important tool, as we see it utilized later in the paper when we discuss solving exact differential equations in Section 5.2.5.

5.2.2 Ratio-Dependent Differential Equations

Now consider a special group of differential equations called ratio-dependent equations. This section introduces the algebraic technique, *substitution*, used to solve different types of differential equations. A ratio-dependent differential equation takes the form

$$\frac{dy}{dt} = f(y/t), \tag{5.23}$$

where the function on the right is explicitly in terms of y/t. In order to solve the ratio-dependent differential equation (5.23), introduce a new unknown function

$$u = y/t.$$

Then we have that $y = t \cdot u$ and by the product rule,

$$\frac{dy}{dt} = 1 \cdot u + t \cdot \frac{du}{dt}.$$

Thus our ratio-dependent differential equation (5.23) becomes:

$$u + t\frac{du}{dt} = f(u),$$

or

$$\frac{du}{dt} = \frac{f(u) - u}{t}. \tag{5.24}$$

We have reduced our ratio-dependent differential equation (5.23) to the separable equation (5.24). We have the tools in order to solve for the solution $u(t, C)$, which will depend on some constant C as a result of indefinite integration. Therefore the general solution to the ratio-dependent differential equation (5.23) is:

$$y = t \cdot u(t, C). \tag{5.25}$$

The key idea to take away is with ratio-dependent differential equations we can rely on separation techniques after a convenient substitution $u = y/t$ in order to determine a solution. Ratio-dependent functions provide a new variety of differential equation which rely on a previous solution method familiar to students in a differential equations course, as well as substitution, a common algebraic tool. In the next section we see another use of substitution when solving differential equations by variation of parameter techniques.

5.2.3 Variation of Parameters for First-Order Differential Equations

A first order linear differential equation takes the form:

$$a(t)\frac{dy}{dt} + b(t)y + c(t) = 0 \tag{5.26}$$

and when $a(t) \neq 0$ equation (5.26) can be rewritten in a more familiar form where after dividing by $a(t)$

$$\frac{dy}{dt} + f(t)y = g(t), \text{ where } f(t) = \frac{b(t)}{a(t)} \text{ and } -g(t) = \frac{c(t)}{a(t)}. \tag{5.27}$$

We already know when $g(t) = 0$, equation (5.27) is called a linear homogeneous equation, otherwise when $g(t) \neq 0$ it is non-homogeneous. Here we utilize a new method to solving linear non-homogeneous differential equations called variation of parameters, where the first step requires that we solve the homogeneous equation:

$$\frac{dy}{dt} + f(t)y = 0. \tag{5.28}$$

This equation is separable, so we can use techniques we've seen before to solve separable equations such that

$$\frac{dy}{dt} = -f(t)y,$$

then

$$\frac{dy}{y} = -f(t)dt,$$

and by the fundamental theorem of calculus:

$$\int \frac{dy}{y} = \int -f(t)dt + C,$$

such that

$$\ln|y| = -\int f(t)dt + C.$$

Solving for $y(t)$ we get

$$y(t) = e^{-\int f(t)dt}e^{C} = Ce^{-\int f(t)dt}$$

and as a result to the homogeneous separable equation (5.28)

$$y(t) = Cv(t), \text{ where } v(t) = e^{-\int f(t)dt}, \tag{5.29}$$

and C is some arbitrary constant. To return now and solve the non-homogeneous linear differential equation (5.27) we replace the constant parameter C in equation (5.29) by an unknown function $C(t)$. This turns our homogeneous solution represented by equation (5.29) into the following:

$$y(t) = C(t)v(t). \tag{5.30}$$

If we substitute this new solution (5.30) into our original non-homogeneous differential equation (5.27), obeying the power rule of derivatives we get

$$\frac{dC(t)}{dt}v(t) + C(t)\frac{dv(t)}{dt} + f(t)C(t)v(t) = g(t)$$

which when rewritten simplifies to

$$\frac{dC(t)}{dt}v(t) + C(t)\Big[\frac{dv(t)}{dt} + f(t)v(t)\Big] = g(t). \tag{5.31}$$

Note that the terms within the brackets in 5.31 sum to zero seen by:

$$\frac{dv(t)}{dt} = \frac{d}{dt}\Big(e^{-\int f(t)dt}\Big) = -f(t)e^{-\int f(t)dt} = -f(t)v(t),$$

after applying the chain rule for differentiation. Equation (5.31) then becomes:

$$\frac{dC(t)}{dt}v(t) = g(t) \text{ or } \frac{dC(t)}{dt} = \frac{g(t)}{v(t)}.$$

This last equation is a simple first-order differential equation solvable by integration. The solution to the unknown function $C(t)$ is then

$$C(t) = \int \frac{g(t)}{v(t)}dt + C_1$$

where C_1 is a new arbitrary constant. Now that we have solved for $C(t)$, equation (5.30) gives the general solution for the linear non-homogeneous differential equation (5.27) as

$$y(t) = \left[\int \frac{g(t)}{v(t)}dt + C_1\right]v(t) \text{ where again } v(t) = e^{-\int f(t)dt}. \tag{5.32}$$

In two steps the variation of parameter method reduces linear differential equations to a problem we know how to solve. The first step is finding a solution of the linear homogeneous differential equation with methods previously discussed (separation of variables, direct integration). The second step is defining a new unknown function C(t), plugging that back into the original linear differential equation, and inevitably solving a simple ordinary differential equation by direct integration. These two steps provided us with a general solution for a linear non-homogeneous solution. The variation of parameter method works as a substitute for the integrating factor method. Thus, we now have another way of solving linear non-homogeneous differential equations that relies on a similar, yet different, mathematical approach.

5.2.4 Newton's Heating/Cooling Law

One physical application of differential equations that interviewed instructors mentioned implementing in their differential equations course can be found in the context of thermodynamics, specifically heating/cooling problems. The rate of change in temperature $T(t)$ of an object is proportional to the difference between the object's temperature and the external temperature $T_{external}$ represented by the following differential equation:

$$\frac{dT}{dt} = -k(T - T_{external}), \quad k > 0. \tag{5.33}$$

Equation (5.33) is known as Newton's heating/cooling law where k is a positive proportionality constant. The practicality of this law is as follows: the temperature of the object T is changing faster when the difference between its temperature and the external temperature is larger. The negative sign in front of the proportionality constant k ensures that the rate of change in the object's temperature:

$$\frac{dT}{dt} < 0 \text{ if } T > T_{external} \text{ (cooling)},$$

and

$$\frac{dT}{dt} > 0 \text{ if } T < T_{external} \text{ (heating)}.$$

There are multiple applications of Newton's heating/cooling law including approximating time of death, estimating how fast ice cream melts or coffee cools, and determining initial temperatures of objects after a period of heating or cooling. Let's first analytically solve Newton's heating/cooling law (5.33) by appealing to the familiar solution method, separation of variables. To separate, divide both sides of the equations by $(T - T_{external})$ such that:

$$\frac{dT}{dt} \cdot \frac{1}{T - T_{external}} = -k.$$

For the sake of simplicity we are going to treat $\frac{dT}{dt}$ as a fraction and multiply both sides of our new equation by dt so:

$$\frac{dT}{T - T_{external}} = -kdt.$$

From here we take the integral of both sides, keeping in mind that $T_{external}$ is constant (unchanging), and thus:

$$\int \frac{dT}{T - T_{external}} = \int -kdt + C,$$

which results in:

$$\ln |T - T_{external}| = -kt + C.$$

From here we exponentiate both sides:

$$|T - T_{external}| = e^{-Kt}e^{C} = Ce^{-kT}$$

and adding over $T_{external}$ we get that our general solution to the differential equation in equation (5.33) is

$$T(t) = T_{external} + Ce^{-kt}. \tag{5.34}$$

To solve for the arbitrary constants C and k we can evaluate equation (5.34) at $t = 0$ and $t = 1$ respectively. Solving for C at $t = 0$ where our initial condition is $T(0) = T_0$ we get

$$T(0) = T_{external} + Ce^{-k(0)} = T_{external} + C = T_0,$$

therefore

$$C = T_0 - T_{external},$$

and our solution becomes

$$T(t) = T_{external} + (T_0 - T_{external})e^{-kt}.$$

To solve for k we utilize the condition that $T(1) = T_1$ so then

$$T(1) = T_{external} + (T_0 - T_{external})e^{-k(1)} = T_{external} + (T_0 - T_{external})e^{-k} = T_1.$$

Through a few steps of algebraic manipulation we solve for k such that

$$k = -\ln\left(\frac{T_1 - T_{external}}{T_0 - T_{external}}\right).$$

Here we notice that as time $t \to \infty$ the temperature approaches the value of $T_{external}$ which behaves graphically as a horizontal asymptote. We get different short-term behavior depending on whether the initial temperature of the object is higher or lower than $T_{external}$.

5.2.5 Exact Differential Equations

This section discusses the differential equations known as exact equations. Exact equations in my courses were first-order, homogeneous, non-linear differential equations. There are also, non-exact differential equations for which we can use an adaptation of the integrating factor technique to determine the solutions. For now, consider the general differential equation

$$\frac{d}{dt}\phi(t, y(t)) = 0 \tag{5.35}$$

where ϕ is continuous. From (5.35) we observe that

$$\int \frac{d}{dt}\phi(t, y(t)) = \int 0$$

which implies that $\phi(t, y(t)) = C$ is constant. By the chain rule for multi-variable functions (Calculus III):

$$\frac{d}{dt}\phi(t, y(t)) = \frac{d\phi}{dt} + \frac{d\phi}{dy} \cdot \frac{dy}{dt} = 0.$$

Let's generate two functions $M(t, y(t))$ and $N(t, y(t))$ such that

$$M(t, y(t)) = \frac{d\phi}{dt} \text{ and } N(t, y(t)) = \frac{d\phi}{dy}.$$

This allows us to rewrite equation (5.35) as

$$M(t, y(t)) + N(t, y(t)) \cdot \frac{dy}{dt} = 0. \tag{5.36}$$

Many differential equations courses discuss the mathematical theory which drives differential equations and their solutions. The following theorem is is an example of the theory seen in such a differential equations course.

Theorem 1. *Let $M(t, y(t))$ and $N(t, y(t))$ be continuous with continuous partial derivatives with respect to t and y in the rectangle $R = \{(t, y) : a < t < b, c < y < d\}$. Then there exists a $\phi(t, y(t))$ such that $M(t, y(t)) = \frac{d\phi}{dt}$ and $N(t, y(t)) = \frac{d\phi}{dy}$. if and only if*

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}.$$

This is a result of Clairaut's theorem taught in Calculus III courses which states that despite order of derivatives, mixed partial derivatives will be equivalent to one another. So, the differential equation (5.36) is *exact* if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial t}.$$

The main idea is once you can determine an $M(t, y(t))$ and $N(t, y(t))$ you can solve for $\phi(t, y(t))$ by solving the two equations

$$M(t, y(t)) = \frac{d\phi}{dt} \text{ and } N(t, y(t)) = \frac{d\phi}{dy}.$$

These two solutions generate a function $\Phi(t, y(t))$ often defined implicitly (when after algebraic manipulation y cannot get be isolated). Based on the nature of exact equations our new function follows:

$$\frac{d}{dt}\Phi(t, y(t)) = 0 \text{ and therefore } \Phi(t, y(t)) = C,$$

where C again is some constant. This provides general solutions to exact equations.

What if equation (5.36) is not exact? Is it possible to find a $\mu(t, y)$ such that

$$M(t, y) + N(t, y) \cdot \frac{dy}{dt} = 0$$

is exact? For this to be exact, in addendum to the previous theorem,

$$\frac{\partial}{\partial y}\Big(\mu(t, y)M(t, y)\Big) = \frac{\partial}{\partial t}\Big(\mu(t, y)N(t, y)\Big),$$

or

$$M\frac{\partial \mu}{\partial y} + \mu\frac{\partial M}{\partial y} = N\frac{\partial \mu}{\partial t} + \mu\frac{\partial N}{\partial t}.$$

A $\mu(t, y)$ satisfying the above equation is an integrating factor for equation (5.36). This relationship holds only if $\mu(t, y)$ is a function of only t or y. Let's assume that $\mu(t, y) = \mu(t)$. Then $M\frac{\partial \mu}{\partial y} = 0$ and so

$$\mu\frac{\partial M}{\partial y} = N\frac{d\mu}{dt} + \mu\frac{\partial N}{\partial t}$$

which can be algebraically manipulated to:

$$\frac{d\mu}{dt} = \mu\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)}{N} = \mu R(t)$$

where $R(t)$ is a function of only t respecting the constraints we placed on μ. Notice now that $\frac{d\mu}{dt} = \mu R(t)$ is a differential equation we should now recognize how to solve from integrating factor techniques in Section 5.2.1. Thus,

$$\mu(t) = e^{\int R(t)dt} \text{ where } R(t) = \frac{\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial t}\right)}{N}.$$

Again we utilized the integrating factor in order to find a solution to a differential equation when the equation didn't present a straightforward approach (e.g non-homogeneous or not exact).

5.2.6 Existence and Uniqueness

We've seen various forms of first order differential equations with an attributed initial condition, which can be summarized as:

$$\begin{aligned} \frac{dy}{dt} &= f(t, y), \\ y(t_0) &= y_0, \end{aligned} \tag{5.37}$$

where $f(t, y)$ is a function of y and t. We call the above expression (5.37) an initial value problem (IVP) where the initial condition removes ambiguity concerning an arbitrary constant in the general solution. While we have discussed multiple methods to solving differential equations of this form, for all possible functions of f typically these are not solvable. Interestingly, it is a lot easier to write a differential equation that has no analytic solution than to generate a differential equation that does. This is where we begin to discuss the ideas of existence and uniqueness. We need to ask

1. Can we prove a solution exists? (Existence)
2. Can we show that there is exactly one solution? (Uniqueness)

These questions are key for the following reasons. Primarily, we must know if a solution exists, otherwise the differential equation cannot be solved. If a solution exists, it's just as important to know that the solution is unique. In fact, it may be more significant to know there exists a unique solution, then knowing the solution itself. Knowing a unique solution exists can provide enough information to understand the behavior of a differential equation without deriving a exact solution.

We start by proving that there exists a solution y(t) to (5.37). To show the existence of a solution we will be using ideas from limits, specifically limits of sequences of functions, as described in the following process.

1. Construct a sequence of functions $y_n(t)$ which come closer and closer to solving equation (5.37).

2. Show that the sequence defined as $\{y_n\}_{n=0}^{\infty}$ has a limit $y(t)$ on a suitable interval (domain) of t,
$$t_0 \leq t \leq \ t_0 + \alpha \quad \alpha > 0.$$

3. Prove $y(t)$ solves (5.37) (even if we don't find an explicit form for $y(t)$).

This process shows existence by demonstrating that iterative functions $y_n(t)$ limit to, or approach, a solution to a differential equation. It's similar to ideas of convergence, where each new iterative function, given an initial guess, gets us closer to the actual solution. If the limit does not converge to a specific function or observable pattern from which we can discern solution behavior, then no solution exists.

We can come up with an approximate solution for equation (5.37) by directly integrating over the interval $[t_0, t]$ such that:

$$\int_{t_0}^{t} \frac{dy}{ds} ds = \int_{t_0}^{t} f(s, y(s)) ds.$$

Note again the conventional change in notation to avoid the variables of the bounds matching the variables of the integrand. Applying the fundamental theorem of calculus on the left side, and rearranging yields:

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s)) ds. \tag{5.38}$$

Equation (5.38) is called an integral equation. From this equation we can construct a sequence of successive approximate solutions, known as Picard iterates. The sequence for our general solution equation (5.38) is constructed as follows:

$$\begin{aligned}
y_0(t) &= y_0, \\
y_1(t) &= y_0 + \int_{t_0}^{t} f(s, y_0(s))ds, \\
y_2(t) &= y_0 + \int_{t_0}^{t} f(s, y_1(s))ds, \\
&\vdots \\
y_{n+1}(t) &= y_0 + \int_{t_0}^{t} f(s, y_n(s))ds,
\end{aligned}$$

where each $y_n(t)$ is a successive approximation. The next step is to show that the Picard iterates converges, that is:

$$\lim_{n\to\infty} y_n = Y(t).$$

The final step is to show that $Y(t)$ solves our differential equation (5.37). Let's do a quick example solving the differential equation below:

$$\begin{aligned}
\frac{dy}{dt} &= y, \\
y(0) &= y_0 = 1.
\end{aligned} \tag{5.39}$$

The solution to this differential equation is e^t, but we are going to prove it using Picard iterates. Constructing our sequence, we get the following:

$$
\begin{aligned}
y_0(t) &= 1 \\
y_1(t) &= 1 + \int_0^t 1 ds \\
&= 1 + t \\
y_2(t) &= 1 + \int_0^t (1+s) ds \\
&= 1 + t + \frac{t^2}{2} \\
y_3(t) &= 1 + \int_0^t (1 + s + \frac{s^2}{2}) ds \\
&= 1 + t + \frac{t^2}{2} + \frac{t^3}{2 \cdot 3} \\
&= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} \\
&\vdots \\
y_n(t) &= 1 + \int_0^t \left(1 + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \cdots + \frac{s^{n-1}}{(n-1)!}\right) ds \\
&= 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^2}{n!} \\
&= \sum_{k=0}^{n} \frac{t^k}{k!}.
\end{aligned}
$$

Now we test for convergence by taking the limit as n goes to infinity; from our Calculus II knowledge of power series expansions we know:

$$\lim_{n\to\infty} y_n(t) = \lim_{n\to\infty} \sum_{k=0}^{n} \frac{t^k}{k!} = e^t.$$

The last step is to check and see that e^t is actually a solution to our differential equation (5.39). Plugging e^t into equation (5.39), we have:

$$\frac{d}{dt}(e^t) = e^t,$$

which is true, so e^t is in fact a solution to equation (5.39). While there were multiple, possibly more intuitive ways (previously discussed) to determine a solution to (5.39), this exercise was intended to present a new solution strategy that is effective for determining whether or not a solution to a differential equation exists. This leads to a theorem of existence and uniqueness for differential equations.

Theorem 2 (Existence and Uniqueness)**.** *Let* f *and* $\frac{df}{dy}$ *be continuous in the rectangle* $R = \{(t, y) : t_0 < t < t_0+a, -b+|y-y_0| < b\}$ *with* $a, b > 0$*. Determine* $M = \max_{(t,y)\in\mathbb{R}} \left|f(t, y(t)\right|$ *and let* $\alpha = \min(a, \frac{b}{M})$*. Then equation (5.37) has at least one solution* $Y(t)$*, the limit of the Picard iterates, on the interval* $t_0 \leq t \leq t_0 + \alpha$ *and that solution is unique.*

Let's run through an example briefly to help make sense of this theorem by showing that the solution of the IVP:

$$\begin{aligned} \frac{dy}{dt} &= t^2 + e^{-y^2}, \\ y(0) &= 0, \end{aligned} \tag{5.40}$$

exists and is unique for $0 \leq t \leq \frac{1}{2}$ and $|y(t)| \leq 1$. To start, we have no known strategies to solve this differential equation currently in our arsenal, but we can check to see if a unique solution exists. First, let's determine the maximum of our differential equation in equation (5.40)

$$M = \max_{(t,y)\in\mathbb{R}} |t^2 + e^{-y^2}|$$

. Notice that t^2 is largest when $t = \frac{1}{2}$ and e^{-y^2} is largest when $y = 0$. Therefore the maximum occurs at coordinates $(1/2, 0)$ and is:

$$M = \left|\left(\frac{1}{2}\right)^2 + e^{-0^2}\right| = 5/4.$$

Now, let's determine the variable $\alpha = \min(a, \frac{b}{M})$ where from our chosen bounds on t and y, we determine that $a = \frac{1}{2}$ and $b = 1$. Hence,

$$\alpha = \min\left(\frac{1}{2}, \frac{4}{5}\right) = \frac{1}{2},$$

and therefore a unique solution $Y(t)$ exists at least on the interval $0 \leq t \leq \frac{1}{2}$. It may not be very satisfying to a differential equation student to simply determine that a solution exists and is unique with no way of expressing that solution. In the next section, we'll discuss a numerical approximation method that will provide ways to determine behavior of a solution for an analytically unsolvable differential equation.

5.2.7 Euler's Method

Not all differential equations have an analytical solution. The term analytical solution is defined as having an exact solution, without having to approximate. Often, there isn't an "exact" solution, and as an alternative to analytical methods, numerical methods were developed to approximate solutions to differential equations. One such numerical technique is known as Euler's Method. We will again consider the initial value problem for a general differential equation

$$\begin{aligned} \frac{dy}{dt} &= f(t, y), \\ y(t_0) &= y_0. \end{aligned} \tag{5.41}$$

For simplicity we will consider a time interval of equally spaced discrete time steps $a = t_0 \leq t_1 \leq \cdots \leq t_N = b$, where the spacing of each discrete time step is defined as:

$$h = \frac{b-a}{N}$$

where N is the number of equally spaced sub-intervals of time. In general, we can define steps of times as:

$$t_{n+1} = t_n + h.$$

We call y_n the value of the solution at a time t_n where:

$$y_n = y(t_n).$$

Finding the approximate values for the different y_n with $0 \leq n \leq N$ is the premise of Euler's method. We already know our solution $y(t)$ at the initial value, $y(t_0) = y_0$. Starting with our initial value, we can construct Euler's scheme in order to approximate the other values of our solution y(t). To construct Euler's scheme let's first recall the limit definition of the derivative from calculus,

$$\frac{dy}{dt} = \lim_{h \to \infty} \frac{y(t+h) - y(t)}{h}.$$

For small values of h we can approximate the limit definition of the derivative as

$$\lim_{h \to \infty} \frac{y(t+h) - y(t)}{h} \approx \frac{y(t+h) - y(t)}{h}.$$

Substituting the approximation into the differential equation in equation (5.41) gives:

$$\frac{y(t+h)-y(t)}{h} = f(t,y)$$

or equivalently

$$y(t+h) = y(t) + hf(t,y). \tag{5.42}$$

Returning to our discrete time steps, allowing $t = t_n$ and $y_n = y(t_n)$, in equation (5.42) we have:

$$y_{n+1} = y_n + hf(t_n, y_n). \tag{5.43}$$

for any $n \in \mathbb{N}$. Thus, we have shown that beginning with the initial value $y(t_0) = y_0$, Euler's scheme is expressed as:

$$\begin{aligned} y_0 &= y(t_0), \\ y_1 &= y_0 + hf(t_0, y_0), \\ &\vdots \\ y_{n+1} &= y_n + hf(t_n, y_n). \end{aligned} \tag{5.44}$$

As an example, let's approximate solutions for the IVP:

$$\begin{aligned} \frac{dy}{dt} &= 1 + (y-t)^2, \\ y(0) &= \frac{1}{2}, \end{aligned} \tag{5.45}$$

from $t = 0$ to $t = 1$ in time steps of size $h = .1$ ($t_1 = .1, t_2 = .2$, etc.). So, $N = 10$ solutions will be used to approximate values of $y(t)$. Following Euler's scheme in (5.44), the approximations for $y(t)$ go as follows:

$$\begin{aligned}
y_0 &= \frac{1}{2} \\
y_1 &= y_0 + hf(t_0, y_0) = \frac{1}{2} + .1\Big(1 + \Big(\frac{1}{2} - 0\Big)^2\Big) \\
y_1 &= 0.625 \\
y_2 &= y_1 + hf(t_1, y_1) = 0.625 + .1\Big(1 + \Big(.625 - .1\Big)^2\Big) \\
y_2 &= 0.7525625 \\
&\vdots \\
y_{10} &\approx 1.9422.
\end{aligned}$$

For the sake of space I have jumped to our desired final approximation. The actual solution to differential equation in equation (5.45) is

$$y(t) = t + \frac{1}{2 - t}$$

and at time $t = 1$ which corresponds to the approximate solution y_{10},

$$y(1) = 1 + \frac{1}{2 - 1} = 2.$$

Our approximation $y_{10} = 1.9422$ is 2.89% off the true solution $y(1) = 2$, showing an accurate estimate.

Due to the nature of approximations, there is inherent error in Euler's method. This error can be lessened if one decreases the step width h between discrete values of time over a given interval. Now if a unique solution exists to a given differential equation (akin to the system in equation (5.41)), then we know how to approximate the solution at discrete values of time. Numerical approximation methods are essential for differential equations in which the existence is known but it cannot be solved analytically.

5.2.8 Introduction to Second-Order Differential Equations

So far we've only dealt with first-order differential equations. It's time to introduce the linear second-order differential equation. Second-order differential equations differ from first-order differential equations based on the highest order of derivative in the differential equation. This presents a few unique aspects of second-order differential equations, which set them apart from first-order differential equations. We will see that second-order differential equations can have more than one solution, and that these solutions can be linearly combined into one general solution. Second-order differential equations typically rely on different solution methods given the mathematical difference of having a higher-order derivative term. In certain cases we use techniques from solving first-order differential equations to solve second-order differential equations. Additionally, the second-derivative term impacts the physical behavior the differential equation can represent. The second derivative, or the rate of change of a rate of change, adds another layer of physical meaning and complexity. Many of the solution methods, similar to the previous sections on first-order differential equations rely on previous mathematical concepts ranging from middle school algebra up through the calculus sequence, as well as ideas from the last few sections such as variation of parameters.

To start, let's examine the linear second-order differential equation which has the general form:

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t) = g(t). \tag{5.46}$$

By definition, equation (5.68) is homogeneous when $g(t) = 0$ and non-homogeneous when $g(t) \neq 0$. Let's consider a simple second-order differential equation where the only terms are the second-derivative and a smooth function of time such that:

$$\frac{d^2y}{dt^2} = g(t). \tag{5.47}$$

To solve this differential equation, we use integration tactics to determine the first derivative $\frac{dy}{dt}$ and then the solution $y(t)$. Integrating equation (5.47) once gives

$$\frac{dy}{dt} = \int g(t)dt + C_1,$$

and by repeated integration,

$$y(t) = \int \Big[G(t) + C_1\Big]dt + C_2 = \int G(t)dt + C_1 t + C_2. \tag{5.48}$$

Equation (5.48) is the general solution to the simple second-order differential equation in equation (5.47), where $G(t)$ is the antiderivative of $g(t)$. Notice this general solution depends on two arbitrary constants, C_1 and C_2. In order to solve for a particular solution to a second-order differential equation, instead of one required initial condition, an additional item of information is needed. There is an initial condition of the function as seen before, $y(t_0) = y_0 \in \mathbb{R}$. Additionally, there is also an initial condition on the first derivative of the function defined as $\frac{dy}{dt}(t_0) = y_0^{'} \in \mathbb{R}$. A general second-order linear initial value problem in*prime* notation, where

$$y^{''} = \frac{d^2y}{dt^2} \text{ and } y^{'} = \frac{dy}{dt},$$

is depicted in the equation 5.49 with the differential equation and initial conditions:

$$\begin{aligned} y^{''} &= f(t, y, y^{'}), \\ y(t_0) &= y_0, \\ y^{'}(t_0) &= y_0^{'}. \end{aligned} \tag{5.49}$$

For a second-order linear homogeneous differential equation IVP of the form

$$\begin{aligned} y^{''} &= r(t)y^{'} + s(t)y, \\ y(t_0) &= y_0, \\ y^{'}(t_0) &= y_0^{'}, \end{aligned} \tag{5.50}$$

the following theorem applies.

Theorem 3 (Existence-Uniqueness Theorem)**.** *If $r(t)$ and $s(t)$ are continuous on $t \in (a, b)$ then there exists a unique solution $y(t)$ satisfying the differential equation in equation (5.50). Further if $y(t)$ solves equation (5.50) for $y(t_0) = 0$ and $y^{'}(t_0) = 0$, then $y(t) = 0$ is the unique solution.*

Before we go over additional solution methods for second-order linear differential equations, we will first discuss linear operators, $L[y]$, where L operates on a function y. In layman's terms, the operator L inputs a function as how functions input independent variables (or numbers). Plainly, it's a 'function' of functions. The operator L has the following two properties:

1. $L[Cy] = CL[y]$, for $C \in \mathbb{R}$ where C is a constant;

2. $L[y_1 + y_2] = L[y_1] + L[y_2]$ where y_1 and y_2 are solutions to the differential equation.

An operator which satisfies the above two properties is defined as a *linear operator*. All other operators are called nonlinear. Let's define define a second-order, linear, homogeneous differential equation as a linear operator such that

$$L[y] = y^{''} + p(t)y^{'} + q(t)y = 0. \tag{5.51}$$

If $y_1(t)$ solves equation (5.51) then $C_1y_1(t)$ solves it for any C_1 by property 1 for linear operators. If both $y_1(t)$ and $y_2(t)$ solve equation (5.51) then $y_1(t) + y_2(t)$ solves it by property 2 for linear operators. Combining the two properties, if $y_1(t)$ and $y_2(t)$ solve equation (5.51) then $C_1y_1(t) + C_2y_2(t)$ solves equation (5.51) and the general solution is written as follows:

$$\phi(t) = C_1y_1(t) + C_2y_2(t);\ C_1, C_2 \in \mathbb{R}, \tag{5.52}$$

where the initial conditions corresponding to equation (5.50) are $\phi(t_0) = y_0$ and $\phi^{'}(t_0) = y_0^{'}$. Now that we have determined the expression for a general solution to a second-order, linear, homogeneous differential equation, we ask: what properties do $y_1(t)$ and $y_2(t)$ have to have in order to be a solution? In order to answer that question, consider the following theorem for linearly independent solutions.

Theorem 4 (Linearly Independent Solutions)**.** *Let $y_1(t)$ and $y_2(t)$ be solutions to a system as in equation (5.51) for* t $\in (a, b)$ *with:*

$$y_1(t)y_2^{'}(t) - y_1^{'}(t)y_2(t) \neq 0, \tag{5.53}$$

for $t \in (a, b)$*, then* $\phi(t) = C_1y_1(t) + C_2y_2(t)$ *is a general solution to (5.51).*

If $y_1(t)$ and $y_2(t)$ are solutions which satisfy the above theorem, they are called a fundamental set of solutions. The left-hand side of equation (5.53) is called the *Wronskian* of $y_1(t)$ and $y_2(t)$ and is defined as:

$$W(t) = W[y_1, y_2] = y_1(t)y_2^{'}(t) - y_1^{'}(t)y_2(t), \tag{5.54}$$

where the above theorem requires the Wronskian $W(t) \neq 0$ for $t \in (a, b)$. From a linear algebra standpoint, the Wronskian is the determinant of the following matrix:

$$\begin{bmatrix} y_1(t) & y_1^{'}(t) \\ y_2(t) & y_2^{'}(t) \end{bmatrix}.$$

Thus, we can express the Wronskian as:

$$\det \begin{bmatrix} y_1(t) & y_1^{'}(t) \\ y_2(t) & y_2^{'}(t) \end{bmatrix} = y_1(t)y_2^{'}(t) - y_1^{'}(t)y_2(t).$$

In order for the above theorem to be satisfied, the determinant of the matrix must not equal zero. Recall from linear algebra that a determinant of vectors not equal to zero implies those vectors are linearly independent. Similarly, if $W(t) \neq 0$, the solutions $y_1(t)$ and $y_2(t)$ are linearly independent, which implies they are not constant multiples of each other, such that $y_1(t) \neq Cy_2(t),\ \forall C \in \mathbb{R}$. If $y_1(t)$ and $y_2(t)$ are linearly independent, $W(t) = 0$, this implies that one of the solutions is zero, or that $y_1(t)$ and $y_2(t)$ are equivalent (not unique) up to a constant term C. This is an important idea to keep in mind as we continue working through solution methods for second-order differential equations: the solutions $y_1(t)$ and $y_2(t)$ must be linearly independent, and we can check for independence using the Wronskian, a technique developed using concepts from linear algebra. If two solutions are not linearly independent, then the solutions are not unique, and this breaks the existence-uniqueness criterion for linear combinations of solutions to higher-order differential equations.

5.2.9 Second-Order Linear Homogeneous DEs with Constant Coefficients

Consider the operator equation

$$L[y] = ay^{''} + by^{'} + cy = 0;\ \ a, b, c \in \mathbb{R}. \tag{5.55}$$

Here $a \neq 0$, otherwise we'd be back to a first-order linear homogeneous differential equation and we already know the methods to solve those. With $a, b, c \in \mathbb{R}$ the coefficients of this operator equation are constant, and thus, it is a constant-coefficient equation. The goal is to determine two solutions, $y_1(t)$ and $y_2(t)$, to construct a general solution, $\phi(t) = C_1y_1(t) + C_2y_2(t)$. Looking at equation (5.55), how can we determine a solution where the second derivative, first derivative, and original function sum to zero? Is there a function from calculus whose derivative and second derivative retain the original function (with an additional multiplicative factor). Besides the constant function 0, there is the exponential function $y = e^{rt}$ where $r \in \mathbb{R}$ is some constant. Let's plug e^{rt} into our operator equation (5.55):

$$L[e^{rt}] = a(e^{rt})'' + b(e^{rt})' + ce^{rt} = 0.$$

Evaluating the derivatives, the above expression becomes

$$ar^2e^{rt} + bre^{rt} + ce^{rt} = 0.$$

Factoring out an e^{rt} from each term yields

$$e^{rt}(ar^2 + br + c) = 0.$$

For any finite time, e^{rt} cannot be zero, which implies the expression in parentheses must be zero. Dividing both sides by e^{rt} results in the quadratic equation:

$$ar^2 + br + c = 0. \tag{5.56}$$

Equation (5.56) is the *Characteristic Equation* for second-order constant coefficient differential equations. Notice the characteristic equation in equation (5.56) is a quadratic equation and its roots can be solved using the quadratic formula:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{5.57}$$

Thus, the two roots are:

$$r_1 = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Now, there are three cases we need to consider when working with these roots and they are as follows:

1. $b^2 - 4ac > 0$ (Real Roots);

2. $b^2 - 4ac < 0$ (Complex Roots);

3. $b^2 - 4ac = 0$ (Repeated Roots).

Let's first discuss the real, non-repeated roots. Our solutions $y_1(t)$ and $y_2(t)$ are simply:

$$y_1(t) = e^{r_1 t} \text{ and } y_2(t) = e^{r_2 t},$$

resulting in the general solution:

$$\phi(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}. \tag{5.58}$$

Given initial values $\phi(t_0) = y_0$ and $\phi^{'}(t_0) = y_0^{'}$, one can determine the exact solution by solving a system of equations for C_1 and C_2.

Now consider the second case, where r_1 and r_2 are complex roots. The quadratic formula yields:

$$r_1 = \frac{-b}{2a} + i\frac{\sqrt{4ac - b^2}}{2a} \text{ and } r_2 = \frac{-b}{2a} - i\frac{\sqrt{4ac - b^2}}{2a}.$$

We expect our general solution to have both real and imaginary parts such that

$$\phi(t) = C_1 u(t) + iC_2 v(t), \tag{5.59}$$

where $u(t)$ and $v(t)$ are real-valued functions. Plugging our expected solution into (5.55), we get

$$a(u + iv)^{''} + b(u + iv)^{'} + c(u + iv) = 0,$$

which, after differentiation and algebra, simplifies to

$$\big(au^{''} + bu^{'} + cu\big) + i\big(av^{''} + bv^{'} + cv\big) = 0.$$

This expression is satisfied if $au^{''} + bu^{'} + cu = 0$ and $av^{''} + bv^{'} + cv = 0$. Therefore, both $u(t)$ and $v(t)$ are solutions to (5.55).

What does e^{rt} look like for complex roots? For notational simplicity, we redefine $\frac{-b}{2a} = \alpha$ and $\frac{\sqrt{4ac-b^2}}{2a} = \beta$. Using the new notation, a general root is $r = \alpha \pm i\beta$, and thus

$e^{rt} = e^{(\alpha \pm i\beta)t}$. This next step may or may not be a new idea to the reader, but we're going to exploit a useful mathematical identity:

$$e^{\pm ix} = \cos(x) \pm i\sin(x); \quad \text{for any} \;\; x \in \mathbb{R}. \tag{5.60}$$

Equation (5.60) is known as Euler's formula. Starting with $e^{(\alpha \pm i\beta)t}$, we have a new expression using Euler's identity which gives:

$$e^{(\alpha \pm i\beta)t} = e^{\alpha t}e^{\pm i\beta t} = e^{\alpha t}\big(\cos(\beta t) \pm i\sin(\beta t)\big) = e^{\alpha t}\cos(\beta t) \pm e^{\alpha t} i\sin(\beta t).$$

The above result matches what is expected in equation (5.59) where $u(t) = e^{\alpha t}\cos(\beta t)$ and $v(t) = e^{\alpha t}\sin(\beta t)$. Therefore our general solution for complex roots is:

$$\phi(t) = C_1 e^{\alpha t}\cos(\beta t) \pm C_2 e^{\alpha t} i\sin(\beta t) = e^{\alpha t}\Big(C_1\cos(\beta t) \pm C_2\sin(\beta t)\Big). \tag{5.61}$$

The constant term C_2 absorbs the $\pm$ argument leaving

$$\phi(t) = e^{\alpha t}\Big(C_1\cos(\beta t) + C_2\sin(\beta t)\Big)$$

as the solution to equation (5.55) when r is a complex root.

The final case is that of repeated roots, when $r_1 = r_2$. If $r_1 = r_2 = \frac{-b}{2a}$, our solutions $y_1(t) = e^{r_1 t}$ and $y_2(t) = e^{r_2 t}$ are no longer linearly independent, and cannot together satisfy as a solution to (5.55). We start with one solution,

$$y_1(t) = e^{\frac{-b}{2a}t}. \tag{5.62}$$

If it's required for our second solution to be linearly independent from our first, let's call our second solution

$$y_2(t) = y_1(t)v(t) \tag{5.63}$$

where $v(t)$ is an unknown function of t. We want to plug our second solution into (5.55), but first let's calculate the derivative and second derivative with respect to time of $y_2(t)$ above. The first derivative is

$$y_2^{'} = y_1^{'}v + y_1 v^{'},$$

and the second derivative is

$$y_2'' = y_1''v + y_1'v' + y_1'v' + y_1v''.$$

Plugging these expressions into equation (5.55) yields:

$$a(y_1''v + y_1'v' + y_1'v' + y_1v'') + b(y_1'v + y_1v') + c(y_1v) = 0.$$

After algebraic rearrangement the above expression can be written as

$$ay_1v'' + (by_1 + 2ay_1')v' + (ay_1'' + by_1' + cy_1)v = 0.$$

Because $y_1(t)$ is a solution to equation (5.55),

$$ay_1'' + by_1' + cy_1 = 0,$$

and so the last term in the previous expression vanishes, leaving us with:

$$ay_1v'' + (by_1 + 2ay_1')v' = 0.$$

In terms of v' this is a first-order differential equation ($v'' = (v')'$). Isolating v'' in the last expression gives:

$$v'' + \frac{by_1 + 2ay_1'}{ay_1}v' = 0. \tag{5.64}$$

We can solve this differential equation using an integrating factor, where:

$$\mu(t) = e^{\int \frac{by_1+2ay_1'}{ay_1}dt} = e^{\int \frac{b}{a}+2\frac{y_1'}{y_1}dt}.$$

Evaluating the indefinite integral,

$$\mu(t) = e^{\frac{b}{a}t}e^{2\ln|y_1|} = y_1^2\, e^{\frac{b}{a}t}.$$

Our solution to equation (5.64) in terms of v' and the integrating factor $\mu(t)$ is

$$v'y_1^2e^{\frac{b}{a}t}.$$

This result is a solution, and so it satisfies that

$$v'y_1^2 \cdot e^{\frac{b}{a}t} = 0.$$

Integrating both sides of the equation,

$$\int v^{'} y_1^2 \cdot e^{\frac{b}{a}t} dt = \int 0 dt = k,$$

for k constant. This resultant constant is arbitrary, so we choose $k = 1$. Hence,

$$\int v^{'} y_1^2 \cdot e^{\frac{b}{a}t} dt = 1.$$

Solving for $v^{'}$ gives:

$$v^{'} = y_1^{-2} e^{-\frac{b}{a}t}$$

and now integrating once more,

$$v(t) = \int y_1^{-2} e^{-\frac{b}{a}t} dt. \tag{5.65}$$

Thus, we have solved for the unknown piece of the second solution to equation (5.55) for repeated roots. The process of changing a second-order differential equation to first-order differential equation is called *reduction of order*. Plugging in equations (5.62) and (5.65) into equation (5.63) gives the second solution as

$$y_2(t) = y_1(t)v(t) = e^{-\frac{b}{2a}t} \int y_1^{-2} e^{-\frac{b}{a}t} dt = e^{-\frac{b}{2a}t} \int (e^{-\frac{b}{2a}t})^{-2} e^{-\frac{b}{a}t} dt.$$

Simplifying the above expression,

$$y_2(t) = e^{-\frac{b}{2a}t} \int (e^{\frac{b}{a}t}) e^{-\frac{b}{a}t} dt e^{-\frac{b}{2a}t} \int e^0 dt = e^{-\frac{b}{2a}t} \int 1 dt.$$

The final result for $y_2(t)$ is:

$$y_2(t) = te^{-\frac{b}{2a}t}. \tag{5.66}$$

Combining equations (5.62) and (5.66) the general solution to equation (5.55) for repeated roots is:

$$\phi(t) = C_1 e^{-\frac{b}{2a}t} + C_2 t \cdot e^{-\frac{b}{2a}t}. \tag{5.67}$$

We've now covered all the bases for solving a second-order linear, homogeneous constant coefficient differential equation. In order to solve these differential equations, we utilized the quadratic formula (middle school up to college algebra), integration, Euler's formula (built from ideas in Calculus II), as well as techniques introduced earlier in this section as taught in a differential equation's curriculum based on instructor interviews and class notes. Once again, through a hierarchy of learning, the new concepts and solution methods just introduced were guided by our experiential (previous and rehearsed) understanding of prior mathematical ideas. Further, the characteristic equation works for differential equations of higher order. As long as you can solve the polynomial governed by the characteristic equation when you input e^{rt} into the operator differential equation of interest your solution will follow the same rules for real, complex, and repeated roots. For roots which are repeated more than twice, the additional solutions have increasing powers of t. For instance, if a root r_3 repeats three times, the part of the solution attributed with that root is:

$$Ce^{r_3 t} + Dt \cdot e^{r_3 t} + Et^2 e^{r_3 t}$$

where C, D, E are arbitrary constants. The most difficult part of solving higher order constant coefficient differential equations is solving for the roots of the polynomials in the respective characteristic equation. Interestingly, the differential equations aspect is less rigorous than the algebra for increasingly high order differential equations once we know the general solution.

With the reduction of order technique, we can now solve the general second-order linear homogeneous differential equation:

$$y^{''} + p(t)y^{'} + q(t)y = 0 \tag{5.68}$$

where if $y_1(t)$ solves equation (5.68), then:

$$y_2(t) = y_1(t)v. \text{ where } v(t) = \int \frac{e^{-\int p(t)dt}}{y_1^2}$$

is the second linearly-independent solution. This one technique has opened the door to solving a plethora of differential equations.

5.2.10 Second-Order Linear Non-homogeneous Differential Equation

The general operator expression for a second-order linear non-homogeneous differential equation is

$$L[y(t)] = y^{''} + p(t)y^{'} + q(t)y = g(t), \tag{5.69}$$

with $g(t) \neq 0$. Similarly to how the general solution to the first-order non-homogeneous differential equation (5.27) relies on the general solution to the first-order homogeneous differential equation (5.28), the general solution to equation (5.69) relies on the general solution in equation (5.52) to the second-order homogeneous differential equation in equation (5.51) such that it can be expressed as:

$$\Psi(t) = \phi(t) + y_p(t) = C_1 y_1(t) + C_2 y_2(t) + \psi(t). \tag{5.70}$$

Here $\psi(t)$ is defined as the *particular* solution to the non-homogeneous differential equation (5.69). Plugging in $\psi(t)$ into (5.69) gives:

$$L[C_1 y_1(t) + C_2 y_2(t) + \psi(t)] = L[C_1 y_1(t) + C_2 y_2(t)] + L[\psi(t)] = 0 + L[\psi(t)] = g(t).$$

Here $L[C_1 y_1(t) + C_2 y_2(t)] = L[\phi(t)] = 0$ because $\phi(t)$ solves the homogeneous differential equation (5.51). Thus, we can conclude that our particular solution $\psi(t) = g(t)$. The following theorem clarifies the above result.

Theorem 5. *Let $y_1(t)$ and $y_2(t)$ be two linearly independent solutions to $y^{''} + p(t)y^{'} + q(t)y = 0$ as in equation (5.51). Then, with a particular solution $\psi(t)$ to the non-homogeneous equation (5.69), the general solution to (5.69) is:*

$$\Psi(t) = C_1 y_1(t) + C_2 y_2(t) + \psi(t).$$

One neat result of this theorem, is given two particular solutions $\psi_1(t)$ and $\psi_2(t)$ to the non-homogeneous differential equation (5.69), their difference $\psi_1(t) - \psi_2(t)$ is a solution to the homogeneous differential equation (5.51). As a brief proof, consider

$$L[\psi_1(t)] = g(t) = L[\psi_2(t)].$$

The difference between them gives:

$$L[\psi_1(t)] - L[\psi_2(t)] = L[\psi_1(t) - \psi_2(t)] = 0,$$

and therefore $\psi_1(t) - \psi_2(t)$ solves the homogeneous differential equation (5.51).

In order to find the general solution to a non-homogeneous differential equation (5.69) we only need to find the general solution $\phi(t)$ to the homogeneous differential equation (5.51) plus *any* one particular solution to the non-homogeneous differential equation (5.69). Additionally, if we find more than one particular solution to the non-homogeneous differential equation (5.69), then we can determine at least one solution to the homogeneous differential equation (5.51). When finding more than one possible solution $\big(y_1(t), \cdots, y_n(t)\big)$ to the homogeneous differential equation (5.51), we can utilize the Wronskian to check for linear independence between the different solutions. We now have all the information required to solve any non-homogeneous constant coefficient differential equation. An issue remains when the differential equation is not constant coefficient. We haven't developed a technique to determine a particular solution $\psi(t)$ to the non-homogeneous differential equation (5.69) outside of making a strategic guess, being given one, or using $g(t)$ in equation (5.69). The following section brings back a familiar solution method from when we discussed solving first-order non-homogeneous differential equations in Section 5.2.3.

5.2.11 Variation of Parameters for Second-Order Differential Equations

One method for solving a second-order linear non-homogeneous differential equation:

$$L[y(t)] = y^{''} + p(t)y^{'} + q(t)y = g(t), \tag{5.71}$$

for a particular solution is using variation of parameters, similar to the variation of parameter process of solving a first-order, non-homogeneous differential equation (5.27) in Section 5.2.3. Introducing two new unknown functions, u_1 and u_2, our particular solution

$\psi(t)$ will be of the form:

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t), \tag{5.72}$$

where $y_1(t)$ and $y_2(t)$ solve the homogeneous second-order differential equation:

$$L[y(t)] = y^{''} + p(t)y^{'} + q(t)y = 0. \tag{5.73}$$

We need to find what exact values of $u_1(t)$ and $u_2(t)$ make $\psi(t)$ a particular solution to (5.71). Let's plug in the proposed solution $\psi(t)$ (5.72) into the non-homogeneous differential equation (5.71):

$$L[\psi(t)] = \psi^{''}(t) + p(t)\psi^{'}(t) + q(t)\psi = g(t). \tag{5.74}$$

Differentiating $\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ gives:

$$\psi^{'}(t) = u_1^{'}y_1 + u_1y_1^{'} + u_2^{'}y_2 + u_2y_2^{'},$$

where $u_1^{'}y_1 + u_2^{'}y_2 = 0$ is a resulting condition because $y_1(t)$ and $y_2(t)$ solve the homogeneous differential equation (5.73). Therefore, we have

$$\psi^{'}(t) = u_1y_1^{'} + u_2y_2^{'},$$

and the second derivative is

$$\psi^{''}(t) = u_1^{'}y_1^{'} + u_1y_1^{''} + u_2^{'}y_2^{'} + u_2y_2^{''}.$$

Plugging in the above derivative expressions and $\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ into equation (5.74) yields:

$$u_1^{'}y_1^{'} + u_1y_1^{''} + u_2^{'}y_2^{'} + u_2y_2^{''} + p(t)\big(u_1y_1^{'} + u_2y_2^{'}\big) + q(t)\big(u_1y_1^{'} + u_2y_2^{'}\big) = g(t).$$

Intense factoring of the expression above gives the following result:

$$u_1\big(y_1^{''} + p(t)y_1^{'} + q(t)y_1\big) + u_2\big(y_2^{''} + p(t)y_2^{'} + q(t)y_2\big) + \big(u_1^{'}y_1^{'} + u_2^{'}y_2^{'}\big) = g(t). \tag{5.75}$$

Note that because $y_1(t)$ and $y_2(t)$ solve equation (5.73),

$$y_1^{''} + p(t)y_1^{'} + q(t)y_1 = 0,$$

and

$$y_2^{''} + p(t)y_2^{'} + q(t)y_2 = 0.$$

Therefore (5.75) simplifies to:

$$u_1^{'}y_1^{'} + u_2^{'}y_2^{'} = g(t).$$

After inputting our solution $\psi(t)$ into equation (5.72), we now know the following conditions must hold:

$$\begin{aligned} u_1^{'}y_1 + u_2^{'}y_2 &= 0; \\ u_1^{'}y_1^{'} + u_2^{'}y_2^{'} &= g(t). \end{aligned} \tag{5.76}$$

This next step is a trick in order to be able to solve for $u_1^{'}(t)$. We are going to multiply the first condition by $y_2^{'}$ and multiply the second condition by y_2. The two conditions are now

$$\begin{aligned} u_1^{'}y_1y_2^{'} + u_2^{'}y_2y_2^{'} &= 0; \\ u_1^{'}y_1^{'}y_2 + u_2^{'}y_2^{'}y_2 &= g(t)y_2. \end{aligned}$$

Next, we subtract the top expression from the bottom one by process of elimination, such that there is one resulting equation:

$$u_1^{'}\left(y_1^{'}y_2 - y_1y_2^{'}\right) = g(t)y_2.$$

Notice the term in the parentheses is the negative of the Wronskian $W[y_1, y_2]$ in equation (5.54) and so

$$u_1^{'} = \frac{-g(t)y_2}{W[y_1, y_2]}. \tag{5.77}$$

Similarly, we can solve for $u_2^{'}$ recalling the conditions in equation (5.76). Multiplying the first condition by $y_1^{'}$ and the second condition by y_1, they can be rewritten as:

$$\begin{aligned} u_1^{'}y_1y_1^{'} + u_2^{'}y_2y_1^{'} &= 0; \\ u_1^{'}y_1^{'}y_1 + u_2^{'}y_2^{'}y_1 &= g(t)y_1. \end{aligned}$$

Once again by elimination, the above expressions simplify to:

$$u_2^{'}(y_1y_2^{'} - y_1^{'}y_2) = g(t)y_1$$

where the term here in parentheses is the Wronskian $W[y_1, y_2]$. Thus,

$$u_2^{'} = \frac{g(t)y_1}{W[y_1, y_2]}. \tag{5.78}$$

Integrating both equations (5.77) and (5.78), we determine the expressions for our unknown functions $u_1(t)$ and $u_2(t)$ such that:

$$\begin{aligned} u_1(t) &= -\int \frac{g(t)y_2(t)}{W[y_1, y_2]}dt; \\ u_2(t) &= \int \frac{g(t)y_1(t)}{W[y_1, y_2]}dt. \end{aligned} \tag{5.79}$$

With the parameters satisfied in equation (5.79),

$$\psi(t) = u_1(t)y_1(t) + u_2(t)y_2(t),$$

from equation (5.72), is a particular solution to equation (5.71). This satisfies the general solution $\Psi(t) = C_1y_1(t) + C_2y_2(t) + \psi(t)$ as in equation (5.70) for linear non-homogeneous differential equations.

Once again we were able to utilize a technique known as variation of parameters in order to solve for solutions to a non-homogeneous differential equation. Most of the scratch work required to reach the desired outcome has utilized concepts from calculus and algebra. Most of the work in differential equations relies on mathematical context from before the course, as is typical in most mathematics sequences. It's fascinating to see that as the ideas become more challenging, the mathematical formalism remains guided by ideas from earlier experiences.

5.2.12 Judicial Guessing

In this section we're going to develop a methodology for *guessing* solutions of the general non-homogeneous constant coefficient differential equation:

$$L[y] = ay^{''} + by^{'} + cy = g(t). \tag{5.80}$$

There's a difference between guessing and judicial guessing. With judicial guessing we'll be making educated choices for our solution based on the function $g(t)$. In my experience, judicial guessing plays a large role in physics differential equations curricula, but for now we're going to show why mathematicians and physicists alike share a fondness in the strategy of guessing.

Consider the following non-homogeneous constant coefficient differential equation

$$L[y] = ay^{''} + by^{'} + cy = a_0 + a_1 t + \cdots + a_n t^n. \tag{5.81}$$

Observing that the right hand side of the differential equation in equation (5.81) is an n^{th} degree polynomial, we want to seek a particular solution $\psi(t)$ such that $a\psi^{''} + b\psi^{'} + c\psi$ is also an n^{th} degree polynomial. Due to the niceness in taking derivatives of polynomials, a strategic guess solution is to let $\psi(t)$ to be an n^{th} degree polynomial itself such that:

$$\psi(t) = A_0 + A_1 t + \cdots + A_n t^n. \tag{5.82}$$

Taking the first and second derivative of (5.82) gives

$$\psi^{'} = A_1 + 2A_2 t + \cdots + nA_n t^{n-1}.$$

and

$$\psi^{''} = 2A_2 + 2 \cdot 3A_3 t + \cdots + n(n-1)A_n t^{n-2}.$$

Plugging in our guess solution (5.82) into the differential equation (5.81) leads to

$$L[\psi(t)] = a\psi^{''} + b\psi^{'} + c\psi = a_0 + a_1 t + \cdots + a_n t^n,$$

which is equivalent to

$$\begin{aligned} a\Big(2A_2 + 2 \cdot 3A_3 t + \cdots + nA_n t^{n-2}\Big) + b\Big(A_1 + 2A_2 t + \cdots + nA_n t^{n-1}\Big) \\ + c\Big(A_0 + A_1 t + \cdots + A_n t^n\Big) = a_0 + a_1 t + \cdots + a_n t^n. \end{aligned} \tag{5.83}$$

The above expression factors as:

$$cA_n t^n + \Big(cA_{n-1} + nbA_n\Big)t^{n-1} + \cdots + \Big(2A_2 a + bA_1 + cA_0\Big) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_0.$$

Matching coefficients in front of equal powers of t yields the following three conditions:

$$cA_n = a_n, \text{ therefore } A_n = \frac{a_n}{c}, c \neq 0,$$

$$cA_{n-1} + nbA_n = a_{n-1}, \text{ therefore } A_{n-1} = \frac{a_{n-1} - nbA_n}{c},$$

where $A_n = \frac{a_n}{c}$ as shown above, and

$$2A_2a + bA_1 + cA_0 = a_0.$$

These three conditions allow for one to solve for all the A_k coefficients, $0 \leq k \leq n$, for the particular solution $\psi(t)$ expressed as an n^{th} degree polynomial in equation (5.82). The second condition is known as a recursion relation, where in order to solve for A_n or A_{n-1} you need to know the other value.

There are two more solutions to consider for the differential equation in equation (5.81), given that either $c = 0$ or b and $c = 0$. Consider first that $c = 0$, such that our differential equation becomes

$$L[y] = ay^{''} + by^{'} = a_0 + a_1t + \cdots + a_nt^n. \tag{5.84}$$

Similar to before, the particular non-homogeneous solution to 5.84 is:

$$\psi(t) = t\Big(A_0 + A_1t + \cdots + A_nt^n\Big). \tag{5.85}$$

where our guess solution is a $(n+1)^{\text{th}}$ degree polynomial. Now consider the case for which both $b = c = 0$. Then (5.81) becomes:

$$L[y] = ay^{''} = a_0 + a_1t + \cdots + a_nt^n, \tag{5.86}$$

and our guess for a particular non-homogeneous solution to (5.86) is:

$$\psi(t) = t^2\Big(A_0 + A_1t + \cdots + A_nt^n\Big), \tag{5.87}$$

which is a $(n+2)^{\text{th}}$ degree polynomial.

Guessing solutions based on the elements of a differential equation is an effective strategy to determining actual solutions. Making this guess relies on a recognition of patterns as well as drawing on former knowledge of functional relationships. In this section, we saw it's important to observe the behavior of the right-hand side of the differential equation, which lead to the guess of a polynomial solution. Other examples include when the right-hand side is a combination of sines and cosines, or a n^{th} degree polynomial multiplied by an exponential term. Based on our knowledge of derivatives involving those different types of functions, we can determine a good guess for a solution. We'll utilize guessing a polynomial solution again in the next section as we discuss power series solutions.

5.2.13 Power Series Solutions

Consider the differential equation:

$$L[y(t)] = P(t)y^{''} + Q(t)y^{'} + R(t)y = 0, \;\; P(t) \neq 0 \text{ for } \; t \in (a,b), \tag{5.88}$$

where we assume $P(t)$, $Q(t)$, and $R(t)$ are continuous polynomial functions with

$$\frac{Q(t)}{P(t)}, \frac{R(t)}{P(t)} \neq 0.$$

The differential equation in equation (5.88) can be expressed as:

$$L[y(t)] = y^{''} + \frac{Q(t)}{P(t)}y^{'} + \frac{R(t)}{P(t)}y = 0. \tag{5.89}$$

We know the general solution to a second-order homogeneous differential equation takes the form $\phi(t) = C_1y_1(t) + C_2y_2(t)$, and because $P(t)$, $Q(t)$, and $R(t)$ are polynomials, we can expect our solution $\phi(t)$ is a polynomial as well. We can express n^{th} degree polynomials as a power series, meaning that our guess solution can take the form:

$$\phi(t) = \sum_{n=0}^{\infty} a_n t^n. \tag{5.90}$$

Recall that the derivative of a series is the series of the derivative, *i.e.,*

$$\left(\sum_{n=o}^{\infty} a_n t^n\right)^{'} = \sum_{n=0}^{\infty} (a_n t^n)'.$$

Therefore, the first derivative of the series solution (5.90) is

$$\phi^{'}(t) = \sum_{n=0}^{\infty} n a_n t^{n-1}.$$

Likewise, the second derivative of (5.90) is

$$\phi^{''}(t) = \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2}.$$

Plugging in the original solution (5.90) and its derivatives into the differential equation (5.89) gives

$$L[\psi(t)] = \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} + \frac{Q(t)}{P(t)} \sum_{n=0}^{\infty} n a_n t^{n-1} + \frac{R(t)}{P(t)} \sum_{n=0}^{\infty} a_n t^n.$$

Beyond here it becomes difficult to demonstrate for a general case how to solve a differential equation with a series solution. After differentiating a power series solution in a differential equation, it is often required re-index the first and second derivative terms as well as the function itself, which cannot be shown generally without knowing the functional dependencies of $\frac{Q(t)}{P(t)}$ and $\frac{R(t)}{P(t)}$. Thus, we examine an example solution. Consider the following differential equation:

$$L[y(t)] = y^{''} - 2ty^{'} - 2y = 0. \tag{5.91}$$

We examine this particular example because it mirrors a physics application discussed later in the thesis. We want to find two linearly independent solutions to equation (5.91). Plugging in our guess solution to equation (5.90) into (5.91) yields:

$$L[\psi(t)] = \sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - 2t \sum_{n=0}^{\infty} n a_n t^{n-1} - 2 \sum_{n=0}^{\infty} a_n t^n = 0.$$

By absorbing the t of the middle term into the series, the above expression becomes:

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - 2 \sum_{n=0}^{\infty} n a_n t^n - 2 \sum_{n=0}^{\infty} a_n t^n = 0.$$

Two of the three series are expressed as n^{th} degree polynomials. Combining those two series as like-terms, the expression above can be rewritten as:

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} - 2 \sum_{n=0}^{\infty} (n a_n + a_n) t^n = 0. \tag{5.92}$$

In order for the left-hand side of the equation to equal zero in general, the first term needs to be converted to an n^{th} degree polynomial by *re-indexing* the series, that is, letting the index n be rewritten as $n = m + 2$. Then the series above becomes:

$$\sum_{n=0}^{\infty} n(n-1)a_n t^{n-2} = \sum_{m=-2}^{\infty} (m+2)(m+1)a_{m+2}t^m$$

where writing out the first two terms gives:

$$(-2+2)(-2+1)a_0 t^{-2} + (-1+2)(-1+1)a_1 t^{-1} + \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}t^m.$$

Notice that the first two terms are equal to zero, and so we're left with the series:

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}t^m.$$

Here m is just an index, so after re-indexing with $m = n$ the above expression becomes:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n.$$

Plugging this series as the n^{th} degree replacement for the first term in (5.92) makes:

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - 2\sum_{n=0}^{\infty}(na_n + a_n)t^n = 0.$$

Combining like terms gives:

$$\sum_{n=0}^{\infty}\Big[(n+2)(n+1)a_{n+2} - 2(na_n + a_n)\Big]t^n = 0,$$

and upon further simplification is:

$$\sum_{n=0}^{\infty}\Big[(n+2)(n+1)a_{n+2} - 2((n+1)a_n)\Big]t^n = 0.$$

In order for this equation to be true, the coefficient of each power of t must equal zero separately; thus

$$(n+2)(n+1)a_{n+2} = 2(n+1)(a_n),$$

such that the recursion relation

$$a_{n+2} = \frac{2(n+1)(a_n)}{(n+2)(n+1)} = \frac{2a_n}{n+2} \tag{5.93}$$

holds. Because this relation is between terms two apart in the series, they hold for even and odd n separately, and thus one needs to have two independent initial conditions, *i.e.*, for

a_0 and a_1, to get two independent solutions. Note that this recursion relation is specific to this particular example and will not work for all series solutions of differential equations, and again depends on the re-indexing of the series solution and it's derivatives as well as the functions $P(t), Q(t)$, and $R(t)$. For the sake of simplicity, let's consider three different conditions on a_0 and a_1.

The first condition is trivial:

$$a_0 = 0 = a_1.$$

Then all other a_n's are zero as well by the recursion relation (5.93).

By letting

$$a_0 = 1 \text{ and } \ a_1 = 0,$$

we get all the a_n's for n even by (5.93). All a_n's for n odd are zero because $a_1 = 0$, and

$$a_3 = \frac{2a_1}{1+2} = \frac{2 \cdot 0}{3} = 0,$$

where this recursion repeats for all odd n. The following are a few of the n even coefficients as a result of (5.93):

$$a_0 = 1, a_2 = 1, a_4 = \frac{1}{2}, a_6 = \frac{1}{6}, a_8 = 1/24, \cdots, a_{2n} = \frac{1}{n!}.$$

Lastly, letting

$$a_0 = 0, a_1 = 1,$$

gives us all the a_n's for n odd by (5.93). The following are a few of the n odd coefficients as a result of (5.93):

$$a_1 = 1, a_3 = \frac{2}{3}, a_5 = \frac{4}{15}, a_7 = \frac{8}{105}, \cdots, a_{2n+1} = \frac{2^n}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1)}.$$

Our solutions $y_1(t)$ and $y_2(t)$ can be built using cases two and three such that:

$$y_1(t) = 1 + t^2 + \frac{1}{2}t^4 + \cdots + \frac{1}{n!}t^2 = \sum_{n=0}^{\infty} \frac{(t^n)^2}{n!} = e^{t^2},$$

and

$$y_2(t) = t + \frac{2}{3}t^3 + \frac{4}{15}t^5 + \cdots + \frac{2^n t^{2n+1}}{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n+1)}.$$

Our final solution $\phi(t)$ is then:

$$\phi(t) = C_1 e^{t^2} + C_2\left(t + \frac{2}{3}t^3 + \frac{4}{15}t^5 + \cdots + \frac{2^n t^{2n+1}}{1\cdot 3\cdot 5\cdot 7\cdots(2n+1)}\right).$$

A few takeaways from this example should be that in order to determine an n^{th} series solution one needs to re-index the second-derivative term and possibly the other terms as well. Re-indexing is a good strategy in order to introduce a recursion relation for the coefficients of your series solution. To supplement the recursion relationship, specific values for certain coefficients may be needed (hence the case by case construction above). It's difficult to develop a general solution breakdown for series solutions based on the different variations each problem can present. Hopefully this example provides a strong enough basis for series solutions; we will explore them further later in a physics context.

5.2.14 Laplace Transforms

In this section we discuss another method for solving second-order, non-homogeneous, constant coefficient differential equations known as the Laplace transform. The Laplace transform allows for one to solve linear constant coefficient differential equations by reducing these differential equations to linear algebraic expressions which can be algebraically manipulated to determine a solution. It's a three step process in which we take Laplace transform of the differential equation of interest, algebraically solve for the Laplace transform of the solution $\mathcal{L}[y]$, and lastly take the inverse Laplace transform to find an exact value for $y(t)$. First, we need to define the Laplace transform.

Consider the following differential equation and initial conditions,

$$\begin{aligned} ay^{''} + by^{'} + cy &= f(t), \\ y(t=0) &= y_0, \\ y^{'}(t=0) &= y_0^{'}. \end{aligned} \tag{5.94}$$

The Laplace transform method becomes exceedingly effective when $f(t)$ in equation (5.94) is discontinuous or almost always zero ($e.g.$, Dirac-Delta function). What is a Laplace transform? Let $f(t)$ exist for $0 \leq t < \infty$. Then the *Laplace transform* of f is:

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)dt. \tag{5.95}$$

The integral in the definition of Laplace transform is an improper integral which is another topic from Calculus II being applied in the context of differential equations. It is this improper integral that makes the Laplace transform effective for $f(t)$ discontinuous or almost always zero. We establish our initial conditions in equation (5.94) at $t = 0$ (as opposed to $t = t_0$) due to the integral definition of the Laplace transform in equation (5.95).

The result of the Laplace transform is sometimes referred to as the Laplace image. As an example let's calculate the Laplace image of $f(t) = 1$.

$$\mathcal{L}[1] = \int_0^\infty e^{-st} \cdot 1 = \lim_{b \to \infty} \left. \frac{e^{-st}}{-s} \right|_0^b = \frac{1}{s} - \lim_{b \to \infty} \frac{e^{-sb}}{s}.$$

If $s > 0$, the last term in the above expression limits to 0. If $s < 0$, the last term in the above expression limits to ∞. In general,

$$\mathcal{L}[1] = \begin{cases} \frac{1}{s} & \text{if } s > 0, \\ \infty & \text{if } s < 0, \end{cases} \tag{5.96}$$

where the $s > 0$ is the image of interest. Let's calculate the Laplace image of a few other recognizable functions. Looking at $f(t) = t$, and applying equation (5.95), we get:

$$\mathcal{L}[t] = \int_0^\infty e^{-st} \cdot t.$$

Using integration by parts (from Calculus II), the above integral becomes:

$$\int_0^\infty e^{-st} \cdot t = \left. \frac{te^{-st}}{-s} \right|_0^\infty - \int_0^\infty \frac{e^{-st}}{-s}.$$

Similar to before, we'll have two cases, $s > 0$ and $s < 0$, and the solution in general is

$$\mathcal{L}[t] = \begin{cases} \frac{1}{s^2} & \text{if } s > 0, \\ \infty & \text{if } s < 0. \end{cases} \tag{5.97}$$

From now on we'll only be considering the finite Laplace images. We can calculate the Laplace image of $f(t) = t^n$ the same way as (5.97) such that:

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \ (s > 0). \tag{5.98}$$

Now that we know the Laplace image for all powers of the independent variable t, let's determine the Laplace image of $f(t) = e^{at}$, where

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{(a-s)t} dt = \left. \frac{e^{(a-s)t}}{a-s} \right|_o^\infty = \lim_{t\to\infty} \frac{e^{(a-s)t}}{a-s} - \frac{1}{s-a}.$$

Thus, the finite Laplace image of e^{at} is

$$\mathcal{L}[e^{at}] = \frac{1}{s-a} \ \text{ for } \ s > a. \tag{5.99}$$

For $s > 0$, the following Laplace images for $\cos(at)$ and $\sin(at)$ can be derived:

$$\mathcal{L}[\sin(at)] = \frac{a}{s^2 + a^2}, \tag{5.100}$$

$$\mathcal{L}[\cos(at)] = \frac{s}{s^2 + a^2}. \tag{5.101}$$

These derivations are a result of taking the Laplace transform of Euler's formula in equation (5.60) from Section 5.2.9 expressed in the current context as:

$$e^{iat} = \cos(at) + i \sin(at).$$

The Laplace transform is a function of the two linear operations, multiplication by e^{-st} and integration. That makes the Laplace transform a linear operator, and in turn it obeys the following properties:

1. $\mathcal{L}[Cy] = C\mathcal{L}[y], C \in \mathbb{R}$ where C is a constant;

2. $\mathcal{L}[f(t) + g(t)] = \mathcal{L}f(t)] + \mathcal{L}[g(t)]$, where $f(t)$ and $g(t)$ are functions representing the right-hand side of equation (5.94).

These properties allow us to find Laplace images of sums of different functions with constant coefficients. For example, taking the Laplace transform of $3t^2 + 5e^{6t}$ yields:

$$\mathcal{L}[3t^2 + 5e^{6t}] = 3\mathcal{L}[t^2] + 5\mathcal{L}[e^{6t}] = 3 \cdot \frac{2}{s^3} + 5 \cdot \frac{1}{s-6} = \frac{6}{s^3} + \frac{5}{s-6},$$

by the above properties and derivations in equations (5.98) and (5.99). The Laplace transform results in functions of s, but we ask how can one determine a solution back in terms of t? The Laplace images are currently in another functional state, and now we need to find a way to convert back in order to solve equation (5.94).

First, let's determine the Laplace transforms for derivatives of $y(t)$. The Laplace image for the first derivative $y^{'}(t)$ is

$$\mathcal{L}[y^{'}(t)] = \int_0^{\infty} e^{-st} y^{'}(t) = s\mathcal{L}[y(t)] - y(0).$$

The integral is computed using integration by parts. The Laplace transform of the second derivative $y^{''}(t)$ is similarly

$$\mathcal{L}[y^{''}(t)] = s\mathcal{L}[y^{'}(t)] - y^{'}(0) = s\Big(s\mathcal{L}[y(t)] - y(0)\Big) - y^{'}(0) = s^2\mathcal{L}[y(t)] - sy(0) - y^{'}(0).$$

Note for the n^{th} derivative, the Laplace image is

$$\mathcal{L}[y^{(n)}(t)] = s^n\mathcal{L}[y(t)] - s^{n-1}y(0) - s^{n-2}y^{'}(0) - \cdots - y^{(n-1)}(0).$$

Let's now solve equation (5.94) by taking the Laplace transform of both sides of the differential equation such that:

$$\mathcal{L}[ay^{''} + by^{'} + cy] = \mathcal{L}[f(t)] = a\mathcal{L}[y^{''}] + b\mathcal{L}[y^{'}] + c\mathcal{L}[y] = \mathcal{L}[f(t)].$$

Using the derivative expressions derived above, this becomes:

$$a\Big(s^2\mathcal{L}[y] - sy(0) - y^{'}(0)\Big) + b\Big(s\mathcal{L}[y] - y(0)\Big) + c\mathcal{L}[y] = \mathcal{L}[f(t)].$$

Solving for $\mathcal{L}[y]$ gives:

$$\mathcal{L}[y] = \frac{(as+b)y_0 + ay_0^{'} + \mathcal{L}[f(t)]}{as^2 + bs + c}.$$

Our solution to equation (5.94) is then the inverse Laplace transform of $\mathcal{L}[y]$ resulting in:

$$y(t) = \mathcal{L}^{-1}\Big(\mathcal{L}[y]\Big) = \mathcal{L}^{-1}\left(\frac{(as+b)y_0 + ay_0^{'} + \mathcal{L}[f(t)]}{as^2 + bs + c}\right). \tag{5.102}$$

The inverse Laplace transform is a tool which allows us to bring our solution out of the Laplace functional form, getting an expression for the solution which solves equation (5.94 in terms of t.

Again, we have simplified to algebra techniques in order to solve complex differential equations.

5.2.15 Systems of Differential Equations

So far we have dealt with solving differential equations one by one. What kinds of situations prompt a system of differential equations? We know we can use a single differential equation to describe the population dynamics of one species. The population of two different species can be described by two separate differential equations. If the two different differential equations influence (depend on) one another, the two differential equations are connected in a *system*. For example, a predator-prey model can be represented by a system of differential equations. A system of differential equations looks like

$$\begin{cases} y_1^{'} = f_1(t, y_1, y_2, \cdots, y_n), \\ y_2^{'} = f_2(t, y_1, y_2, \cdots, y_n), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ y_n^{'} = f_n(t, y_1, y_2, \cdots, y_n). \end{cases} \tag{5.103}$$

Here there are n unknown functions. In the context of this thesis, we'll be considering the maximum of 2 unknown functions where in general we deal with systems like

$$\begin{cases} y_1^{'} = f_1(t, y_1, y_2), \\ y_2^{'} = f_2(t, y_1, y_2). \end{cases} \tag{5.104}$$

Consider a second-order differential equation:

$$y^{''} = g(t, y, y^{'}). \tag{5.105}$$

It turns out that we can represent this second order differential equation as a system of first-order differential equations. To do this, we introduce the unknown functions $u_1 = y$

and $u_2 = y^{'}$ such that $u_1^{'} = u_2$. Then equation (5.105) turns into a system of differential equations

$$\begin{cases} u_1^{'} = u_2, \\ u_2^{'} = g(t, u_1, u_2). \end{cases} \tag{5.106}$$

Now, we can use methods for solving systems of first-order differential equations to determine u_1 and u_2, which in turn will lead us straight to the solution $y(t)$ to equation (5.105). These methods discussed in the next section use ideas from linear algebra, with which students may or may not have experience. This is the first time in the thesis I present a mathematical technique that isn't necessarily prior (experiential) mathematical knowledge for a student in a sequence of mathematics courses.

5.2.15.1 Eigenvalue and Eigenvector Method

Consider the following system of first-order linear differential equations

$$\begin{cases} \frac{dx}{dt} = ax + by, \\ \frac{dy}{dt} = cx + dy. \end{cases} \tag{5.107}$$

In vector notation the system in equation (5.107) is equivalent to

$$\vec{u}^{\,'} = A\vec{u} \text{ where } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } \vec{u} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

In order to solve our system of differential equations we have to determine two linearly independent eigenvectors $\vec{v}_1, \vec{v}_2$ of the matrix A defined above. These eigenvectors correspond to eigenvalues λ_1 and λ_2 respectively.

The general solution to the system in equation (5.107) is governed by the following theorem.

Theorem 6. *If a matrix A of order n has n linearly independent eigenvectors $\vec{v}_1, \cdots, \vec{v}_n$ with the eigenvalues $\lambda_1, \cdots, \lambda_n$, then*

$$\vec{y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + C_n e^{\lambda_n t}\vec{v}_n$$

is the general solution to a system of first-order linear homogeneous differential equations.

In order to determine the eigenvectors for matrix A we can set up the following equivalence:

$$A\vec{v} = \lambda\vec{v}, \tag{5.108}$$

where in this context $\vec{v}$ is a two-dimensional vector. Rewriting the right-hand side in matrix notation gives:

$$A\vec{v} = \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{v} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \vec{v}.$$

Subtracting the right side from the left side,

$$A\vec{v} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \vec{v} = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \vec{v} = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \vec{v} = 0.$$

We define the resulting matrix as:

$$B = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix}.$$

The determinant of matrix B has to be zero and so,

$$\det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = \lambda^2 - (a+d)\lambda + (ad - bc) = 0.$$

Solving this quadratic equation for roots λ_1 and λ_2 results in the desired eigenvalues. We can then plug these eigenvalues back into equation (5.108) to solve for their corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$. If there are repeated roots ($\lambda_1 = \lambda_2$), then there is only one linearly independent eigenvector which solves equation (5.108). After determining the eigenvalues and eigenvectors which solve equation (5.108) the general solution to equation (5.107) is:

$$\vec{y}(t) = C_1 e^{\lambda_1 t}\vec{v}_1 + C_2 e^{\lambda_2 t}\vec{v}_2.$$

5.2.15.2 Straight Line Solution Approach

As discussed in the background literature section, as a reinvention of the eigenvalue method students developed the *straight line solution* approach as a way to overcome the mathematical complexity/unfamiliarity of linear algebra. The students' reinvention is influenced by their past, experiential knowledge of slopes, as opposed to the less commonly seen or understood concepts concerning eigen-anything.

We consider again the system of differential equations

$$\begin{cases} \frac{dx}{dt} = ax + by, \\ \frac{dy}{dt} = cx + dy. \end{cases} \tag{5.109}$$

Solutions to this system can be graphed in the $x - y$ plane, or *phase plane*. The solutions graphed on the phase plane are called straight-line solutions because they lie along the straight line of tangent vectors. The equation for a line on the $x - y$ plane that passes through the origin in slope-intercept form is

$$y = mx$$

where m is the slope of the line defined as"

$$m = \frac{y}{x}. \tag{5.110}$$

If the solutions to equation (5.109) are vectors, then the ratio of the derivatives is the slope of that vector. Therefore we can define slope as:

$$m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = cx + \frac{dy}{ax} + by. \tag{5.111}$$

Equating equations (5.110) and (5.111) yields:

$$\frac{y}{x} = cx + \frac{dy}{ax} + by.$$

Substituting $y = mx$ into the above expression gives a quadratic expression in terms of m. Depending on the result of the quadratic formula, there are either two, one or no real values for m. Once m is determined, we can plug $y = mx$ back into equation (5.109) and solve for the solutions $x(t)$ and $y(t)$ using separation of variables (Section 5.1.3). These are the general solutions, which lie along the straight line. An initial condition along the straight line would provide a specific straight-line solution. With two unique slopes corresponding to two linearly independent straight-line solutions, $(y_1(t), x_1(t))$ and $(y_2(t), x_2(t))$, the general solution for (5.109) is any linear combination of the straight-line solutions.

5.2.16 Instructor's Thoughts: Differential Equations

There is a lot of material in a differential equations course, from the types of differential equations to the methods with which to solve them. This course is limited to ordinary differential equations; while instructors in interviews implied they may mention the existence of partial differential equations, but "typically solution techniques are not discussed." Many of the methods for solving ordinary differential equations require a variety of tricks. The instructors I interviewed that taught a differential equations course acknowledged that there are a lot of applications that could be taught in a differential equations course, but "it's hard to implement them due to time constraints." One instructor brings up the concern that "any one application won't necessarily pique the interest of all the students", implying an added difficulty in presenting applications of differential equations in a mathematics course. Over the course of the interviews the instructors mentioned different types of physical applications that could be covered such as free-fall motion (with air drag), oscillations, electronics, and resonance. One instructor mentioned that they usually like to implement "biological or financial models... typically models with growth or decay." Additional applications an instructor included were Newton's Law of Heating/Cooling and mixing problems. Many of these applications are all described by first-order differential equations. One instructor mentioned that "in a course evaluation, students wrote that they appreciated having applications for the mathematics." A few students wrote wanting more applications to see how the work they're doing applies to the real world.

One instructor in their interview demonstrated a favor for the theory aspects of a differential equations course and placed emphasis on "what makes a solution a solution", for example the existence and uniqueness theorems. The instructor suggests that in differential equations a lot of the work comes from recognizing forms of differential equations; whether or not they are separable, linear, ratio-dependent, or constant coefficient; or if they are even analytically solvable. This is where numerical strategies, like Euler's method, are introduced in the classroom, and When asked how they introduce more complex material, instructors would imply that every new concept builds from mathematical concepts before it, as “typical of a mathematics course.” One instructor said they “expect students to recall previous concepts as they’re applied to the newer material in hopes that the students can make those connections.” The instructor later comments that “at times new differential equations are special cases of a previous differential equation and consequentially the solution technique is similar.”

In the interviews, each instructor talked about introducing linear differential operators in a differential equations course to formalize the set up of solving a differential equation. As for types of differential equations, an instructor siad that “the linear operators allow for functional coefficients, but generally for simplicity, many problems and exercises are done with constant coefficients, due to the complexity of functional coefficients.“ For second-order constant coefficient differential equations, another instructor suggests that “it simply comes down to determining roots of a quadratic known as the characteristic equation, or characteristic polynomial... the solution type depends on whether the roots are distinct, complex, or repeat.”

In two interviews, instructors discussed the importance of the Laplace operator. Taking the Laplace transform of the differential equation brings the problem into, as an instructor puts it, “the Laplace World.” The instructor added that the “Laplace transform takes the problem from solving a differential equation to solving an algebraic expression. Laplace World allows for the problem to be simplified; once the business of simplification is complete, we can leave Laplace World, go back to the real world, and we have the solution. “ In an interview with the second instructor, they provided that the Laplace transforms are additionally useful when given a discontinuous function.

When asked about invoking initial and boundary conditions, instructors mention in general that they may explain the difference between these conditions, but that it is “not a main focus for the course.” Exercises or problems may require initial or boundary conditions, and the conditions help solve for constants in the general solution. One instructor states that “the purpose of the course is to establish the mathematical rigor to prepare students for applications in their fields.” Another instructor implied that it’s important for students to understand how the differential equation relates to the given problem, and that they understand the solution process, and the solution itself. One instructor comments that “it is important to establish the significance of notation and recognition of mathematical expressions that will come up again in the future.”

One instructor mentioned the importance of showing students more than one solution method. ”Students may favor one method over the other, and the method they favor may benefit their overall understanding.” Another instructor suggests that a differential equations class is an ”opportunity to build student intuition on guessing solutions.” When I asked instructors if they had heard of the straight-line solution method [3] for solving systems of linear first-order differential equations, none of the instructors I interviewed recognized the name or had previously implemented similar characteristics of the technique in their classrooms. This demonstrates that there are alternative viable options to solutions that instructors are unaware of; students inadvertently miss out on the opportunity to discover these methods as well.

Chapter 6

Physics Courses

This chapter describes the variety of physics topics found in courses that I took as an undergraduate (with the exception of one general education physics course which was brought up in an interview with one of the faculty). It is written in an order to help describe the physics content in a sequence that mimics my experience as an undergraduate physics major, as well as demonstrate how mathematical ideas in physics build off of one another. The lessened inclusion of mathematical formalism in this chapter is reflective of my experience in the physics classroom, and lends itself to the previous chapter where we have already done the mathematical rigor. While I argue that mathematics should be a significant aspect of a physics course to foster the transfer of knowledge from the mathematics classroom, it's important to not misrepresent how students are learning differential equations in a physics context.

6.1 General Education Physics Courses

In this section we consider topics from a beginner course for physics students as well as a course offered to students in and outside of the physics and mathematics departments, as an offered science elective. These following sections are designed to demonstrate how to critically think about differential equations without students necessarily having a background in mathematics and physics.

6.1.1 Creating Differential Equations: The Zombie Apocalypse

How can instructors discuss differential equations in classes that don't necessarily have any mathematics pre-requisites, and with students who certainly have had no exposure to differential equations? One question to answer for students is, where do differential equations come from? In the next few sections we will discuss how differential equations arise from various physics concepts, but differential equations can be used to describe just about everything. How then can we convey to students how to create a differential equation? One interviewee introduced the idea of the zombie apocalypse. Students were asked to develop a differential equation in order to prevent a total zombie uprising. The purpose behind creating a differential equation is to determine a solution to a problem, in this case the threat to human existence. Letting students develop their own mathematical expression for saving the world is an enticing method to demonstrate the fundamental use of differential equations. Having to consider the rate at which zombies are created, how quickly they are killed, how many of them return when they're presumed dead, etc. presents a really physical (yet imaginary) perspective for building differential equations, as opposed to the resulting equation which formally is just a chain of mathematical symbols. The differential equation students create is a population model similar to the logistic model discussed in Section 5.1.2.

6.1.2 Graphical Analysis: "Curviness"

For differential equations, there is typically more than one way to represent the solution. So far we've discussed analytical, graphical, and numerical solutions to various differential equations. In a course not necessarily centered around mathematics and physics, it may be helpful to use the graphical approach of studying solutions. Consider one form of

Schrödinger's equation (which will be discussed in more detail in Section 6.4):

$$\frac{-\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = (E-V)\Psi, \tag{6.1}$$

which can be rewritten in as

$$\Psi^{''} = \frac{2m}{-\hbar^2}(E-V)\Psi.$$

Let

$$k = \frac{2m}{\hbar^2}$$

be a proportionality constant. Then this expression becomes:

$$\Psi^{''} = -k(E-V)\Psi. \tag{6.2}$$

The goal is to understand how this differential equation predicts the behavior of Ψ without overthinking the basic construct. Note that no explanations are being presented for this particular equation as to what exactly it physically represents. What we can tell is that Ψ is proportional to its own second derivative. Using our knowledge of solutions, in reference to (6.2), if $E > V$ the solution would be a sine or cosine function. If $E < V$ the solution will be an exponential function. These conclusions come based on the sign difference between Ψ and its second derivative.

If you didn't understand differential calculus, how would you be able to interpret this equation? One way to do this is to denote the second derivative of Ψ, $\Psi^{''}$, as the "curviness" function for Ψ, curv(Ψ). (6.2) becomes

$$\text{curv}(\Psi) = -k(E-V)\Psi. \tag{6.3}$$

It's no surprise that the second derivative is defined as curviness, because the second derivative determines information about the concavity of the original function. Again, for a general education course, the goal is to avoid as much mathematical formalism as possible. In this scenario, the curviness of a function is described by picturing oneself driving along the graphical solution of Ψ, and thinking about how far one has to turn the steering wheel at a point along the curve. The sharper the turn being made on the curve, the more curviness at that point along the solution. For example, picture yourself driving along a sine curve. When the function is zero, that is, when it crosses the x axis, the curviness is also zero; the steering wheel would be (momentarily) unturned. The further from the x axis you get, the more you have to turn the wheel to steer the car back towards the axis, until you turn around, at which point you start straightening out the wheel until you reach the axis again. Driving along the curve also demonstrates the changing levels of curviness. At minima and maxima, the steering wheel is turned as far as it can go. For a sinusoidal function, this motion repeats for all time.

To apply some physical relevance, Schrodinger's equation can be used to determine the quantized energy levels in a potential well. The curviness method works well at determining whether a specific energy fits inside a finite well. As a visual aide, Figure 6.1 depicts a finite potential well with a few determined energy levels and their corresponding wave function solution.

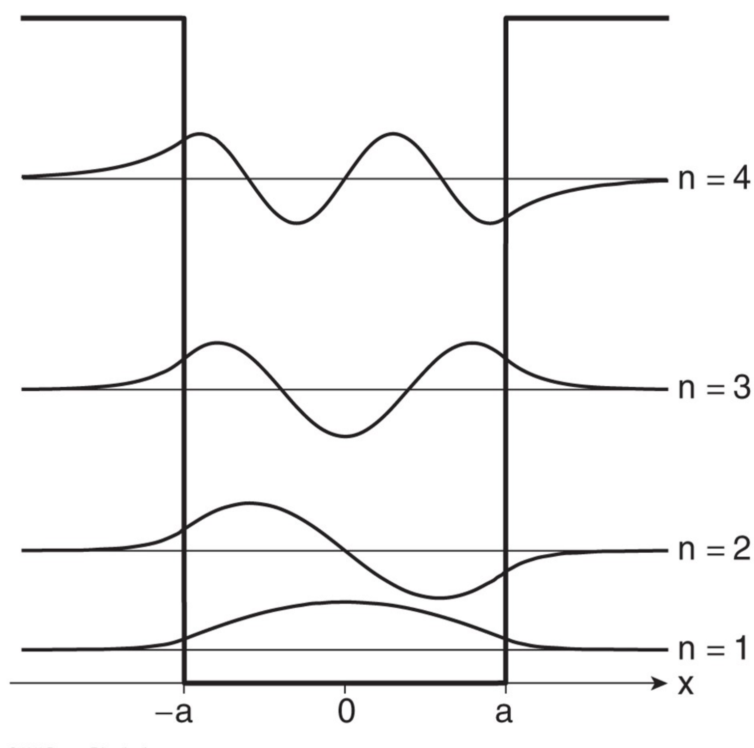

Figure 6.1: Finite Potential Well with Four Energy Levels (from [25])

.

The curviness method becomes rather difficult to use for wells of infinite size, which we discuss later in Section 6.4.1. Focusing back on wells of finite size, outside the well leading into the boundary, there are exponential functions which connect to sinusoidal functions within the well. We picture it like two opposite side ramps (exponential) getting on and off a twisty highway (sinusoid). Curviness plays a role in determining which energy levels actually exist within a finite well. One rule is that getting off and on the exponential ramps, there can be no corners taken, only *smooth* driving. Any energy solutions with corners at the boundary of the well are not possible. Mathematically, we'd say the derivatives at the boundaries must be continuous. The area beneath the functions that make up the road in and out of the well must also be finite. If the space beneath the functional path can be filled with a finite amount of paint, that constitutes a possible solution, it meets the previous curviness requirements. The energy values which meet the criterion are quantized energy states. Using a non-mathematical approach, the curviness function was able to determine quantized energy states governed by the Schrödinger equation. For students not typically fond of physics and math this is a analogous way to come to the same conclusion working through the mathematical formalism of solving equation (6.1). We will discuss how to actually solve variations of the Schrödinger equation in Section 6.4.

6.1.3 Instructors' Thoughts: General Education Physics Courses

For students knowing no calculus or differential equations, one instructor said they "focused on the idea of creating a differential equation as opposed to finding solutions that solve them." One issue an instructor had with differential equations was wondering "where do differential equations come from?" The instructor mentions that in general, "physics gives us the differential equation, which describes some physics behavior, and we're expected to solve it." The instructor goes on to add that "learning to create a differential equation is a way to introduce students to differential equations, building their intuition in a less formal mathematical, more qualitative way."

For students not comfortable with mathematics such as calculus, the curviness graphical representation of the Schödinger equation is an effective visual cue of solution behavior. In an interview the instructor describes it as “the visual strategy of driving down the solution curve, and the curviness of the function is given a value based on how much the steering wheel is turned.” The instructor suggests that there are further applications of the driving analogy including ideas such as ‘smooth driving’, a finitely painted area beneath the road for potential well, and tunneling systems. In fact, the instructor mentioned that “general education students taught explicitly using the curviness method outscored senior students coming out of a quantum mechanics course when tested on tunnelling.”

6.2 Classical Mechanics

Classical Mechanics is an area of physics that describes the behavior of macroscopic systems, typically of an object in motion. Key concepts from Classical Mechanics include Newton’s Laws, the relationships between work, kinetic energy, and potential energy, and oscillatory motion. Much of the physical motion in classical mechanics can be described by differential equations, for example Newton’s Second law is a second-order differential equation as a function of position. In the following few sections we explore the applications of free-fall motion and classical harmonic motion and how knowledge of mathematics aides in student understanding and allows students and instructors alike to differentiate behavior of physical systems.

6.2.1 Newton’s Second Law

One typical differential equation in a traditional classical mechanics course is Newton’s Second Law in one-dimension, where the net force, or sum of the forces, acting on an object is equivalent to that object’s mass m multiplied by the object’s acceleration a. Mathematically, Newton’s Second Law in one dimension is expressed as

$$\sum F = ma. \tag{6.4}$$

Acceleration is a measurement of the change in an object's velocity over the time it takes for the object to experience that change. In other words, the acceleration of an object is the derivative of the velocity with respect to time. Now Newton's Second Law (6.4) can be written as:

$$\sum F = m\frac{dv}{dt}. \tag{6.5}$$

Velocity is a measurement of the change in an object's position over the time it takes for the object to achieve that displacement. Similarly, the velocity of an object is the derivative of the position with respect to time:

$$v = \frac{dx}{dt} = \dot{x}.$$

Notice this change in notation. When taking a derivative with respect to time, it is common in mechanics to denote the first derivative with a single dot above the quantity of interest. For the second derivative, there are two dots, so that acceleration becomes

$$a = \frac{dv}{dt} = \frac{d}{dt}(v) = \frac{d}{dt}(\dot{x}) = \ddot{x}.$$

To summarize, the velocity is the first derivative of the displacement with respect to time because it describes how the displacement changes over time. The acceleration is the derivative of the velocity with respect to time because it describes how the velocity of an object changes over time. In turn the acceleration is the second derivative of displacement with respect to time. Newton's Second Law (6.4) can be rewritten as

$$\sum F = m\ddot{x} \tag{6.6}$$

At first glance, Newton's Second Law doesn't appear to be a differential equation, when in fact it can be expressed as a first-order differential equation for velocity (6.5), and a second-order differential equation for position, (6.6). We can use an object's acceleration to derive a solution in terms of position $x(t)$. Let's first solve (6.5) considering a constant net force F_0 and consequently a constant acceleration $a = \frac{F_0}{m}$. Notice

$$\sum F = F_0 = m\ddot{x} = \frac{dv}{dt}$$

is a first-order separable differential equation. Using methods from Section 5.1.3 we can readily solve this for velocity $v(t)$. Separating the differentials and integrating we get

$$\int dv = \int \frac{F_0}{m} dt.$$

While we could simply leave these as indefinite integrals, it's important to establish initial (and later boundary) conditions to provide an accurate physics interpretation of the mathematics. At time $t = 0$ we say an object has initial velocity (speed) $v = v_0$. At some later time t the velocity of the object is v. The above integrals can then be evaluated at our initial and final conditions such that

$$\int_{v_0}^{v} dr = \int_0^t \frac{F_0}{m} ds.$$

Solving yields

$$v = v_0 + \frac{F_0}{m} t = v_0 + at. \tag{6.7}$$

This is one of the kinematics equations for systems undergoing constant acceleration typically discussed in an introductory physics course. It's comforting that our solution matches a previously defined physics concept. Notice that (6.7) can be written as

$$v = \frac{dx}{dt} = v_0 \frac{F_0}{m} t. \tag{6.8}$$

This is yet another first-order separable differential equation; this time we're solving for position $x(t)$. Establishing initial conditions is important, so at time $t = 0$, the position of the object is at a point $x = x_0$. At a later time t, call the position x. Now, separating (6.8) and integrating gives

$$\int_{x_0}^{x} dr = \int_0^t \left(v_0 + \frac{F_0}{m} t \right) ds.$$

The solution solves to be

$$x = x_0 + v_0 t + \frac{1}{2} \frac{F_0}{m} t^2 = x_0 + v_0 t + \frac{1}{2} a t^2. \tag{6.9}$$

This is a second kinematics equation for systems with constant acceleration typically seen in an introductory physics course. Once again the solution to the differential equation gives a result familiar to a student in a stereotypical physics sequence for undergraduates. The third kinematic equation can be solved for by exploring the relationship between acceleration and velocity. Recall that acceleration is the derivative of velocity,

$$a = \frac{dv}{dt}.$$

which can be expressed as

$$\frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt}$$

expressed as a total differential. Plugging in for values we know,

$$a = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{1}{2}\frac{d}{dx}(v^2).$$

and so

$$\frac{F_0}{m} = \frac{1}{2}\frac{d}{dx}\big(v^2\big).$$

This is a first-order differential separable equation, using the same initial and final conditions as before for both velocity and position, separating the above expression and integrating gives

$$\int_{v_0^2}^{v^2} d\big(v^2\big) = \int_{x_0}^{x} \frac{2F_0}{m} dx.$$

The solution is

$$v^2 - v_0^2 = \frac{2F_0}{m}(x - x_0) = 2a(x - x_0), \tag{6.10}$$

and that is the work-energy theorem for a particle. Applying the differential equation from of Newton's Laws we were able to derive the three kinematics equations for constant acceleration taught in typical introductory physics courses. These strategies show where the kinematic equations come from using mathematics through the physics context.

6.2.2 Free Fall Motion with Air Resistance

In order to further explore the differential equation representation of Newton's Second Law, let's consider free fall, where we measure vertical motion in terms of position $y(t)$ conventionally. For our purposes, we will consider any upward displacement as positive, and any downward displacement as negative. In an introductory physics course, a free fall problem examines motion of an object through the air in one-dimension considering only the force due to gravity $F_g = mg$ acting on the object. In this particular case Newton's Second Law for free fall is:

$$\sum F = mg = m\ddot{x}, \tag{6.11}$$

where g is the acceleration due to gravity, defined as $g = 9.81m/s^2$. We've already solved this differential equation above, where here our constant force is the force due to gravity, and the constant acceleration is g. The solution to (6.11) for vertical position $y(t)$ is

$$y(t) = y_0 + v_0 t + \frac{1}{2}at^2 = y_0 + v_0 t + \frac{1}{2}gt^2. \tag{6.12}$$

Upper-division classes, such as an Intermediate or Advanced Mechanics course, introduce air resistance, or the *drag* force. The drag force on an object is proportional to that object's velocity, $F_{\text{drag}} = bv$, where b is a proportionality constant. The drag force always opposes motion (like friction) and depends on the medium through which an object moves (in our case air) as well as the object's cross-sectional area and shape. Adding the the drag force to (6.11), the second-order differential equation for Newton's Second Law for free fall becomes

$$\sum F = mg - bv = m\ddot{y}. \tag{6.13}$$

The sign attributed to the force due to gravity and the drag force depends on our established coordinate system. If a force opposes the direction of motion, it's sign is the negative of the velocity. If velocity is negative, then force is positive. Therefore, both the drag force and force due to gravity are negative for upward motion. The force due to gravity always points toward Earth, and the drag force points opposite the direction of motion. For downward motion, the force due to gravity is in the same direction as the velocity (which is negative), and so it's still a negative force. The drag force always opposes the motion of the object and so as the object travels downwards, the drag force is positive.

It's important to keep in mind the overall motion (trajectory) of the object in order to account for proper sign changes on the forces. From Newton's Second Law, the net force will bear the same sign as the acceleration. If the force is negative, that force is causing the object to accelerate in the negative direction. If you're moving in the negative direction, and you have a negative acceleration, you will speed up. Similarly, if the force is positive, it causes the object to accelerate in the positive direction.

In the downward motion of free fall, there is a moment when an object reaches what is called its *terminal velocity*. This occurs when the force due to gravity and the drag force are equal in magnitude, but opposite in direction. This implies the net force on the object is zero,

$$\sum F = mg - bv = 0,$$

which means the acceleration of the object is also zero, so it moves with constant (terminal) velocity. In order to analyze the motion of a free fall system for which air resistance is taken into account, we need to solve the differential equation (6.13). Rewriting (6.13) in terms of time-derivatives of y, we get

$$m\ddot{y} = mg - b\dot{y}. \tag{6.14}$$

Notice that (6.14) is a first-order differential equation in terms of the first derivative of position. Let's instead consider this equation in terms of velocity, such that $\dot{y} = v$ and $\ddot{y} = \dot{v}$. This gives

$$m\dot{v} = mg - bv. \tag{6.15}$$

This reduces (6.14) to a first-order linear differential equation that we know to solve (Section 5.1.1).

Before doing any formal mathematics, let's discuss what requirements must be met to abide by the laws of physics. We know an object in free fall may reach a terminal velocity, as discussed above. An object reaches terminal velocity when its acceleration is zero, $i.e.$,

$$\frac{dv_{\text{term}}}{dt} = 0. \tag{6.16}$$

This serves as a boundary condition. So far, we've only discussed initial conditions that remove ambiguity from general solutions. Boundary conditions serve the same purpose, by providing conditions placed on the derivatives or the solution to fit a given physical context.

Here, based on terminal velocity, we add a boundary condition on the derivative of velocity. At increasing values of time, the velocity of the object will approach its terminal velocity. Plugging our boundary condition (6.16) into our differential equation (6.15) gives

$$m \cdot 0 = 0 = mg - bv_{\text{term}}$$

where v_{term} is the terminal velocity of the object. Solving for the terminal velocity yields

$$v_{\text{term}} = \frac{mg}{b}.$$

Our boundary condition provided a value for the terminal velocity of the object, this information will be useful later on.

For now, let's solve (6.15). It may not be quite clear how to arrange this equation in order to see how our solution methods from Section 5.1.1 can be utilized here. With a little algebra, we can get a recognizable form from which we can proceed.

Factoring out a b from the right-hand side of (6.15) gives

$$m\dot{v} = b\Big(\frac{mg}{b} - v\Big) = b(v_{\text{term}} - v).$$

For mathematical convenience we want $(v - v_{\text{term}})$. factoring out a -1 from the last expression gives

$$m\dot{v} = -b(v - v_{\text{term}})$$

Let's establish an initial condition on velocity such that $v(0) = v_0$. Solving the previous expression using methods from Section 5.1.1 or 5.1.3 gives

$$v(t) = v_{\text{term}} + (v_0 - v_{\text{term}})e^{-\frac{b}{m}t} = v_0 e^{-\frac{b}{m}t} + v_{\text{term}}\big(1 - e^{-\frac{b}{m}t}\big). \tag{6.17}$$

In the limit as time increases to infinity,

$$\lim_{t\to\infty} v(t) = v_{\text{term}}.$$

This matches what is expected physically in a free fall system.

Now that we know the solution for velocity as a function of time, we can determine the solution $y(t)$ by integrating (6.17) with respect to time. Defining the initial position as $y(0) = y_0$, the solution is

$$y(t) = y_0 + v_{\text{term}} \cdot t + \frac{m}{b}(v_0 - v_{\text{term}})\big(1 - e^{-\frac{b}{m}t}\big). \tag{6.18}$$

Notice here that as time increases, at first there will be exponential growth in the position, but eventually the $v_{\text{term}} \cdot t$ term dominates, and the change in position becomes linear. From ideas in calculus relating graphs of the function and of its derivative, when the change in position becomes linear, the velocity is constant $\frac{dv}{dt} = 0$, which implies the object has reached terminal velocity (see Fig. 6.2).

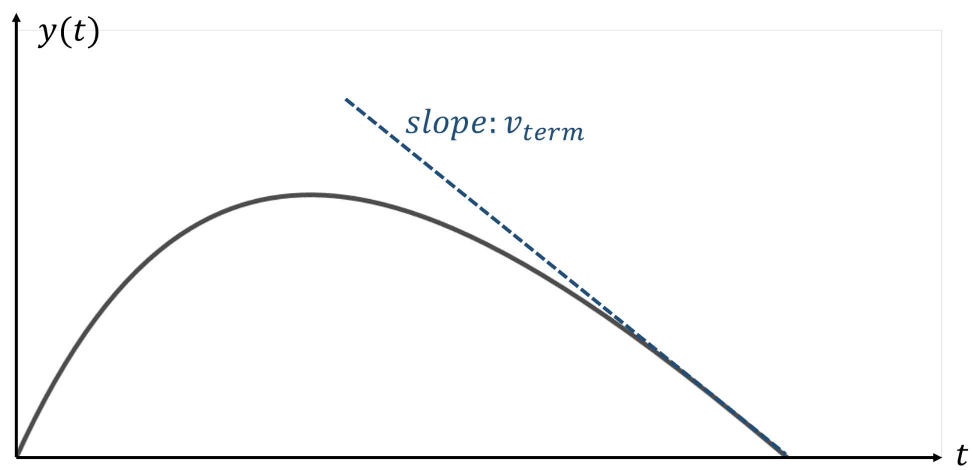

Figure 6.2: General Graphical Solution for Free Fall Motion with Air Resistance.

Our solution fits the physical laws which define free fall motion with air resistance. Comparing solutions (6.12) and (6.18) notice that by adding a drag force we get a more accurate representation of motion for an object in free fall. Defining the boundary condition governed by terminal velocity provided a solution with more physical relevance.

6.2.3 Classical Harmonic Oscillator

To explore another differential equation discussed in mechanics, let's consider oscillations. One familiar system to explore is the mass-on-a-spring experiment. The force on the mass by the spring is governed by Hooke's Law:

$$F = -kx \tag{6.19}$$

where k is the spring constant (experimentally determined) and has units of Newtons per meter. The x denotes how far the system is from equilibrium: if the string is stretched, the displacement is positive and if the spring is compressed, x is negative. Hooke's Law describes the force on an object by a spring, and by Newton's third law, the force on the spring by the object. If the only force acting on the object is due to the spring, then Newton's Second Law is written as

$$\sum F = m\ddot{x} = -kx. \tag{6.20}$$

Dividing both sides of (6.20) by m gives

$$\ddot{x} = -\frac{k}{m}x. \tag{6.21}$$

This is a second-order differential equation with constant coefficients as discussed in Section 5.2.9, and we could use the characteristic equation to determine the solutions to this differential equation. The roots of the characteristic equation for this particular differential equation will be complex, so the solutions are functions of sine and cosine. If we didn't already have that tool, how else could we have determined a solution to (6.21)? Notice that the second derivative $\ddot{x}$ is a negative constant away from its original function x. Are there any functions we know of whose second derivative is the negative of itself (aside from a constant)? Three functions come to mind: sine, cosine, and $e^{\pm i\omega t}$. It seems intuitive that we would choose periodic functions like sine and cosine to describe oscillatory behavior. Let's guess a solution to (6.21) such that the position of an object can be described as

$$x(t) = A\cos(\omega t - \phi_0), \tag{6.22}$$

where ϕ_0 is the initial phase of the oscillation.

The velocity is the derivative of position with respect to time, which gives

$$v(t) = \dot{x} = -A\omega\sin(\omega t - \phi_0). \tag{6.23}$$

Similarly, the acceleration is the derivative of velocity such that

$$a = \ddot{x} = -A\omega^2 \cos(\omega t - \phi_0). \tag{6.24}$$

What do all of these unknown variables $A,\ \omega$, and ϕ_0 represent? Plugging (6.22) and (6.24) into (6.21) we get

$$-A\omega^2 \cos(\omega t - \phi_0) = -\frac{k}{m} A \cos(\omega t - \phi_0).$$

Dividing out common terms we find the angular frequency

$$w = \sqrt{\frac{k}{m}} = 2\pi f.$$

where f is frequency, measured in cycles per second. A is a boundary condition which represents the maximum displacement from the origin or rest point. The maximum displacement occurs at the turn-around points, or when $\dot{x} = 0$. The term ϕ_0 is the phase shift, which adjusts the solution function if the maximum displacement A doesn't occur at $t = 0$. The phase shift inherently is an initial condition that describes the position x_0 of the object at $t = 0$. Checking the initial condition for (6.22),

$$x(0) = A\cos(\omega(0) - \phi_0) = A\cos(-\phi_0) = x_0.$$

If we start at the maximum displacement from equilibrium, $A = x_0$, then our phase shift ϕ_0 is zero. Again, this model describes oscillation for when only one force is present along the direction of motion, $F = -kx$. The result predicts that the oscillatory motion will continue for all time.

We can add one layer of realism to this scenario by considering a drag force that impedes the motion of the oscillator, similar to air resistance in free fall. The resulting motion is described as *damped* oscillation. We already have

$$\ddot{x} + \omega_0^2 x = 0 \tag{6.25}$$

where we redefine $\sqrt{\frac{k}{m}} = \omega_0$ as the natural angular frequency. Adding a drag term $-b\dot{x}$, by Newton's Second Law,

$$m\ddot{x} = -b\dot{x} - kx$$

where b is a proportionality constant. Then (6.25) becomes

$$\ddot{x} + \frac{b}{m}\dot{x} + \omega_0^2 x = 0 \tag{6.26}$$

This is a second-order constant coefficient differential equation; we can use the characteristic equation from Section 5.2.9 to determine unique solutions. Let's first define for notational convenience and later physical relevance that $\frac{b}{m} = 2\beta$. The roots of the characteristic equation

$$r^2 + 2\beta r + \omega_0^2 = 0$$

are

$$r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2} \text{ and } r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}.$$

Our general solution follows as

$$x(t) = e^{-\beta t}\left(C_1 e^{(\sqrt{\beta^2-\omega_0^2})t} + C_2 e^{(-\sqrt{\beta^2-\omega_0^2})t}\right).$$

How does the solution depend on the relative values of β^2 and ω_0^2? There are three specific types of damping with occur, under-damped, over-damped, and critically damped. If $\beta^2 < \omega_0^2$, then the amplitude of the oscillation will decay over time. Additionally, if $\beta^2 < \omega_0^2$ then $\sqrt{\beta^2 - \omega_0^2}$ is imaginary and the solutions will be of the form

$$x(t) = Ce^{-\beta t}e^{\pm i\omega_1 t}$$

where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ is the *observed angular frequency*. With no damping, the observed frequency is the natural frequency ($\omega_1 = \omega_0$). For a reminder on solutions for complex roots of the characteristic equation, see Section 5.2.9. We know by Euler's formula (5.60) that

$$e^{i\omega_1 t} = \cos(\omega_1 t) + i\sin(\omega_1 t)$$

and from Section 5.2.9 we know either $\cos(\omega_1 t)$ or $\sin(\omega_1 t)$ are solutions to (6.26) for complex roots. Hence, our solution is

$$x(t) = Ce^{(-\beta + i\sqrt{\omega_0^2-\beta^2})} = Ce^{-\beta t}e^{i\omega_1 t} = Ce^{-\beta t}\cos(\omega_1 t).$$

Applying initial and boundary conditions for phase shift and amplitude, the solution becomes

$$x(t) = Ae^{-\beta t}\cos(\omega_1 t - \phi_0). \tag{6.27}$$

This looks familiar to our previous solution (6.22) for a system without damping, but with an additional decay term $e^{-\beta t}$, which has high physical relevance. $e^{-\beta t}$ behaves like an *envelope* around the typical oscillatory function $A\cos(\omega_1 t - \phi_0)$, decreasing the amplitude of the oscillations over time, coming to a stop in the limit of very large t (see Fig. 6.3). The quantity $beta$ (β) is called the decay parameter. The case where $\beta^2 < \omega_0^2$ is known as

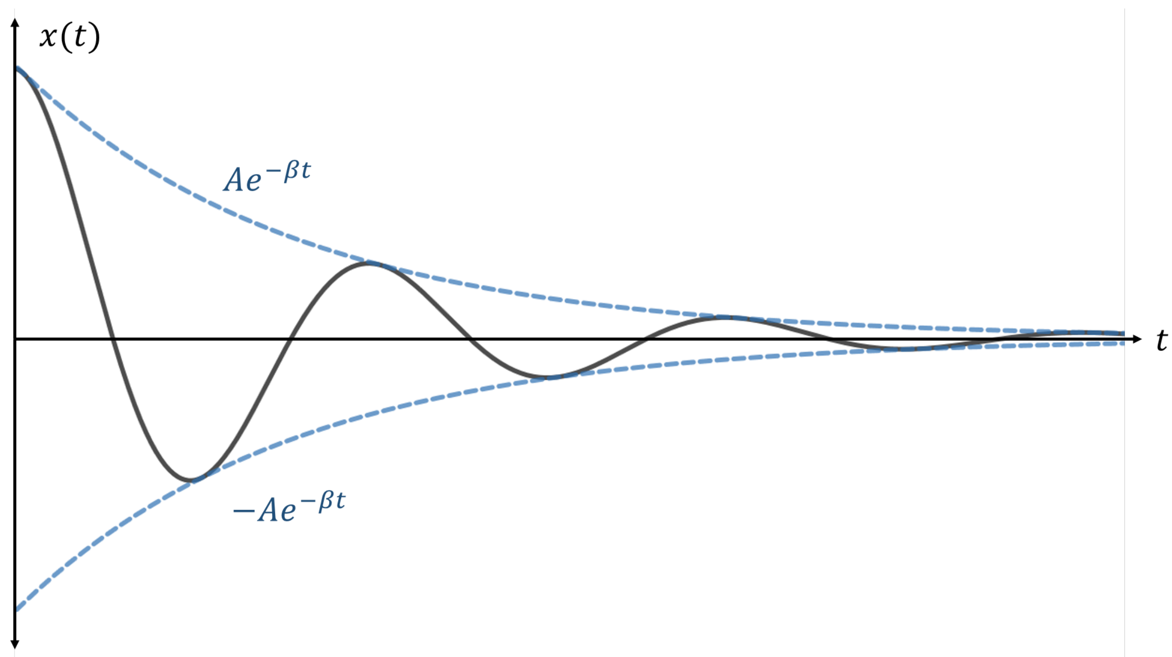

Figure 6.3: General Graphical Solution for an Under-damped Oscillator (Equation (6.27))

.

under-damping, and represents an amplitude decay for oscillations over time. Additionally, β effects the observed angular frequency: $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$, so for higher β, there is a lower observed frequency.

What if $\beta^2 > \omega_0^2$? Then our roots are real and our solution no longer contains sines and cosines. Our solution is

$$\begin{aligned} x(t) &= \left(C_1 e^{(-\beta+\sqrt{\beta^2-\omega_0^2})t} + C_2 e^{(-\beta-\sqrt{\beta^2-\omega_0^2})t} \right) \\ &= e^{-\beta t}\left(C_1 e^{(\sqrt{\beta^2-\omega_0^2})t} + C_2 e^{(-\sqrt{\beta^2-\omega_0^2})t} \right) \end{aligned} \tag{6.28}$$

The question then remains as to which exponential term dominates as time $t \to \infty$. The exponential with the root of smallest magnitude will dominate the motion of the system. The smallest root in this case is $r = -\beta + \sqrt{\beta^2 - w_0^2} = -(\beta - \sqrt{\beta^2 - w_0^2})$. This root is the decay parameter for this system. When $\beta > \omega_0$ the system experiences *over-damping*, and the position solution has exponential behavior.

The last case to consider is when $\beta^2 = \omega_0^2$. This gives repeated roots $r_1 = r_2 = -\beta$. We know from Section 5.2.9 that our solution for repeated roots (ensuring two unique solutions) is

$$x(t) = C_1 e^{-\beta t} + C_2 t e^{-\beta t} = e^{-\beta t}(C_1 + C_2 t). \tag{6.29}$$

Again, β is the decay parameter, which governs the system's motion. For repeated roots, $\beta^2 = \omega_0^2$, there is *critical damping*.

6.2.4 Instructors Thoughts: Classical Mechanics

In an interview, an instructor says "mechanics courses are designed as a refining of concepts from introductory physics courses, adding new physical realities such as air resistance and damping to physical systems to which students have previously been exposed." Instructors mention using the separation of variables technique to solve Newton's second law, which they anticipate students have seen before, either in Calculus II or a differential equations course. In the context of oscillations, where the solution method requires the characteristic equation; instructors suggest that "students say that they have not seen or do not remember seeing the characteristic equation, sometimes even after being in differential equations course." One instructor implied that time would have to be taken out of class in order to rederive the characteristic equation technique for solving second-order differential equations.

One instructor states that "initial and boundary conditions in mechanics are essential, especially with air resistance." The instructor adds later that "it's easy to solve a differential equation analytically, students instead have difficulties integrating a definite integral with initial and boundary conditions... specifically students have trouble identifying the correct limits over which to integrate now that they must pay attention to the physical laws which govern the mathematics." Another instructor stressed that "physically a solution doesn't make sense without initial and boundary conditions; without them, the solution would just mirror a mathematics course." One mechanics instructor that I interviewed acknowledged that the first and second derivatives, depending on the original function, may appear strange, but admitted to not spending time exploring that concept in class with students.

6.3 Electrostatics and Circuits

6.3.1 Laplace's Equation

In electrostatics one can determine the electric field $\vec{E}$ given by a stationary charge distribution using Coulomb's Law in integral form:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\hat{\mathscr{r}}}{\mathscr{r}^{\,2}}\rho(\vec{r}\,')d\tau'. \tag{6.30}$$

$\mathscr{r}$ is the magnitude of the separation vector $\vec{\mathscr{r}} \equiv \vec{r} - \vec{r}\,'$, where $\vec{r}$ and $\vec{r}\,'$ describe the locations of a test charge and a single point charge, respectively. Also, $\rho(\vec{r}\,')$ is the volume charge density ($i.e.$, the charge per unit volume) and $d\tau'$ is the differential volume element. Integrating expressions such as (6.30) can be difficult to do analytically. It's a common tactic in electrostatics to first find the potential, governed by the expression

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{1}{\mathscr{r}}\rho(\vec{r}\,')d\tau'. \tag{6.31}$$

This integral often remains to be difficult to compute analytically as well. Instead let's consider a function of potential in differential form know as Poisson's equation:

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho. \tag{6.32}$$

The *Laplacian* operator $\nabla^2 \equiv \vec{\nabla} \cdot \vec{\nabla}$ is a linear operator that transforms a function into partial derivatives with respect to the coordinate system of interest. In Cartesian coordinates for 3-dimensions,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}.$$

It is typical in electrostatics to want to determine the potential of a region in which the charge per unit volume, ρ, is zero. For this case in particular,

$$\nabla^2 V = 0. \tag{6.33}$$

The above expression is known as Laplace's equation, which is the homogeneous version of Poisson's equation (6.32). Written out in Cartesian coordinates for three dimensions, Laplace's equation is

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}. \tag{6.34}$$

For much of electrostatics and other physics topics (*e.g.,* magnetism, gravitation, thermodynamics), Laplace's equation plays a central role. Let's first consider the case for Laplace's equation in one dimension,

$$\frac{d^2 V}{dx^2} = 0. \tag{6.35}$$

The general solution to (6.35) will be the equation for a straight line

$$V(x) = mx + b \tag{6.36}$$

with m and b unknown constants. Two unknown constants, as we saw in Section 5.2.8, is standard for second-order differential equations. These particular constants are fixed by the boundary conditions of the problem. For concreteness let's consider a case for which the potential at the boundaries $x = 1$ and $x = 5$ are $V(1) = 8$ and $V(5) = 0$, then the constants solve to be $m = -2$ and $b = 10$.

With predetermined boundary conditions, it comes down to solving two equations for two unknowns. For these particular boundary conditions, our exact solution to (6.35), plugging the solved constants into (6.36), is

$$V(x) = -2x + 10, \tag{6.37}$$

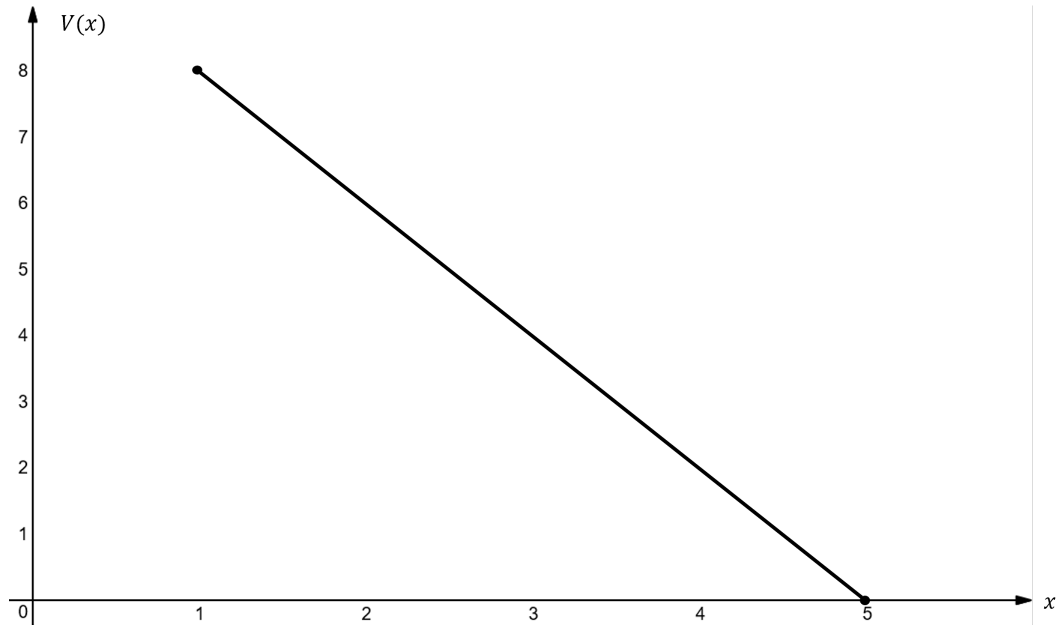

Figure 6.4: Solution Graph of Potential for Equation 6.37

represented graphically by Figure 6.4.

Given two specific boundary conditions, we can now solve for that potential at any point in one-dimension for the case where there is a known charge-per-unit-volume. Our solution $V(x)$ can be thought of as an average between two nearby points $V(x-a)$ and $V(x+a)$ for any value a such that

$$V(x) = \frac{1}{2}[V(x-a) + V(x+a)].$$

Because the potential at any point is the average of two points nearby it, the maximum and minimum potentials must occur on the boundaries. Hence, there are no local extrema (minimums and maximums) in terms of potential in one-dimension. One takeaway here is that a solution to Laplace's equation cannot be determined without boundary conditions set on the potential. This demonstrates how some mathematics relies on physical contextualization in order to generate a solution. Boundary conditions may come in many different forms, so far we've seen the function defined on two ends. We could also define a value for the derivative at either end to pair with a function value at either end. There is never need for more than two conditions when solving for a second-order differential equation, because there are only two constants to solve for. On the other hand, we need more than one in order to determine an exact solution. A boundary condition on the derivative immediately defines the value for m in (6.36). Having more than one boundary condition on the derivative at once gives no new information, because in order for the solution to exist, both must equal each other, otherwise the value for m would be inconsistent.

6.3.2 Circuits

Certain relationships between different elements of circuits can be described using differential equations. One such relationship is the following differential equation

$$\frac{dV_c}{dt} = -\frac{1}{RC}(V_c - V_0), \tag{6.38}$$

which describes the voltage across the capacitor in an RC-circuit. This is a first-order differential equation which we know how to solve using separation of variables. Separating the above expression like so

$$\frac{dV_c}{V_c - V_0} = -\frac{1}{RC}dt$$

our solution is

$$V_c(t) = V_0[1 - e^{-\frac{1}{RC}t}]. \tag{6.39}$$

Let's focus now on where differential equation (6.38) comes from. Picture a circuit (see Fig. 6.5) with a battery, resistor, and capacitor in series with the capacitor initially uncharged (open switch).

The battery has an emf $\epsilon = V_0$. Applying Kirchoff's Loop Rule around this circuit we get

$$V_0 - IR - Q/C = 0. \tag{6.40}$$

Because the elements are all in series, we know that the current through each element is the same. Due to the nature of RC circuits, the current is time dependent and represented by the change in charge over time,

$$I = \frac{dQ}{dt}.$$

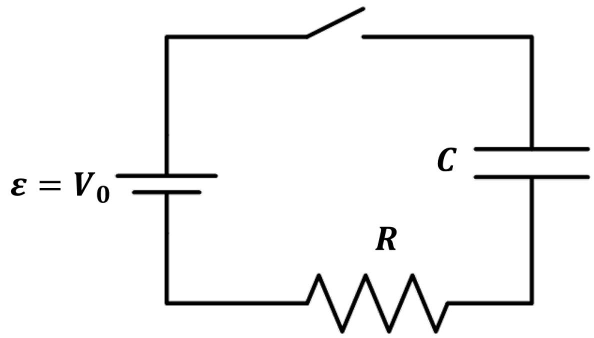

Figure 6.5: RC Circuit Diagram

Plugging this expression for current into (6.40) we get

$$V_0 - \frac{dQ}{dt}R - Q/C = 0. \tag{6.41}$$

The charge across the capacitor is defined as

$$Q = V_c \cdot C$$

Taking the derivative with respect to time we get that

$$\frac{dQ}{dt} = \frac{dV_c}{dt} \cdot C.$$

Plugging these two expressions back into (6.41) yields

$$V_0 - \frac{dV_c}{dt} \cdot RC - V_c = 0 \tag{6.42}$$

Solving for $\frac{dV_c}{dt}$ returns us to the differential equation (6.38). The reason for deriving the differential equation returns to the challenge of understanding where many of these expressions come from. Having a physical contextualization provides a means to determining exactly what many differential equations model and represent.

Looking back at (6.41) we can additionally determine a solution for the charge as a function of time. Solving

$$\frac{dQ}{dt} = -\frac{1}{RC}(Q - Q_0)$$

using separation of variables we get

$$Q(t) = Q_0[1 - e^{-\frac{t}{RC}}] \tag{6.43}$$

where $Q_0 = CV_0$ is the inital charge acorss the capacitor. Differentiating (6.43) gives the expression for current while charging a capacitor,

$$I(t) = I_0 e^{-\frac{t}{RC}} \tag{6.44}$$

where $I_0 = \frac{Q_0}{RC} = \frac{V_0}{R}$ is the maximum current for the circuit.

Through an identical process, the equations for potential, charge, and current could be determined for a circuit with the capacitor initially fully charged. The solutions respectively are as follows

$$V_c(t) = V_0 e^{-\frac{t}{RC}}, \tag{6.45}$$

$$Q(t) = Q_0 e^{-\frac{t}{RC}}, \tag{6.46}$$

and

$$I(t) = -I_0 e^{-\frac{t}{RC}}. \tag{6.47}$$

Notice that (6.44) and (6.47) differ by a negative sign. This implies that the current direction when charging a capacitor is opposite that when discharging a capacitor.

Another important aspect of the solutions derived in this section on circuits is that we can plug in our limits of time $t = 0$ and $t = \infty$ to check that the solutions match our physics intuition. For instance, looking at the solution to (6.38), where the capacitor in the circuit was initially uncharged, we have

$$V_c(t) = V_0[1 - e^{\frac{t}{RC}}].$$

Note that $V_c(0) = 0$, which is expected because the capacitor is uncharged initially. As we let time get large, V_c approaches V_0. The potential across the capacitor eventually is equivalent to the potential across the battery.

6.3.3 Instructors' Thoughts: Electrostatics and Circuits

The electrostatics instructor I interviewed stated that "the main focus is the Laplace equation due to its centrality in related physics concepts, but most specifically solving for potential." The instructor added that "in some years there's not enough time to solve Laplace's equation mathematically.” Beyond one dimension, Laplace’s equation is solved by, as an instructor puts it, “a physicist’s favorite solution method, separation of variables for partial differential equations.” The instructor stressed the importance of boundary conditions and stated that “they’re critical to the physics... in terms of a mathematical solution meaning something, there has to be a boundary condition.” The instructor later adds that “having a context is easier to see a differential equation as a tool, especially knowing where the differential equation comes from.” Further the instructor implied that the “physical meaning can get lost in the mathematics” if student’s don’t know where differential equations come from.

For circuits, the instructor I interviewed said “the primary focus is not on the mathematical formalism of solving the differential equation”, but that “the most important aspect is testing the limits of the differential equations’ solutions to see if the behavior matches our intuition.” The instructor iterates that due to time commitments, “the primary solution method in circuits instruction is guess and check (notice I use separation of variables techniques above in Section 6.3.2 for mathematical clarity and to avoid the follow-the-leader aspect of guessing). The instructor implied that they expect that students have seen the solution methods before in a sequence where mechanics courses precede circuit courses. Further, the instructor says “the focus is directed on the derivation of the differential equation using ideas from circuit analysis and Kirchhoff’s Loop Law.” The instructor suggests that “the derivation of the differential equation building from a physics perspective of circuits is critical for students physical understanding of the differential equation.” The instructor implies that by removing the ambiguity of mathematical symbolism, this students generally have an increased understanding.

6.4 Quantum Mechanics

Quantum mechanics is the area of physics that describes the behavior of microscopic systems, typically an electron in some electric potential energy landscape. There are conditions under which a system is considered to be in a classical or quantum state, which are not relevant here. Quantum systems are probabilistic rather than deterministic. A system is described by the probability that a measurement of some property will yield a particular value. The basic mathematics that describe quantum systems is actually linear algebra and eigentheory. But for continuous quantities (*e.g.,* position, momentum), the system behavior can be described by a differential equation, known as the Schrödinger equation, that relates the "wave function," $\psi(x)$, to the potential energy function of the surroundings and the discrete values of system energy (which happen to be the eigenvalues of the *Hamiltonian* operator). The wave function can be complex. The square of the wave function, $|\psi(x)|^2$, is the (position) probability density of the system. This is integrated over the spatial region of interest to determine the probability of a measuring the particle's position in that region.

The full Schrödinger equation is a function of position and time. But in the standard system, which is closed and thus conserves energy, the equation is separable, and the time function, which carries the energy eigenvalues with it, is an imaginary exponential, or a sinusoidal function. This function oscillates with a frequency proportional to the energy level of the system. The other side of the equation is a function of position but not time, and can be rearranged to be the time-independent Schrödinger equation

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E - U(x))\psi(x), \tag{6.48}$$

where $U(x)$ is the potential energy, E is the total energy of the system, and $\hbar$ is a constant known as Planck's constant, divided by 2π.

This chapter provides examples of the use of differential equation methods in three canonical quantum mechanical systems.

6.4.1 Particle in a Box

One concept from quantum mechanics is attempting to understand particle behavior when it is trapped in a "box." In physics, this box is an infinite square potential well of base length L.

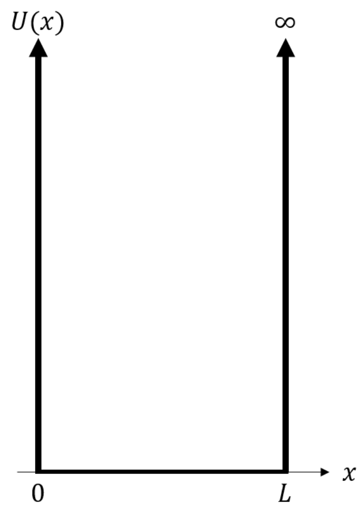

Figure 6.6: Infinite Square Potential Well

Inside of the well (see Figure 6.6), the potential energy U is zero, while the walls of the well have infinite potential energy. We introduce the infinite potential well in order to determine a solution for the wave function $\psi(x)$ from the time-independent Schrödinger equation.

For the infinite potential square well, we consider $U(x) = 0$. For this particular case, (6.48) becomes

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi(x). \tag{6.49}$$

Let's define

$$k = \sqrt{\frac{2mE}{\hbar^2}} \tag{6.50}$$

for convenience. Then the expression above becomes

$$\frac{d^2\psi}{dx^2} = -k^2\psi(x).$$

This is a second-order constant coefficient differential equation with complex roots. Therefore our general solution to (6.49) is

$$\psi(x) = A\sin(kx) + B\cos(kx). \tag{6.51}$$

There are boundary conditions on both sides of the well, $x = 0$ and $x = L$. On both sides $U(x) = \infty$ which implies that there is no wave function at the boundaries, because the wave function must be finite in order to be normalizable. Therefore we cannot find a particle on the boundaries and the wave function at the boundaries is zero, *i.e.,* $\psi(0) = \psi(L) = 0$. Using these boundary conditions we can determine the constants A and B. For the boundary condition at $x = 0$ we get

$$\psi(0) = A\sin(0) + B\cos(0) = 0$$

which implies that $B = 0$. Our solution (6.51) simplifies to

$$\psi(x) = A\sin(kx).$$

Testing the second boundary condition at $x = L$ gives

$$\psi(L) = A\sin(kL) = 0.$$

This is where "quantization" of the energies comes in: because of the constraints on the wave function at the edges of the well, only particular values of k are allowed that solve the equation. The values for which $\sin(kL) = 0$ are $k = \frac{n\pi}{L}$, where n is a positive integer. A is the amplitude of the wave function, so we will write it as $C_1 = A$. The solution (6.51) then becomes

$$\psi(x) = A\sin(\frac{n\pi}{L}x) = A\sin(kx) \tag{6.52}$$

Earlier (6.50) we defined k in terms of the energy E. With our new expression for k as a result of the second boundary condition, $k = \frac{n\pi}{L}$ we can determine an expression for the total energy,

$$E = \frac{n^2\pi^2\hbar^2}{2mL^2} = \frac{n^2h^2}{8mL^2}. \tag{6.53}$$

Now we have quantized energy states for different values of n, where $n = 1$ is the lowest energy, or ground, state. The final task now is to "normalize" our wave function solution (6.52) to determine a value for the amplitude A. To normalize the wave function is to ensure that the probability $P(x)$ of finding the particle at some point in the allowed spatial region is 100%. Mathematically, we integrate the square of the wave function over all space. The space in which the particle may exist with respect to the potential well is between $x = 0$ and $x = L$, and so integration yields

$$\int_{-\infty}^{\infty} P(x)dx = \int_{0}^{L} |\psi(x)|^2 dx = \int_{0}^{L} |A^2| \sin^2(kx) dx = 1.$$

Computing the integral and solving for A gives

$$A = \pm\sqrt{\frac{2}{L}}.$$

While it is possible for the amplitude A to be complex, in order to keep things as simple as possible we are only going to consider the real case. The exact solution to (6.49) is

$$\psi(x) = \sqrt{\frac{2}{L}} \sin(kx). \tag{6.54}$$

6.4.2 Step Potential Regions

After determining the solution for the infinite potential well, let's now explore what is known as a "potential step" region. This models an abrupt change in the potential energy of the system as a instantaneous change in the value of $U(x)$. For $x < 0$, the potential is constant at $U(x) = 0$. For $x > 0$, the potential jumps (steps) up to a constant value $U(x) = U_0$.

The behavior of the solution to Schödinger's equation (6.48) for the step potential depends on the value of the total energy E with respect to the step U_0. Note that this is not a bound state like the infinite potential well. We still solve for individual energies, but they are not quantized here. The reason being is that there is only one "boundary condition" at the step $x = 0$ (See Figure 6.7). After solving for the individual energies, we need the overall solution to be a linear combination of solutions to be normalized to obey physics. The first case to consider is $E > U_0$. Then for the region $x < 0$ where $U(x) = 0$ we get sines and cosines (or positive and negative imaginary exponential) for $\psi(x)$ with the same k as for the infinite square well (6.50). These can be considered as plane waves in the negative-x region.

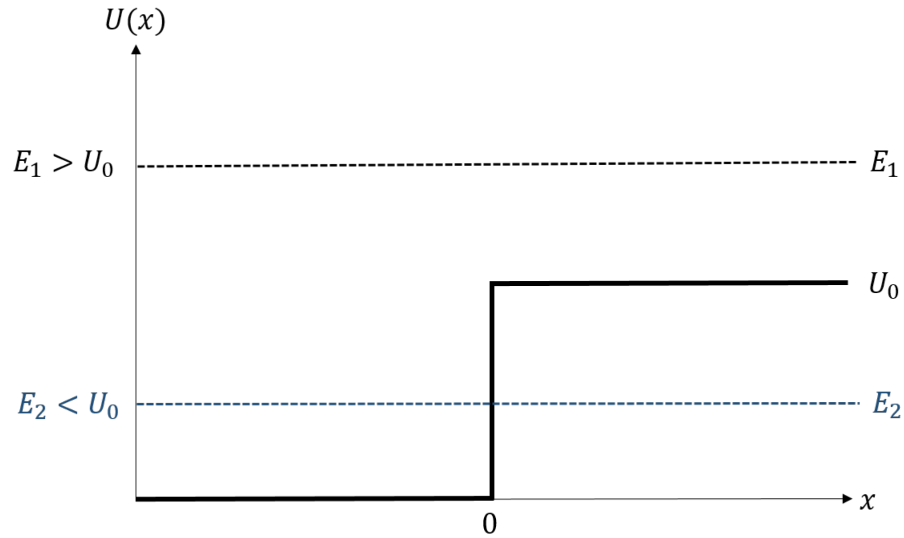

Figure 6.7: Step Potential Region with Two Energy Levels

For $x > 0$, (6.48) takes the form

$$\frac{d^2\psi}{dx^2} = -\kappa^2\psi(x)$$

where

$$\kappa = \sqrt{\frac{2m(E - U_0)}{\hbar^2}}.$$

Notice that $\kappa < k$. The solutions for $\psi(x), x > 0$ are still sines and cosines, but the wavelengths are longer, because there is a lower kinetic energy $E - U_0$ as opposed to the total energy E being entirely kinetic for $x < 0$.

The second case to consider is $0 < E < U_0$. The total energy is between the energy step U_0 and zero potential energy. For $x < 0$, as before, $\psi(x)$ is sinusoidal (a plane wave) with the same k as before (6.50). For $x > 0$, (6.48) is expressed now as

$$\frac{d^2\psi}{dx^2} = K^2\psi(x)$$

where K is a new constant defined as

$$K = \sqrt{\frac{2m(U_0 - E)}{\hbar^2}}.$$

The possible solutions to this equation are

$$e^{+Kx} \text{ or } e^{-Kx}.$$

Both are exponential functions; one describes exponential growth and the other exponential decay. The exponential growth solution e^{Kx} cannot be the solution because then $\psi(x) = e^{Kx}$ would go to infinity for large values of x. Therefore, the solution for $x > 0$ must be the exponential decay expression e^{-Kx}. Thus function matching must occur at the boundary $x = 0$, $i.e.$, the wave function transforms from a sinusoidal function into an exponential decay for the case $0 < E < U_0$.

As mentioned earlier, the wave function is related to the position measurement probability. When the energy is within the region of the step, the probability of finding the particle is exponentially decaying, but non-zero. This is different from classical physics, which would predict the probability finding a particle in a wall would be zero. Picture throwing a tennis ball at brick wall, would you expect it to travel through the wall? No, and that's why this result for quantum mechanics is so fascinating.

6.4.3 Quantum Harmonic Oscillator: Power Series Solution

The utility of the harmonic oscillator as a model for physical systems is not limited to classical mechanics; it is a powerful model in quantum mechanics as well. Thus solving the Schrödinger equation for a harmonic oscillator potential is extremely relevant. We explored the classical harmonic oscillator in Section 5.4.3. For the quantum harmonic oscillator, the potential energy function is written as a function of the angular frequency ω rather than the "spring constant" k, so that $U(x) = \frac{1}{2}m\omega^2x^2$; the differential equation corresponding to the Schrödinger equation here becomes

$$\frac{-\hbar^2}{2m}\frac{d^2\varphi_E(x)}{dx^2} + \frac{1}{2}m\omega^2x^2\varphi_E(x) = E\varphi_E(x). \tag{6.55}$$

The goal of this section is to determine a solution to (6.55) using power series. To begin, let's make the clever variable change,

$$y = \sqrt{\frac{m\omega}{\hbar}}x,$$

which simplifies (6.55) to

$$\frac{-\hbar^2}{2m}\frac{d^2\varphi_E(y)}{dx^2} + \frac{\hbar\omega}{2}y^2\varphi_E(y) = E\varphi_E(y). \tag{6.56}$$

By the chain rule,

$$\frac{d^2\varphi_E(y)}{dx^2} = \frac{d}{dx}\left(\frac{d\varphi(y)}{dy}\cdot\frac{dy}{dx}\right) = \sqrt{\frac{m\omega}{\hbar}}\frac{d}{dx}\left(\frac{d\varphi(y)}{dy}\right) = \sqrt{\frac{m\omega}{\hbar}}\left(\frac{d^2\varphi(y)}{dy^2}\cdot\frac{dy}{dx}\right) = \frac{m\omega}{\hbar}\frac{d^2\varphi(y)}{dy^2}$$

where

$$\frac{dy}{dx} = \sqrt{\frac{m\omega}{\hbar}}.$$

Substituting the above expression back into (6.56) yields

$$\frac{-\hbar^2}{2m}\frac{m\omega}{\hbar}\frac{d^2\varphi_E(y)}{dy^2} + \frac{\hbar\omega}{2}y^2\varphi_E(y) = E\varphi_E(y)$$

which further simplifies to

$$-\frac{\hbar\omega}{2}\left(\frac{d^2\varphi_E(y)}{dy^2} - y^2\varphi_E(y)\right) = E\varphi_E(y).$$

Solving for the second derivative term gives

$$\frac{d^2\varphi_E}{dy^2} = (y^2 - \frac{2E}{\hbar\omega})\varphi_E = (y^2 - k)\varphi_E \tag{6.57}$$

where $k = \frac{2E}{\hbar\omega}$. Rewriting this expression so that the right side is zero gives

$$\frac{d^2\varphi_E}{dy^2} - (y^2 - k)\varphi_E = 0.$$

The solution is going to have multiple components, but some constraints on the wave function properties can help determine one or two of these components. In order to identify some components, the solution is considered for extreme values of y.

The first constraint will be implemented when we determine that the solution becomes asymptotic for large y, in which case k becomes negligible in comparison, leaving the second-order differential equation

$$\frac{d^2\varphi_E(y)}{dy^2} - y^2\varphi_E(y) = 0.$$

The solution to this second-order differential equation is a variation of the exponential function. The solution for this equation is

$$\varphi \approx Ae^{-\frac{y^2}{2}} + Be^{\frac{+y^2}{2}}. \tag{6.58}$$

Notice that

$$\frac{d^2}{dy^2}(e^{-\frac{y^2}{2}}) = (y^2 - 2)e^{-\frac{y^2}{2}} \approx y^2 e^{-\frac{y^2}{2}}.$$

The last equivalence is a result of our asymptotic assumption where $y^2 - 2$ approximates to y^2 for $y >> 1$. Referring back to (6.58), in order for the wave function solution φ to be normalizable – $i.e.$, for φ to asymptotically approach zero at large values of y, so that the integration over the square of the wave function is finite – the positive exponential must vanish, so B must be zero. The adjusted approximate solution is

$$\varphi \approx Ae^{-\frac{y^2}{2}}.$$

The next component of the solution is labeled $h(y)$ and is multiplied with the exponential, so that the "full" solution takes the form

$$\varphi_E(y) = h(y)e^{-\frac{y^2}{2}}. \tag{6.59}$$

Substitution of (6.59) into (6.57) will give the differential equation in terms of $h(y)$; the exponential terms will survive on both sides of the equation and can be eliminated, leaving a second-order differential equation of $h(y)$.

First we need to know the second derivative of (6.59). The first derivative, as a result of the product rule for differentiation, is

$$\frac{d\varphi(y)}{dy} = \frac{dh}{dy}e^{-\frac{y^2}{2}} + h(y)e^{-\frac{y^2}{2}}(-y).$$

The second derivative follows similarly as

$$\frac{d^2\varphi(y)}{dy^2} = \frac{d^2h}{dy^2}e^{-\frac{y^2}{2}} + \frac{dh}{dy}e^{-\frac{y^2}{2}}(-y) + \frac{dh}{dy}e^{-\frac{y^2}{2}}(-y) + h(y)e^{-\frac{y^2}{2}}(-1) + h(y)e^{-\frac{y^2}{2}}(-y)^2$$

which can be simplified to

$$e^{-\frac{y^2}{2}}\left(\frac{d^2h}{dy^2} - 2y\frac{dh}{dy} + (y^2 - 1)h\right).$$

Substituting φ from (6.59) and its second derivative into (6.57) yields

$$e^{-\frac{y^2}{2}}\left(\frac{d^2h}{dy^2} - 2y\frac{dh}{dy} + (y^2 - 1)h(y)\right) = (y^2 - k)h(y)e^{-\frac{y^2}{2}}.$$

After factoring and cancelling out common terms, including the exponentials, the above expression becomes

$$\frac{d^2h}{dy^2} - 2y\frac{dh}{dy} + (k-1)h(y) = 0. \tag{6.60}$$

This is a new second-order differential equation for $h(y)$. Notice that the coefficients in front of the first derivative and $h(y)$ are both polynomials. Recall from Section 5.2.13 that a strategic guess solution is a polynomial: let

$$h(y) = a_0 + a_1y + a_2y^2 + a_3y^3 + \cdots = \sum_{j=0}^{\infty} a_jy^j. \tag{6.61}$$

The first and second derivative of our guess solution are as follows:

$$\frac{dh}{dy} = a_1 + 2a_2y + 3a_3y^2 + \cdots + na_nt^{n-1} = \sum_{j=1}^{\infty} ja_jy^{j-1};$$

$$\frac{d^2h}{dy^2} = (2\cdot 1)a_2 + (3\cdot 2)a_3y + \cdots = \sum_{j=2}^{\infty} j(j-1)a_jy^{j-2}.$$

Substituting the derivatives and (6.61) into (6.60) gives

$$\sum_{j=2}^{\infty} j(j-1)a_jy^{j-2} - 2y\sum_{j=1}^{\infty} ja_jy^{j-1} + (k-1)\sum_{j=0}^{\infty} a_jy^j = 0.$$

Reindexing the second derivative term yields

$$\sum_{j=2}^{\infty} j(j-1)a_jy^{j-2} = \sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}y^j.$$

The first derivative term can be reindexed also. The y term in front of the summation can be put inside the sum, raising the power of y by 1. Then the index can start at zero, since the first term (for $j = 0$) is zero:

$$y\sum_{j=1}^{\infty} ja_jy^{j-1} = \sum_{j=1}^{\infty} ja_jy^j = \sum_{j=0}^{\infty} ja_jy^j.$$

Rewriting the expression above with the new reindexed summations gives

$$\sum_{j=0}^{\infty}(j+2)(j+1)a_{j+2}y^j - 2\sum_{j=0}^{\infty} ja_jy^j + (k-1)\sum_{j=0}^{\infty} a_jy^j = 0.$$

Since all three sums are in terms of the same power of y ($i.e., y^j$), we can combine the sums into one large sum:

$$\sum_{j=0}^{\infty} \Big[(j+2)(j+1)a_{j+2} - 2ja_j + (k+1)a_j\Big] y^j = 0.$$

As pointed out in Section 5.2.13 for this result to hold for all y values, the coefficient of each power of y must individually be zero, thus the expression in the brackets must be equal to zero:

$$(j+2)(j+1)a_{j+2} - 2ja_j + (k+1)a_j = 0.$$

Solving this equation for a_{j+2} gives the recursion relation

$$a_{j+2} = \frac{2j-k+1}{(j+2)(j+1)} a_j. \tag{6.62}$$

Here we will get even and odd solutions for a_j, that is, solutions of exclusively even powers of y or exclusively odd powers of y. (This makes sense that there are two solutions, given that it's a second-order differential equation.) These functions have symmetry (even or odd symmetry, respectively) about $y = 0$. a_0 determines a_2, which determines a_4 and so on. a_1 determines a_3, and so on.

However, there is a concern at this point. If $h(y)$ is an infinite power series with the nonzero coefficients, then it can be approximated as

$$h(y) = \sum_{j=0}^{\infty} a_j y^j \approx e^{y^2},$$

and our full solution for $\varphi(y)$ (6.59) has the functional form

$$\varphi(y) = e^{y^2} e^{-\frac{y^2}{2}} = e^{\frac{y^2}{2}}.$$

This is a problem, because once again the wave function φ would not be normalizable, because for increasing values of y, the solution would become infinitely large, and the square integral would not be finite. This implies that in order for the solution to be normalizable, the power series solution $h(y)$ must terminate at some maximum power of y, which is labeled $n, i.e., j_{max} \equiv n$. These are additional limits imposed on the power series due to normalization. The conditions of termination can be expressed n the recursion relation:

$$a_{n+2} = \frac{2n+1-k}{(n+2)(n+1)} a_n = 0.$$

Solving this expression for k gives

$$k = 2n + 1.$$

Note that this is where quantization comes from: n is an integer representing some power of y, and k has values that depend on n. Recall originally that

$$k = \frac{2E}{\hbar\omega}.$$

Setting the last two expressions equal to one other gives an expression for allowed energies

$$E_n = \hbar\omega(n + \frac{1}{2}). \tag{6.63}$$

These are the energy values that allow $\varphi(y)$ to satisfy the Schrödinger equation for the harmonic oscillator potential.

Let's try and determine the first few solutions. Let's first look at h_0. This is an even solution ($n = 0$) which implies $a_{\text{odd}} = 0$. Seeming as $n = 0$, $h_0(y) = a_0$, and our solution $\varphi(y)$ (6.59) is

$$\varphi_0(y) = h_0(y)e^{-\frac{y^2}{2}} = a_0 e^{-\frac{y^2}{2}}.$$

This is the wave function as a function of y. Rewriting the wave function in terms of x gives

$$\varphi_0(x) = a_0 e^{-\frac{m\omega}{2\hbar}x^2}.$$

We find a_0 by recalling that the integral over all space of the wave function squared must be equal to 1 by normalization:

$$1 = \int_{-\infty}^{\infty} \left| a_0 e^{-\frac{m\omega}{2\hbar}x^2} \right|^2 = \int_{-\infty}^{\infty} a_0^2 e^{-\frac{m\omega}{\hbar}x^2} = 2a_0^2 \int_0^{\infty} e^{-\frac{m\omega}{\hbar}x^2}.$$

Computing this integral and solving for a_0 yields

$$a_0 = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}.$$

Therefore the "ground state" solution for (6.55) is

$$\varphi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}. \tag{6.64}$$

To continue determining solutions for higher energy states, it's important to note that the a_j's are different for different energy levels n. This is a result of normalizing the wave function solution for each energy level. Let's go through and determine the solutions for the $n = 1$ and $n = 2$ energy states.

For the first excited state ($n = 1$), $a_0 = 0$, $a_1 \neq 0$, so $h_1(y) = a_1 y$ (6.59). The wave function is

$$\varphi_1(y) = a_1 y e^{-\frac{y^2}{2}}$$

which becomes, as a function of x,

$$\varphi_1(x) = a_1 \sqrt{\frac{m\omega}{\hbar}} x e^{\frac{m\omega}{2\hbar}x^2}.$$

Normalizing this wave function gives

$$a_1 = \sqrt{2}\left(\sqrt{\frac{m\omega}{\hbar\pi}}\right)^{\frac{1}{4}}.$$

The first energy state solution to (6.55) is

$$\varphi_1(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \sqrt{2} \cdot \sqrt{\frac{m\omega}{\hbar}} x e^{-\frac{m\omega}{2\hbar}x^2} = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \sqrt{\frac{m\omega}{2\hbar}} (2x) e^{-\frac{m\omega}{2\hbar}x^2}. \tag{6.65}$$

Lastly, let's look at the $n = 2$ energy state. Here there are no odd solutions, and $h_2(y) = a_0 + a_2 y^2$. By the recursion relation (6.62),

$$a_2 = 2\frac{-2(2-0)}{(2)(1)} = -2a_0,$$

and so

$$h_2 = a_0 - 2a_0 y^2 = a_0(1 - 2y^2).$$

The second excited state wave function is

$$\varphi_2(x) = a_0\left(1 - 2\left(\sqrt{\frac{m\omega}{\hbar}} x\right)^2\right) e^{-\frac{m\omega}{2\hbar}x^2}.$$

Normalizing this wave function gives

$$a_0 = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{3\frac{m\omega}{\hbar} - 2\sqrt{\frac{m\omega}{\hbar}} + 1}}.$$

(Notice this is not the same a_0 as for the $n = 0$ state.) The solution to (6.55) for the second excited state is

$$\varphi_2(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{3\frac{m\omega}{\hbar} - 2\sqrt{\frac{m\omega}{\hbar}} + 1}} \left(1 - 2\frac{m\omega}{\hbar}x^2\right) e^{-\frac{m\omega}{2\hbar}x^2}. \tag{6.66}$$

6.4.4 Instructors Thoughts: Quantum Mechanics

In the interview with the instructor who has taught quantum mechanics, they say that "the primary differential equation used in quantum mechanics is the Schödinger equation. The Schrödinger equation becomes a different differential equation to solve depending on the functional form of the potential, and so the Schrödinger equation in theory represents numerous differential equations." The instructor adds that "the potential term in the differential equation is typically time-independent", and so that is how they teach the Schrödinger equation throughout a course. Further the instructor states that "it's a key relevance that the solutions correspond to quantized energy levels, and only certain solutions for the wave function or energy eigenfunction work with given boundary and initial conditions. The solutions and their derivatives have to be continuous at the boundaries, unless the potential is infinite at those points", like for the infinite potential well in Section 6.4.1. The instructor mentioned how much of the analysis comes down to "looking at the system relationship between the energy and the potential." They continued by acknowledging that "while a function could satisfy the general solution, after plugging in boundary conditions, the function must also obey the continuity laws at the boundary." Additionally, it was implied that the function must be square integrable and normalizable, which is a "reoccurring theme utilized in determining solutions in quantum mechanics."

The instructor said "the applications come down to finding the quantized energies and determining where a particle is most likely to be found at a given energy using probability distributions." The goal is to locate a particle in space and determine how the particle propagates in time, and "predict the future." The wave function is known as the probability amplitude, but has no true physical meaning, and is a mathematical construct. The probability density, the square of the wave function, is the aspect of the wave function in which the physics plays a role. The instructor implied that they emphasize to their students the significant difference between probability amplitude and density to "differentiate the mathematics and the physics of probability." Generally, it was said that instructors implement the Schrödinger equation when potential is constant over specific intervals and can be expressed as a second-order homogeneous constant coefficient differential equation. "These forms of the Schödinger equation are either solved by real or complex exponential functions (or sines and cosines)." When the potential is no longer constant (*e.g.,* quantum harmonic oscillator), the instructor mentioned that "there are additional methods to solve the differential equation, either the power series solution or operator method", which rely on different mathematical concepts.

The instructor argued that "key to the power series solution is the relationship between the mathematical solution and the physical conditions imposed by the physics system." They continued by adding that "these conditions lead to mathematical approximations such as termination of the power series to prevent the solution from blowing up, which creates a recursion relationship for the coefficients of the polynomial described by the power series." The instructor highlights the approximations in the power series solution as "necessary to abide by physical laws." The supplementary operator method uses ideas from linear algebra and as the instructor mentioned, "removes the aesthetic look of the differential equation, converting it into momentum and position operators." The instructor went on to describe the process of the operator method. "By manipulating the operators back into the differential equation gives an operator equation. From the operator equation one can determine the ground state, and from the ground state one can use new operators to raise and lower the energy states... The operators can be re-expressed as first-order differential equations... The solutions to these differential equations translate back to the operators to determine the quantized energies." In conclusion, the instructor pointed out the "elegance" of this operator method over the power series solution, although "each hold their own weight as solutions to the Schrödinger equation for the harmonic oscillator potential."

Chapter 7

Discussion

The last two chapters present the parallels and differences in presentation of differential equations in mathematics and physics classrooms. In the mathematics classroom the material is implemented with some use of applications, but primarily focuses on the mathematical rigor of differential equations: being able to solve differential equations using the various solution methods, understanding what properties a solution needs to have, and developing a sense for the notation common of differential equations. Instructors correlate the lack of applications to a lack of instructional time over the length of a course. Furthermore, mathematics instructors don't consistently use initial or boundary conditions to lessen the ambiguity of problems unless a particular exercise requires it.

In physics, the differential equations only describe physical applications such as free-fall motion, harmonic oscillation, properties of electric circuits, etc. The mathematical formalism becomes less of the focus. Instructors expect students to have seen the solution methods before, or simply lead students to make an educated guess for a convenient solution. Differential equations and their solutions in physics are often coupled with initial and boundary conditions. Instructors claim that without initial and boundary conditions the differential equations returns to traditional mathematics. In physics, because a differential equation relates to a real physical situation, there is more focus on showing where the differential equation comes from to help make sense of the mathematical symbolism.

Education research literature would suggest that mathematics instruction should include more experientially real situations to help students develop more formal mathematics [3]. While implementing applications may be difficult due to time pressures, "experientially real" for students can be linking new mathematics to previous concepts, similar to how students developed a slope-focused approach to overcome difficulties using eigenvalue ideas from linear algebra [3]. As instructors mention, every new concept is built from previous ideas covered in a typical sequence of mathematics courses, from algebra, to calculus, to differential equations. In differential equations, if there is a new style of differential equation, it's typically a special case of one seen before.

Mathematics instruction often consists of a mix of analytical, graphical, and numerical solution methods, but primarily the focus is on analytical solutions. In my experience, tests primarily asked to solve for analytic solutions or recall theory, but did not focus on numerical and graphical concepts. Research shows that what is primarily emphasized is what students come away with, so when instructors give multiple representations, but only really go into depth on one- including assessments- then students won't place the importance on others [15]. Yet, central to a student's mathematical work is the interplay between numerical, graphical, and analytical work [1].

Instructors should find more time to include graphical solutions corresponding to the analytical solutions, as well as add an additional focus to numerical methods to approximate solutions to differential equations. It's too easy to write a differential equation that cannot be solved analytically and thus requires a numerical solution; therefore numerical methods should be included more in differential equations studies. Certainly not everything described by differential equations in the real world is refined to a simple analytic solution. In Quantum Mechanics for example only the Hydrogen atom is solved exactly analytically (using separation of variables for partial differential equations). Anything more sophisticated has to use either an approximation technique or numerical methods, and that may be simple in comparison to developing the analytic solution to the Hydrogen atom.

In physics, students struggle with differential equations because the mathematics is now being applied to a specific context. In other words, the application of knowledge in different contexts is limited by students' understanding of the conditions under which knowledge applies [11, 21]. If a student first sees a type of differential equation in a mathematics context, they may not immediately recognize a similar differential equation in a physics context. The same argument applies to solution methods. Simply because a student may have learned a solution method in a mathematics context, or even another physics course, does not imply students will make the connection when it comes up again in a new setting. This is also true for connections between old and new mathematics. One role of the instructor is to make those connections apparent to students.

Let's now break from the general discussion and talk about specific aspects of the mathematics and physics classes.

7.1 Calculus Techniques

When students are first introduced to differential equations, the primary technique to solve them requires the students to take an integral or determine the antiderivative of a function, a tool introduced in Calculus I and II. Student difficulties with integration or antidifferentiation is its own research topic. Broad difficulties include the difference between indefinite and definite integration, the two parts of the fundamental theorem of calculus, and how the antiderivative relates to the to the function being integrated [16]. Despite student difficulties, solving a first-order ordinary differential equation, or a simple second-order differential equation (Section 5.2.8) utilizes no new mathematical techniques, only a new context in which students have to apply those integration techniques. This is where it's important to explain the components of a differential equation and its solution, because solving for a function rather than a number may be a new idea for students in differential equations [2].

As we saw in Sections 5.1 and 5.2, outside of general solutions to differential equations there are initial value problems. Initial value problems are defined by set initial conditions, based on the order of the differential equation. For first-order differential equations, there is one initial condition typically set on the solution itself. Solving first-order differential equations requires direct integration, but adding an initial condition changes the process in one of two ways. For a general solution, after indefinite integration there is a left over constant C. With an initial condition, the ambiguity of the constant is replaced with a numerical value. One way to remove the ambiguity is to perform definite integration when determining the solution; this prevents an unknown constant from appearing as a result of indirect integration. The other way is to determine the general solution, and then plug in the initial condition and solve for the unknown constant algebraically. If students have difficulties with definite integrals, evaluating the antiderivative at two bounds, the algebraic approach may be a more intuitive way of solving for the arbitrary constant. Solving for the constant using definite integration only works for first-order differential equations. Second- and higher-order solutions using algebraic methods determine any arbitrary constants. Remember, there must be an initial or boundary condition for every order of the differential equation. This provides a system of equations in which to solve for any ambiguities.

7.1.1 Separation of Variables

Separation of variables is a very common technique for solving first-order linear homogeneous differential equations. As discussed in Section 5.1.3, there are two different methods of solving separable differential equations. One method treats terms like $\frac{dy}{dt}$ as a derivative that cannot be separated into differentials dy and dt. This method requires integration over the derivative term, which results in the solution function. A substitute for the calculus technique is the algebraic method, which allows the differential terms dy and dt to be treated as algebraic quantities. This allows the differential equation to be separated, with one side a product of dy and a function of y and the other side a product of dt and a function of t. Once the equation is separated, then one can proceed with integration to determine the solution. The algebraic method, which I refer to as a short cut, is commonly used in physics courses, as opposed to the integral technique. I consider the algebraic method to be a short-cut because solving for values algebraically is less abstract for me. On the other hand, it does not feel natural to break down a derivative into its respective differentials. Every student can have a preference, and if both methods provide the same conclusion, I believe it's important that students get the opportunity to work with both, and recognize the connection between them. As an instructor, this provides two pathways of experiential content to draw from to demonstrate separation methods.

In physics, separation of variables is probably one of the most utilized techniques for solving differential equations [9, 10]. There are two different types of separation of variables, one for ordinary differential equations, the other for partial differential equations. In this paper I have only discussed separation of variables for ordinary differential equations. In an undergraduate physics curriculum partial differential equations appear in numerous contexts including waves on a string, thermodynamics, Laplace's equation for multiple dimensions and the Schrödinger equation. Partial differential equations deserve their own paper in order to cover the range of content and context which they cover. Separation of variables for partial differential equations refers to the technique of guessing a general solution with a functional form that allows the partial differential equation to be separated into several ordinary differential equations and then solving these ordinary differential equations individually with appropriate boundary conditions [10]. I focus on the separation of variables technique for ordinary differential equations. In Sections 6.2.1, 6.2.2, and 6.3.2. we solved differential equations for a few physics contexts. Despite the centrality of the separation of variables technique in physics, there are student difficulties due to the procedural aspects of algebraic problem solving [9].

7.1.2 Calculus III and Exact Equations

Exact differential equations are special. In order for a differential equation to be exact it has to have a specific functional form dependent on M and N as discussed in Section 5.2.5. The theory that governs the solution requires that the mixed partial derivatives of the solution are equivalent to each other. This is a consequence of Clairaut's theorem, typically taught in a Calculus III class when partial derivatives are introduced. The hope is that when students recognize the calculus, the differential equations content seems less unfamiliar. Students that correctly utilize the concepts from Calculus III may refine the formal mathematics of exact differential equations and develop strategies for determining whether a differential equation is exact and further finding a solution to match the unique characteristics of the exact equation.

Calculus III would also play a significant role for partial differential equations not discussed in this thesis.

7.1.3 Integrating in the Integrating Factor

The integrating factor, discussed in Section 5.2.1, as its name implies, involves concepts from calculus. It is one of the first methods I learned for solving non-homogeneous first-order differential equations. To understand where the integrating factor comes from requires the product rule for differentiation, as demonstrated. The integrating factor can be solved for using techniques discussed for solving first-order linear homogeneous differential equations, such as separation of variables or direct integration. Once the integrating factor is determined, the original differential equation is solved with techniques from first-order homogeneous differential equations.

The integrating factor technique is used solving other types of differential equations as well. In Section 5.25 when discussing exact equations, the integrating factor plays a role in determining solutions to differential equations that were not exact. The integrating factor is a chosen function such that the differential equation becomes exact, similar to how the integrating factor turns a non-homogeneous differential equation into a homogeneous differential equation in which the techniques exist in order to solve. We saw the integrating factor again when solving for repeated root solutions in Section 5.2.9. Here the integrating factor helps determine the second linearly independent solution for second-order constant coefficient differential equations.

While at first the integrating factor is a tool for solving first-order non-homogeneous differential equations, it can be used across multiple facets of differential equations. The integrating factor is a choice function that enforces the ideas of calculus in differential equations. If you need a specific solution, the integrating factor can be implemented to adapt the differential equation to be solvable. The integrating factor provides a recognizable differential equation for which students should already have the techniques to solve. If there are first-order differential equations in physics which are non-homogeneous or non-exact, the integrating factor is one strategy which can provide a solution. The solution for the critically damped oscillator in Section 6.2.3 is determined using the integrating factor technique to get the second linearly independent solution. In my experience, the second solution was taught as a guess which happened to solve the differential equation for the classical harmonic oscillator. With the integrating factor, we can demonstrate why that solution makes sense.

7.2 Intuitive Guessing

In both mathematics and physics it is common to first guess a solution if it is not obvious. Guessing a solution is how many solutions are derived in differential equations. For instance, in Section 5.2.12 we discussed judicial guessing of a polynomial solution based on the behavior of the right-hand side of the differential equation. Guessing is also useful when the right-hand side is the product of a polynomial and exponential, or a sinusoidal function. In Section 5.2.13 we guessed a power series solution based on the polynomial coefficients in the differential equation. We then utilized this strategic guess to determine the power series solution to the quantum harmonic oscillator in Section 6.4.3. In Section 6.3.1 we guess a solution to Laplace's equation in one-dimension, which can also be solved for using direct integration and naming constants conveniently.

A problem arises with guessing when there is not enough time in instruction to re-visit all the historical possibilities of solutions that *don't* work. In some settings *guessing* the solution becomes more of knowing the answer ahead of time. One consequence of this is that students think that getting the answer is sufficient and aren't expected to understand why the result makes sense.[3] While the guess solution can be proven to solve the differential equation of interest, that does not constitute understanding; it just feeds more into the rule-based explanations to which students are accustomed.

7.2.1 The Characteristic Equation and the Quadratic Formula

One critical guess occurs solving the second-order, homogeneous, constant-coefficient differential equation in Section 5.2.9. The guess that the solution is the exponential function comes from ideas from calculus. Knowing that the derivative of the exponential function always maintains exponential form makes it a valid guess for a solution. Guessing this solution brings about the characteristic equation or characteristic polynomial. The characteristic equation for second-order equations mirrors the quadratic formula. It allows us to solve for the roots of the exponential solutions. This iconic guess simplifies differential equations down to an algebraic formalism for all orders of constant coefficient differential equations.

The characteristic equation, like separation of variables, is another common technique to solving differential equations in physics. In Sections 6.2.3 and 6.4.2 we use the characteristic equation to solve differential equations for a few physics contexts. The characteristic equation is useful if there is different behavior under different conditions, because there are three cases for solutions: real roots, complex roots, and repeated roots. In Section 6.2.3 we determined the behavior for under-damped oscillations, over-damped oscillations, and critically damped oscillations manipulating the characteristic equation. Similarly, in Section 6.4.2 the solution behaves differently based on the relationship between total energy E and potential energy U_0. The mathematics lends itself to the physical behavior for a given solution.

7.3 Laplace World

Instructors in differential equations consider the Laplace transform as a way of simplifying complicated differential equations. The Laplace transform converts differential equations into an algebraic expression, using techniques from improper integration typically taught in Calculus II. The Laplace transform may be a new concept for many students, but its function is governed by former techniques students have studied. When the differential equation is transformed to an algebraic expression, this is what instructors called the Laplace World: algebraic techniques determine the solution. Often with Laplace transforms of any non-trivial functions, solving the algebraic expression requires partial fractions. This algebraic method is a strategy commonly introduced in complex integration in a Calculus II context. Once the algebraic expression has been solved in Laplace transform, we can convert back to the differential equation form and determine the solution to the original differential equation.

In physics, Laplace transforms can be useful whenever the system behavior has a discontinuity or is almost always zero. The Laplace transform works well with situations that include Dirac delta functions, piecewise functions, and pulse-like behavior. Laplace transforms are also a generalization of Fourier series/transforms which often occur in upper-level physics courses, including Optics and Quantum Mechanics.

7.4 The Role of Linear Algebra

Linear algebra plays a huge role in differential equations, especially second-order equations. In Section 5.2.8 we discussed the Wronskian, which implements the determinant of a matrix from linear algebra to demonstrate whether or not two solutions are linearly independent. Linear algebra backs the theory for unique solutions. This is the foundation of determining multiple solutions for higher-order differential equations: they must be linearly independent. Linear algebra techniques are a central part of solving systems of linear differential equations.[3] Solving for eigenvalues and their corresponding eigenvectors determines solutions for systems of linear differential equations, as discussed in Section 5.2.15.

The challenge with linear algebra concepts is that depending on the sequence of undergraduate mathematics courses, not all students have a background in linear algebra. It then falls on the professor to take time out of a differential equations or physics class to teach linear algebra. For instance, I learned about linear operators in linear algebra after seeing it in the context of a differential equations course. This is one reason why the straight line solution method is a useful substitute to the eigenvalue method for systems of linear differential equations. The straight line solution method, or ”slope first approach,” utilizes students' understandings of slopes to solve systems of linear differential equations.[3]

As mentioned in Section 6.4 much of quantum mechanics is mathematically described using linear algebra and eigentheory. In Section 6.4.3, I discuss the power series solution for the harmonic oscillator using differential equations to determine the wave function for quantized energy. In McIntyre's Quantum Mechanics text [25], there is a different solution method, considered to be more elegant, called the operator method. The operator method is a linear algebra based approach on solving the harmonic oscillator for wave functions for quantized energy states. This particular method was taught to me in class, while I never saw the power series solution. The power series solution requires multiple approximations and it's not quite clear how the intermediate steps fit in until the solution is reached. The operator method provides a cleaner step-by-step analysis building up to quantized energies and how to go up in energy levels or go down using raising and lowering operators. Depending on a student's preference, either method derives the behavior of energies in a harmonic oscillator system. Personally I found the power series solution worked well with the boundary conditions of the system, and helped make sense of the physics through the reasons behind the mathematical formalism. I too acknowledge that the operator method provides a slick derivation, and students may have more experiential ties with linear algebra.

7.5 Lacking Numerical and Graphical Methods

There were not many numerical or graphical methods implemented in the mathematics and physics courses I took as an undergraduate, and my thesis reflects this. Instructor's correlate this to the time pressures associated with instructing new material for students. Students prefer analytical/algebraic methods over graphical and numerical methods despite being in classes that emphasize graphical and qualitative analysis, and few studies emphasize a strong understanding of the interplay between graphical and analytic solutions. Yet, central to students' mathematical work is interplay between numerical, graphical, and algebraic work.[1] The question is whether students prefer analytic methods because that's all they've known prior to differential equations content.

In Section 6.1.2 we discussed the construct of "curviness" for graphically solving the Schrödinger equation. An instructor commented that general education students, using ideas of curviness, outperformed undergraduates in an upper-division undergraduate quantum mechanics course when tested on quantum tunneling solutions. The straight-line solution method, as discussed in Section 5.2.15.2, uses concepts from the phase plane as well as graphical identities such as slope and tangent vectors to derive the solutions to systems of linear differential equations. This graphical approach was a reinvention of the algebraic approach solving for eigenvalues first, then finding the corresponding eigenvectors. The collection of solution graphs in the phase plane (referred to as the phase portrait) represents an emerging new mathematical reality (for students).[3]

Look at the sections where we solved differential equations analytically and coupled those solutions with a graphical representation ($e.g.$, sections 5.1.2, 5.1.4, 6.2.2, 6.2.3, 6.3.1, 6.4.1, 6.4.2). Do the graphical forms of the solutions aid in the understanding of the analytic solution? That is what we need to find out from students. Does it help determine where the differential equation comes from? In my experience, I'd be given a differential equation without context and would then solve for a solution to that differential equation, again without context. The idea behind supplementing the analytic solution with a graphical representation is to provide additional contextualization for students. In many cases, the graphical representation is easier to manipulate in order to determine behavior for different parameters. Take the population modelling in Section 5.1.2, for example. The graphical guide helps to determine how the population is changing over time for different population values. It's not as easy to see the range of behavior when simply plugging in values to an analytic solution.

Let's discuss the importance of numerical methods. Given the ease with which one can write down a differential equation with no analytic solution, it is reasonable to assume that numerical solutions are at least necessary. Yet in my studies, numerical solution methods were scarce in both mathematics and physics. This may be that instructional material is idealized to have convenient solutions for differential equations discussed in mathematics and physics curricula. Additionally, doing numerical methods out by hand is long and tedious, where most numerical methods should be done on a computer program. Instructors would then need to implement computational strategies in the courses for which they already don't have enough time to cover all desired material. (And some instructors may not be familiar with computational numerical methods, and thus not comfortable teaching it to their students.) Often there may exist a numerical analysis course which would cover a lot of these gaps seen in other courses.

7.6 Applications

There were not many applications discussed in my core differential equations class; other instructors admit that applications are difficult to implement in the classroom due to time constraints or overall student interest. I argue that mathematics courses should include contextualized examples to ease the transfer of knowledge when students later apply differential equations in physical contexts. On the other hand, in applied courses, such as physics, it cannot be assumed that students already know and understand the mathematics. The transfer of knowledge from mathematics to a physics context does not ensure immediate recognition or comprehension from students.[11]

Applications remove ambiguity of general solutions by providing relevant initial and boundary conditions. All interviewed physics instructors agree that without these conditions, the differential equation solution methods and solutions would be physically irrelevant, just mathematics. I want to point out that although applications may be an effective tool to increase student understanding, the particular applications should be chosen carefully, especially in a mathematics course. For instance, electric potential alone can be a difficult subject for students in a physics classroom.[10] Therefore, it wouldn't be a strategic choice for a general application in a differential equations course which may have non-physics students. Introducing physical applications in a mathematics context however may help with students' transfer of knowledge from generic to context-specific differential equations in their respective areas of study [8, 11].

7.7 Hierarchy of Differential Equations

As more than one instructor mentioned, content in mathematics courses is sequenced so that new material builds off of previous content. I argue that differential equations is no exception. I hope that I've made it clear that much of differential equations can be simplified using an existing knowledge of calculus and algebra. A differential equations course should be structured to demonstrate a smooth interconnectedness between ideas. Consider the first-order linear homogeneous differential equation solution methods, once students understand how to the solve the homogeneous form, the non-homogeneous solution method (integrating factor and variation of parameters) uses the homogeneous solution techniques. The ratio-dependent differential equations as discussed in Section 5.2.2 are a special case of first order differential equations and adapt no new techniques outside of algebraic manipulation to get the change of variables desired. After the integrating factor is introduced, exact and non-exact equations can be implemented.

Theory is an integral part of differential equations. While I spent little time in this thesis proving theoretically why many of the techniques work for certain differential equations (*e.g.*, exact differential equations), ideas like existence and uniqueness are central to differential equations. Existence and uniqueness are especially important if there is no easily recognizable analytic solution or solution method to a given differential equation. The theory establishes the criteria based on which numerical methods approximate or even determine solutions. In Section 5.2.7 I discussed Euler's method, which uses Taylor series approximations to determine solutions. Other numerical methods include Fixed-Point method and Newton's method, which similarly rely on the idea of Picard iterates. Newton's method is sometimes taught/seen in the calculus sequence.

For second-order differential equations, there is an addendum to the theory for existence and uniqueness. Now there's the possibility for two solutions, which introduces the Wronskian to check for linear independence between solutions. Like with first-order, one starts with linear homogeneous differential equations, but more specifically, differential equations with constant coefficients. This introduces the characteristic equation, which determines the solution set for any linear homogeneous constant coefficient differential equation. Solutions to the non-homogeneous differential equations are the sum of two solutions to the homogeneous equation plus a particular solution to the non-homogeneous differential equation. When finding more than one solution to the non-homogeneous or homogeneous differential equation, the Wronskian can be used to determine whether or not the solutions are linearly independent.

The challenge arises in determining a particular solution to the non-homogeneous differential equation. Returning to a technique from first-order non-homogeneous differential equations, in Section 5.2.11, there is variation of parameters. The particular solution is determined using variation of parameters and depends on the Wronskian of the homogeneous solutions. In variation of parameters we also introduce the idea of reduction of order. Another effective method to determining a solution is guessing. Guessing relies on students' intuitions, which have hopefully developed thus far in the differential equations course. Guessing is particularly useful when the coefficients are no longer constant. Guessing leads to solutions in terms of polynomials, sines, cosines, as well as series/power series. It can be incredibly difficult to determine the solution to a non-homogeneous differential equation, but knowing how to guess a reasonable solution is an essential first step. If the non-homogeneous differential equation has a discontinuous function, the Laplace transform is an effective integration operator to convert differential equations into algebraic expressions to provide non-homogeneous solutions.

The last topic in my undergraduate differential equations course was systems of linear equations. By reduction of order, second-order differential equations can be broken down into a system of two first-order differential equations, which can be solved using either the typical eigenvalue method or the straight-line solution method[3], as discussed in Sections 5.2.15.1 and 5.2.15.2, respectively.

7.7.1 The Intellectual Pathway Through Differential Equations

Figure 7.1 is a map depicting the intellectual pathway for differential equations content in mathematics and physics courses, as presented in this thesis. Observe the interconnectedness of ideas and concepts while noting the way the various topics build off one another. Topics are connected by a sequence of arrows. The lighter shaded regions are the particular techniques used to solve the type of differential equation and the darker circled regions are applications attributed to the various differential equation types. The box in the upper-right is the portion of general topics not covered in my research that would expand that certainly belong in a more global map of differential equations. The map can be utilized as a guide for students and instructors to determine what prior mathematical content is required to problem solve later, possibly more complex concepts.

7.7.2 Higher-order, Non-linear, and Partial Differential Equations

This map above is only a fraction of the scope that differential equations encompass. Other areas of differential equations to consider are qualitative analysis of non-linear differential

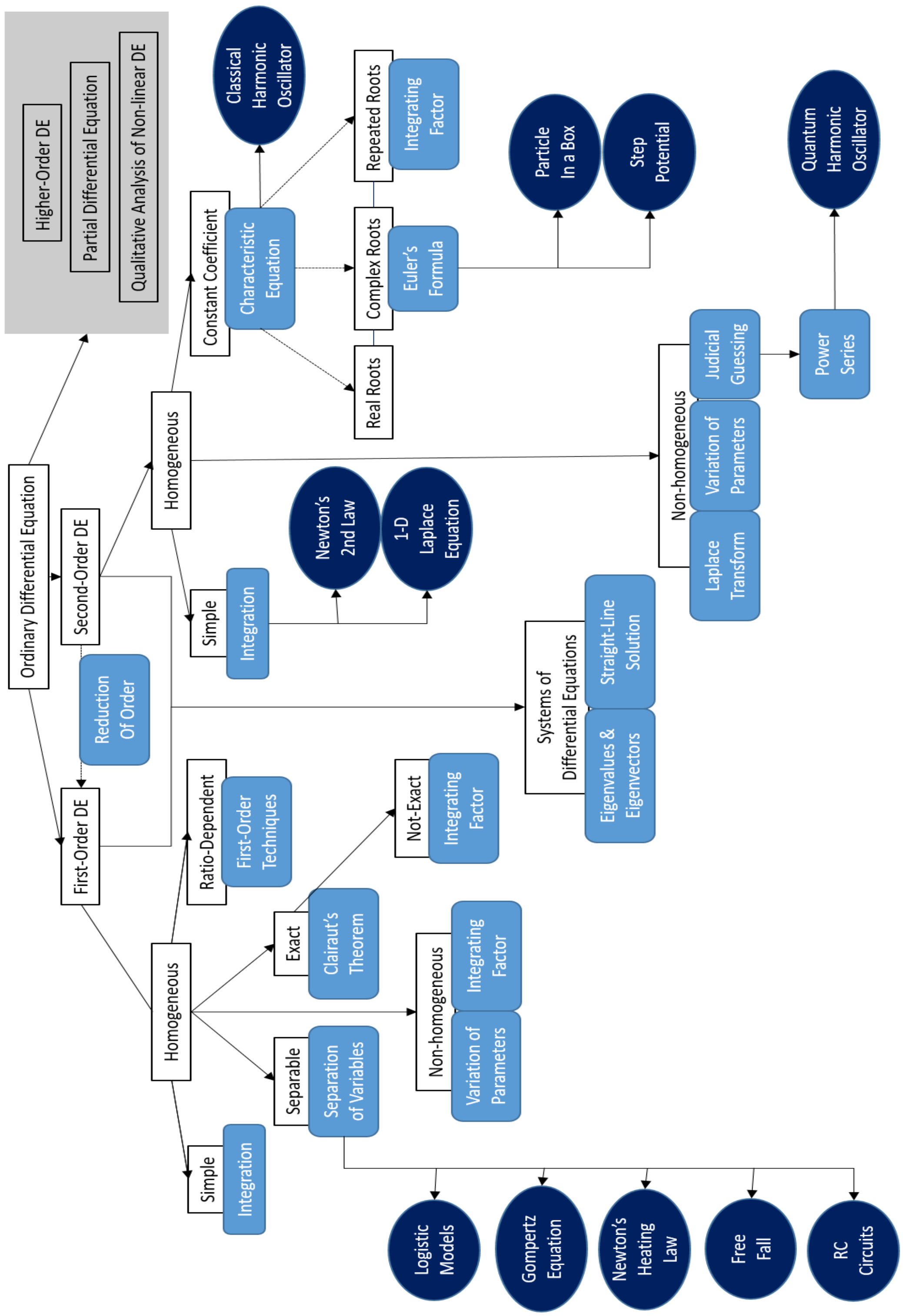

Figure 7.1: An Intellectual Pathway Through Differential Equations (The Map)

equations, partial differential equations and ordinary differential equations of higher order. Given equations of higher order, we utilize reduction-of-order techniques to reduce the differential equation to a form for which we can determine a solution. The characteristic equation works for all orders of constant coefficient differential equations. The difficulty comes in solving for roots of n^{th} degree polynomials.

Given a non-linear differential equation which cannot be solved using techniques discussed in this paper (e.g. separation of variables and exact equations), there exist different techniques. Certain non-linear differential equations must be linearized in order to use techniques we already know for solving linear differential equations, which is the primary focus of this thesis. Other non-trivially solvable non-linear systems of equations arise in conserved quantities, such as Hamiltonian and gradient systems, Lyapunov Functions, and bifurcation theory.

Partial differential equations rely on their ordinary differential equation counterparts. The second separation of variables technique requires guessing a functional solution that allows the partial differential equation to be separated into several ordinary differential equations and then solving these ordinary differential equations individually with appropriate boundary conditions.[10] Partial differential equations rely on ideas from Calculus III such as partial differentiation as well as integrating with respect to more than one variable. Partial differential equations describe many systems in physics, including higher dimensions of the Laplace equation, the hydrogen atom, as well as waves and thermodynamics. Partial differential equations rely on ideas from Calculus III such as partial differentiation as well as integrating with respect to two or more variables.

For future research, ideas from these different types of differential equations should be included to better demonstrate the complete interconnectedness of differential equation content for students and instructors alike.

References

[1] Rasmussen, C., & Wawro, M. (in press). Post-calculus research in undergraduate mathematics education. In J. Cai (Ed.), *The Compendium for Research in Mathematics Education.* Reston, VA: National Council of Teachers of Mathematics.

[2] Rasmussen, C.L. (2001) New directions in differential equations: A framework for interpreting students' understandings and difficulties. *Journal of Mathematical Behavior* (20) 55–87.

[3] Rasmussen, C., & Blumenfeld, H. (2007). Reinventing solutions to systems of linear differential equations: A case of emergent models involving analytic expressions. *The Journal of Mathematical Behavior*, 26(3), 195-210.

[4] Holton, D. (Ed.) (2001). The teaching and learning of mathematicsat university level. *Kluwer Academic Publishers.*

[5] Rasmussen, C., Kwon, O.N., Allen, K., Maroongelle, K., & Burtch, M. (2006). Capitalizing on Advances in Mathematics and K-12 Mathematics Education in Undergraduate Mathematics: An Inquiry-Oriented Approach to Differential Equations. *Asia Pacific Education Review*, 7(1), 85-93.

[6] Kuster, G. E. (2016). On the Role of Student Understanding of Function and Rate of Change in Learning Differential Equations. Unpublished doctoral dissertation, Virginia Polytechnic Institute and State University.

[7] Rasmussen, C., Stepahn, M., & Allen, K. (2004) Classroom mathematical practices and gesturing. *Journal of Mathematical Behavior* 23.

[8] Czocher E. A. (2017) How can emphasizing mathematical modeling principles benefit students in a traditionally taught differential equations course? *Journal of Mathematical Behavior* 45.

[9] Black, K. E., & Wittmann, M. C. (2015). Mathematical actions as procedural resources: An example from the separation of variables. *Physical Review Special Topics – Physics Education Research* 11, 020114.

[10] Wilcox, B. R. & Pollock, S. J. (2015). Upper-division student difficulties with separation of variables. *Physical Review Special Topics - Physics Education Research* 11, 020131.

[11] National Research Council, Committee on the Status, Contributions, and Future Directions of Discipline-Based Education Research. (2012). Discipline-Based Education Research: Understanding and Improving Learning in Undergraduate Science and Engineering. *Discipline-Based Education Research: Understanding and Improving Learning in Undergraduate Science and Engineering.* Washington, DC: The National Academies Press.

[12] Smith, T. I., Mountcastle, D. B., & Thompson, J. R. (2015). Student understanding of the Boltzmann factor. *Physical Review Special Topics – Physics Education Research* 11, 020123.

[13] Smith, T. I., Christensen, W. M., Mountcastle, D. B., & Thompson, J. R. (2015). Identifying student difficulties with heat engines, entropy, and the Carnot cycle. *Physical Review Special Topics – Physics Education Research* 11, 020116.

[14] Smith, T. I., Mountcastle, D. B., & Thompson, J. R. (2013). Student understanding of Taylor series expansions in statistical mechanics. *Physical Review Special Topics – Physics Education Research* 9, 020110.

[15] McDermott, L.C. (2001). Oersted Medal Lecture 2001: "Physics Education Research—The Key to Student Learning". *American Association of Physics Teachers* 69 (11).

[16] Yeatts F. R., & Hundhausen, J. R. (1992). Calculus and physics: Challenges at the interface. *American Journal of Physics* 60, 716.

[17] oton, A. (1983). Students' Understanding of Integration. *Educational Studies in Mathematics* 14 (1).

[18] Nguyen D. & Rebello N. S. (2011) Students' difficulties with integration in electricity. *Physical Review Special Topics – Physics Education Research* 7, 010113.

[19] Hall, Jr. W. L. *Student Misconceptions of the Language of Calculus: Definite and Indefinite Integrals*. Proceedings of the 13th Annual Conference on Research in Undergraduate Mathematics Education

[20] Sealey, V. (2006). Definite Integrals, Riemann Sums, and Area Under a Curve: What is Necessary and Sufficient? *PME-NA* 2 (46).

[21] Mestre, J. (Ed.) (2005). *Transfer of learning from a modern multidisciplinary perspective.* CT: Information Age Publishing.

Course Textbooks

[22] Braun, M. M. (1983). *Differential equations and their applications: An introduction to applied mathematics.* New York: Springer-Verlag.

[23] Lvin, S. (2014). *Introduction to Differential Equations and Linear Algebra.* Printed course notes, University of Maine.

[24] Griffiths, D. J. (2012). *Introduction to electrodynamics*. New York: Pearson.

[25] McIntyre, D. (2012). *Quantum Mechanics: A Paradigms Approach*. New York: Pearson.

Author's Biography

Brandon L. Clark was born in Lewiston, Maine on July 22, 1994. He was raised in Greene, Maine and graduated from Leavitt Area High School in 2013. Double majoring in Mathematics and Physics, Brandon is a member of Pi Mu Epsilon, Sigma Pi Sigma, and Phi Beta Kappa. He has received the Frank H. Todd and Frederick M. Viles '43 Scholarships.

Upon graduation, Brandon plans to continue his post-secondary education as a graduate student in a Mathematics Masters program at West Virginia University advancing toward a PhD in Mathematics Education.