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Investigating Student Understanding of Vector Calculus in Upper-Division Electricity and Magnetism: Construction and Determination of Differential Element in Non-Cartesian Coordinate Systems

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INVESTIGATING STUDENT UNDERSTANDING OF VECTOR CALCULUS IN UPPER-DIVISION ELECTRICITY AND MAGNETISM: CONSTRUCTION AND DETERMINATION OF DIFFERENTIAL ELEMENTS IN NON-CARTESIAN COORDINATE SYSTEMS

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An Abstract of the Dissertation Presented
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Differential length, area, and volume elements appear ubiquitously over the course of upper-division electricity and magnetism (E&M), used to sum the effects of or determine expressions for electric or magnetic fields. Given the plethora of tasks with spherical and cylindrical symmetry, non-Cartesian coordinates are commonly used, which include scaling factors as coefficients for the differential terms to account for the curvature of space. Furthermore, the application to vector fields means differential lengths and areas are vector quantities. So far, little of the education research in E&M has explored student understanding and construction of the non-Cartesian differential elements used in applications of vector calculus. This study contributes to the research base on the learning and teaching of these quantities.

Following course observations of junior-level E&M, targeted investigations were conducted to categorize student understanding of the properties of these differentials as

they are constructed in a coordinate system without a physics context and as they are determined within common physics tasks. In general, students did not have a strong understanding of the geometry of non-Cartesian coordinate systems. However, students who were able to construct differential area and volume elements as a product of differential lengths within a given coordinate system were more successful when applying vector calculus. The results of this study were used to develop preliminary instructional resources to aid in the teaching of this material.

Lastly, this dissertation presents a theoretical model developed within the context of this study to describe students' construction and interpretation of equations. The model joins existing theoretical frameworks: symbolic forms, used to describe students' representational understanding of the structure of the constructed equation; and conceptual blending, which has been used to describe the ways in which students combine mathematics and physics knowledge when problem solving. In addition to providing a coherent picture for how the students in this study connect contextual information to symbolic representations, this model is broadly applicable as an analytical lens and allows for a detailed reinterpretation of similar analyses using these frameworks.

DEDICATION

For the students who never give up

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CHAPTER 1

INTRODUCTION

"So let us then try to climb the mountain, not by stepping on what is below us, but to pull us up at what is above us, for my part at the stars."

-M.C. Escher

Those embarking on the endeavor of learning physics at any level, seeking to understand or shape the universe, are sure to find the strong mathematical undercurrents that influence reasoning, deepen understanding, and model the nature of physical systems. Modeling, in particular, is intricately tied to how physics is understood through conceptualizations of the underlying mathematics. In introductory physics, students from a variety of disciplines regularly engage with concepts of algebra and calculus. For those that advance further within a physics curriculum, the physics becomes more varied and sophisticated, and the associated mathematics follows suit: junior-level electricity and magnetism (E&M) involves vector calculus, vector differentials, and multivariable coordinate systems; upper-division thermodynamics includes manipulations of partial derivatives of multivariable functions of interdependent variables and Taylor series approximations; quantum mechanics incorporates many aspects of linear algebra with complex variables. Much of physics, especially at the upper division, exists at the interface of physics and mathematics.

For over thirty years, physics education research (PER) has been carrying out detailed scientific investigations of how students learn, understand, and apply physics concepts across various topics in introductory physics (see [1] for an overview). This work has included, but is certainly not limited to, an in-depth focus on introductory student understanding of mechanics [2,3], waves [4,5], and electric fields and circuits [6,7].

As the field progressed, more research began to explore students' conceptual understanding in upper-division physics courses, answering calls for more upper-division/interdisciplinary work [8,9]. Research at this level has included mechanics [10,11], electricity and magnetism (E&M) [12–16], quantum mechanics [17–20], and thermodynamics [21–23]. While much of the focus of PER has been an inquiry into the nature of students' conceptualization of physics, the caveat of working in upper-division physics is that both procedural and conceptual mathematics understandings are much more intricately tied to conceptual understanding than in some introductory courses. Given the ubiquity of mathematics in these upper-division courses, much of this work has involved specific investigation into student understanding of related mathematical topics [12,14,24–28].

Notably, there are many cases in which the mathematics instruction relevant to these courses occurs in the physics department before it appears in a standard mathematics course sequence. One solution to this involves many departments supplementing their undergraduate physics curriculum with a "mathematical methods for physics" course to teach the relevant aspects of a myriad of mathematical concepts and procedures (*e.g.*, complex variables, line integrals, diagonalization and change in basis, sequences and series, ordinary and partial differential equations), allowing upper-division content courses to focus on the physics and the ways in which the content incorporates the mathematics, rather than spending time developing the relevant mathematical formalism. This speaks largely to the importance placed on students' mathematical competence within the physics curriculum.

The incorporation of mathematics into physics extends beyond calculation, as mathematics plays a role in reasoning about relationships between physical quantities or the state of the system to depiction and conveyance of these relationships with graphs or equations. Several physics education researchers have sought to describe and represent the way students incorporate mathematical concepts and reasoning in physics (Fig. 1.1). An early instantiation separated the mathematics and physics domains into two distinct spaces that students cycled between: the physical system and mathematical representation [29]. Within this framing, "modeling" appears as the action that moves students from the physical system into a mathematical representation space (e.g., setting up an integral, abstracting a relationship between quantities). This representation is then processed within the mathematical domain (e.g., calculating an integral). Interpretation of this new representation brings one out of the physical system and back into the physics domain.

Uhden and colleagues developed a more sophisticated representation that considers a blended space of mathematics and physics [30]. Each level in this portrayal of the mathematics-physics interface represents a degree of mathematical modeling, which has also been referred to as mathematization. Moving up to a higher level corresponds to an abstraction from the physical system. As students model the physical system by defining proportionalities, writing equations to connect variables, or using various laws, theorems, or physics relationships, the level of mathematization increases. Interpretation of these

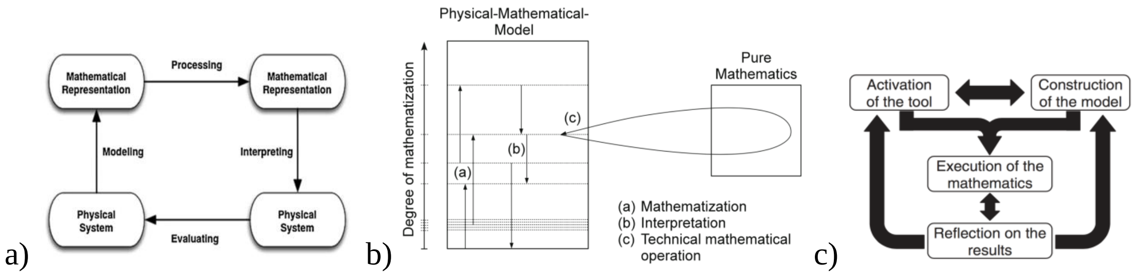

Figure 1.1. Models of mathematization in physics. (a) Model from Redish and Kuo [29]. (b) Model from Uhden and colleagues [30]. (c) Model from Wilcox and colleagues [31].

results corresponds to movement to a lower degree of mathematization. A third model of students' use of mathematics resulted as a framework from work in upper-division E&M [31]. The ACER framework designed a more student-centered script in which the arrows in the previous two diagrams are now where steps in problem solving occur. This framework designates spaces for the "activation of a tool" (tool referring to the choice of an equation), "construction of the model," "execution of mathematics," and "reflection on the results." While each diagram represents students' use of mathematics in a different way, they all include features to account for modeling, calculation, and interpretation.

For the purposes of this project, we explore students' mathematization in terms of their understanding and application of the underlying mathematics in upper-division electricity and magnetism, one course in particular where an understanding of physics is mediated by relevant and sophisticated mathematics. E&M is traditionally the first content course where students are reasoning with vector fields and using elements of vector calculus to develop and understand relationships between electrical charges, currents, electric and magnetic fields, and electric and magnetic potentials. Additionally, since many of the electric and magnetic fields have differing types of symmetry, students must often employ one of two multivariable non-Cartesian coordinate systems.

Recognition and use of symmetry often relieves the burden of heavy calculation, especially in relation to problems employing vector calculus. The caveat of curving coordinate planes to suit spherical and cylindrical symmetry, however, means that differential units take on scaling factors to account for the new mapping of three-dimensional space, rather than maintaining the standard form of dx for a change in the x-coordinate direction. Appendix A discusses the mathematics surrounding the three coordinate systems employed in E&M – Cartesian, spherical, and cylindrical – including the nature of the systems and how differential elements are constructed and appear in each.

Research on student understanding of mathematics in E&M found general student difficulties with setting up calculations, interpreting the results of calculations, and accounting for underlying spatial situations (symmetry) [12]. Other work, upon which this study was built, has explored students' applications of Gauss's and Ampère's Laws [12,15,16,24,32,33] or broadly addressed student understanding of integration and differentials [14,25,34]. Despite this, few studies have explored student understanding of differential line, area, or volume elements as they are constructed or determined in the non-Cartesian coordinate systems employed in E&M. Relevant literature and pertinent theoretical frameworks are discussed in Chapter 2.

Given the importance of the understanding and application of coordinate systems and differential vector elements to developing calculational proficiency and conceptual understanding throughout the whole of E&M, this research seeks to address the following questions:

- To what extent do students understand the multivariable coordinate systems used for vector calculus in E&M?

- In what ways do students build and/or make determinations about differential vector elements (*i.e.*, line, area, and volume elements) in these multivariable systems?
    - To what extent does student understanding of the symbolic expressions and conceptual aspects of differential vector elements, more specifically in non-Cartesian coordinate systems, impact element construction?

Offering qualitative answers to these questions begins to address student understanding of multivariable coordinate systems and construction of differential vector elements in E&M and sets the groundwork for future study. Additionally, the results of such an analysis can be used to inform the instruction of differential elements within generic coordinate systems and for particular physical symmetries.

The context of the research, methodologies, and discussions of applied frameworks is presented in Chapter 3. Chapter 4 describes preliminary investigations of the study relating to student performance and understanding of homework, quizzes, and tests given as part of regular course instruction. From this work, the author designed tasks to further probe student understanding of particular topics. Specific task design, implementation, and results related to student understanding of differential line elements are presented in Chapter 5*, while research related to differential volume and area elements is discussed in Chapters 6* and 7, respectively. Since student understanding of particular coordinate systems is often closely tied to their choices of differential elements, results of this nature are discussed across these chapters. One result of this work includes in-class group activities with explicit focus on methods for construction of length and area elements. These efforts are elaborated on in the relevant chapters. Further analysis of students'

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* Chapters 6 and 8 represented self-contained portions of this study and are included here as manuscripts in preparation for publication. A portion of Chapter 5 (5.1.5) is from a draft of a manuscript being submitted for publication.

construction of differential elements has led to the development of a model for students' construction of equations from the combination of two theoretical frameworks. This model and its affordances are detailed in Chapter 8*, and discussed in terms of the current literature utilizing these frameworks. Lastly, Chapter 9 presents conclusions and discussions of the topics, tying together specific themes found across investigations.

CHAPTER 2

REVIEW OF RELEVANT LITERATURE

“History is a race between education and catastrophe.”

–H. G. Wells

One area of focus in physics education research has been to understand the difficulties students have with the mathematics in upper-division electricity and magnetism courses (E&M). On a broader scale, research addressing student difficulties with mathematics in E&M has outlined several categories of difficulty including: (i) assessing the underlying physical symmetry, (ii) interpreting physical situations when setting up calculations, (iii) accessing the appropriate mathematical tools and (iv) interpreting results of calculation in terms of the given physical situation [12]. These difficulties spanned contexts from Gauss’s law to divergence and electric potential.

This project adds to this literature base by exploring student understanding of differential vector quantities that appear in numerous calculations in E&M. Understanding and applying a differential vector element in E&M involves a consolidation of understanding of differentials, an understanding coordinate system geometry, and an ability to interpret underlying symmetry. While little work has previously addressed differential lengths, areas, and volume elements, there have been studies addressing the three areas of differentials, coordinate systems, and underlying symmetries of E&M systems.

In an effort to gain a broader picture of what has been previously studied, the following sections address prior research. Section 2.1 addresses students’ attention to and understanding of underlying symmetry as part of applying Gauss’s law or Ampère's law, and as part of interpreting vector fields in terms of gradient, divergence, and curl. The literature in this section represents the majority of literature addressing student understanding of vector calculus topics.

Other research within E&M has attended to student understanding of integration and differential quantities (see section 2.2). This majority of this work has primarily dealt with one-dimensional systems or cases where the quantities being integrated are resistivity or capacitance. However, these works contribute to a larger body of literature which has addressed student understanding of differential quantities. Understanding this literature provides insight into the ways in which students within our study will likely approach integration or construction of differential quantities.

Lastly, this chapter presents research on student understanding of coordinate systems (section 2.3). As much of the literature regarding students' coordinate system understanding in E&M is subsumed with student application of symmetry in physical situations, little work has addressed students' general understanding and use of three-dimensional non-Cartesian coordinate systems. This section is thus supplemented with discussion of work addressing student use of polar coordinates to provide insight into how students in E&M will use and think about non-Cartesian coordinate systems.

2.1 Student difficulties with vector calculus in electricity and magnetism

When using vector calculus in E&M, the first step of problem solving involves recognizing the appropriate symmetry of vector fields. This has ramifications for choice of coordinate systems and associated differential elements. There has been considerable research addressing student understanding of symmetry in application of Gauss's law and

Ampère's law, two common vector calculus expressions [12,15,16,32,33]. The section further addresses student understanding of differential vector operators (gradient, divergence, and curl) where it is connected to interpretation of vector fields [35,36].

Manogue and colleagues highlighted several aspects of Ampère's law, an equation often used to solve for the magnetic field in highly symmetric situations that could be the source of students' difficulties [15]. Unfortunately, while a high degree of symmetry makes Ampère's law a viable solution pathway, the desired information (the magnetic field) is part of a dot product quantity comprising the integrand in a line integral. Thus students have to unpack the dot product and constancy of the field under integration to solve the given task to isolate the targeted magnetic field. The authors classify this as an inverse problem. Analyzing students' reasoning when solving Ampère's law problems, Wallace and Chasteen found that students often choose Ampèrian loops based on whether or not they enclose charge rather than on arguments of symmetry or the direction of the field, as one would expect of an expert physicist [16]. In particular, students had difficulty breaking the integration of Ampère's law into parts along rectangular paths. Both of these papers discuss issues of recognizing symmetry as a student difficulty in E&M.

Gauss's law, which often involves solving for the electric field from within an electric flux integral, is another example of an inverse problem, requiring students to make appropriate symmetry arguments based on the physical situation to solve for the electric field. Research on student understanding of Gauss's law has also found student difficulty with recognizing and appropriately applying symmetrical surfaces during problem solving [32,33]. The particular inverse nature of Ampère's and Gauss's laws is unique to how vector calculus is used in physics, but they are also pervasive and nearly ubiquitous in the E&M course. Additionally, students' attention to symmetry often requires them to utilize non-Cartesian symmetry when working in these cases.

Using questionnaires and interviews to highlight the similarity of student difficulties with the two laws, Guisasola and colleagues found that students tend to believe only the charges and currents enclosed by Gaussian surfaces or Ampèrian loops are responsible for the unknown fields [24]. They also found that students tend to conflate ideas related to fields with those related to the integral of fields, or fluxes. This finding is consistent with interview results, where students were asked to find the electric field for a point within a "non-uniform blob" of constant charge density [12]. Specifically in the context of electrostatics, Pepper and colleagues identified students equating the electric field (the integrand) with the electric flux (the integral). Students in these interviews also incorrectly attempted to use Gauss's law by drawing a Gaussian surface within the uneven shape and arguing that only the enclosed charge was responsible for creating the electric field at the desired point.

Other research has investigated student understanding related to vector differential operators (e.g., gradient, divergence, and curl) and how these properties connect to physical representations of vector fields. Students often responded to tasks with a belief that divergence was a property of a field, either zero or non-zero everywhere, rather than only true for points within a field [36]. Additionally they would connect spreading field lines to a positive divergence, even if no source was present within the defined field. Bollen and colleagues conducted further observations to probe students' conceptual meaning of the operators, interpretation of vector fields, and calculational proficiency related to the vector differential operators [37]. Utilizing the concept image framework [38] from research in undergraduate mathematics education, they found that very few students were able to evoke a complete or correct concept image, claiming “the divergence is a measure of how the field is changing” or “the gradient of A is the vector normal to the plane.” However, when it came to calculation, more than half of the students could solve for the correct expressions (allowing for minor errors). Thus students’ ability to carry out correct mathematical procedures was not an indicator of their sensemaking abilities, which is consistent research at the introductory level. Further work explored how students tied together the physical, mathematical, and conceptual understandings related to divergence of vector fields by utilizing conceptual blending [39]. Results showed that while multiple students were able to give appropriate descriptions of divergence and curl, they could not always link these understandings to graphical representations of fields. Despite relevant and correct elements being imported from the input spaces, incomplete or partial blending suggests a less robust understanding of the relationships between the mathematics and physics concepts. One student eventually recognized the need for an enclosed charge density (source of field) in order to measure a flux, but still struggled when connecting this idea to vector field diagrams, connecting positive flux to spreading lines within a region. This shows that improper blending of these conceptual and mathematical input spaces may be a source of student difficulties rather than lack of prior knowledge.

2.2 Student understanding of coordinate systems

The majority of research in E&M has addressed underlying symmetry as a means of choosing an appropriate coordinate system. This section attends to research of student understanding of coordinate system representation.

When addressing non-Cartesian coordinates, Dray and Manogue highlight a large concern as being the lack of standardization of polar, cylindrical, and spherical coordinates [40]. The presentation of non-Cartesian coordinate systems in most mathematics sequences begins with polar coordinates. Here, θ is used as the azimuthal angle (rotating about the z-axis) and r is used for the radial direction. When moving to a three-dimensional coordinate system, mathematics notation keeps θ as the azimuthal angle and uses ϕ as the polar angle (measured from the z-axis) and ρ for the three-dimensional radius. This constrasts with physics convention, which uses r for the three-dimensional radius and swaps the labels for the angles. While Dray and Manogue do not highlight any student work in particular, results from work published in 2010 on students' abilities to write $\vec{r}$ in spherical coordinates for six points, each located on a Cartesian axis, revealed this as an aspect of student difficulty [41]. Of the 28 volunteers, no student was able to correctly answer the original question by writing $\vec{r} = r\hat{r}$, and only slightly fewer than half of the students were able to identify the correct r, θ, and ϕ for each point. The most common mistakes were with the writing of the angles: 20% switched the values for θ and ϕ.

Sayre and Wittmann used Hammer's *resources* perspective [42] to analyze the plasticity of students' understanding of two-dimensional polar coordinates and Cartesian coordinates in sophomore mechanics [10]. The authors break down the *coordinate* *system* resource into groupings that describe general properties of coordinate systems, when to use a particular system, and the specifics of each system. The plasticity of a particular resource is determined by the number of connections to other resources and the durability of the internal structure. Students were asked to derive an equation of motion for a simple pendulum during the fourth and tenth week of the semester. Results show that while one student recognized the ease of applying polar coordinates, the second made an attempt to apply Cartesian in both cases. Thus, this work highlights how even after explicit instruction, students maintain a preference for Cartesian coordinates, even when another system may be easier.

Vega and colleagues further developed resources for unit vectors and coordinate systems from analysis of a task asking students to identify the direction of polar unit vectors on a spiral path [43]. They found students were conflicted between the use of a *position* resource, which determines $\hat{r}$ as away from the origin, or a *motion* resource, where the inward motion of the path cued students to think of the direction of $\hat{r}$ as toward the origin. Students had similar difficulty with $\hat{\theta}$, attempting to direct it tangent to the path or as a curling vector from the x-axis to $\vec{r}$ describing the point. This speaks to the difficulty for physics students in articulating the conventions of non-Cartesian coordinate systems where vector direction is a prominent piece of understanding, and further heralds the salience of path to students' choices of unit vectors and motion in the context of line integration.

Research in undergraduate mathematics education has predominantly addressed students' covariational understanding of functions plotted on polar coordinate grids [44–46]. While students were not seen to use Cartesian coordinates to make sense of how r

and θ changed together, researchers found students often treat these graphs as pictorial objects, rather than as relationship between two variables. Furthermore, students often had trouble determining properties of the function, such as whether it was a linear relationship. These students were identified as being unable to translate graph and function meanings rooted within a Cartesian coordinate system to a polar coordinate system in which shapes and representational conventions are changed.

2.3 Student difficulties with differentials and integration

As mentioned above, little work has previously addressed student understanding of differential length, area, and volume elements. While investigating various aspects of student understanding in E&M, Pepper and colleagues cited two mistakes with differential elements from observations in homework help sessions [12]. One group of students incorrectly wrote a spherical differential area as $da = d\theta d\phi,$ without the necessary scaling factor $r^2 \sin\theta$ to account for the curving of space in spherical coordinates. Another group used $dxdydz$ as a length element when calculating a line integral and became confused when recognizing that the result resembled a volumetric integral. These instances speak to the larger concerns of students' understanding of how differential elements are represented within coordinate systems, as well as difficulty with the dimensionality of differential elements.

The majority of research on mathematics in E&M has attended to various aspects of student understanding of integration, including how students think about differential quantities as they are used to set up integrals. Using the resources framework [42] and symbolic forms [47], Meredith and Marrongelle identified the cues that led students to integrate for a particular problem [48]. They found students were often cued to integrate based on recognizing similarity to other problems, recognizing the need to accumulate multiple parts, or seeing the dependence of one quantity on another. After adapting a concept image framework, Doughty and colleagues found that the recognition of dependence was the strongest cue for taking flux and surface integrals [14].

Nguyen and Rebello found that while students were able to recognize the need for integration, they had difficulty during computation due to an inability to interpret the physical meaning of symbols [34]. In particular, Nguyen & Rebello found that within the E&M context, the accumulation model of an integral, the adding up of parts of terms such as elements of charge, was more productive to students than area under the curve. They identified additional difficultly with discerning the meaning of the differential area element.

Hu and Rebello adapted conceptual blending to address students' mathematical understanding of integral and differential abstracted from the physics concepts and variables [25]. Here they identified the how understanding of the differential as a small amount or variable of integration affected a blended understanding within the context of physics. This expanded upon earlier studies identifying resources and conceptual metaphors students used for differentials in E&M. While it was common for students to treat the differential as a small amount or as a cue for procedural differentiation, in many cases, students interpreted the differential as an indicator of which variable to integrate with respect to. Notably, treating the differential as a variable of integration did not attach any further physical meaning to the differential for students. This disregard for the true meaning of the differential when performing integration is a common finding in literature

[28,34,49–51]. Very little work has addressed how these conceptions carry into understanding multivariable vector differentials.

Work within the mathematics community outside the context of E&M has looked at students' understanding of single and multivariable integrals. Similar to the finding of Nguyen and Rebello, in a comparison of mathematics and physics contexts, Jones found that an "adding up pieces" model of integration was more productive for solving physics problems than thinking of integrals in terms of areas or antiderivatives [52]. Generalizing to multivariable integrals, Jones and Dorko extended this work to categorize student conceptions of integrals of functions over two variables [53]. Rather than area under the curve from a Riemann sum interpretation, students invoked a volume under the plane representation where integrating involved adding up "rectangles," or sometimes accumulating an infinite number of slices or strips as they integrated along one of the axes in the *xy*-plane.

The ideas of symbolic forms were also used to interpret calculus students' ideas when making sense of integrals [50]; students' exposed conceptual understandings often included graphical representations of given functions.

Condensing the process of setting up a Riemann sum for definite integrals within a layers framework, Sealey identified four layers: *product, summation, limit,* and *function* [54]. Students were given problems with a physics context, such as the force water exerts on a dam, which involved elements of pressure and area. Sealey identified an *orienting* pre-layer to correspond to students' sense-making and construction of the integrand, $f(x)$, and Δx terms. Physics education researchers looking at integration in electricity and magnetism expanded the layers to include direct attention to the differential term dx, which is commonly used by physicists as an infintesimal Δx [27]. This additionally accounted for summing discrete tangible amounts of quantities such as charge (Fig. 2.1). Explicitly addressing the idea of the differential as a small physical quantity in physics, Roundy and colleagues expanded upon Zandieh's layers framework [55] to include other contexts that are important for physical scenarios (numerical, experimental) but aren't relevant in mathematics [56]. This connects the mathematical understanding of derivatives to the way derivatives are conceptualized in physics, specifically calculation and measurement as part of experimentation (Fig. 2.2). This adds the conceptualization of the derivative as a ratio of small changes.

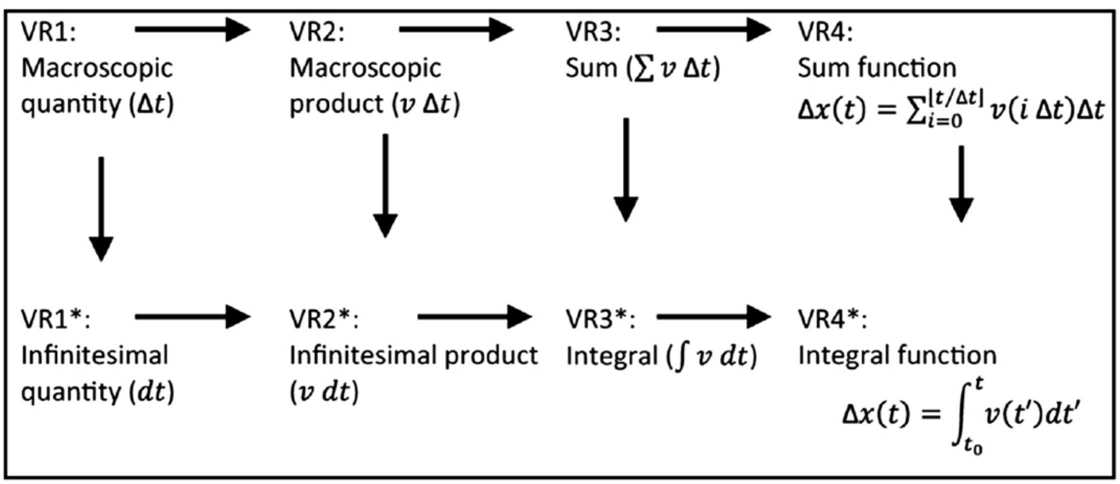


Figure 2.1. Extended layers framework of integration, representing possible routes for construction of the integral as a function. Image reproduced from Von Korff and Rebello [27].

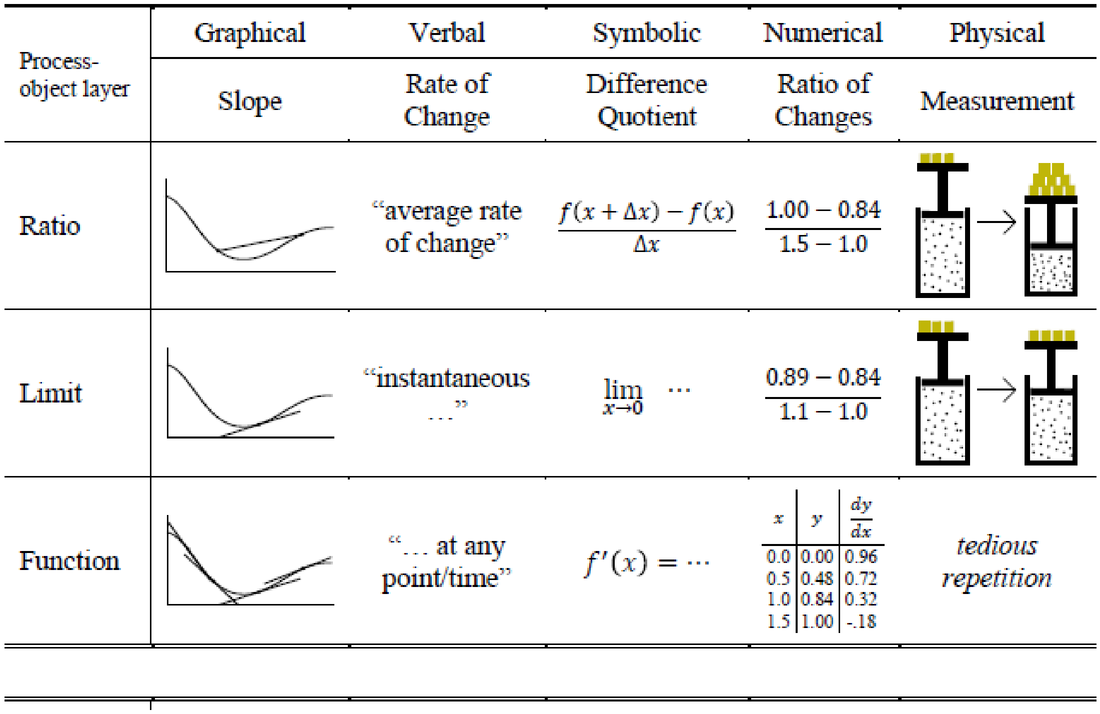


Figure 2.2. Extended layers framework for derivatives. The original process-objects layers, graphical, verbal, and symbolic [55] and two more columns, numerical and physical, to account for others uses of derivatives in physics. Image reproduced from Roundy and colleagues [56].

2.4 Summary and the gaps in the current literature on student understanding

The most common hindrances for students in upper-division E&M lie in relating conceptual physics understanding to mathematical argumentation and in articulating complex symmetry arguments relating to vector calculus. Work on integration and differentials has shown two predominant conceptions: the first almost inherently procedural, where the differential is merely a variable of integration; and the second where differentials are small quantities that are added in the context of integration. In the context of vector calculus, the literature speaks to difficulty with interpretation of vector fields, including the conflation of conceptual understanding of the field with the results of related integrals. These difficulties have regularly appeared in the contexts of Gauss's and Ampère's laws, which require complex symmetry arguments to determine the field within the integral and dot product, but have not branched into other aspects of vector calculus.

Despite the attention of much of the vector research to symmetry, there has been little to no work addressing student understanding of the specific differential line, area, and volume elements as they are constructed or determined in the non-Cartesian symmetries of E&M or interpreted as vector quantities. Investigations of student understanding of these topics are the focus of the following work.

CHAPTER 3

RESEARCH CONTEXT AND METHODOLOGY

"We can't understand what students are thinking unless we're doing the mental equivalent of bombarding them with high energy photons."

–Dr. Kevin Van De Bogart

Research on student understanding of vector calculus concepts occurred over a variety of courses, employing clinical interviews for further qualitative analysis exploring student understanding as they constructed or determined differential elements in multivariable coordinates.

At the outset of this project, data collection and analysis were focused on course observations to identify any specific difficulties [57] students encountered as they used vector calculus in non-Cartesian coordinate systems. This first phase of the project, which is detailed in Chapter 4, led to the development of targeted research questions, which then spurred further investigation during which specific interview protocols were designed to isolate student understanding around these difficulties. Student interviews conducted during the second and more extensive phase of this project compromise the main body of this work and are the data from which the larger conclusions are derived.

This chapter provides an overview of the courses studied (section 3.1), types of data collected (section 3.2), and analytical methodologies (section 3.3). The guiding theoretical frameworks, concept images [38] and symbolic forms [47], are elaborated upon in section 3.3, with particular focus on how they are used to gain insight into student work. The task-specific details pertaining to the particular style of interview, the specific population of students interviewed, and the guiding theoretical lens(es) for interpreting data are identified with the descriptions of each task (See Chapters 5-7).

3.1 Overview of relevant courses

In order to cover the breadth of vector calculus concepts, investigations and data collection were carried out over four courses at one university, University A. Three of these courses were physics courses, including both semesters of the two-semester sequence of Electricity and Magnetism (E&M), and one semester of Mathematical Methods in Physics. This study also involved course observations in a special topics course covering vector calculus topics in the mathematics and statistics department.

To supplement interview data, investigations also involved several interviews from the second semester of E&M at a second university, University B. While it is known that the course structure and sequencing within the curriculum is similar to that at University A, no formal course observation was conducted, so we omit further discussion of this course from this section. Both courses used Griffiths' *Introduction to Electrodynamics* textbook [58].

3.1.1 Electricity and Magnetism I

Within the physics curriculum at the University A, E&M I is the first course where students are introduced to a working understanding of spherical and cylindrical coordinates in the context of differential vector elements and unit vectors. As such this course is the primary source of data collection; extensive course observations were conducted here as well.

E&M I is typically taken in the fall of the junior year for three credits towards the physics major. Over the course of the project, enrollment ranged from 10-25 students, with the majority to all of students majoring in physics or engineering physics (approximately 50% were engineering physics majors). Occasionally graduate students are enrolled in the course for credit upon the discretion of the graduate coordinator.

The course uses a standard textbook: Griffiths, *Introduction to Electricity and Magnetism* [58]. The first chapter of the textbook is a review of mathematical content utilized throughout the rest of the book, including vector products, differential vector operators, vector calculus theorems, and coordinate systems. The course itself covers material starting in the second chapter of electrostatics up through Chapter 4, "Electric Fields in Matter," returning to the relevant mathematics as needed. Homework was generally assigned on a weekly basis and consisted of problems from the text. The course included two exams and a final. While the final was non-cumulative, the ideas within the course are continually built upon what is taught before (*i.e.,* calculation of electric field using methods from the beginning of the course, is relevant to problem solving of other quantities later in the course).

Spherical coordinates are introduced early and used for a couple weeks. Specific emphasis is given to the construction of the spherical differential length vector and students are quizzed on this coordinate system following instruction. Spherical coordinates are then used for Coulomb's Law ($\vec{E} = k\int dq(\vec{r}-\vec{r}')/(\vec{r}-\vec{r}')^{\frac{3}{2}}$), which represents a first-principles approach where the effects of differential charges, dq, are added over a given distribution. After several more classes, Gauss's Law ($\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enclosed}}{\epsilon_o}$) is introduced as a secondary approach for solving for the electric field when there is an appropriately symmetric charge distribution (i.e., constant or spherically/cylindrically symmetric). Cylindrical coordinates are introduced within this context.

3.1.2 Mathematical Methods for Physics

As is common practice in undergraduate physics programs, the physics curriculum at University A includes a Mathematical Methods course. The goal of a typical Mathematical Methods course is to prepare students with much of the sophisticated mathematical knowledge (conceptual and procedural) that goes into the teaching of the content in upper-division courses. Therefore, this course covers a wide variety of mathematical topics essential to upper-division content, including aspects of vector calculus and coordinate systems.

Mathematical Methods is taken for 3 credits and is a major requirement of the major. During the span of this project, the course was regularly offered in the fall of students' junior year as a co-requisite with E&M I and there is typically significant overlap between student populations.

The course textbook is standard and widely used: Boas, *Mathematical Methods in the Physical Sciences.* Coordinate systems and vector calculus concepts are taught in Chapters 5 and 6, respectively, and thus covered later in the semester. There are some differences here in representations when compared to the E&M text, which are discussed in Chapter 5 of this dissertation.

Given the timing of Mathematical Methods with respect to E&M I, which were taken during the same semester, the course content is covered asynchronously, with students having learned and used vector calculus and non-Cartesian coordinate systems for the better part of a semester in E&M I before the content is covered in Mathematical Methods. This, coupled with the overlap of students enrolled, made E&M I an appropriate environment for focus.

3.1.3 Electricity and Magnetism II

In the semester following E&M I, students typically enroll in E&M II for three credits as a requirement of the major. This course begins with Chapter 5 of Griffiths, "Magnetostatics." Course observations were conducted in E&M II, up through Chapter 7, "Electrodynamics." The remainder of the course focused on electromagnetic waves and involved little use of differential vector elements and non-Cartesian coordinate systems.

Course enrollment in E&M II typically mirrors that of E&M I, given the sequencing of the course, and also occasionally includes graduate students at the discretion of the graduate coordinator.

The introduction of magnetic fields and currents offers both new applications of vector calculus and different vector field symmetries, which affect the choice of coordinate systems and differential elements. As such, the course served as the primary data source for these topics.

3.1.4 Special Topics: Vector Calculus

Offered as an 400-level elective with the mathematics and statistics department, Vector Calculus is a three-credit course colloquially considered "Calculus IV" and typically offered during alternate fall semesters. At the time of course observations, nine students were enrolled in the class. There was no overlap between students taking this course and students enrolled in the targeted physics courses at the time of this study. However, there were some physics and engineering physics majors registered in the course. While Vector Calculus is not a major requirement for the physics, students wishing to complete a mathematics minor need only one additional mathematics course beyond what the physics department requires. Vector Calculus is a commonly considered an option given the ties to upper-division physics.

It should be noted that this course does not emphasize or explicitly teach the use of coordinate systems outside of the traditional Cartesian coordinates. However, the class does cover relevant mathematical concepts that are often expected to be in the repertoire of upper-division physics students: gradient, divergence, curl, and related theorems; motion along lines, and calculus of level surfaces, including multidimensional scalar functions and flux integrals. Due to the differences in the use of vector calculus discovered during course observations, no interviews were solicited from this population of students. Rather, I draw upon this course to illuminate differences in disciplinary conventions and practices given the specific mathematical focus of this project.

3.2 Data Sources and Collection

During the first phase of research, data collection involved extensive field notes from course observations in the E&M I and Vector Calculus courses described in the previous section. Analysis of these field notes and of the content presented by the textbooks provided a sense of what students are expected to be familiar with. Field notes were also taken in Vector Calculus to provide a record of the way vector calculus is presented as a topic in a mathematics course.

Students' homework, quizzes, and exams given as part of the regular course were collected and scanned for later data analysis before being graded by the instructor to allow for an unbiased analysis. The problems given to students as part of regular instruction provided a range of content that the students are expected to have learned and be familiar with.

During the second phase of this project, interviews were solicited from students in both E&M courses. Interviews provide more insight into student responses and choices when compared with written data because they offer a means to capture students' procedural and conceptual understanding and reasoning as they think about and solve tasks in physics, whereas written data only provides a final result with no opportunity to follow up in the moment and extract additional information from a student. Interviews conducted in the fall semester were solicited from students in E&M I after the relevant content on coordinate systems and vector calculus had been covered in the corequisite Mathematical Methods course. Interviews solicited during the spring semester were of the population in E&M II after students covered the relevant material through dynamic magnetic fields. Students took part in the interviews on a volunteer basis.

Clinical interviews [59] were conducted with tasks primarily designed around typical E&M problems, with the protocol targeting specific areas of interest, including coordinate systems and choice of differential elements, to draw out student understanding of associated concepts as well as the influence of physics context. Other questions were designed to have students work within a given coordinate system and construct the related differential elements. Detailed discussions of the design, solution, and target of each interview task are provided in the chapter in which the resulting data is discussed (Chapters 5-7). While some tasks involve students determining various vector calculus expressions from particular situations/geometries, there are no numerical calculations. This is typical of many E&M tasks that ask students to derive expressions for quantities. This design also allows us to track students' use and treatment of variables used during problem solving. The full suite of interview tasks is presented as part of Appendix B.

Interviews are particularly useful in exploring student understanding of mathematics in upper-division physics since for problems seen earlier in the course sequence, rote memorization may take the place of conceptual understanding without hampering students' ability to arrive at the correct answer. This is reminiscent of findings presented by Bollen and colleagues, in which students were able to correctly solve calculations involving differential vector operators but were unable to recall the meaning of the result conceptually [37].

Interviews were conducted using a think-aloud protocol [60–62]. As a part of this protocol, students are presented with a task and asked to work through the task while explaining their thought processes. The think-aloud nature allows the researcher to make the assumption that the student completely shares their thoughts while engaging with the task. However, the interviewer may prompt the student for explanations in the absence of a spontaneous response and may ask students to clarify statements or actions without affecting the students' line of reasoning.

For this particular study, student interviews were designed to involve the administration of three to four tasks over an hour. In practice, the length of a few interviews were shorter given the level of a student's familiarity and understanding of the material. Students are solicited after the material has been covered in the class in order to determine what was learned as a result of the typical course. Coupled with analysis of the field notes, this shows what specific concepts are difficult for students and need to be supplemented with additional instructional material. Analysis here also shows what ideas help students access requisite ideas and productively respond to tasks.

Pair interviews [60–62] were sometimes used to allow for a more authentic interaction and sharing of ideas between students with minimal influence from the interviewer. Students were paired for interviews primarily based on availability, but sometimes matched on course performance (strong, medium, weak) based on analysis of course observation data. Matching students by course performance kept strong students from overshadowing others who might have had more difficulty with course material. Pair interviews treat students as a unit within which information and understanding can be shared between students, consistent with a social constructivist perspective [63]. This style of interviews has been used extensively within physics education research, including studies on students' mathematical understanding (*e.g.,* [10,47,64]). For this project, pair interviews were used explicitly with some presumably difficult tasks or those being piloted to be incorporated with later instructional development, such as construction of a differential length element in an unconventional (made-up) spherical coordinate system, as the task was atypical and more to gauge structural understanding of coordinate systems (see Chapter 5). The vision for instructional development included tutorial design [65,66] which focuses on small-group work so students can share and build ideas together.

Individual interviews were preferred when the selected tasks more closely resembled problems seen in E&M. Here the emphasis of research is gaining a larger breadth of what individual students understand and what choices they make in regards to coordinate systems and differential elements. Interviewing students as individuals on these more procedural tasks allows for a greater number of responses and for subtle variations to be attributed to the individual student.

Interviews were videotaped and audio recorded. Transcripts were then created as a record of student interaction including relevant nonverbal aspects such as gestures, drawings, and written expressions. The analysis methodologies are described in the following section. As interviews are the primary source of data for this project, these methodologies and frameworks are given in more detail.

3.3 Analysis Methodologies

The data collected as part of this project are primarily qualitative as we are investigating and categorizing students' conceptual understanding as they reason about and construct differential vector elements. Furthermore, the limitation of working within upper-division courses is a small student population prevents large-scale quantitative analysis. Thus, instead of being able to make claims of the likelihood or frequency with which students have a certain idea, this work addresses the existence of common responses, understanding, and treatments of differential vector elements within and without physics contexts.

Student understanding is fundamentally approached from a constructivist perspective [67,68] in which the student is not a blank slate when solving a task, but instead continually builds upon their own experiences. In this case, as students encounter unfamiliar experiences, these new aspects are reconciled with previous understanding. Furthermore, the system in which construction occurs is subject to certain laws, transformations, and self-regulation [67]. A specific facet of constructivism includes reflective abstraction, in which meaning is learned by drawing out similarity in objects (i.e., learning the concept of red by being shown a red ball, red shirt, red block, etc.) [67]. In an integration context, students could learn the meaning of particular components of the definite integral by performing multiple integrations and recognizing the particular role of each component over multiple iterations in different calculations [69]. With this in mind, the goal of analysis using more targeted frameworks is to identify the understanding of target concepts that students have constructed as part of course instruction.

A first pass at analysis during both course observations and interviews used a modified grounded theory approach [70,71] with open coding to identify commonalities and trends in students' choices of coordinate systems and differential elements. Grounded theory focuses on categorization of what students are doing in response to a task. Codes evolved as data were interpreted and were combined along common themes. However, where pure grounded theory starts from a blank slate with no preconceptions of student understanding, the modified analysis was informed by relevant literature within the area of focus.

Initial analysis also grouped students based on specific difficulties [57], which represent incorrect or inappropriate ideas expressed by students. This method of analysis was used for written data given as part of course instruction, where only the students' final responses were able to be analyzed. By classifying these responses, common difficulties could be seen to emerge from the data, which suggested similar patterns of thinking exhibited by the students. Similarly, as some of the material, such as differential elements, is used progressively throughout the course, changes in student responses and use of differential elements from assignment to assignment could be tracked longitudinally through the term. This analysis draws a comparison of students' ideas within the context of this project to previous literature and contributes new findings to the current research base.

Beyond identification of student difficulties, data analysis of the interviews was informed by relevant approaches and frameworks already established in the literature at the interface between mathematics and physics, specifically *concept image* [38] and *symbolic forms* [47], which are outlined in the following subsections. These frameworks focus on identifying elements of students' conceptual and representational understanding as they work within a particular context and construct equations. Thus they provided suitable categorizations for qualitative analysis and address the research questions targeting student understanding and construction of differential vector elements.

3.3.1 Overview of the concept image framework and application

Similar to the use of resources [42] or knowledge-in-pieces [72] in physics education research, mathematics education research offers a broader frame for studying conceptual understanding through a *concept image* [38]. Originally developed as a way to examine student understanding of limits, a students' concept image is a multifaceted construct that represents a student's entire cognitive structure about a particular idea. This can include properties, processes, mental pictures and any other aspects of a concept a student may access. Unlike the *concept definition*, which accounts for formal textbook definitions and theorems, a concept image is a dynamic construct, in that it can accumulate additional ideas and interpretations from relevant contexts as new information is learned or old understanding is applied in new context. In many cases, a concept image can contain elements that are contradictory or false, much in the way a resources perspective can be productive or unproductive. The concept image framework was chosen over a resources approach to better enable comparison with the previous research in mathematics education and physics education in this domain.

By analyzing the *evoked concept image* that is elicited within a specific context, researchers can gain specific insights how students think about that concept. For example, an integration task may elicit one of several concept images, such as a Riemann sum or the area under the curve depending on the task being administered (Fig. 3.1) [14]. While particular concept images of integration may contain similar elements, knowing whether the students' evoked concept image is something reminiscent of Riemann sums or area under the curve tells one how students interpret particular problems. Likewise, it is also telling if a student's concept image for integration only involves procedural aspects, such that the integral of $nx^{n-1}dx$ is $x^n + C$, without being able to recall the specific meaning of the process. As per the concept image being a multifaceted construct, one student may express both of the above ideas given two distinct contexts (e.g., graph vs. formula).

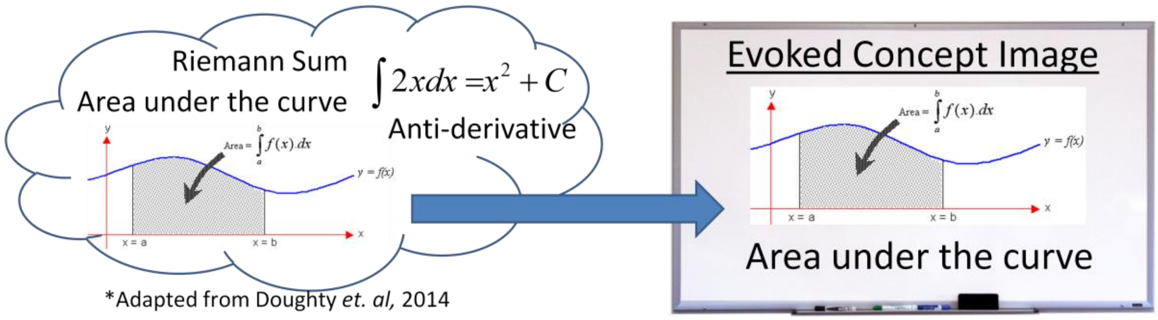


Figure 3.1. Diagram of an evoked concept image. Representation of students' evoking an *area under the curve* concept image for integration within a given context, despite having multiple other concept images for integration.

According to Tall and Vinner, a restricted concept image can develop when students work for long periods repeatedly applying a given conceptual idea in a formulaic manner. While students may initially be presented with the formal definition or other approaches, they may be unable to evoke a more appropriate concept image aspect when met with a broader context. For example, students regularly calculating derivatives of functions may dissociate *dy/dx* from a ratio of small changes, or how *y* changes with respect to a differential change along the x-axis .

The idea of concept image has recently been adopted by physics education researchers as a way to gain insight into student understanding of mathematics concepts in E&M, particularly in the context of integration [14] and vector differential operators [37].

The concept image framework [38] comprises the base of the theoretical analysis for this project addressing differential vector elements and non-Cartesian coordinate systems. Chapter 7 describes analyses identifying students' concept images of differential area elements when solving two physics tasks. In this case, I describe the different ways students treat or invoke differential areas when problem solving, similar to the example describing a student's invocation of Riemann Sums, area under the curve, or a rule/procedure.

A concept image analysis is also employed while analyzing students' construction of differential length vectors in an unconventional coordinate system (see Chapter 5). However, rather than address the treatment or invocation of differential length elements, the analysis of this chapter seeks to identify what common properties or ideas that students invoke during the construction with differential length vectors in non-Cartesian coordinate systems. This analysis then looks at how students use and make sense of these concept image aspects in the context of element construction in order to gain insight into students' understanding of the differential length vector and curvilinear coordinates as a whole.

Each property or associated idea was made a code as it was identified as being commonly used across multiple interviews. These were then refined through discussion and rereading of the interview transcripts.

3.3.2 Overview of the symbolic forms framework and application

Utilizing the perspective of the knowledge-in-pieces model, *symbolic forms* [47] identifies students' representational understanding of the structural components of equations as they construct and interpret expressions. Sherin's initial study involved interviews of students in a third-semester introductory physics course, in which students were provided with several word problems modeling physical situations common to introductory physics. The equations students constructed for given situations involved scalar quantities and the mathematics was limited to basic algebraic manipulation. Sherin found that rather than trying to derive an expression by manipulating known equations, students built or attempted to build equations from a sense of what they wanted the equation to express. The development of symbolic forms was driven by the analysis of student work within these interviews in an effort to provide a critical lens for the investigation of students' construction and sense-making of equations at the introductory level.

The specific nature of a symbolic form comes from the combination of a *symbol template* with a *conceptual schema*. A symbol template is an externalized structure of an equation. A student's conceptual schema is the intuitive internalized mathematical idea that the student expresses in the template.

One example of a symbol template is $\square + \square + \square$; each box represents one or more variables and/or numbers, depending on what a student deems fit. The template belongs to the *parts-of-a-whole* symbolic form, which has a conceptual schema in which parts of a substance or quantity are summed to contribute to the whole. This means that one term can change and would affect the whole but not necessarily the other parts.

Sherin defines the conceptual schemata as simple structures, similar to phenomenological primitives [72]. Furthermore, these schemata can vary for the same mathematical operation. One reason to add quantities is when the sum represents a whole quantity and each term in the addition – each box – represents one component of that quantity. For example, in the expression for the surface area of a cylinder of radius R and length L, there is an area term for the side ($2\pi RL$) and a term for the (two) ends ($2\pi R^2$). The symbolic form associated with this particular template-schema pair is known as *parts-of-a-whole* [47]. A student could also interpret the expression $v_o + at$ as an initial velocity quantity plus some increase or decrease depending on the acceleration. The schema behind this addition is identified as *base + change*, and has the associated template $\square + \Delta$ [47]. In short, the conceptual schema is what informs how students need to write particular expressions and accounts for their understanding of the template. The symbol template is then the manifestation of the conceptual schemata as a reified, or physicalized, symbolic pattern.

An understanding involving symbolic forms buys students the ability to "(a) construct expressions, (b) reconstruct partly remembered expressions, (c) judge the reasonableness of a derived expression, and (d) extract implications from a derived expression" [47] (pg. 499). In the knowledge-in-pieces tradition, the correctness of the equation is irrelevant. It is important to recognize that symbolic forms analysis only considers a structural understanding of the equations, as defined by Sherin, and not the context in which they are being used.

As such, symbolic forms analysis lives almost entirely in the structural realm of the equations; the conceptual schema is conceptual in the sense of justifying the mathematical operation, but not the conceptual understanding of the physical scenario that leads to it. In other words, symbolic forms were not developed to interpret student understanding of the physics represented by a particular equation.

This work utilizes a symbolic forms framework to give specific focus on the construction of differential elements. A symbolic forms approach allows the identification of the specific structures (symbol templates) students created as well as insight into the mathematical understanding that students attach to the structures (conceptual schemata) as they are combined to be represented in the final equation.

Symbolic forms were identified by isolating the smallest units of structure that students wrote during equation construction and by finding explicit attachment of that structure to students' mathematical understanding as expressed in the associated transcript.

Notably, a strict symbolic forms analysis neglects the content basis for choices, using only procedurally based mathematical justifications for the symbolic arrangements that indicate only that a student needs a particular structure in their expression. In these cases, the concept image analysis is used to provide a depiction of the content understanding connected to students' invocation of symbolic forms.

By combining these two frameworks for the study of various tasks involving students' construction of differential elements, analysis illustrates both the mathematical and physics understandings that go into students' construction of these expressions.

CHAPTER 4

PRELIMINARY INVESTIGATIONS

"Do what I do. Hold tight and pretend it's the plan!"

-The Doctor, *Season 7, Christmas Special*

In this chapter, I summarize preliminary findings and observations from the beginning phases of the project, specifically attending to student understanding of non-Cartesian coordinate systems and construction of the subsequent differential elements which become the focus of my later study. While this chapter does not represent a formal presentation of research, it provides the context to understand how material is presented to students and leads to the development of the research questions addressed in chapters 5-7.

Section 4.1 discusses course observations in the mathematics department to outline differences in disciplinary conventions within the departments at University A; the disciplinary differences represented here have been previously outlined in the literature [73,40]. I shed further light on these differences here to discuss the treatment of material at the institution at which this project was undertaken and as further evidence for why student understanding of the specific instantiation of vector calculus used in E&M is something to be studied by physicists and in physics classes. These differences show that it is in physics classes and not mathematics classes that students are learning the specific mathematics of vector calculus used to model E&M systems.

In section 4.2, there are informal discussions of both the course textbooks from E&M I and Mathematical Methods to give a sense of the basic treatment of non-Cartesian coordinate systems and differential elements in the two physics courses. Several discrepancies are presented here that suggest students are learning material in two different ways within the same curriculum structure. This further isolated E&M as the main course of study as this course is where the targeted content is first presented and given the most context.

Following this discussion, preliminary findings from course observations conducted in E&M I during the fall of 2015 are discussed in section 4.3 to provide background for the motivations of the larger project. Findings show that while students' performance with writing spherical differential elements improves over E&M I, they still have difficulties with element construction in cylindrical coordinates, even at the end of a full semester of E&M. This contradicts results of earlier course quizzes showing students were more successful when writing differential length vectors in cylindrical coordinates. The subsequent development of research questions and transition to the full study is synthesized in section 4.4.

4.1 Observations of the Vector Calculus course

As stated in Chapter 3, Vector Calculus is offered as an upper-division mathematics special topics course in alternating fall semesters. Course observations were conducted during the fall of 2015, concurrently with observations in E&M I. The class was lecture-based and was taught three times a week for 50 minutes.

The material covered for roughly the first month of the course included an introduction of vectors as quantities, using vectors to define a plane, and the conceptualization and calculation of vector products. Following this, students began to discuss curve parameterization and vector fields, which come into use later when calculating line integrals. This occurred well into the semester and notably, the material was taught in Cartesian coordinates. The course then covered area integrals, using scalar differential area elements and an $\hat{n}$ to specify the direction of a particular surface. Vector differential operators were also taught in Cartesian coordinates, with specific focus given to a conceptual understanding of what gradient, divergence, and curl mean. Lastly, tying all of these concepts together, the course covers vector calculus theorems (e.g., Divergence Theorem and Stoke's Theorem).

The presentation of line integrals using curve parameterization and the explicit use of Cartesian coordinates verify earlier claims of a "vector calculus gap" mentioned by Dray and Manogue [74,73], identifying these areas among the differences between the mathematics and physics disciplines. However, the use of Cartesian coordinates in a mathematics discipline makes sense, as the variables and unit vectors remain fixed and independent of position in space and since application of vector calculus here could be considered more universal (i.e., for any instantiation of line or surface), whereas cylindrical and spherical coordinates only make calculation "easier" for the specific subset of situations that are common to E&M.

In this case, while students often incorporate other calculus ideas that are taught within a mathematics curriculum, such as an understanding of differentials, derivatives, and integrals, the specific use of differential length and area vectors in spherical and cylindrical coordinates is something unique to a physics conceptualization of mathematics. This is at least the case at the institution in which this project was conducted, although given such publications as those identifying a "vector calculus gap," it is unlikely that this discrepancy is localized to a few departments.

As such, this project explores the specific instantiation of mathematics used in E&M, focusing investigations on the physics curriculum where vector calculus is almost uniquely applied to non-Cartesian coordinate systems; we also address at physics students' understanding of the mathematics as they connect the ideas to physics concepts.

4.2 Treatment of coordinate systems in physics course texts

In this section, there is discussion of the common course texts used within the physics curriculum used at University A, *Introduction to Electrodynamics* [58] and *Mathematical Methods in the Physical Sciences* [75], used for the E&M course sequence and mathematical methods course, respectively. This does not represent a formal analysis, but instead is provided as a context of how material is presented to students as part of course instruction and as a means to present differences in presentation between courses taught in a physics curriculum.

The first chapter of Griffiths, *Introduction to Electrodynamics,* [58] includes a plethora of mathematical background relevant to the student and learning of E&M concepts (e.g., vector analysis, integral calculus, vector fields, etc.). After presenting the Cartesian coordinate and unit vector transformations, the text gives the differential length component in each spherical direction and provides a depiction of the changes within the coordinate system (Fig. 4.1). However, the text does not explicitly connect these constructions to the ideas of arc length and projection that go into the component determination (see Appendix A). Next, the text presents the construction of a differential volume element as the product of three differential lengths and offers two examples of

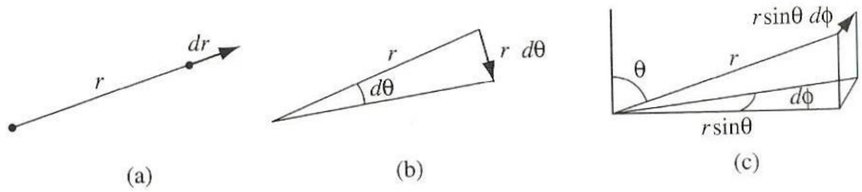


FIGURE 1.38

Figure 4.1. Construction of differential length components in spherical coordinates. A differential change in each variable produces a differential length component traced by the vector, $\vec{r}$. Image reproduced from E&M course text [58].

differential areas in spherical coordinates that result from a product of two differential lengths chosen based on analysis of the geometry (Fig. 4.2).

When it comes to a presentation of cylindrical coordinates, the text only provides the variable and unit vector transformations and a statement of the differential length vector and volume element. What is lacking in this section is a discussion of the differential area vectors offered in spherical coordinates. Arguably, the inclusion of the construction of differential areas in this system is of more importance given that each of the three differential areas in cylindrical coordinates is used in various E&M equations (Fig. 4.3). When students are first introduced to cylindrical coordinates in problem solving, it is

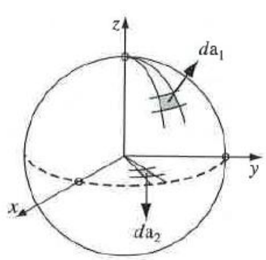
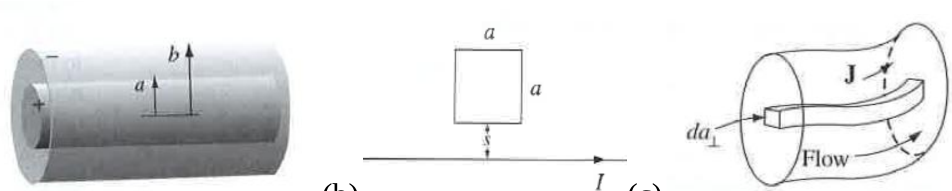


FIGURE 1.39

$$d\mathbf{a}_1 = dl_\theta \, dl_\phi \, \hat{\mathbf{r}} = r^2 \sin\theta \, d\theta \, d\phi \, \hat{\mathbf{r}}.$$

$$d\mathbf{a}_2 = dl_r \, dl_\phi \, \hat{\boldsymbol{\theta}} = r \, dr \, d\phi \, \hat{\boldsymbol{\theta}}.$$

Figure 4.2. Two differential areas in spherical coordinates. da_1 and da_2 depict differential areas for the surface of a sphere and one in the xy-plane, respectively. Each is constructed as a product of two differential length components representing changes in each of the angles. Image reproduced from E&M course text [58].



(a) (b) (c)

Figure 4.3. Images for tasks that use different cylindrical differential areas. (a) The curved cylindrical surface has a differential area of $sd\varphi dz\hat{s}$. (b) The curling magnetic field resulting from the current, I, dictates a differential area for the square of $dsdz\hat{\varphi}$. (c) Current can be determined from integrating the current density, J, through a cross sectional area. The differential area for a cylindrical wire is $sdsd\varphi\hat{z}$. Images reproduced from E&M course text [58].

in the context of Gauss's Law, where now the writing of the differential area is made superfluous by symmetry arguments (Figure 4.4). Upon further review, there is no example that involves writing a cylindrical differential area until current density is introduced, in the fifth chapter.

Mathematical Methods is the second place within a physics curriculum, at University A and many other universities, where students encounter non-Cartesian coordinate systems. In contrast to the E&M text, Boas [75] introduces the coordinate systems prior to the discussion of vector analysis. As such, the differential lengths and areas are presented as scalar quantities (Figure 4.5) due to their future representation in vector calculus. The differential length element is first

Now, symmetry dictates that **E** must point radially outward, so for the curved portion of the Gaussian cylinder we have:

$$\int \mathbf{E} \cdot d\mathbf{a} = \int |\mathbf{E}|\, da = |\mathbf{E}| \int da = |\mathbf{E}|\, 2\pi s l,$$

while the two ends contribute nothing (here **E** is perpendicular to $d\mathbf{a}$). Thus,

Figure 4.4. Example of application of Gauss's law within the course textbook. The symmetry of the problem means the differential area doesn't need to be written out if the final surface area of the shape is known. Image reproduced from E&M course text [58].

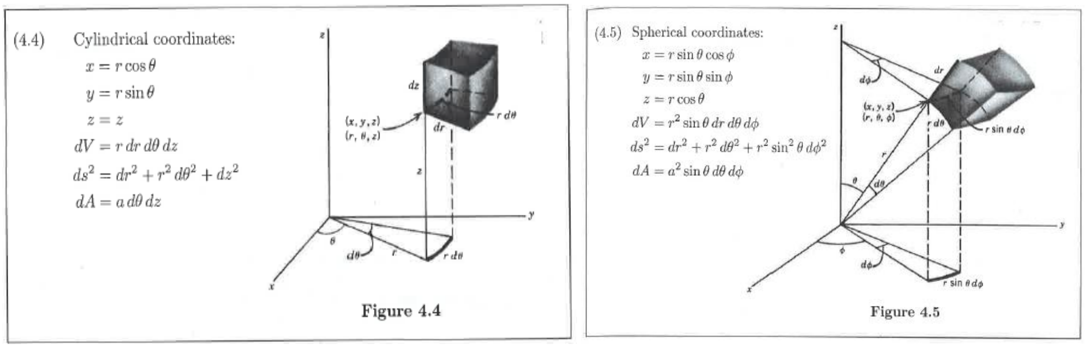


Figure 4.5. Comparative coordinate system in Mathematical Methods textbook, showing notational differences between variable use and representation as vectors. Image reproduced from [75].

defined as ds via a Pythagorean expression for Cartesian differentials as a ds^2.This is defined this way as an arc length for multivariable path integrals before the introduction of vector calculus in a later chapter. This particular representation obscures the underlying construction of the length components as vectors, which is how they are employed in E&M. In fact, the construction of the differential length vector in this manner is absent from the text.

Rather than building the length elements within the coordinate system as is done in Griffiths [58], Boas presents the Cartesian terms and determines the new coordinate differential elements via a Jacobian transformation rather than from the differential length elements [75]. Noticeably, the text presents a single differential area element for each coordinate system, where again, all three differential elements for cylindrical coordinates are eventually used.

This depiction of vector quantities as purely magnitudes extends to representation of integration. Integration involving the effects of vector fields over an area is presented in the typical mathematics fashion. Rather than embedding the unit vector in the differential area vector, $\overrightarrow{da}$, the unit vector describing any given surface of interest is represented as an independent part of the expression (e.g., $da\,\hat{n}$). Upon defining the surface, $\hat{n}$ is specified in a given coordinate system. This provides a distinct difference from Griffiths's treatment of differential area as a vector in its own right. In mathematics, this is a sensible representation as it accounts for any possible case. However in E&M, the high symmetry allows for the change in representation and the choice of one differential area element to represent a highly symmetric surface. Yet the conventional differences are, once again, another area to be on guard for student difficulties.

4.3 Course observation and preliminary data collection in E&M I

Observation in E&M I took place during the Fall 2015 semester. Class met twice a week for an hour and fifteen minutes. Information was generally presented to students via Power Point slides, but students were often sent to the board to work through problems in small groups. Extensive field notes were taken and all assignments were scanned before being graded by the instructor. This section addresses the presentation of non-Cartesian coordinates and differential elements, which subsequently became the focus of the project.

Spherical coordinates were introduced around the second week of class after time was spent familiarizing students with the concepts of electric fields. The introduction of spherical coordinates followed closely with the section of course text. Students were then shown an example of how spherical coordinates are applied in the context of Coulomb's Law. After the introduction of the spherical coordinate system, students were quizzed on a number of mathematical aspects presented so far. This included labeling the variables of a spherical coordinate system and writing the system's differential length vector. The results of this quiz are presented in section 4.3.1 as preliminary data.

Over the next several classes students continued to work with spherical coordinates as they used it to find the electric field due to spherical surfaces and volumes, as well as to construct vectors for the calculation of Coulomb's Law. In the third or fourth week, students were introduced to Gauss's Law and explicitly shown when and how to make the appropriate symmetry arguments to isolate and solve for the electric field. Students used this new solution method for earlier charge distributions having the appropriate symmetries as a way to show the relative ease of Gauss's Law compared to the more general Coulomb's Law (Fig. 4.6). Around this time cylindrical coordinates were formalized in accordance with the course text.

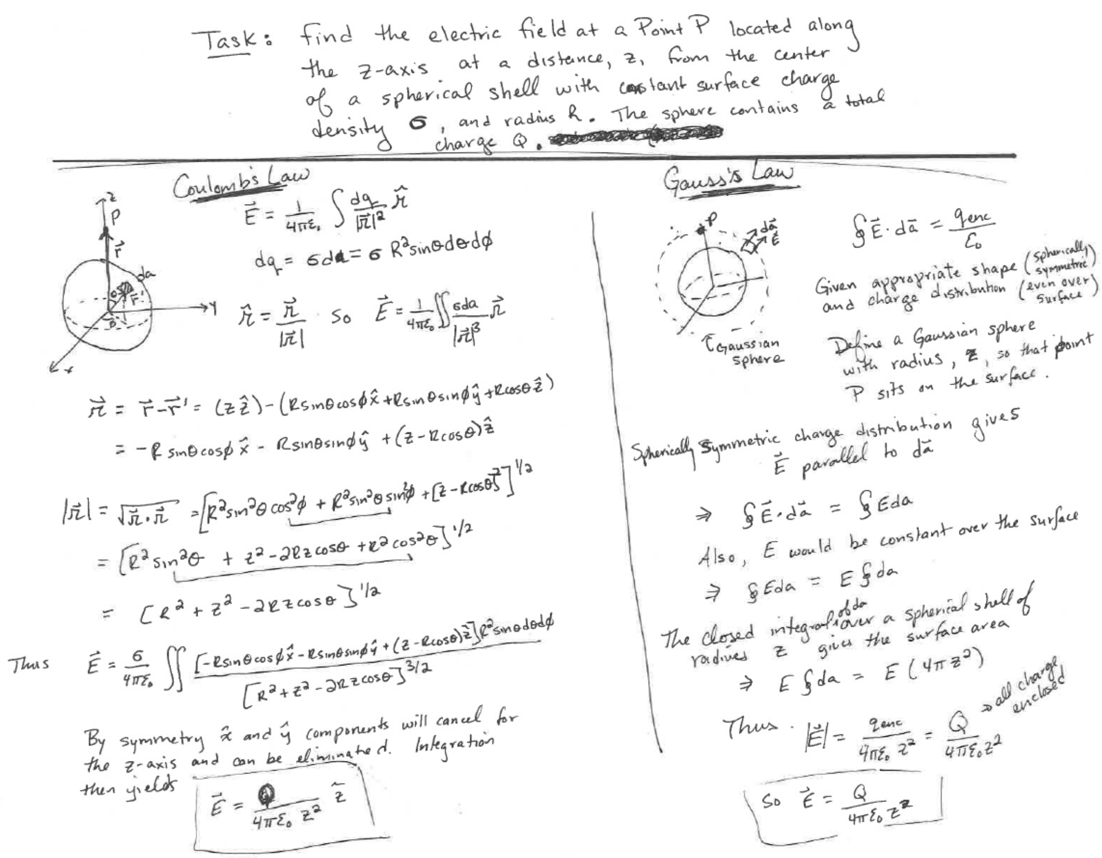


Figure 4.6. Comparison between application of Gauss's law and Coulomb's law. Coulomb's law involves several mathematical steps, vector decomposition, and symmetry argumentation. By comparison, Gauss's law is primarily solved using symmetry argumentation.

Following these classes, students were given a second quiz, as part of regular course instruction, which entailed students drawing a representation for Cartesian, cylindrical, and spherical coordinates as well as writing the differential length vector and differential volume elements for each. The results of the second quiz are presented in section 4.3.2 as preliminary data to compare with earlier results.

As the students progressed throughout the rest of the semester, they used the coordinate systems and differential elements in almost every problem given as homework or on an exam. Section 4.3.3 discusses an overview of students' written work throughout the semester in terms of expressing differential elements.

The results of these course observations and open coding of students' written work led to the development of the research questions presented again at the end of this chapter in section 4.4. As with the textbook review, the following sections do not represent a formal presentation of research but a background for the reader to provide the context of student understanding upon which this study was developed.

4.3.1 First quiz given on spherical coordinates and the differential length vector

The last question of the math quiz given after the introduction of spherical coordinates included a picture of the coordinate system as given in the text, with both the variables and unit vectors replaced with empty boxes. Students were asked to fill in each box with the appropriate coordinate variables or unit vector. Lastly students were asked to construct a generic differential length vector for the system.

Of the twenty-one students present for the quiz, only twelve correctly labeled the physics coordinate system variables (Fig. 4.7a). All but one of the remaining students used the mathematics representation where the angles are switchted (Fig. 4.7b). As the quizzes were returned, one student mentioned "Oh, I was confused with the way I learned it in math," referencing the differences in convention between the two disciplines. This has been identified as a possible obstacle to student learning in physics in the literature [40].

Student responses for the spherical differential length vector proved to be significantly variable, with only one student writing a correct vector. Most notable however was the initial disconnect between the dimensionality of terms and the number of components needed. Only about half of the class was able to write a term with the correct dimensions of length, while others were a mixture of lengths, areas, and volumes. Looking at the class as a whole nine students included multiple terms in their vector. Of these students, three constructed dl as a magnitude of Cartesian elements. This construction generally included the Cartesian-to-spherical transformations.

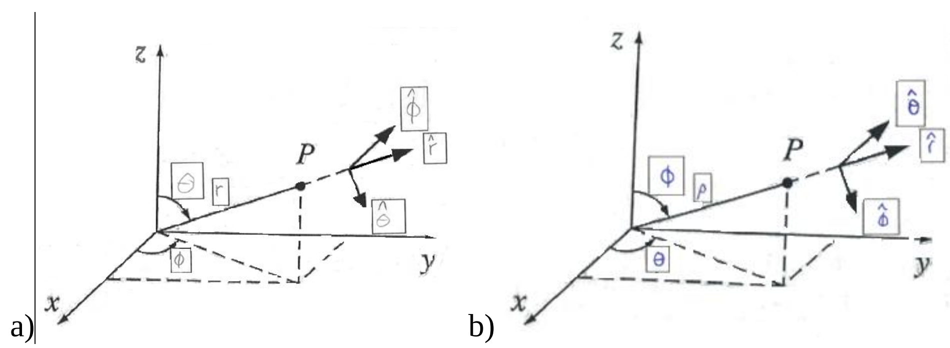


Figure 4.7. Two most common student responses for labeling spherical coordinates. (a) Correct (physics) spherical coordinate system representation. (b) Mathematics representation of the spherical coordinate system with swapped angles, and most common incorrect response on coordinate system quiz.

4.3.2 Second quiz given on coordinate system understanding and differential length construction

Two weeks after the initial quiz on spherical coordinates, students were given a second quiz in which they drew each coordinate system by hand and wrote differential length and volume elements. Results here show marked improvement on spherical coordinate notation, yet construction of length elements in any coordinate system, while better, still remained somewhat mixed.

Twenty-two students were present for the administration of the second quiz. Fifteen students correctly represented the coordinate angles (Fig. 4.8). Of the remaining students, only two used the conventional mathematics representation. Four students depicted the angle theta as being measured from the xy-plane (Fig. 4.8); this response arose only when the students had to draw the coordinate systems from scratch rather than just label the angles. All but three students correctly depicted cylindrical coordinates.

For the differential length vector, all but one student (who wrote a differential volume element instead) accounted for the fact that there needed to be three terms. Additionally, most attended correctly to the dimensionality of each term. However, only ten students (approximately half) had an appropriate differential length vector expression. Common difficulties included writing length elements with Cartesian unit vectors and attempting to make unnecessary projections to specific Cartesian axes while still using spherical unit vectors (Table 4.1). These mistakes suggest that students were uncomfortable working within spherical coordinates independent of Cartesian and are most likely trying to recall the decomposition of radial vectors when they had written "script-*r*" for spherical symmetry problems using the general method. Just as Sayre and Wittmann [10] have identified in sophomore-level mechanics, students have a strong preference for Cartesian coordinates, even after explicit instruction in problem solving.

When it came to writing a differential length vector within the cylindrical and Cartesian systems on the second quiz, we see students performing no better than with spherical coordinates. Almost half of the twenty-two students wrote the correct differential length element in Cartesian, with the most common difficulty being not including the differentials themselves. In cylindrical coordinates, only nine students could reproduce a representation of the coordinate system and write the differential length element correctly. This speaks to student difficulty solidifying these concepts as tools for future problem solving, even after practice with drawing coordinate systems and explicit instruction on constructing differential elements.

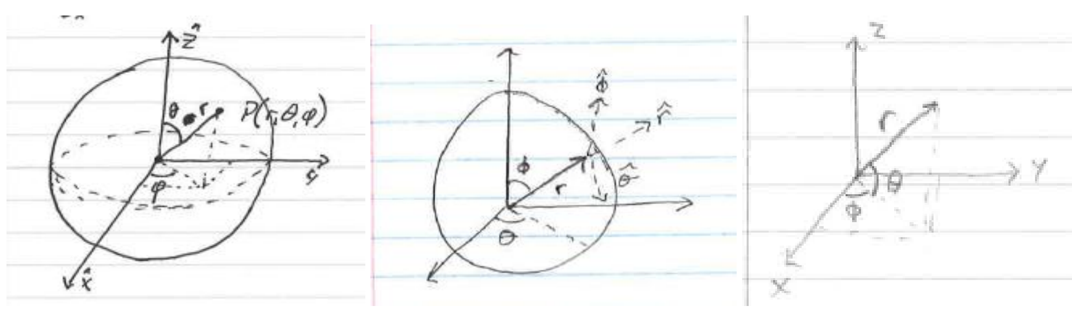


Figure 4.8. Sample student responses for depicting spherical coordinates on a later quiz.

Type of response	(# /22)	Example of student response
Correct	10	$d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta\, d\phi\hat{\phi}$
Partial axis projection	4	$rdr + r\cos\theta\, d\theta + r\sin\phi\, d\phi$
Cartesian elements	3	$d\vec{l} = dr\cos\theta\cos\phi\,\hat{x} + dr\cos\theta\sin\phi\,\hat{y} + dr\sin\theta\,\hat{z}$
Differential as a vector	1	$d\vec{l} = d\vec{r} + r\sin\theta\, d\vec{\phi} + rd\vec{\theta}$
Volume	1	$d\vec{l} = r^2\sin\theta\, drd\theta d\phi$
Angle confusion	1	$d\vec{l} = dr\hat{r} + rd\phi\hat{\phi} + r\sin\theta d\theta\hat{\theta}$
No projection	1	$d\vec{l} = d\vec{r} + rd\vec{\theta} + rd\vec{\phi}$
Only differentials	1	$d\vec{l} = dr\hat{r} + d\phi\hat{\phi} + d\theta\hat{\theta}$

Table 4.1. List of student responses for spherical differential length element on the second quiz

4.3.3 Student use of differential elements during problem solving throughout the remainder of the semester

Qualitative analysis of student homework and test data over the semester highlights an increased percentage of correct differential element use for spherical coordinates, with almost all students using the correct spherical volume and area elements by the end of the semester.

Correctness of students' cylindrical elements over the course of the semester also increased, but fewer students were able to write correct cylindrical elements when compared to student writing of correct spherical elements. On an early homework assignment, only nine of 22 students constructed a differential area element, while eight skipped the writing of the differential area, as is done in the example in the text (Fig. 4.9). On the first exam, fewer than half of students were able to write the correct cylindrical differential area when using Coulomb's law. The most common difficulties included writing only the differentials without the scaling factor(s) (e.g., $d\phi dz$) or writing the differential area for the end cap of the cylinder, $sdsd\phi$, when the problem needed the differential area for a curved shell, $sd\phi dz$. On a later problem, 17 of 21 students wrote the correct differential area element for spherical coordinates.

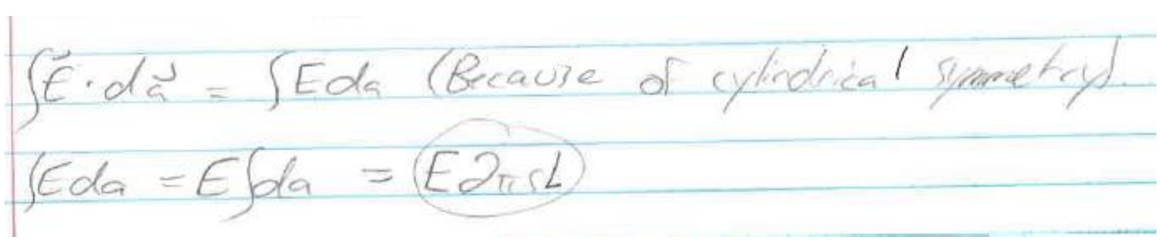


Figure 4.9. Student application of Gauss's law on a homework assignment. Here, one student bypasses the writing of the differential area by taking advantage of coordinate symmetry.

The percentages of students writing correct differential area elements remained low at the end of the semester, with only 60-75% of the students using the correct differential area across later homework assignments and tests. Notably, while working on homework, students could easily access the coordinate system information in the text. Despite this, they still underperformed on cylindrical coordinates: only 12 of 19 wrote the correct differential area on a later homework. The difficulty with cylindrical coordinates seen in the remainder of the semester contradicted the results of the second quiz on multivariable coordinate systems, which showed more students writing the correct cylindrical differential element. This juxtaposition, as well as the general difficulty students displayed with writing differential elements within non-Cartesian coordinate systems, motivated the development of the project.

4.4 Conclusions and Transition to Further Investigations

A review of courses in which vector calculus is taught and of common textbooks show a variety of differences in the way differential length, area, and volume elements are used and taught at University A, and likely other universities. The discussion presents several disparities in the language of vector calculus between mathematics and physics curricula and between physics courses themselves. Following the larger overview, the study focused on the E&M I course, where vector calculus concepts were most commonly being applied in a physics contexts.

After initial course observations in E&M I, students' facility with coordinate systems emerged as a particular area of interest. Cylindrical coordinates are arguably easier than spherical coordinates, given that there is still a single Cartesian component and thus only one angle to work with. However, answers to the quiz early in the semester coupled with the use of incorrect differential elements over the progression of the semester suggest that cylindrical coordinates are the harder system for students to use. One difficulty could come from the selection of appropriate differential area elements. While students typically only integrate over one area in spherical coordinates (the surface of the sphere at a fixed radius), there are three possible areas used when it comes to integration in a task with cylindrical symmetry.

As only a few students have been documented as using area elements for the wrong surface area, it seemed more likely that there was difficulty working within both systems. The supposed "ease" with spherical coordinates was then hypothesized to be due to the repeated use over similar tasks given over a long period. Repeated use would then lead to memorization of the elements abstracted from understanding, which is consistent with the development of a restricted concept image resulting from repetitive use of a formal definition [38]. A student having a restricted concept image is unable to work in a broader context (e.g., cylindrical coordinates), due to the focus on memorization. The suggestion is then that students have difficulty recognizing the origin of differential elements even in spherical coordinates.

At this stage the main research questions were determined. Following the subsequent immersion into the previously described theoretical frameworks, the questions were developed into the broadened ones described at the outset of this dissertation:

- To what extent do students understand the multivariable coordinate systems used for vector calculus in E&M?

- In what ways do students build and/or make determinations about differential vector elements (*i.e.*, line, area, and volume elements) in these multivariable systems?
  - To what extent does student understanding of the symbolic expressions and conceptual aspects of differential vector elements, more specifically in non-Cartesian coordinate systems, impact element construction?

Additionally, such questions marked a need to depart from the analysis of written data. Typically a solution to a vector calculus problem in E&M does not require students to express their reason for coordinate system choice or why a differential element is expressed in a particular way. While written data provide some idea of students' ability to arrive at the right answer, the quizzes and problems generated as part of the course were not optimized to extract student thinking about differential elements. Even more, students' use of coordinate systems and differential vector elements is often more peripheral to problem solving, as it is typically only a step in the process of a larger problem. Thus, interviews become the primary source of student data. Within an interview, students are given the space to discuss the motivations and underlying ideas that ultimately lead to a choice of coordinate system and the final representation of these differential elements.

Several interview tasks are outlined in the chapters that follow. These have been developed to further explore student understanding of these topics and answer the research questions.

CHAPTER 5

STUDENT CONSTRUCTION AND DETERMINATION OF DIFFERENTIAL LENGTH VECTORS

"Just when you think you know something, you have to look at it another way. Even though it may seem silly or wrong, you must try."

-Robin Williams, *Dead Poets Society*

For vector calculus use in E&M, the differential length vector, $d\vec{l}$, is a fundamental quantity in the sense that while it is used individually in problems for change in potential,

$$V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{l},$$

Ampère's law,

$$\mu_o I_{enclosed} = \oint \vec{B} \cdot d\vec{l},$$

or Biot-Savart's law,

$$\vec{B} = \frac{\mu_o}{4\pi} \oint \frac{I d\vec{l} \times \hat{r\prime}}{|r\prime|^2},$$

to name a few examples, the components of a differential length vector within a given coordinate system are used to determine the representation of differential areas depending on which variables are changing and which are held constant, as detailed in Appendix A. For example, multiplying $dl_\theta = r d\theta \hat{\theta}$ and $dl_\varphi = r \sin\theta \, d\varphi \hat{\varphi}$, two differential length components in spherical coordinates, yields the differential area for the surface of a sphere, a differential area commonly used in Coulomb's and Gauss's Laws. Problems necessitating non-Cartesian differential elements (as scalars or vectors) appear early in a typical E&M course text and are used consistently throughout (e.g., Griffiths [58]). Therefore, an understanding of the differential length vector across each coordinate system emerges as a fundamental mathematical construct in the application of vector calculus in our upper-division electricity and magnetism courses.

The determination of an appropriate differential length element for tasks that involve equations such as those above is predicated largely on two aspects: the relevant coordinate system and the direction of the associated field or targeted quantity. The relevant coordinate system selects the subset of differential length elements, typically expressed as the three-component differential length vector. The direction of the field or current then isolates the component needed for the integration as an application of the embedded vector product.

Given the importance of the differential length vector in problem solving and its use in determining the other differential elements, the following research questions were identified as areas for investigation:

- In what ways do students build and/or make determinations about differential vector elements (*i.e.*, line, area, and volume elements) in these multivariable systems?
    - To what extent does student understanding of the symbolic expressions and conceptual aspects of differential vector elements, more specifically in non-Cartesian coordinate systems, impact element construction?

In order to address these research questions, I discuss the analysis of data from two tasks in the following sections. The first provides students with an unconventional spherical coordinate system and asks them to construct a generic differential length vector (section 5.1). This allows us to isolate student understanding of the construction of these elements within curved space coordinates, providing a picture of student level of understanding of the origin of differential length components in each coordinate direction. The second task involves students solving for a change in potential over a spiral path (section 5.2). This was designed to provide insight into students' understanding of differential length vectors as part of problem solving within a physics context. Analysis of student understanding of differential length construction within and without context provides a larger picture of students' conceptual understanding of mathematics and how it is applied in physics, as well as help identify students' difficulties [57] and successes when employing non-Cartesian coordinate systems in problem solving.

Data analysis employs two theoretical perspectives: concept image [38] and symbolic forms [47]. The former addresses students conceptual understanding related to construction of differential length elements while the latter attends to the mathematical understanding of equation construction. As each framework addresses aspects of construction in a complementary fashion, results of this work led to theoretical development, fully detailed in Chapter 8, which ties the individual analyses together using a conceptual blending framework [76].

5.1 Construction of a differential length element in an unconventional spherical system

5.1.1 Research task design

In order to investigate student understanding of how differential vector elements are constructed in non-Cartesian coordinate systems, I developed an interview task based on an unconventional spherical coordinate system (Fig. 5.1, Appendix B1). As part of the task, students were asked to conclude whether the system was feasible, and to build and verify the differential line and volume elements. The goal of using an unconventional coordinate system are to be able to determine students' abilities to work with the underlying conceptual ideas, rather than their ability to recall a memorized answer.

The unconventional system, which I will hence call "schmerical coordinates," is designed with several features to distinguish it from traditional spherical coordinates. Firstly, it is a left-handed coordinate system, with the x- and y-axes swapped from their usual representations. The left-handed nature allows us to determine if any Cartesian elements presented by students are the result of recall or accurate (but unnecessary) projections within the Cartesian system. The swapped location of Cartesian axes also means that the polar angle, β, is placed differently than the analogous ϕ in spherical coordinates. This shift, however, does not impact the expression for the length element.

Likewise, the placement of the polar angle α is different than that of θ. However, this change in coordinate representation does influence the expression for the differential length. As discussed in the mathematical background (Appendix A), the $\sin\theta$ in the $\hat{\phi}$-component results from a projection of the radial vector into the xy-plane. This projection

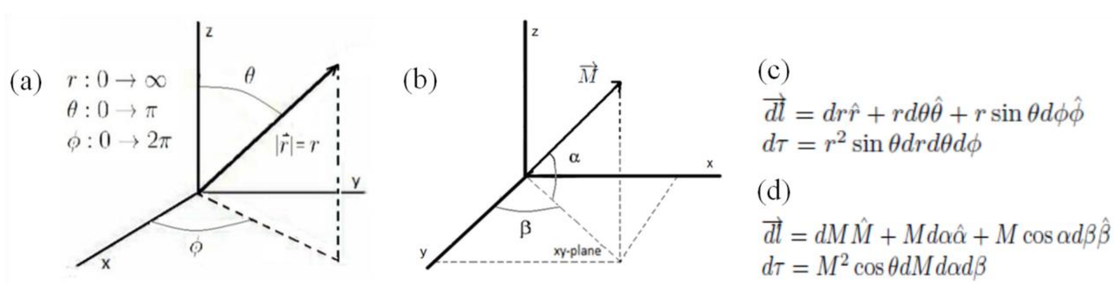


Figure 5.1. Comparison of spherical coordinates and unconventional system given to students. (a) Conventional (physics) spherical coordinates; (b) an unconventional spherical coordinate system ("schmerical coordinates") given to students, for which they were to construct differential length and volume elements. The correct elements for each system are in (c) and (d), respectively.

is the radius used in the definition of the differential arc length for a differential change in angle, $d\phi$. Thus within schmerical coordinates, the $\cos\alpha$ term is needed to describe the new differential length component.

The variation in the placement of the angles from spherical coordinates sought to require students to critically assess and employ the various techniques of building differential elements.

5.1.2 Methodology for analysis of the schmerical task

Clinical think-aloud interviews were conducted at two (public) universities with students enrolled in junior-level E&M. Both universities teach E&M as a two-semester sequence following the same textbook [58]. Four pairs of students (N=8) were interviewed at University A at the end of the first semester and two pairs and a single student (N=5) at University B at the beginning of the second semester of E&M. As described in section 3.3, pair interviews facilitated more authentic student discussion and allowed them access to each other's conceptual understanding, thus minimizing the input and influence of the interviewer. In some cases, it may be noted where ideas are introduced by one student and not understood by the other. However, in general, knowledge is treated as belonging to the pair as a whole. Groups are identified as AB, CD, EF, and GH for the first university and PQ, RS, and T for the second. These identifiers signify pairings of students with pseudonyms Adam and Bart, Carol and Dan, etc. The interview population included two graduate students, Adam and Bart, who were enrolled in the course for credit.

Each interview was videotaped and transcribed; transcriptions at University A were analyzed to compile elements of students' evoked concept images [38] of the differential length vector (see section 3.3.1 for overview of concept image framework and methodology). Elements were identified as belonging to a concept image of a differential length vector if they appeared across multiple groups (productively or unproductively) and were used by students to construct some aspect the differential length vector expressions. Once aspects of the concept image were identified, the data were re-examined to determine the order and/or grouping of these ideas over the course of the task. The specific ordering of ideas is described later in this chapter.

Analysis of transcripts from University B did not involve a progression or grouping of concept image aspects, as these interviews were performed just over a year after those at University A and because students at University B had greater difficulty with the task, relying more heavily on aspects of recall and less on aspects of construction.

In order to provide a larger picture of students' understanding of the mathematical representation of the differential length vector, students' expressions was analyzed throughout the stages of construction to identify uniform templates that might be connected to symbolic forms [47], either those identified by Sherin [47] or new forms specific to this context (see section 3.3.2 for overview of symbolic forms framework and methodology). Symbolic forms were identified as invoked by pairs if students included the template within their expression and discussed some level of mathematical justification for the structuring of that part of the expression in that way.

All transcripts were analyzed for students' invocation of symbolic forms and the concept images associated with the moments focused on construction.

Analysis of evoked concept images in the unconventional coordinate system allowed us to develop a clearer picture of student understanding, as well as to identify specific student difficulties [57] and successes when working with coordinate systems that they apply to particular problems throughout the semester.

5.1.3 Overview of Results

All students were able to complete the first aspect of the task, which discusses the feasibility or validity of the system. Each group identified schmerical coordinates as similar to spherical coordinates and at least one of the required properties of a coordinate system (e.g., span all space, unique mapping to points). Upon recognizing that α and θ covered the same range of π radians, students easily claimed similarity between the two systems. As such, the students were able to recognize schmerical coordinates as a non-Cartesian coordinate system; they were then asked to construct a differential length element for the unconventional system.

The remainder of this section focuses only on the analysis of the initial stages of construction of the differential length element.

Upon completing their first attempt at constructing a differential length vector, prior to being asked to construct a differential volume, no group was able to construct an appropriate expression due to inattention or misapplication of certain ideas such as arc length or dimensionality.

Three pairs of students at University A (AB, CD, EF) explicitly discussed their construction during the interview, elaborating on their choices of how they structured the equation and their inclusion or exclusion of certain terms, while others used recall from other coordinate systems. Due to the focus of the research questions on students' conceptual ideas associated with construction, the data corpus presented here is primarily derived from these three groups that attend to the properties needed to build the differential element. However, common elements of reasoning did appear for other groups as they constructed terms, and thus these interviews provide additional supporting data to the existence of particular concept images and symbolic forms.

Despite emphasis on construction, none of the initial three groups constructed a correct differential length vector: they either included a $\sin\theta$ following mapping to spherical coordinates (AB, CD), or excluded the trigonometric function altogether (EF). Students in the remaining groups had more significant difficulty reasoning about the construction of the differential length vector, despite being able to connect the unfamiliar system with spherical coordinates. PQ, as well as the fourth group at the first university, GH, relied on recall. In these interviews students spent a significant amount of time trying to remember the forms of equations learned in classes. Both groups ended up working within the structure of a recalled Cartesian differential length. The other two groups had difficulty with ideas of arc length or failed to recognize the need to express multiple components.

The remainder of this section will present the findings of both the concept image analysis at University A (section 5.1.4) and the symbolic forms analysis. The symbolic forms analysis is accompanied with a discussion of the concept image aspects that warranted the inclusion of a particular template.

5.1.4 Results of Concept Image Analysis

Analysis of students' concept images allowed us to identify four particular aspects that students commonly associated with the construction of a differential element as part of our interviews. Table 5.1 defines each aspect and provides an example of how students attended to and drew upon these aspects during construction. Elements were identified as belonging to a concept image of a differential length vector if they appeared across multiple groups and were used by students to construct some aspect the differential length vector expressions.

The *component and direction* aspect involved students' attention to the summation of three different components as well as the idea that each component of the vector equation is displaced independently. Many of the students placed emphasis on the aspects of *dimensionality,* specifically attending to the need of each component to have units of length. Students used the aspect of *differential* to talk about needing small displacements

Concept Image Aspect	Specific Idea	Example (in bold)
Component & Direction	Recognition of multiple components, each displaced independently	Frank: Yeah, so like there, dl, there are **three different $\boldsymbol{dl}$'s**. There is dl with respect to M, dl with respect to a, α, and dl with respect to β…
Dimensionality	Each term needs units of length	Adam:… This **doesn't have any units of length**…so, it needs to have **some $\boldsymbol{M}$ term**.
Differential	Small changes (of displacements)	Carol: Right. So you have **a change in your $\boldsymbol{\hat{M}}$** is going to be your **$\boldsymbol{dM}$, it's your change in your $\boldsymbol{M}$**.
Projection	Use of cosine/sine explicitly (not rote recall)	Elliot: …but if we're pointed way up here, then **we need to take the cosine** so that we're, we have the **component of $\boldsymbol{r}$ that is actually in the $\boldsymbol{\beta}$ plane**.

Table 5.1. Aspects of students' concept image of a differential length vector in a non-Cartesian coordinate system.

or changes in specific directions. Due to the nature of the coordinate system, the aspect of *projection* (obtaining a component of a vector in a particular plane) is relevant to appropriately explain the need for a $\cos\alpha$ in the β-component. However, many students did not apply this last aspect to their construction.

In a number of groups, emphasis was put on matching terms to differential elements in known coordinate systems. Because of the variability in student responses, analysis needed to expand beyond identifying only the properties that students associated as belonging to the differential length vector. In addition to identifying necessary concepts for building, there were several actions that students took during the interviews: *rote recall* of length elements from other systems*; mapping* of the variables to spherical or Cartesian coordinates; and *grouping* of elements, typically based on variable (Table 5.2). Actions are distinguished from aspects in that, while they are still seen commonly across groups, they are not properties students associated with the differential length vector. Instead, an action is defined as something students did during construction as a means to

Construction action	Specific Idea	Example (in bold)
Grouping	Combining elements by like variables or terms	Harold: You've got $rdr\,\hat{r}$ plus, **is it $\sin\boldsymbol{\theta}\, \boldsymbol{d\theta}$ or is there an $\boldsymbol{r}$ in there?**
Rote Recall	Writing or remembering elements from Cartesian or spherical coordinate systems	Greg: **dτ in spherical** is $r^2\ sin\theta$=... =$d\theta\ dr$=...=$d\varphi$.
Transliteration	Direct matching of variables from existing coordinate system	Bart:...so now we have just to compare so we have **$\boldsymbol{r}$ it is $\boldsymbol{M}$, $\boldsymbol{\theta}$ is $\boldsymbol{\alpha}$=...=$\boldsymbol{\varphi}$ is $\boldsymbol{\beta}$.**

Table 5.2. Actions taken by students during construction of a differential length vector for schmerical coordinates.

build and understand components. *Grouping* as we identify it here is distinguishable from the grouping resource identified by Wittmann and Black [64], where terms in a differential equation are combined into a single combined term.

In order to illustrate what concept image aspects and building actions students invoked as they progressed through construction of the various differential components, flow charts were designed for the analysis of the order in which concept image aspects appeared and were connected for students. (e.g., Fig. 5.2). These flow charts further allowed for a juxtaposition of construction from conceptual ideas with the use of recall to determine the schmerical length element. The use of these diagrams also aided the

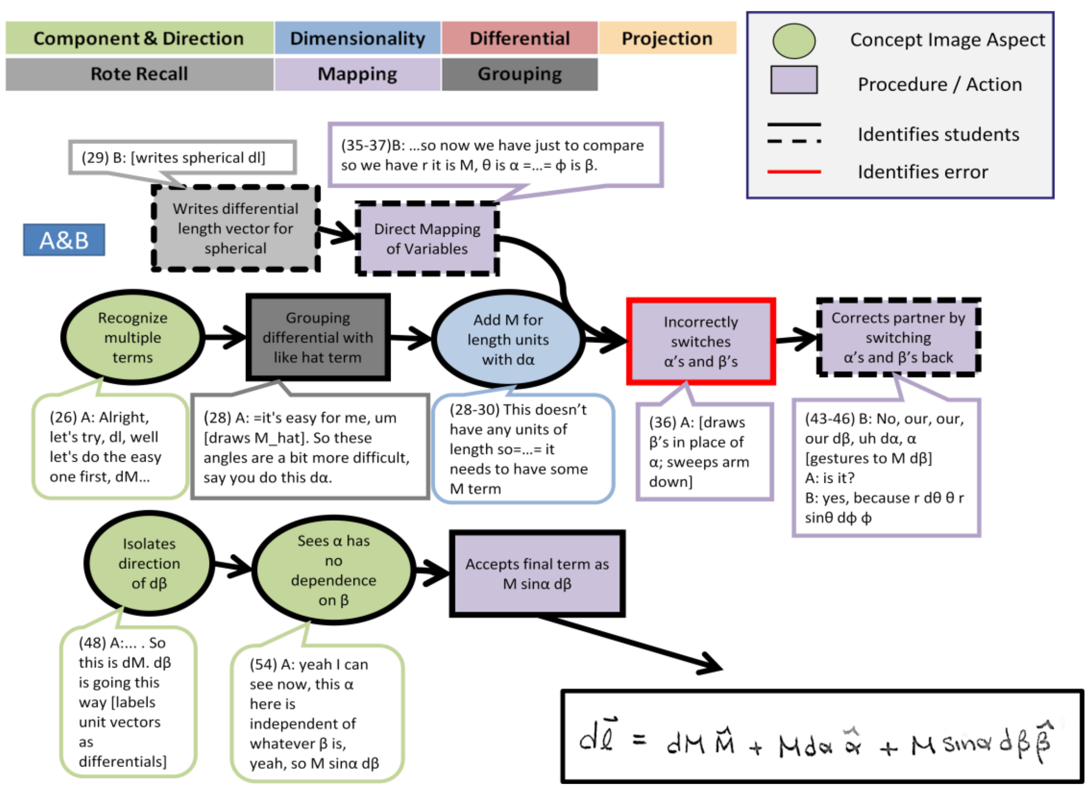


Figure 5.2. Concept image flow chart for Adam (solid outline) and Bart (dotted outline). Excerpts from transcripts are provided to show coding for elements. The final element to the right is followed uninterrupted by the first element on the left in the next row.

discussion of themes identified within students' construction. In these representations, concept image aspects are identified using circles and building actions as squares. Each aspect and action is color-coded. Solid and dotted lines are used to distinguished which student is using the action or aspect at a given point in time. When ideas or actions were used incorrectly or produce an incorrect element in the expression, the lines around the shape are colored red. Each element or grouping of elements represents a complete sentence or phrase pertaining to a section of construction. As a proof of concept, the chart is illustrated in Figure 5.2 with connections between the transcript excerpt and the abstracted concept image component or building action. In the diagrams presented in the remainder of this section, I remove these elements to show only the introduction and progression of ideas.

The remainder of this section discusses students' approaches to differential construction (section 5.1.4.1) as well as themes across groups in terms of the way concept image aspects were invoked and applied (5.1.4.2). Notably, there was high variability in the extent to which students constructed a differential length vector by building in terms of concepts or matched terms to a recalled differential element. Concept image aspects, such as *component and direction, dimensionality,* and *differential* were used in common ways across groups.

5.1.4.1 Student application of recall and mapping versus building of length terms

Each group of students appeared to approach the problem in a different way. Some attempted to reason about the length elements through direct mapping from spherical or Cartesian coordinates. Whether a student chose to build the differential length element from the necessary concepts and ideas or recalled and mapped from previous differential elements provided insight into how students approach multivariable differential elements in integration in E&M. All but one group at University A began by working with the unfamiliar coordinate system and purposefully building components. Each of these groups eventually experienced difficulty centered around the *projection* aspect, in terms of whether or how to include a trigonometric function. At this point, two groups switched to making comparison to spherical coordinates. The fourth group began by incorrectly recalling a Cartesian differential element and mapping the schmerical differential element to this form.

When asked to construct a differential length element, the graduate students (AB) each initially took a different approach.

Adam: Alright, let's try, dl, well let's do the easy one first, dM, and I know you don't like this but=

Bart: Yes. [laughs]

Adam: =it's easy for me, um [draws $\widehat{M}$] So these angles are a bit more difficult, say you do this $d\alpha$. This doesn't have any units of length=

Bart: [independently writes differential length element from spherical coordinates]

Adam: =so, it needs to have some M term. I think it is just like that, isn't it [writes $Md\alpha$]. For α? [sweeps arm down as if covering the space of the angle] Yeah.

Bart: You can, you can check from this, um…

Adam: For α it doesn't have any dependence on this other angle over here, but when you're talking about β, um [looking at the spherical $\overrightarrow{dl}$ that B wrote]

Bart: So this is dl [gestures to spherical differential he wrote], okay? $dr\ r$ [hat], $rd\theta\ \theta$ [hat],=

Adam: No, I have this backwards. (erases α terms)

Bart: = $r\sin\theta\ d\varphi\ \varphi$ [hat], so now we have just to compare so we have r it is M, θ is α=

Adam: (writes β's in place of α terms)

Bart: =φ is β.

We see from this exchange that Adam attempted to reason using the aspects of *component and direction* and *dimensionality*, while Bart made use of the existing spherical coordinates using *recall* and *mapping*. Once Bart articulated the direct mapping, the two students worked together and finished the construction of the differential element so that it mirrored the spherical length element and includes $\sin\alpha$ (Fig. 5.2).

It is notable that the actions made by Adam in the last few lines of the transcript were later illuminated as confusion between mathematical and physical convention for spherical coordinates. This would have been acceptable as long as the angles were also changed in the description of the differential element, which was not the case for Adam. Using limits for the angles from the mathematical convention of spherical coordinates coupled with a physics interpretation of the spherical differential volume element results in a value of zero for integration (due to the integral of $\sin\theta\, d\theta$ from 0 to 2π) along with potential for several conceptual inconsistencies, as seen here. The two students drew a spherical coordinate system and Bart instituted the physics convention, allowing Adam to fix his mistake. Adam then isolated the β-component in his diagram to reason about motion in that direction before agreeing with Bart about the use of $\sin\alpha$.

Carol and Dan initially progressed through the task by reasoning about the building aspects, but spent more time discussing the choices and reasons for their actions than AB. The pair began building using all four aspects, relying on ideas of *dimensionality* and *component and direction* (Fig. 5.3).

> Carol: So we're going to have, um, we're going to have this [writes $\hat{\alpha}$], this [leaves space and writes $\hat{\beta}$], and some $\widehat{M}$ [writes $\widehat{M}$]. That's what we usually do and then they each need to be a length. You need a length vector…This is, there is going to be a plus here [writes "+" after first two unit vectors].

Dan: [Writes M with $\widehat{M}$ as shown in Fig 5.3]

This attention allows CD to structure the differential length vector as three components with a unit vector for each direction. They did not attend appropriately to aspects of projection or the differential later, when constructing the β-length-component.

Dan: I mean, it's like $M \cos \alpha$ would put us where we're =... = down in the b[β]-hat range. And so judging by what you're saying is we just need that there [writes a "d" in front of $M \cos \alpha$ to make a dM].

After further difficulties in building and difficulty determining the expressions for the angular components, Carol and Dan recalled the differential volume element from

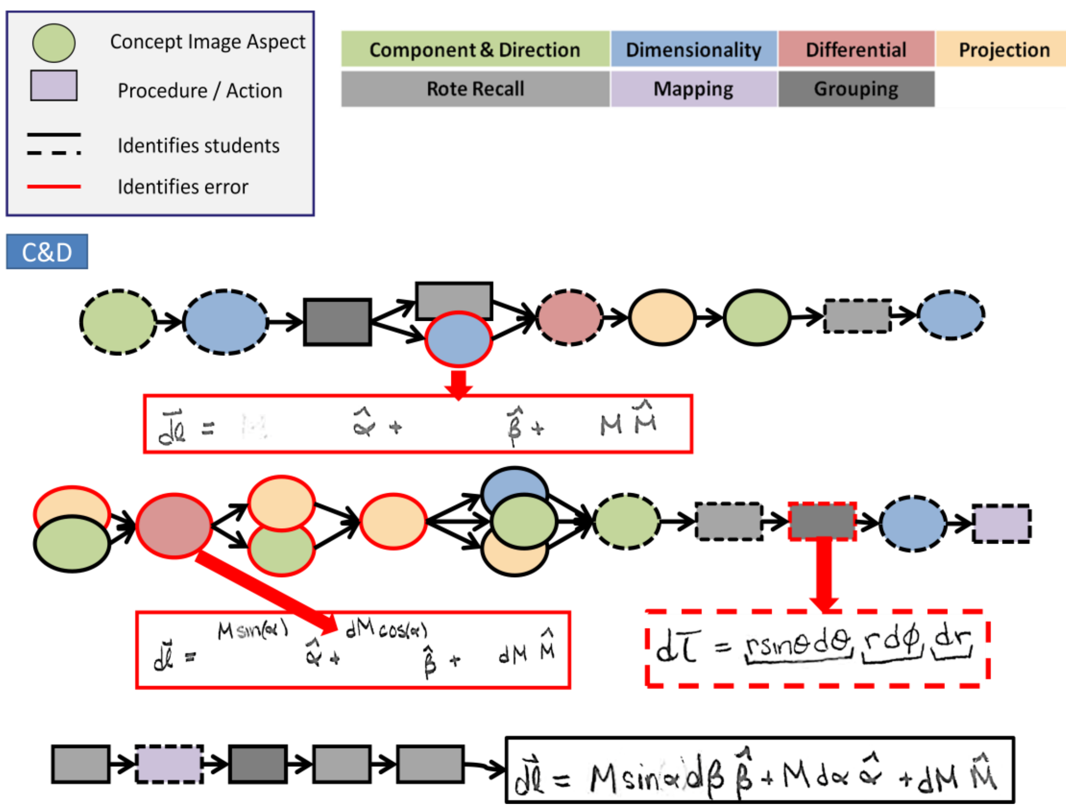


Figure 5.3. Concept image flow chart for Carol and Dan. Student began with building elements, but difficulty with the differential and projection aspects (coded with red outline) lead to the pair switching to recall and mapping.

spherical coordinates to reason about the components of the differential length element for schmerical coordinates. While they had previously recognized the appropriate term for projection, the direct mapping resulted in the incorrect use of $\sin\alpha$ in the β length component, as it had for the graduate student pair AB.

EF provided a contrast to the previous two groups. While still focusing largely on building terms within the schmerical geometry, the two students resolved to build the integral from scratch and made a deliberate choice to not "fog their minds with preconceived notions of how things should work." They spent the interview weaving together aspects of *component and direction, differential,* and *dimensionality,* building each component of the length vector independently; later they added each component together to represent the entire differential length element (Fig. 5.4). Upon recognizing that spherical coordinates had a trigonometric function, the pair chose to forgo using the familiar coordinate system. As a result, the aspect of *projection* was entirely absent from their reasoning, and thus does not appear in the concept image flow chart for this group. At one point they made a comparison to spherical but agreed that they should not include a $\sin\alpha$ term, given that they could not justify the need. As a result, their differential element lacked any trigonometric function.

The final pair, GH, focused entirely on *rote recall* and *mapping*. Neither student, however, could appropriately construct a spherical differential length element, due to lack of consideration of *dimensionality* coupled with the *grouping* of terms by variable (as is done in integration) rather than by appropriate length component. This grouping difficulty pushed them toward building an element in Cartesian coordinates using the form $xdx\hat{x} + \ ydy\hat{y} + \ zdz\hat{z}$. They then decomposed $\vec{M}$ into x-, y-, and z-components

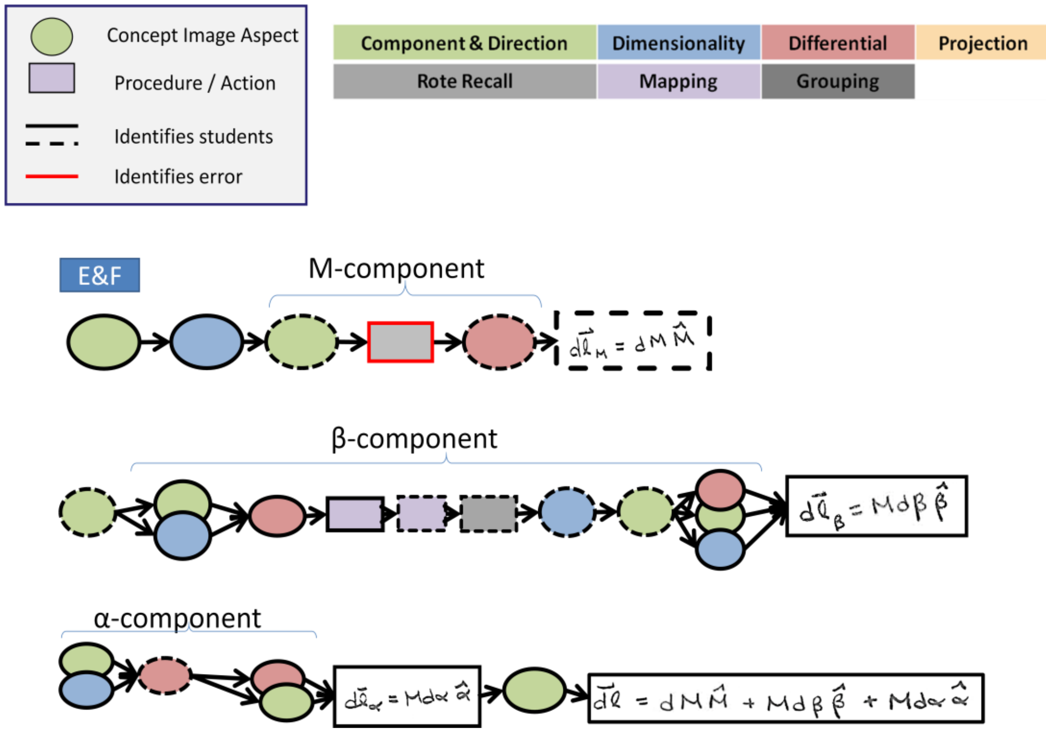


Figure 5.4. Concept image flow chart for Elliot and Frank. The pair methodically constructed each component but failed to elicit the *projection* aspect (as shown by the absence of that code).

for a right-handed system, rather than the given left-handed coordinates. Recognizing that the determined differential element was in Cartesian coordinates and not in schmerical coordinates, the students returned to the idea of building the differential length element later in the interview by recalling the method of construction they had learned in class at the beginning of the semester.

5.1.4.2 Themes in differential element construction

Identification of these four building aspects and three actions afforded us the ability to determine the order and grouping of these aspects as students progressed through the interviews. Generalizations across the interviews led to the observation of recurring patterns in students' construction. This focus addresses the research questions of the project, by attempting to identify the extent that the identified conceptual aspects impacted the construction of the differential length vector.

We identified aspects or combination of aspects that were used productively, in that attention to the aspects led students towards construction of a correct differential length element. The absence, or misapplication, of particular aspects also commonly hampered further construction. Analysis across all of the interviews identified specific difficulties [57] faced by individual groups or incorrect ideas that were commonly held by several students.

The following subsections address three themes in the findings from interviews. The first subsection addresses the productive combination of *component and direction* and *dimensionality* concept image aspects. Students commonly invoked these elements together or in sequence as they focused in on each component. The remaining two subsections address the common ways in which students invoked the *dimensionality* and *differential* concept image aspects. In some cases students employed the concept image aspects correctly, but in other instances students knew they needed to incorporate these aspects and did so in incorrect ways, such as including a dM in the $\hat{\beta}$-component. More attention is given to these ideas in section 5.1.5, where the concept images aspects are connected to the mathematical structures invoked during construction.

5.1.4.2.1 ***Productive combinations:*** **Component and direction *and* dimensionality**

Analysis across groups identified that the use of *component and direction* coupled with *dimensionality* was very productive for students in the first three pairs when considering the differential length element as a whole. For the third pair of students, the combination of these two aspects was additionally beneficial when constructing each individual components of the differential.

> Frank: So then if you have β/
> Elliot: $d\beta$.
> Frank: Oh, yeah.
> Elliot: So you're going to have a length component in the β-hat direction.

For each term, the pair would isolate a specific direction of movement and then discuss what a length element in that direction was comprised of. As such, the Concept Image Flow Chart depicts several instances of these ideas being used together, especially when the students turn to the next component (Fig. 5.4).

5.1.4.2.2 The role of dimensionality

In general, students invoking *dimensionality* were very explicit in checking that each component contained appropriate units of length. Carol and Dan were particularly adamant about accounting for units of length.

> Carol: ...it's going to be like, so if it's going to be some trig thing but sine of something isn't a length so we're going to have to also have something else in there.

Carol and Dan used the aspect of *dimensionality* to reason about the variables of each term, to such an extent that later in the interview they could not recall whether or not differential angles or unit vectors gave units of length to their vector components. While the pair made a comparison to the spherical volume element, the concern persisted as they continued to construct terms. Other students often did not provide additional reasoning for including an M in their construction, as was seen in early transcripts.

> Adam: … This doesn't have any units of length…so it needs to have some M term.

However, Elliot specifically addressed the idea of arc length, combining aspects of *direction, dimensionality,* and *differential,* which made using the radius of length M apparent (Fig. 5.4).

Elliot:	So it's M times some Δ, I think it's M times $\Delta\beta$, a small β, because it's like if you take r times its small θ then that is the arc length=
Frank:	Yeah.
Elliot:	=around a circle.
Frank:	Yeah, okay.
Elliot:	Right, so like $rd\theta$ would be like the length component around a circle, so this would be $Md\beta$.

The final pair of students did not attend to dimensionality and subsequently had difficulty with early recall from spherical and Cartesian coordinate systems.

5.1.4.2.3 The role of differential

Not surprisingly, students' concept image of a differential length element involved a discussion of ideas related to the differential. Particular ideas of differentials were important to students' reasoning approaches. The treatment of differentials in terms of small amounts of motion [27,49,56] was helpful to the building of terms. This idea is trivial for students here, but other views may be coming into play. Carol and Dan had particular trouble constructing the α and β components due to difficulties reasoning about the differential, thinking only in terms of changes rather than small motions applied to the $\vec{M}$, and more specifically not attending to the need to have a $d\beta$ with the $\hat{\beta}$-term. This is discussed more in a following section while highlighting the differential symbolic form.

5.1.5 Results of symbolic forms analysis

To further explore student understanding of the construction and understanding of differential length vectors, analysis incorporated a symbolic forms perspective [47] (see section 3.3.2 for detailed overview). While the concept image [38] analysis provided insight into students' conceptual understanding, symbolic forms provide a means to analyze student understanding of the mathematical representation in terms of the structures students incorporated to construct the differential length vector.

Analysis of interview data revealed several emergent symbolic forms (Table 5.3). Symbolic forms were identified by attending to common elements of structure (symbol template) included in students' written expressions, as well as common mathematical justification leading to structuring of the expression in that way (conceptual schema). Some of the symbolic forms invoked by students during differential length vector construction were consistent with forms previously identified at the introductory level [47]: *parts-of-a-whole*, *coefficient*, and *no dependence*. Additionally, we identified other forms that represented novel template-schema pairings: *magnitude-direction*, and *differential*. The newly identified symbolic forms account for the increase of mathematical sophistication with the need to express vectors and calculus concepts absent from the introductory problems given in the original literature.

The remainder of the section provides the details of each symbolic form as well as student data to support its invocation within the context of differential length construction. Students' invocation of symbolic forms is addressed by isolating the symbolic forms into two generalized stages of construction, consistent with student work.

Symbolic Form	Symbol Template	Conceptual Schema
Parts-of-a-whole	$\square + \square + \square$	Accounts for multiple components that contribute to a larger whole (Sherin, 2001)
No dependence	$[\ldots]$	Indicates an expression is independent of, or not a function of, a specific variable (Sherin, 2001)
Coefficient	$[\square\ldots]$	Represents a quantity seen as just a number or a constant (possibly having units) put in front of an expression (Sherin, 2001)
Magnitude-direction	$\square\hat{\square}$	Used to denote a vector expression including the magnitude of a quantity (having units) and a unit vector to indicate a specific direction
Differential	$d\square$	Represents taking a small amount of or infinitesimal change in a quantity

Table 5.3. Existing and novel symbolic forms identified in students' construction of a differential length element.

In the beginning of construction, most groups attended to the vector/component nature of the differential length element. At this stage, groups constructed templates consistent with *parts-of-a-whole* and *magnitude-direction* forms. Subsequently, groups typically discussed the structure of each component, attending to the ideas related to the magnitudes of each component, which involved developing the templates associated with the *differential*, *coefficient*, and *no dependence* symbolic forms.

At various stages students' concept images motivated the need for various symbolic forms as well as helped students determine the particular variables needed to complete construction (Table 5.1). Analysis, described in the previous section (5.1.4), has identified four aspects of student's concept images associated with constructing a non-Cartesian differential length vector: *component and direction, dimensionality, differential*, and *projection*. Similarly, three processes were also identified across student work that played a role in construction: *grouping* of like terms, *transliteration*, and *rote recall* (Table 5.2).

This section has three purposes: presenting the symbolic forms that students invoke during construction; introducing and arguing for the adoption of the two newly identified symbolic forms; and connecting students' invocation of symbolic forms with students' application of concept image aspects. Combining the two theoretical frameworks in this manner provides a more complete picture of the things students are doing and understand about a non-Cartesian differential length vector.

5.1.5.1 Symbolic forms related to vector properties

As shown in the concept image analysis, the majority of student groups at the outset of construction attended to the *component and direction* aspect of differential length elements, highlighting the need for a summation of three different components as well as the idea that each component of the vector equation is an independent displacement of the vector *M* in each of the variable directions. In each group, the *component and direction* aspect manifested as a combination of two symbolic forms: *parts-of-a-whole* [47], which accounts for the inclusion of multiple terms, and *magnitude-direction*, which expressed the direction associated with each component term.

Students were generally successful with construction of these larger templates. Almost all groups recognized the need to express multiple components and expressed vectors in terms of a magnitude and direction.

5.1.5.1.1 Parts-of-a-whole

The need for multiple components to completely express a differential length vector resulted in the invocation of the *parts-of-a-whole* symbolic form by almost all groups. Frank demonstrated a requisite conceptual schema when starting construction:

> Frank: There are three different dl's. There is dl with respect to M, dl with respect to α and dl with respect to β.
>
> [pair constructs components independently]
>
> Elliot: You sum them, so $\overrightarrow{dl}$ is those added together: $dM\widehat{M} + Md\beta\hat{\beta} + Md\alpha\hat{\alpha}$.

Elliot and Frank worked on each component independently; Elliot then summed these components to express their full (incorrect) vector differential at the end of their construction. Similarly the pair AB built their differential length term-by-term.

> Adam: Alright, let's try, dl, well let's do the easy one first, dM =...=it's easy for me, um (writes $\widehat{M}$) So these angles are a bit more difficult, say you do this $d\alpha$. This doesn't have any units of length.

As a slight contrast, CD started by writing the overall structure, accounting for the unit vector of each component, and subsequently filled in each term (Fig. 5.5). Each of these groups recognized the need for and express the multiple components associated with the differential vector element in this coordinate system. The expression of multiple terms with the conceptual schema of "three different dl's" that must be summed or "added together" makes this consistent with Sherin's *parts-of-a-whole* symbolic form [47].

Perry and Quinn recognized the need to sum multiple components but were unable to disentangle themselves from Cartesian coordinates. They initially structured their differential length as the addition of three dl's for each Cartesian direction (Fig. 5.6), invoking the *parts-of-a-whole* template but for the incorrect coordinate system.

Figure 5.5. Beginning stages of construction for Carol and Dan showing the coupling of the *parts-of-a-whole* and *magnitude-direction* symbolic forms.

Figure 5.6. Perry's and Quinn's final expression for a differential length vector showing the invocation of *parts-of-a-whole.*

RS, having first decided that $d\vec{M}$ was sufficient to describe the differential length element, later remembered having also used dl as a description of circular paths and recognized the need for multiple terms.

Rachel: ... it's like a path along something so like that is fine if the path is like in the dM direction but if it is not then [dl] is not very generic... there would have to be three components to it...because it has three dimensions.

Rachel and Silas then represented this new dl using bracket notation for vectors (Fig. 5.7). While the group encodes their length vector using bracket vector notation, their conceptual schema is illustrative of *parts-of-a-whole* and explicitly explains students' summation of only three terms.

Following construction of the template for the full differential length element, several groups attended explicitly to the *dimensionality* of each component.

Carol: ...and then they each need to be a length.

Elliot: ...and each of them need to be a length.

This need for dimensionality, while recognized early in construction, became increasingly relevant as students made decisions about what terms belonged in each component.

$$d\lambda = \langle dM, d\alpha, d\beta \rangle$$

Figure 5.7. Rachel's and Silas's final expression for a differential length vector including three components.

5.1.5.1.2 Magnitude-Direction

Either following or coupled with the *parts-of-a-whole* symbolic form, students' attended to the vector nature of components. Students split each component into a pair of two distinct parts, one that displays the magnitude of the differential length term, and the other the direction each component is associated with. We identify this particular product as the *magnitude-direction* symbolic form with the template $\square\hat{\square}$. Group CD's work displays this explicitly, as they left space to write the magnitudes of the components in their expression (Fig. 5.5). During a second attempt to construct a differential length element motivated entirely by *rote recall*, GH completed their expression by adding a unit vector to each of the summed differential length magnitudes (Fig. 5.8).

While some students inherently included the vector nature when constructing components, other students paid particular attention to the unit vector of the component, using it to reason about the preceding magnitude in that direction.

Carol: So, dl is like you just have some path. So I'm trying to think, like, if I was going to walk in the $\hat{\alpha}$-direction...

Elliot: So you're going to have a length component in the $\hat{\beta}$-direction…

a) $dl = dM + Md\alpha + M\cos\alpha\, d\beta$ b) $\vec{dl} = dM\hat{M} + Md\alpha\hat{\alpha} + M\cos\alpha\, d\beta\hat{\beta}$

Figure 5.8. Greg's and Harold's differential length vector (a) before and (b) after recognizing the need to include unit vectors.

Each student here isolates the specific direction or unit vector and then attends to the magnitude of the component as a second entity. After reasoning about the nature of what is included in the magnitude of the component, students automatically write the magnitude of the vector component as preceding the unit vector as it is typically expressed in physics.

5.1.5.2 Elements related to construction of the magnitude of the components

After developing a sense of the overall structure for the equation, groups attended to the individual components, accessing various concept image aspects to fill the *magnitude-direction* template. Most specifically, this involved a combination of Sherin's *coefficient* [47] and the newly labeled *differential* symbolic forms. While the differential symbolic form involved reasoning about small changes and infinitesimally sized quantities, the coefficient form had more varied justification for its invocation, involving attention to *dimensionality* and geometrical reasoning as well as *rote recall* and *mapping*. Several students also invoked the *no dependence* symbolic form to distinguish which variables depend upon the others in the coordinate system (i.e., the arc length in the $\hat{\beta}$-direction being dependent on the angle α).

5.1.5.2.1 Differential

In addition to the identification of *differential* as an aspect of students' concept image for a differential length vector, students expressed a common template with the differential. This depicting was connected to student attention to needing small displacements or small changes in specific directions, as seen in the following excerpts.

Carol: Right. So you have **a change in your $\boldsymbol{\hat{M}}$** is going to be your **$\boldsymbol{dM}$, it's your change in your $\boldsymbol{M}$**.

Rachel: Um, [$d\vec{M}$] represents a tiny portion of like, a length, or a change in the radial component of the vector.

Given the importance of the differential and the distinct meanings students associated with it, we identify a *differential* symbolic form, $d\square$, from students' work. The form itself is similar to what appears in graphically oriented symbolic forms for integration, where students describe dx as a "small portion of each graph," width of rectangle in a sum, a specific shape depending on the shape of the function (e.g., circle or square), or commonly just a cue for integration [50]. For students constructing differential vector elements, the differential is not (yet) associated with a particular integral expression, and thus is treated as a standalone quantity with its own attached schemata as a need to represent a small quantity. When removed from the context of integration, there are a number of other conceptual ideas attached to differentials, especially in E&M [25]. The treatment of differentials in terms of small amounts of motion or changes of a quantity [25,27,49,56] was helpful to the building of terms. This idea is trivial for a number of students, while for others different views of the differential impact the construction of their differential lengths.

Tyler initially represented $\vec{r}$ as a pattern-matched form of a vector in Cartesian coordinates [77], then attempted to determine partial derivatives from particular components.

Tyler: So any vector r, that's an $M\hat{M} + \alpha\hat{\alpha} + \beta\hat{\beta}$...so is, I mean/ we're not looking for like the total dr but like dr/dM?

With particular difficulty, Tyler begins to express this as $d\vec{r} = Md\hat{M} + \beta$, explaining his $d\hat{M}$ as a need to take the derivative of the unit vector to account for any "phase or time dependence." This type of view of the differential as a cue to take a derivative is consistent with treatment of the differential as a "machine" that outputs another function [25]. After being assured there was no time or phase dependence, he attempted to recall to specific coordinate transformations between spherical and Cartesian coordinate systems.

In some cases, difficulty reasoning about how to incorporate the differential led to students forcefully trying to insert a differential into their expressions. After recognizing $M \cos \alpha$ as a projection into the xy-plane, CD wrote a "d" in front of the whole expression (Fig. 5.9a). Soon after, they labeled this an incorrect expression, and turned to recall of spherical coordinates to complete the task. Similarly, Frank tried to express an infinitesimal arc length as $r \sin d\theta$ as a way to also explain where the differential and trigonometric function would appear (Fig. 5.9b). Elliot corrected him by defining arc length for a differential change in angle.

> Elliot: There's actually a little bit on the circle; there is a little curvature. This length is $rd\theta$.

Following this, the pair EF focused their construction on having a differential length component in a particular direction containing a differential with that variable.

> Frank: so then if you have β/
> Elliot: $d\beta$.
> Frank: Oh, yeah...
> Elliot: So you're going to have a length component in the β-hat direction…so, basically we're going to need… an M… so it's M times some Δ, I think it's M times $\Delta\beta$, a small β, because it's like if you take r times its small θ then that is the arc length (Fig. 5.10).

EF finally articulated this as the length component, $\overrightarrow{dl}_\beta = M d\beta \hat{\beta}$, which now only lacked the needed trigonometric term, but correctly connected the expression of $d\beta$ with needing a small change in the angle.

a) $d\vec{l} = M\sin(\alpha)\,\hat{\alpha} + dM\cos(\alpha)\,\hat{\beta} + dM\hat{M}$ b) $r\sin d\theta$

Figure 5.9. Students' incorrect insertion of differentials into their components. (a) Carol and Dan incorrectly incorporating the idea of a differential by writing "*d*" before their $\hat{\beta}$ term. (b) Frank attempting to account for the arc length of a small angle and forcibly inserting both a differential and trigonometric function into his expression.

Elliot and Frank's discussion here highlights another aspect of students' attention to the differential that ties into the *magnitude-direction* symbolic form. As part of students' conceptual schema during construction, students eventually used the same variable from the differential symbolic form (M, α, or β) as the variable corresponding to the unit vector (i.e., $dM\ \widehat{M}$, $d\alpha\ \hat{\alpha}$, and $d\beta\ \widehat{\beta}$). Greg and Harold do this inherently as they attend to the *magnitude-direction* symbolic form (Fig. 5.8), while Carol and Dan explicitly recognize the need for pairing this after correcting a grouping error in a recalled spherical volume element. Both GH and CD initially combined the $r\sin\theta$ with the $d\theta$-term, resulting with an $\hat{\alpha}$-component having $M\cos\alpha\, d\alpha$. After recognizing this mistake, they first switched only the differentials for the terms before recognizing the unit vectors would need to be switched as well, in order to keep the $d\alpha$ term with the $\hat{\alpha}$-component.

a) $\vec{dl}_{\beta} = \quad d\beta$ b) $\vec{dl}_{\beta} = M\, d\beta\, \hat{\beta}$

Figure 5.10. Pair EF constructing the beta component of the differential length. (a) Initially they leave space to write the needed coefficient and unit vector. (b) After discussion they include a coefficient lacking the projection term $\cos\alpha$.

5.1.5.2.2 Coefficient

The appearance of the *coefficient* symbolic form as a prefix to the *differential* form was most often predicated by the need for appropriate dimensions, recognition of arc length, or some level of rote recall to the more familiar spherical coordinate system. The coefficient form is generally invoked to include a space for specific factors or constants that appear in typical physics equations [47]. Students will often treat coefficients as a parameters that "define circumstances under which [physics] is occurring." [47] This symbolic form manifests physically in the equation as a term multiplied on the far left of a product of terms. While functionally similar to the *scaling* symbolic form [47], the *coefficient* form is used to account for quantities with specific units. This distinction makes the *coefficient* symbolic form more applicable to describe students' construction because of the explicit attention to *dimensionality.*

The most prominent and prescient evoked concept image was the need to include dimensionality, as seen in the following two (independent) excerpts.

> Adam: …This doesn't have any units of length, so it needs to have some *M* term. (Fig. 5.11)
>
> Carol: …So, if it's going to be some trig thing but sine of something isn't a length so we're going to have to also have something else in there.

Students accessing of the dimensionality concept image aspect, both for the coefficient symbolic form and when discussing the magnitude portion of their

$$\vec{dl} = dM\hat{M} + d\alpha\hat{\alpha} \qquad \vec{dl} = dM\hat{M} + M\,d\alpha\hat{\alpha}$$

Figure 5.11. Adam's inclusion of "*M*" based on dimensional reasoning.

components, resulted in the inclusion of an M or a dM term. Recognizing that the dM term satisfied the dimensionality, differential, and component and direction aspects, students did not include any more terms in the $\widehat{M}$-component.

Group EF was the only group that invoked arc length as the actual physical justification for the M and $M\cos\alpha$ in the $\hat{\alpha}$ and $\hat{\beta}$ components respectively.

> Elliot: Just like when you get the circumference, it's equal to $2\,\pi r$, well it's taking all of the radians, instead [you take] a tiny amount of radians, which would give you a tiny arc length.

Elliot and Frank then constructed the two angular components, but failed to recognize that for the $\hat{\beta}$ component they needed to account for the projection of M in the xy-plane and end up with $\overrightarrow{dl}_{\beta} = Md\beta\hat{\beta}$ as shown above. While for the $\widehat{M}$-term, M is the important dynamic variable that the component depends on, for the angle components where only the angles are changing it is a static variable representing a radius in an arc length.

Rachel and Silas expressed arc length when constructing sides for a differential volume as dM, $Md\beta$, and $Md\alpha$. They make no attempt at reconciliation between the volume element and their single differential length component, dM, and fail to recognize the need to do so. This is most likely due to a restricted concept image, where only the radial component of the differential length is used to account for line integrals in radial fields common in electrostatics. Upon recalling that differential lengths are used to describe circular paths in magnetostatics, RS decide three terms are needed. However, their new components no longer include scaling factors to account for arc length (Fig. 5.7).

> Rachel: I think it would be like, the first if it’s in r would be dr. Right? So you want it in Cartesian or in spherical?
>
> Interviewer: I want it for this coordinate system.

> Rachel: So I think dl is just $dM, d\alpha, d\beta$, like commas in between those because that is how you figure out path...you have your change in your M direction, then you have your change in your [α] direction....

The expression of dl as $\langle dM, d\alpha, d\beta \rangle$, is sufficient for them since it accounts for the change in each direction. It is likely the students were attempting to map to a Cartesian representation of a differential length element, where the individual components are solely expressed as the differential for a variable and its corresponding unit vector.

Rote recall and transliteration often occurred when students faced difficulties with the application of concept images or when geometric ideas were inaccessible. This is reminiscent of a symbolic forms analysis of physical chemistry students' construction of partial differential equations in the context of thermodynamics [78]. In these cases recall mediated students construction of equations in terms of particular processes, such as taking the total derivative, or as recall of specific concepts, such as $dX = 0$ if X is a constant.

While group EF chose to avoid recall to spherical coordinates and focused construction specifically within the schmerical system (with subsequent lack of attention to the projection aspect), groups AB and CD incorrectly included a $\sin\theta$ due to heavy reliance on spherical coordinates to complete the differential length vector.

After initial difficulty with construction, Tyler decided that "length is really only the radial component," and expressed $\overrightarrow{dl}$ as MdM.

> Tyler: ...Yeah, because it's the amount of M for every little dM that I move… It's so much easier in Cartesian...but I think the only reason the M is there because when you transform coordinate systems your length is no longer just dM.

Tyler then justifies his extraneous invocation of the coefficient form by citing the scaling factors gained by the spherical volume element when making the transformation from Cartesian coordinates.

In many cases the *coefficient* symbolic form appeared as a means to complete an expression, driven most often by what Carol expressed as a "need to have something else in there." To accommodate for the need for further terms, students commonly left blank spaces in the equation as if calling forth a particular template to fill in later. Specifically we see this for AB's (Fig. 5.11) and EF's (Fig. 5.10) inclusion of M as the coefficient, but also earlier with CD (Fig. 5.5) as they separated out the necessary components when invoking the *parts-of-a-whole* symbolic form.

5.1.5.2.3 No Dependence

The no dependence symbolic form appears when students explicitly address the absence of a variable in an expression. Frank and Elliot invoke this symbolic form while constructing the radial component.

> Frank: If you change [$\vec{M}$] a little bit, α, and β doesn't change at all. This is just M because it's just the radius.

Here Frank, is articulating that a differential length in the radial direction is independent of the angles and thus writes $\vec{dl}_M = \mathrm{d}M\hat{M}$ without inscribing either angle into this component.

While the invocation of *no dependence* may seem trivial for a radial component, it played a larger role for Adam and Bart during the construction of the angular components. The need to project our vector $\vec{M}$ into the plane of β to get the requisite arc length results in the $\hat{\beta}$ component being a function of the angle α. In comparison, the arc length of the $\hat{\alpha}$-component uses the full radius, M, and ignores the coordinates system's polar angle. Adam explicitly addressed this during construction of the $\hat{\alpha}$-component.

Adam: For α, it doesn't have any dependence on this other angle.

Here Adam recognized and addressed that constructing the arc length term resulting from a change in α is independent of the angle β. As a result, students explicitly omit a β term in the component.

5.1.6 Summary of findings from the schmerical task

Analysis of student construction of a differential length vector through the symbolic forms and concept image frameworks enabled the identification of specific structures that students associated with vector expressions as well as of the concepts students connected to these structures and the associated variables. Our results suggest students do not have a robust understanding of how to build non-Cartesian differential elements. When working in an unconventional spherical coordinate system, students used a mixture of approaches to construct differential length and volume elements. Some attempted to reason about the length elements through direct mapping from spherical or Cartesian coordinates. We found students could implement successful strategies using necessary concepts. Particular attention to *component and direction* as well as *dimensionality,* both individually and combined, allowed students to think productively about terms. Using *differential* to think in terms of small changes was also useful to students.

Interviews also highlighted a number of difficulties students faced when working with differential length elements, including an overreliance on rote recall and mapping without underlying understanding. It was also noted that students had particular difficulty grouping terms within recalled spherical length and volume elements. Students' inattention to *dimensionality* and *projection* hampered construction of terms. The successes and difficulties surrounding dimensionality speak to the importance of reasoning about units and dimensions when it comes to modeling physical quantities in terms of mathematical representation.

The explicit context of vectors and the increased mathematical sophistication of the upper-division content led to the identification of new symbolic forms in addition to forms previously identified. The symbol templates and associated schemata for the new *differential* and *magnitude-direction* symbolic forms were consistent across groups, but the ideas motivating the invocation of the symbolic forms varied. For example, students often explained the need for the differential as having to account for a change or small amount of a quantity.

Further analysis identified that students at University A were able to recognize the general structure needed for the equation and invoke the correct template. The primary difficulties here were connected to the conceptual information needed to express the appropriate terms in the symbol templates. For example, students constructed an appropriate expression for the β-component in terms of dimensional and differential considerations, but the projection aspect of the concept image responsible for introducing the $\cos\alpha$ term was either misapplied or inaccessible. Students interviewed from University B were less successful invoking and combining necessary symbol templates and had difficulty accessing or applying ideas related to dimensionality or component and direction. As discussed in previous chapters, classroom observations at the first institution suggest students were able to arrive at the general structure due to explicit and repeated emphasis on construction of differential length elements early in the semester. However, students still were unable to connect the necessary ideas for differential length construction at this university. The exact nature of the difference in performance between the two universities is beyond the scope of this study, as we do not possess comparable data from classroom instruction at each site. Furthermore, limitations in the number of participants prevent any large-scale claims about differences between courses.

Dimensionality and geometric reasoning were especially prominent in the more successful efforts. In these cases, dimensionality and component and direction were closely tied, appearing when discussing overall structure and when isolating the change in each individual component. While reasoning about dimensionality and units was relevant to student construction, in some cases students struggled to determine the units of certain terms, such as angles and unit vectors. Findings suggesting the generalized use of units to support expression construction are especially important as previous research on symbolic forms does not address how students' attention to units impacts their problem solving [47].

Geometric reasoning proved to be a more productive approach during construction. In many cases, students attempted to visualize the paths traced by $\vec{M}$ as small changes were made to individual variables in the coordinate system. Most groups recognized the need for multiple components to properly express the differential length vector and appropriately connected the differentials to unit vectors of the same variable.

In cases where segments of construction proved difficult for students, recall mediated expression construction, similar to upper-division physical chemistry findings dealing with partial derivatives [78]. In our study, however, recall of spherical coordinates, despite having the potential to be productive, led students to construct expressions that incorrectly included a sin α term. In several instances, students attempted recall from Cartesian coordinates or tried to find the component of $\vec{M}$ in the direction of a Cartesian axis. While this was in many cases only an attempt to understand the nature of the unconventional system, two groups explicitly constructed elements with Cartesian unit vectors. This supports earlier literature that students have more familiarity with Cartesian coordinates [10,46] and further suggests students have difficulty isolating ideas needed to construct differential vector elements in non-Cartesian coordinate systems.

Chapter 6 has a discussion of students' understanding of differential volume elements and their connection to students understanding of the length vector. A few groups recalled a spherical volume element in an attempt to reason about components during construction. More importantly, some students constructed the differential volume element from the terms in their length vector; the checking of a differential volume led these students to correct their initial mistakes.

Results indicate instructional changes should focus on the concepts associated with the building of the differential, specifically making explicit the connection from the coefficients for the angle components to the idea of arc length and coordinate system geometry. Findings of this task has led to the development of a student-centered tutorial [65], to be used as part of instruction in E&M and/or mathematical methods of physics courses. The tutorial is designed as a more structured version of the schmerical task focusing on differential length and volume construction (see Chapter 6 for discussion of volume element construction). More detail on the specifics of each tutorial component is in section 9.5 and Appendix C. This tutorial is the first of a two-tutorial sequence building off of the findings from this task and from student construction and determination of differential area elements discussed in Chapter 7.

Continued analysis of student construction of these equations has integrated the *symbolic forms* and *concept image* frameworks further using a conceptual blending framework [76], to more completely account for students' integration of conceptual understanding with symbolic expression during differential length construction. Connecting the frameworks in this way provides structure for the use of blending to interpret student application of mathematics in physics. Chapter 8 presents the theoretical model derived from the empirical data analysis in the context of this work.

5.2 Student differential length construction for a spiral task in a physics context

Previous work on generic differential length construction in an unfamiliar system (as described in the previous section) gives specific insight into students' fundamental understanding of the differential length vector. However, students rarely encounter such an abstracted task in typical course instruction. The construction and determination of differential elements is often mediated by the given physical systems, which include charge distributions or current densities, and associated vector fields. As such, the research questions were extended to include the construction and determination of differential length elements within a physics context. This provides insight into the extent to which the physics influences the expression of differential elements as well as what features of the context influence construction.

5.2.1 Research Design and Methodology

In order to investigate students' performance on more typical E&M problems, a task was designed involving the change in potential due to a point charge, Q, centered at the origin (Fig. 5.12). Students were asked to find the differential length vector for a spiral path given by $r = 2\theta/\pi$ in the xz-plane and to find the change in potential experienced by a test charge as it moved along the path from the point (4,0,0) to (0,0,-7) around the central point charge. The electric field due to a point charge is a highly symmetric case where change in electric potential depends only on changes in position in the radial direction. Any task involving a purely radial field only needs the $dr\hat{r}$ term and can exclude any angular components for the purposes of computing this line integral. By using a spiral path and explicitly asking students first to construct the generalized differential length vector, the task required both differential length components to describe it completely:

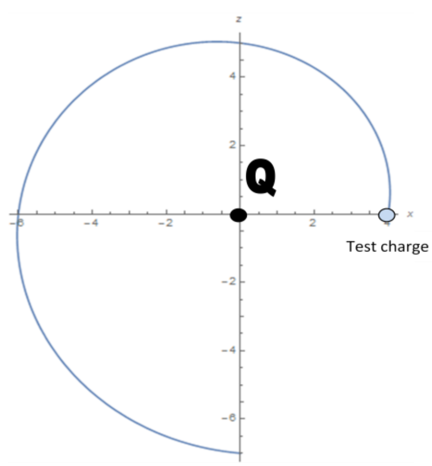


Figure 5.12. Image of the spiral task provided to students, depicting the charges and spiral path of the test charge. The figure shows the section of the path along with the test charge travels.

$d\vec{l} = dr\hat{r} + rd\theta\hat{\theta}$. Incidentally, expressing the differential as a sum of vector components is relatively independent to physics problem solving, as vector calculus in mathematics typically taught with parameterization of the path [79].

The task was administered as part of two clinical think-aloud settings; first with two pairs of students (B&H, D&V) and again the following year as part of a different interview protocol with six individual students (J, K, L, M, N, O) at University A and one individual (T) at University B. All students were enrolled in the second semester of a two-semester, junior-level E&M sequence. Pseudonyms are provided for students corresponding to their identifying letter (*i.e.*, Jake for J). (Repeated letters from above indicate the same students as for the schmerical coordinates tasks.) This particular question took students about 10-20 minutes in interviews. As before, Bart is a graduate student enrolled in the course for credit.

This section focuses mainly on students' construction of $d\vec{l}$ within a physics context to make comparison to generic $d\vec{l}$ construction. Video interview data were transcribed, taking student writing and drawing into account. The transcripts were analyzed under the same methodological guidelines as the schmerical coordinate system task with the goal of identifying student attention to *symbolic forms* and the associated aspects of students' *concept images* in line with previous findings. Analysis additionally looked for new aspects now appearing because of the applied context.

5.2.2 Results in comparison to schmerical data

Data analysis showed attention to many of the relevant symbolic forms and concept images identified in the schmerical differential length task, but among fewer students.

Surprisingly, a number of students wrote a differential length vector accounting for the *angular* motion as the sole component and neglected the inclusion of dr, which is the only component dictated by the physics. We draw on discussions of students' invocation of *parts-of-a-whole*, *magnitude-direction*, and *differential* symbolic forms explicitly as a means to discuss the results of this task with differential length construction in the schmerical task in the previous section. We attribute differences in student responses not only to the physics nature of the task, but also other features, such as the spiral path. The inclusion of a specific path means the task is not isomorphic to schmerical coordinates but still provides a different context for students' differential length construction.

In particular, *parts-of-a-whole* and *magnitude-direction,* both prominent in the acontextual task, did not appear as often during students' construction in the spiral task. Five students invoked *parts-of-a-whole*, described earlier as students' recognition of parts summing up to a whole with the template $\square + \square + \square$. However, only one student applied a polar coordinate system and initially included *magnitude-direction*. *Magnitude-direction* accounts for the magnitude and unit vector parts of a vector quantity and is associated with the template $\square\hat{\square}$. Both these symbolic forms are associated with the *component and direction* concept image, where students would recognize that differential length vectors need multiple components, and that each component corresponds to motion in a specific direction. The following transcript illustrates a correct response and highlights the *component and direction* aspect needed for differential length vector construction.

> Molly: Yeah, and then you go a little bit…I'm picturing you go from this point to this point …So first I travel in the r direction so I go dr in the $\hat{r}$, and then I travel in the $\hat{\theta}$ direction and the arc length of a circle is the radius times the angle that you move so that is $rd\theta$, here in the $\hat{\theta}$. (Fig. 5.13a)

Molly appropriately separated each component as two distinct motions ("I travel"), then encodes each length as the magnitude and the corresponding direction as the unit vector, resulting in a correct $d\vec{l}$.

Two other students invoked *parts-of-a-whole* without encoding components with a *magnitude-direction* template. Neither student specifically attended to the directions each component traced out, resulting in differential length components absent of unit vectors (Figure 5.13b, 5.13c). Kyle's transcript demonstrates this.

> Kyle: We stay in the one plane… so we're only changing by θ and r, so it we have some $d\theta$ or let's say $\Delta\theta$, then Δr is going to be $2\Delta\theta/\pi$, so the actual length is the change in the radius and the change in the angle times the radius so that we stay in units of length.

Upon recognizing a need to account for a dot product during the later integration, both students added unit vectors to each of their terms.

Both of the above transcripts also highlight multiple concept images of the *differential,* accounting for "a little bit" of or "changes" in variables, consistent with students' ideas of differentials identified in the literature [25,27,49,56]. These ideas cue

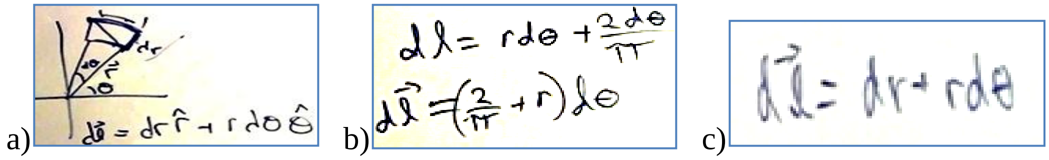


Figure 5.13. Various responses of students who expressed two components. (a) Molly's correct differential length elements. (b) Kyle's and (c) Jake's differential length elements absent of unit vectors.

students' invocation of the differential symbolic form: representing a differential quantity with template d☐.

The last two students to invoke the *parts-of-a-whole* template used Cartesian coordinates. They both mentioned needing small changes in x and y, rather than starting in the more appropriate polar coordinate system. Oliver attempted to differentiate coordinate transformations for x and y with respect to θ in order to express dx and dy. Tyler began similarly but then suggested that a spherical transformation would produce $dl = r^2 \sin\theta \, dr d\theta$. He reduced his dl down to one component without addressing a need to maintain a sum of two components, or directionality.

The remaining interview subjects only attended to one component, neglecting both the *parts-of-a-whole* and *magnitude-direction* symbolic forms. Dan and Victor addressed just the change in the r direction, addressing the change in θ as irrelevant to calculation of the electric potential (Fig. 5.14a). While this does lead to the correct solution for the potential difference, the length element for the path is incomplete without the θ-component.

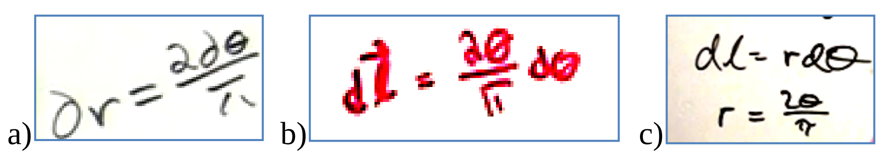


Figure 5.14. Various responses of students who expressed one component. (a) Dan and Victor's accounting for only change in r-direction and converting to terms of θ. (b) Nate's dl, with function replacing r in $rd\theta$. (c) Bart and Harold's dl, where the function for r is written with the term to account for changes in r along the path.

The three remaining students only accounted for the θ-component (Figs. 5.14b, 5.14c), correctly including the r in the arc length and including the functional relationship to write the length component in terms of θ.

> Nate: I think I'm going to move just a tiny bit. This point changes, and so r is going to change and [θ] is going to change… r is going to be obvious because I think it's going to be [$2\theta/\pi$] and then [θ] would just change some $d[\theta]$… To me it makes sense, because you're moving some infinitesimal amount in θ and then you have that r change.

This reasoning appeared across multiple interviews in which students only expressed the θ-component. Students still recognize the need for change in particular variables, an evoked concept image that results in the differential symbolic form. Here students use the functionality of r on θ and the inclusion of r in arc length to account for r changing. This appears to supersede their need to include change in r as a separate component of the differential length. The need to include a dr is entirely absent from their constructions.

Notably, as one of these students, Lenny, was asked to find the change in potential experienced by the test charge, he immediately switched to a thinking dominated by the electric field.

> Interviewer: Okay. How do you account for the change in the radius there?
> Lenny: That would just be the r being a function of θ, so as θ goes from 0 to 3/2 π. Yeah, so as θ increases, r increases which is what we see here in that figure.
> Interviewer: Ok, …what is the change in potential experienced by the test charge?
> Lenny: Well, so I guess if I call that the $\hat{r}$, r-direction, even though it is spinning and getting bigger, the potential on that charge would only change in that direction.

Once shifting from the mathematical formalism of determining the expression for the differential length vector to the physics context, Lenny immediately addresses the directionality of the field and makes an argument as to why only the radial change of the path is relevant to calculation. However, he does not connect this reasoning back to his expression for the differential lengths and incorrectly attempts another solution pathway. This appears to be a point of disconnect between Lenny's expression of mathematical formalism and the given context of the task, as he does not connect any of the physics argumentation to the construction of the differential length vector.

5.2.3 Conclusions of spiral task

Analysis of student interviews on differential length construction in a more typical E&M task reveal that the reasoning that students employ changes with task structure. In the previous "schmerical" task, students were asked to construct a generic differential length vector in the absence of a path and physics context. Here, students easily recognize the need for multiple components for the general expression of the differential length vector, most likely due to the more formal mathematical nature of the task. Results from the spiral task, which includes an embedded physics context and includes a specific path for which students are asked to determine the differential element, suggest that students have difficulty recognizing that the path accounted for multiple component directions.

In general, students' attention to θ was prominent across all interviews, not just for students who constructed a single component in the θ-direction. In both Molly's, and Dan's and Victor's interviews, the students correctly determined that only the radial component is necessary for calculation of potential, but continued to write and carry out integration in terms of θ (which is more complicated given the substitution of $(\frac{2\theta}{\pi})^2$ in place of r^2, and $2d\theta/\pi$ in place of dr).

While calculation in terms of theta still yields the right expression, a number of students interviewed on the spiral task only included the θ-component in their differential length. This points to student difficulty recognizing the possibility for multiple components, but also with attending to the underlying physics; the latter was an area of difficulty noted for students' use of mathematics in E&M [12].

The specific attention to the theta direction can possibly be attributed to a number of factors. The curvature of the spiral path and functional representation of r in terms of theta appear to be salient distracting features [80]. As such, they attract student focus and result in attention to those quantities.

Additionally, the focus on theta may be due to the typical instantiation of the high symmetry for many tasks in E&M that allow students to select one component of a length or area vector and disregard others. For a task involving a spherically symmetric electric field, students would usually select the $\hat{r}$-component. However, as the students were all enrolled in E&M II, which predominately involves cases with circular symmetry (e.g. Ampère's Law for curling magnetic fields), this could be the reason some students only expressed the theta component.

Notably, these students additionally recognize the need for a change in r given that the path terminates at a higher value for radius. However, because of the focus on the functional dependence of r in terms of θ and the existence of r in the $rd\theta$, students can further justify their original expression of single differential length component.

Future work is needed investigate students' work on these tasks and to investigate the influence of providing an explicit function for the path as well as whether attention to the theta direction is as prominent for students enrolled in E&M I. These extensions to the investigation would result in the generalization of these claims and student difficulties [57] in this context.

5.3 Summary of student understanding of differential length construction in non-Cartesian coordinates

The previous chapter has outlined two efforts to investigate student understanding of differential length vectors in terms of how they are constructed within non-Cartesian coordinate systems. In the first interview task, students were provided with an unconventional spherical coordinate system and asked to construct a generic differential length vector. The second interview task involved students expressing the differential length vector for a spiral path with an additional context of electric potential experienced by a test charge due to a point charge.

Findings from the generic task show pervasive difficulty connecting the curvature of coordinate geometry to the expression of differential components. This most commonly appeared as a failure to account for the meaning of the trigonometric function as a projected radius. Other difficulties included expressing the differential in terms of Cartesian unit vectors and only including one component as a change in the radius.

The expressing of the differential as a single component was more prevalent in the second task. However, rather than expressing only a radial component, which was the only component necessary to calculate change in potential in a radial field, students expressed the angular component instead. This is most likely attributed to the use of circular paths in E&M II and/or the salience of the spiral path. Additionally, students in the spiral task were more successful in connecting arc length to the $rd\theta$ expression, whereas during the generic construction task only one group used this idea explicitly.

Notably, the high symmetry of E&M means that when working in the context of a specific problem students only need to attend to one component of a differential length vector. In E&M I students commonly work with radial fields and often only need the radial component, while E&M II involves curling magnetic fields and thus necessitates the angular component of a differential length vector. This most likely accounts for some student responses in both tasks, given that the generic construction task and the spiral task were given in E&M I and E&M II, respectively.

Findings suggest that instruction should focus more on the connection of geometry of coordinate systems to the writing of the generic differential length vector as well as connecting the generic expression to the choice of components within a context. To address the building of generic differential length vectors, an instructional task was designed around the interview task (Appendix C) as a means to explicitly connect changes on a three-dimensional spherical surface to the scaling factors appearing in differential length components (see section 9.5 or Appendix C for more details).

CHAPTER 6

PHYSICS STUDENTS' CONSTRUCTION AND CHECKING OF DIFFERENTIAL VOLUME ELEMENTS IN AN UNCONVENTIONAL SPHERICAL COORDINATE SYSTEM

"Many a small thing has been made large
by the right kind of advertising."

-Mark Twain

This chapter presents a continuation of the "schmerical" coordinates task (see section 5.1). Following the construction of a differential length vector, students' were asked to construct a differential volume element and then subsequently check the correctness of the element. This portion of the task addressed student understanding of non-Cartesian differential volume elements, specifically as a product of differential length elements.

Volume element construction occurred either by combining associated lengths, an attempt to determine sides of a differential cube, or mapping from the existing spherical coordinate system. None of the students were able to arrive at a correct differential length element in the initial task; however, students who constructed volume elements from differential length components corrected their length element terms as a result of checking the volume element expression by integration. Students relying heavily on spherical coordinates displayed further difficulty connecting dimensionality and projection ideas to differential construction. This work continues to add to the literature on students' understanding of differential elements and student understanding of the geometry of multivariable coordinate systems in E&M.

This chapter is being submitted as an article for publication as a companion article to a paper presenting a concept image [38] and symbolic forms [47] analysis of students' differential length construction in the schmerical coordinate system (see section 5.1.5 for overview of these results).

6.1 Introduction

An understanding of mathematical systems, equations, and expressions is often key to the foundational understanding of upper-division physics. Research on student learning in electricity and magnetism (E&M) has indicated several categories of difficulty related to student use of mathematics, including accounting for underlying physical symmetry, extracting information from physical situations for calculation, and interpreting the results of calculations physically [12]. Vector calculus, including vector integration and vector differential operators, is ubiquitous across the E&M curriculum, often providing the underlying representation for relationships between various concepts. A crucial aspect of problem solving in E&M is setting up the mathematical expressions for desired quantities, often in integral or differential form, based on the physical scenario. The prominent role of multivariable calculus operators requires students to have a reasonable command of differential quantities in a two- or three-dimensional space. Additionally, due to the high instantiation of non-Cartesian symmetry, understanding of these differential quantities is often mitigated by an understanding of spherical or cylindrical coordinate systems and the associated differential length, area, and volume elements.

The variation in the use of coordinate systems is one of the key factors in the "vector-calculus gap" [74,73], which represents the pedagogical and conceptual differences between mathematics and physics. Among the differences is the idea that mathematics courses predominantly use Cartesian coordinates, whereas physicists often choose a coordinate system from the symmetry of the physical scenario. Other work in this area notes a large concern over the lack of standardization of variable labeling conventions in non-Cartesian coordinates between disciplines [40]. For this work, we will use the physics convention for spherical coordinates, which labels the azimuthal angle as θ and the polar angle as ϕ.

Beyond this, volume integration in mathematics typically unfolds from thinking about the area between two functions and finding the volume of rotating that area about a specific axis, or finding the volume enclosed between two planar surfaces. In E&M, volume integration is commonly used to determine the total charge of a given object (e.g., sphere or cylinder) with a given charge distribution. In these tasks, students are expected to integrate the product of the charge density and a differential volume element expressed in the appropriate coordinate system. As many of the physical scenarios in E&M are most easily solved in a non-Cartesian coordinate system, differential volume elements include scaling factors that account for the curving of spherical ($d\tau = r^2 \sin\theta \, dr d\theta d\phi$) or cylindrical ($d\tau = s ds d\phi dz$) space, rather than the straightforward $dxdydz$ from a rectangular coordinate system.

While scaling factors can be determined through a Jacobian/coordinate transformation, they can also be constructed less formally with an understanding of the underlying geometry. The latter involves recognizing that the curvature of the space necessitates arc lengths to represent some of differential length components and that the resultant volume element is composed of a product of the magnitude of the length components. The length component and subsequent volume component for spherical coordinates are shown below.

$$d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta\, d\varphi\hat{\varphi}$$

$$d\tau = (dr\hat{r}) \cdot \left(rd\theta\hat{\theta} \times r\sin\theta\, d\varphi\hat{\varphi}\right) = (dr)(rd\theta)(r\sin\theta\, d\varphi) = r^2 \sin\theta\, drd\theta d\varphi$$

However, as shown in the final form of the volume element above, most conventions for writing the differential volume element involve the scaling factors written in front of the set of differentials, obscuring the origin of the terms as differential lengths.

Previous research has addressed student use and understanding of many aspects of vector calculus quantities in the context of E&M, including differential elements [25], integration [14,48,81], applications of symmetries for Gauss's Law and Ampère's Law [12,15,16,24,32,33], and vector differential equations in mathematics and physics settings [37,82]. However, despite the centrality and ubiquity of non-Cartesian symmetry in E&M problems requiring vector calculus operations, little attention has been given to student understanding of differential elements in non-Cartesian coordinate systems, and the extent to which these elements are used in a rote procedural fashion or whether the structure of the expressions has meaning to students when employed. As part of a broader study to investigate these issues, we developed an interview task in which students were asked to construct a differential length vector and a differential volume element for a spherical coordinate system where variable labels and placement are changed from standard conventions. Findings from the differential length construction part of the task are presented in the previous chapter (see section 5.1 for broad overview, 5.1.5 for specific results). The results presented here address the second portion of the task, students' construction of differential volume elements to gain insight into student treatment of this type of differential element used commonly in E&M.

6.2 Context for Research

Course observations were conducted in the first semester of junior-level E&M at the first of two universities (University A). Informal review of student written data on homework and quizzes throughout the semester showed discrepancies in students' performance when writing differential elements for spherical and cylindrical coordinate systems (see section 4.3). It is in this course that students first encountered these multivariable coordinate systems and differential vector elements. Spherical coordinates were introduced and used for several class periods before the introduction of cylindrical coordinates. An in-class quiz was subsequently administered as part of regular instruction. At this point in the class, more students were able to construct differential length vectors in cylindrical coordinates in comparison to spherical coordinates; as the course progressed, homework and exam data suggested students were more proficient with spherical differential elements when solving various integration tasks. This suggested underlying difficulties in students' understanding of how differential elements are constructed and used in particular coordinate systems, and suggested that performance on spherical coordinates was due to extended use early in the semester.

These observations prompted further investigations into students' conceptual and symbolic understanding of differential elements in non-Cartesian coordinate systems within and without physics context. As reported in the previous chapter (see section 5.1.5 for results), analysis of differential length construction showed student attention to various conceptual aspects and symbolic structures needed to construct a three-dimensional differential length vector. However, no student was initially able to completely construct a correct length element. In the following sections, students' construction and checking of the differential volume element for the unfamiliar system is explored in terms of the ideas accessed during the initial length construction, as well as the connections made between the differential length vector and differential volume element for the given coordinate system. This provides further insight into the ways in which students construct and understand this type of differential element that is commonly used in E&M, as well as the ways in which students understand the geometry of non-Cartesian coordinate systems in which these elements are often expressed.

6.3 Relevant literature

Research on student understanding of vector calculus in E&M has addressed topics in several key areas. Much of this work has explored student understanding of Gauss's and Ampère's laws, expressed as a flux and line integral, respectively [12,15,16,24,32,33]. These laws are frequently employed in E&M in the abundance of highly symmetric cases. Thus, much of the literature in either case focuses on students' recognition and/or application of symmetry. It is common for students to overgeneralize the use of either law to include cases where symmetry is not present, or attempt to apply any given coordinate symmetry as long as the Gaussian surface or Ampèrian loop encloses the desired charge or current.

Other work within the realm of vector calculus has explored student understanding of vector differential operators and students' interpretations of vector fields [37,82].

Researchers found students were adept at the procedural calculation when provided tasks of gradient, divergence, and curl, but were unable to appropriately express the conceptual meaning of the operations [37]. These difficulties speak to the larger encompassing difficulties students have with the application and interpretation of mathematics at this level, as categorized by Pepper and colleagues: assessing underlying physical symmetry, establishing mathematical representations of physical situations for the purpose of calculation, and interpreting the results of calculation in terms of the given physical situation [12].

Pepper and colleagues also briefly noted two cases of difficulties with construction of differential elements. In one case, students neglected to include the necessary scaling factors when writing spherical differential areas, using $da = d\theta d\phi$, rather than $da = r^2 \sin\theta \, d\theta d\phi$. This is reminiscent of students' attempts to pattern-match a product of two differentials in a non-Cartesian system from their understanding in Cartesian coordinates [44,77]. Students at various levels are less comfortable when working within polar coordinate systems [10,44,83]. In a second example presented by Pepper and colleagues, a group attempted a three-dimensional line integral using $dxdydz$ as a path length element [12]. These types of errors speak to a larger difficulty with students' understanding and construction of differential elements in multivariable coordinate systems that has been relatively unexplored before now.

Student understanding of calculus concepts has been another area of focus in E&M. Hu and Rebello have investigated student understanding of differentials in the context of integration of charge or resistivity along one dimension [25]. Several resources and conceptual metaphors were used by students across these tasks, establishing four common treatments of differential quantities: derivatives as small amounts, as unitless points, as a cue to differentiate a formula to derive a second differential quantity, and as an indicator of the variable of integration. The identification of the differential as a small amount can be connected to a specific cue for students to integrate, where students identify the need to add up “little chunks” using an integral [14,48]. However, research in mathematics education has commonly reported student treatment of the differential as a meaningless quantity that only serves to identify the variable of integration [28,49–51]. The sum of this work highlights the fact that many students do not connect the differential quantity to a physical meaning, even when given a specific context. While addressing larger concerns about students’ treatment of integration and differentials, these studies primarily focused on integration in one dimension, or on quantities such as resistance or capacitance.

Therefore, despite significant forays into various levels of mathematical understanding, little work has explored student understanding of the differential vector element, in particular as expressed in the non-Cartesian coordinate systems used in physics problems. This work takes a next step toward analysis of student understanding of one of these elements – the differential volume element – as it appears in non-Cartesian coordinate systems used in E&M.

6.4 Theoretical Perspectives

Building largely off of work on student construction of differential length elements within the same task, we analyze student construction of differential volume elements using a concept image framework [38] to make explicit connections to earlier work as well as address new ideas related specifically to differential volume elements.

A student's concept image is the multifaceted cognitive structure that includes all the properties, processes, mental pictures, or ideas that students associate with a particular topic. For example, students may have multiple ways to think about integration: with a Riemann sum, area under the curve, or anti-derivative approach. The sum of these ideas that the student associates with integration make up the student's full concept image; however, a specific task or context may only elicit one of these approaches [14]; this is referred to as the *evoked concept image* for that task or context. While a student may have other ideas related to integration, determining a student's evoked concept image for a particular task (e.g., area under the curve) allows insight into how a student approaches a problem in a given context. Likewise a student's evoked concept image may only have a rule-based understanding, e.g., the integral of $nx^{n-1}dx$ is $x^n + C$, without an understanding of the underlying meaning.

Notably, as a student continues to apply and extend an idea, their concept image grows and may pick up ideas that are false or contradictory with earlier aspects. In some cases, a restricted concept image can develop if a student learns and applies a concept in a very specific way for an extended period of time. When this occurs, a student later meeting a broader context is unable to extend the concept to cope with the change. For example, a student learning Coulomb's law who then spends several weeks using Gauss's law may develop a restricted concept image of integration of electric fields, and may attempt to apply Gauss's law in a case where symmetry is absent, a situation well documented in the literature [12,24,33]. The formation of a restricted concept image is a reasonable way to describe procedural knowledge without conceptual understanding. In these cases, students have only learned a particular concept as a computational entity (e.g., integrals as antiderivatives) and have not been asked to interpret or make sense of the computation.

The use of concept image as an analytical perspective has recently been adopted by physics education researchers studying students' mathematical reasoning in the context of integration [14] and differential vector operators in electromagnetism courses [37], as well as to identify the specific properties and associations students used (or neglected to use) when constructing the differential length element for an unconventional coordinate (see section 5.1).

6.5 Research Design and Methodology

In order to investigate student understanding of associated differential elements, a task was developed in which students were asked to construct expressions for differential elements of an unconventional spherical coordinate system that we called "schmerical coordinates" (Fig. 6.1). The use of an unconventional coordinate system enabled observation of conceptual exposition in the construction process and reduced the effect of recall of memorized quantities as static knowledge. While schmerical coordinates are left-handed, the most noticeable difference in the system from spherical coordinates is the placement of the polar angle: while θ is measured down from the z-axis and ranges from 0 (the z-axis) to π, schmerical coordinates measures alpha up, ranging from $-\pi/2$ to $\pi/2$, with $\alpha = 0$ corresponding to the xy-plane. This necessitates the use of $\cos\alpha$ rather than $\sin\theta$ to describe the projection used to construct the azimuthal component. This change then carries through to the construction of the differential volume element, but becomes abstracted from its origin as a projection. In the first part of the task, students were asked to judge the reasonableness of the coordinate system and to construct a differential length vector (see section 6.1). The second part of the task had students construct a differential volume element and subsequently check the correctness of that element.

Clinical think-aloud interviews were conducted with students in a junior-level E&M sequence at two universities. Four pairs of students (N=8) were interviewed at one university (University A) at the end of the first semester of a two-semester sequence; two pairs and a single student (N=5) were interviewed at a second university (University B) at the beginning of the second semester of this same sequence. The use of pair interviews facilitated authentic discussion between students where they could arrive at a single answer with minimal input or influence from the interview. Groups are identified as AB, CD, *etc.*, with individual students given pseudonyms associated with the letters (e.g., Adam and Bart for AB).

Interviews were videotaped and fully transcribed. Analysis used open coding to identify common actions and recurring ideas across interview groups. This highlighted the ways students treated and constructed these non-Cartesian differential volume elements. Analysis additionally sought to address student understanding of differential volume elements in terms of previously identified concept image aspects associated with differential lengths. This initial analysis categorized these ideas as aspects of students' concept images [38]. Concept image aspects associated with differential length construction include

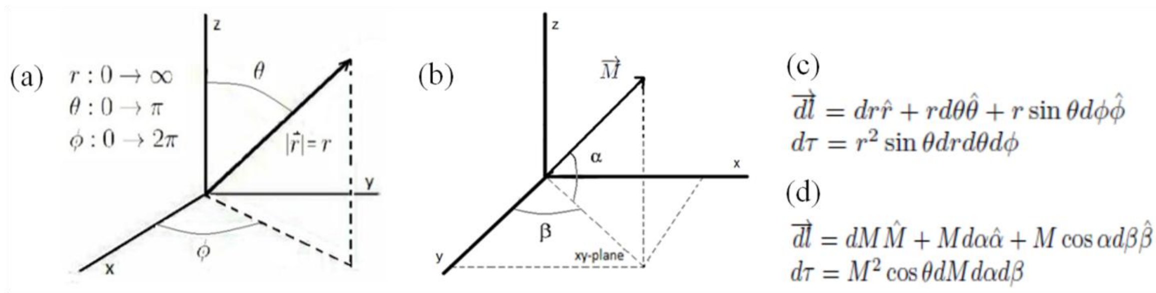


Figure 6.1. Comparison of spherical coordinates and unconventional system given to students. (a) Conventional (physics) spherical coordinates; (b) an unconventional spherical coordinate system ("schmerical coordinates") given to students, for which they were to construct differential length and volume elements. The correct elements for each system are in (c) and (d), respectively.

component and direction, dimensionality, differential, and *projection* (see section 5.1.4 for definitions). Building actions involved *recall* of and *mapping* from other coordinate systems, as well as *grouping* of specific terms.

6.6 Results and Discussion

The schmerical coordinates differential volume, $d\tau$, task took place after completion of a task where students were asked to construct the differential length vector, $\vec{dl}$, for the system. As mentioned above, there were two segments to the volume element task: element construction and checking of the expression.

Groups constructed the schmerical differential volume elements in three distinct ways. Some pairs recognized $d\tau$ as the product of their previously established length vector components, making this a relatively quick process. With mixed results, two of these student pairs had previously attempted to capitalize upon this product understanding by recalling a spherical differential volume element and extracting the length components for comparison to their schmerical $\vec{dl}$ construction. Utilizing a different approach for the construction of a differential volume element, one group attempted to determine the differential volume element by expressing the sides of a differential volume within the geometry of the coordinate system. We distinguish this as a separate approach because this group had not accounted for multiple components in their differential length vector and had not connected the sides of their constructed differential volume to the need for three components of a differential length vector. Lastly, the remaining groups could not exploit the "product of length components" understanding at all, typically either expressing a length element in Cartesian components or expressing the differential length as a single component in the $\hat{M}$-direction. They determined $d\tau$ by mapping to the more familiar spherical volume element.

The last phase of the task involved the checking of the differential volume element. This most often involved integration to obtain the expression for the volume of a sphere of constant radius, but in some cases additionally involved a dimensional analysis. Students were asked to check their differential volume element if they used terms associated with their (incorrect) differential length vector or mapped incorrectly from spherical coordinates and thus had an incorrect term within their differential volume. Students who mapped correctly were not asked to check their differential volume, as the connection between their differential volume and length elements was weaker and a correct differential volume would not likely lead towards reconciliation between the terms.

For students with differential length elements in which only the trigonometric function was missing or incorrect, the checking of the differential volume elements led to the eventual correction of the differential length vector and solidification of the connection between the trigonometric function and the *projection* aspect. Groups using recall and transliteration to construct the differential volume element were still not able to recognize the need to invoke projection: the use of cosine remained a mathematical transformation rather than acquiring a geometric justification. This further supports student difficulty found in the differential length study where students had specific difficulty with understanding the role of the trigonometric function

6.6.1 Construction of a schmerical differential volume element

6.6.1.1 Volume as a product of differential length components

When asked to construct a volume element for schmerical coordinates, AB, CD, and EF immediately knew to take a product of differential length magnitudes.

Interviewer: Okay, so can you make a differential volume element?
Adam: Sure just multiply them all together.

Each of these groups had constructed a differential length vector with three components based on the unit vectors of the unconventional system (see section 5.1). However due to errors with differential length construction, the constructed differential volumes included an incorrect trigonometric function or lacked the trigonometric function entirely.

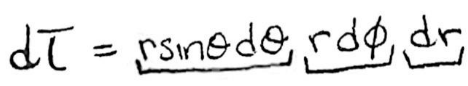
While the creation of a differential volume as a product may seem trivial, during length construction (the second of four tasks), students having difficulty with direct recall to a spherical differential length vector struggled to isolate the length components from the more easily recalled spherical volume. For example, after recalling the spherical differential volume expression, Carol explicitly recognized that the differential volume element is constructed from a product of length components and that the terms are grouped differently in the volume element.

Carol: ... I was trying to figure out which, I guess, um, I don't know, vector direction each come from, um, because I feel like, right? This is right, right? We just write it r^2 for convenience, right? It comes from separated out [terms].

Carol and Dan then began to check the units (*dimensionality*) of terms to confirm their choices for the separated components. Similarly, Greg and Harold recalled the spherical $d\tau$ in an attempt to reconstruct the spherical length vector.

However, rather than recreating the appropriate length components, both pairs grouped angular terms based on variables (Fig. 6.2), pairing the $\sin\theta$ with the $d\theta$ similar to how the terms would appear in multivariable integration. Because this is what the differentials are typically used for in solving E&M problems, the typical expressions for differential volume elements (e.g., $r^2 \sin\theta \, dr d\theta d\varphi$ for spherical coordinates) involve a grouping of terms in a way that dissociates the variables from their particular length component. Students' coupling of the theta terms and ease of recalling the spherical volume element over the assembly of the volume element from the differential length components supports the idea that students do not have the fundamental understanding of non-Cartesian systems necessary for interpreting vector calculus in E&M.

After some time, Carol and Dan were able to correct the grouping error, when Dan made the explicit connection to length vector construction in spherical coordinates and

(a) 

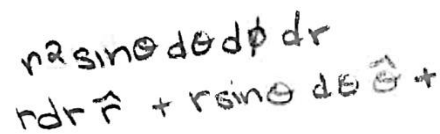
(b) 

Figure 6.2. Two examples of incorrect recall of a spherical differential volume element. (a) Incorrectly distributed length terms in a spherical differential volume written by Carol and Dan. (b) Unsuccessful attempt to reconstruct differential spherical length element by Greg and Harold.

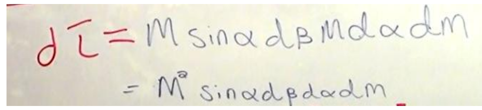


Figure 6.3. Final differential volume constructed by Carol and Dan including incorrect trigonometric function.

connected the $\sin\theta$ to a projection into the plane of the polar angle. Due to *transliteration* of terms, this lead to a $\sin\alpha$ in their length component that carried over into their $d\tau$ as they multiplied length terms (Fig. 6.3).

For Greg and Harold, the dissociation from length components was much more complicated, as neither student attended to the necessary dimensionality.

> Harold: You've got $r\ dr\ \hat{r}$ plus is it $\sin\theta\ d\theta$ or is there an r in there?
>
> Greg: I think there is an r there, it's an r because you want, you want at that radius uh, plus a small angle.

Harold seemed to have a concept image in which the grouping of terms based on like variables rather than the grouping based on correct ideas for each length component. If he had only been concerned with the grouping of variables, all the r terms in the differential length component would have been grouped with $\hat{r}$. As they decomposed the volume element, they ran out of components to be able to express the remaining φ-component. The pair then abandoned this method of construction and began to express the differential length in terms of Cartesian unit vectors (see section 5.1). This goes further to show how a lack of reasoning about *dimensionality* can hamper problem solving in E&M.

6.6.1.2 Volume as product of sides of a differential cube

Rachel and Silas entered the volume construction phase of the task after first constructing a differential length vector as a single component accounting only for change in the radial direction. Without the three components, which pairs AB, CD, and EF relied upon, Rachel and Silas started their volume construction attempt by drawing a small volume at the end of $\vec{M}$ (Fig. 6.4a). This construction elicited a discussion of arc length to account for the sides of the volume element, but did not cause the students to reflect upon the single-component differential length vector constructed in the earlier phase of the task.

> Rachel: That is like the differential volume element right here with dM as the thickness. So if alpha changes you have this arc length.

This shows that students' difficulties with length construction may not have been due to lacking the prerequisite ideas, but to having a limited concept image of the differential length vector as a whole. Given that the majority of problem solving in the electrostatics portion of E&M involves calculating a change in potential over a radial field, the predominance of such problems early in E&M may restrict students' concept image to only needing to account for the radial component of the differential.

Yet the ideas of dimensionality and arc length – ideas that other groups correctly attributed to the length component – were elicited from Rachel and Silas once they were

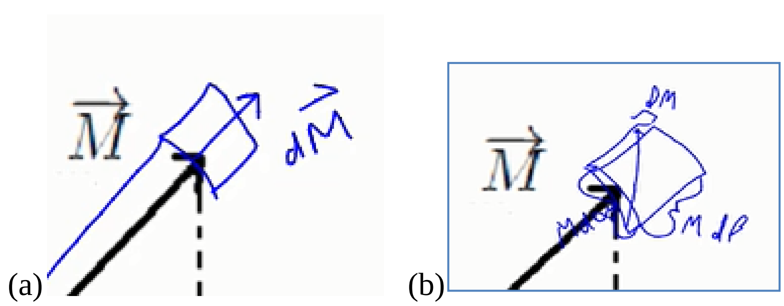


Figure 6.4. Physical construction of a differential volume element by Rachel and Silas. (a) Beginning of volume construction where $d\vec{M}$ represents the pair's $\vec{dl}$. (b) Final differential volume, where location of dM has changed. Students do not connect the sides of this volume to the $\vec{dl}$ components.

able to build the differential volume geometrically. As RS continued in their construction, they correctly represented $Md\alpha$ as the side resulting from a small change in alpha, but placed $Md\beta$ where dM had previously been on their diagram. As a result, dM took the role of the "thickness" into and out of the page rather than in the radial direction, as previously depicted (Fig. 6.4b). This highlights a difficulty of visualizing the geometric directions of the schmerical unit vectors. This difficulty could be connected to a student difficulty reasoning about three-dimensional objects within a two-dimensional space, something sparsely studied in mathematics education research [84,85]. At the end of this differential volume construction, Rachel and Silas were unsatisfied with their differential volume lacking a trigonometric function, and began to build a volume by making a comparison of variables (*transliteration*) to spherical coordinates.

6.6.1.3 Construction of volume by comparison to spherical coordinates

Students who had difficulty with length construction, either constructing a differential length vector with one component (RS, T) or without scaling factors (RS), or who represented the differential length vector in terms of Cartesian symmetry (GH, PQ), could not draw on the same product of terms as the first three groups.

Rather than finding a solution pattern to determine the volume element in schmerical coordinates, students recalled the spherical volume element and then mapped the schmerical variables to the spherical terms. This problem-solving approach is consistent with the "transliteration to mathematics" epistemic game [86]: students identified the task target quantity, found a related solution pattern, mapped new quantities into the related solution, and ended by evaluating the mapping.

After attempting to construct a physical volume and expressing a need to include trigonometric function in their schmerical differential volume, RS began to match variables to the spherical coordinate system (Fig. 6.5a). Here they appropriately accounted for the relationship between theta and alpha, as [$\alpha = -\theta + \pi/2$]. The pair then connected the differentials and rewrote the spherical volume in terms of the associated schmerical variables. They recognized mathematically that the $\pi/2$ shift of alpha from the original theta turns $\sin\theta$ to $\cos\alpha$, but they did not connect the change or original trigonometric function to the physical justification of projection.

> Rachel: Well okay, so if we have it down in this plane then wait, set alpha equal to 0 right? So it's down in [xy] plane. I can convince myself that this is cosine. No, no, that's beta. Hold on. I don't even know.
>
> Silas: Well I know that is right. I know that much.
>
> Rachel: Yeah, ... I just don't know why it is right.

Here Rachel and Silas are able to arrive at the correct expression for the differential volume element by a change in variable but do not recognize that the trigonometric function scales the specific arc length of the beta component. Without being able to connect the cosine to a physical justification, their epistemological stance is to trust the

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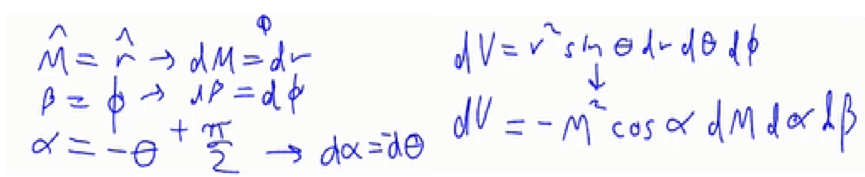

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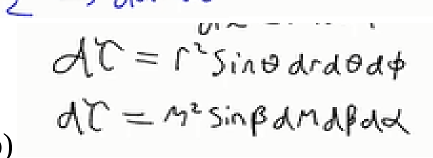


Figure 6.5. Student work constructing a differential volume by comparison to spherical coordinates. (a) Work of Rachel and Silas accounting for the changes in the variables. (b) Work of Tyler directly replacing variables with mathematics conventions.

mathematics [87]. This lack of understanding of the reason behind the projection is pervasive across all groups, especially during length construction (see section 5.1).

After arriving at a correct volume element, RS returned to their differential length vector, but again due to the lack of a trigonometric function in the drawn volume element, they did not connect the length and volume differential expressions. Rachel and Silas then augmented their length vector to include a $d\alpha$ and $d\beta$, in their respective directions, but failed to recognize the need for arc length discussed previously during the construction of the volume. Additionally they did seek to reconcile the differences between the differential elements as the previous groups did during the checking phase of the task.

Individual subject Tyler and group PQ also attempted to map onto a spherical differential element but did so unsuccessfully, connecting the physics variation of the differential element with the mathematical conventions for the spherical coordinate system (θ as polar angle, ϕ as azimuthal). Compounded with the missing idea of projection in the polar length component, this resulted in differential volume elements that include a $\sin\beta$ instead of a $\cos\alpha$ (Fig. 6.5b).

Having had particular trouble with construction of a differential length vector, Greg and Harold quickly constructed their new $d\tau$ from a direct mapping of the previously recalled spherical differential element. Greg initially accounted for the different placement of alpha by writing $(\pi/2 - \alpha)$ as the argument of the sine function, but then decided a direct replacement of the variable would be sufficient.

> Greg: Actually, if you just said $\sin\alpha$ I think it would work. You would just have to know that it points in a different direction.

At this point, they returned to the differential length element upon request of the interviewer and eventually reconstructed a correct differential length element based on the process in the course text [58] (see Fig. A.6). When asked if they were still satisfied with their differential volume element, they had difficulty recognizing the need to reconcile the cosine in their length vector with the sine in their volume element.

Harold:	I still like our volume element=
Greg:	Yeah, I think so.
Harold:	= I don't know about you, this one over here, I still think that/
Greg:	They're the same, yeah.
Interviewer:	Okay, and can you check that that volume element is correct?
Greg:	Isn't that kind of the same question?
Harold:	Oh, you want us to actually do this integral out.
Greg:	Oh. No, but see in down here we've gone with the $\cos\alpha$.
Harold:	Oh, we've gone cosine, oh yeah.
Greg:	And so we might want cosine. Yeah, I think we do, oh wait, let's see. Oh no, that's, alright, yeah we do want these, we want these to agree so they need to be, this needs to be a cosine [in the volume element].

Despite GH’s attempt to deconstruct the volume element as a product of terms, their hesitancy to connect the length and volume terms, coupled with the difficulty deconstructing the volume element due to misuse of the grouping of terms and inattention to dimensionality, show that Greg and Harold did not have a strong understanding of the structure of these differential elements.

Generally, students who struggled with differential length construction were better able to recall the form of the differential volume element in spherical coordinates but had further difficulty connecting the geometry of the coordinate system to the terms in the differential volume element. This appeared specifically as a difficulty associating the trigonometric function as a projection.

6.6.2 Checking of the schmerical differential volume

6.6.2.1 Checking volumes from products of differential length components

Upon checking their differential volume elements, both AB and CD easily recognized that integration of their differential volume would give the expression for the volume of a sphere of radius *M*, but due to their incorrect trigonometric function, integration over the bounds of α yielded a volume of 0 for both groups. This alerted the groups to an error in their length components, which they quickly traced to the $\sin\alpha$ term. Adam immediately recognized the mistaken projection that resulted from directly substituting alpha for theta during their mapping. He articulated that the change in the placement of the angle meant a $\cos\alpha$ was needed to obtain the appropriate length component. Carol and Dan were able to recognize that $\sin\alpha$ was the cause of their unexpected result, but did not immediately connect this to the idea of projection.

> Carol: 0, which means our volume is wrong. Which means, should this be cosine? No, we need.
>
> ...
>
> Dan: I mean, well our trig might be wrong but we also could be running into the problem that we were incorrect about. Oh... when you assumed $\sin\alpha$, you assumed you were basing it off $\sin\theta$ where theta was on a different part of the graph.

Carol first suggested cosine as a way to make the mathematics work. It is not until after a couple of incomplete exchanges that Dan connected the mathematical implications of change in trigonometry to the physical difference in the geometry of schmerical coordinates. The construction and checking of the volume component cued *projection*, the absence of which had previously led to a shift to recall during length construction.

When asked to check the volume element, Frank reasoned using *dimensionality*, saying that integration of the M terms would give units of length cubed and therefore it didn't matter what the remaining integrals gave as a result. Unconvinced, Elliot suggested integration of the full differential volume element, $M^2 dM d\beta d\alpha$. As their expression contained no trigonometric function, their integration yielded a result with π^2 in their answer.

> Frank: π^2, so –
> Elliot: We needed that sine in there.
> Frank: We need a sine or a cosine so we can get rid of a pi.
> Elliot: But I don't know where it comes from.
> [...]
> Elliot: [audible gasp]Oh, I remember where it comes from... like if r is pointing way up here, then we need to get the component that's in the flat plane and then that is times d beta.

The pair recognized they need a trigonometric function to get the appropriate mathematical result, but as with their difficulty during length construction, they could not figure out the particular reason for the inclusion of the term. Shortly after this discussion, Elliot recognized the need for a cosine function to account for the necessary projection and the group corrected their length vector. Just as with CD, EF recognized the mathematical need for cosine but was not immediately able to connect it to the radius term in the $\hat{\beta}$-component.

For students constructing a differential length vector with the three components of schmerical coordinates, the checking of the differential volume element provided students not only with the correction to their earlier differential length element, but led to the connection of the trigonometric function to the idea of projection.

6.6.2.2 Checking volumes constructed from recall and transliteration

As the pair GH checked their differential volume element, Greg became unsure about the reason for the cosine term, despite earlier work during their second attempt at length construction.

> Greg: Why did we change it to cosine?
> Harold: I'm sorry?
> Greg: Actually wait, no, because the negative sign, the negative $\sin\frac{\pi}{2}$ is one=

This further suggests that projection is not strongly tied to this pair's understanding of the differential elements here. It was upon seeing that the computation resulted in the expected answer that Greg regained comfort with the use of the cosine function.

The result of Tyler's checking of his volume element, $M^2 \sin\beta \, dMd\beta d\alpha$, via integration yielded 0, but he was unable to connect this to the discipline-specific variable label conventions or to the projection. At this time the interviewer conveyed the physics convention for the spherical coordinate system and Tyler changed the $\sin\beta$ to $\sin\alpha$. A second attempt at integration still yielded 0, which Tyler connected to the difference in how theta and alpha are defined. However, Tyler still did not connect this to his differential length element or recognize the need for the length vector to have three components. Tyler further drew upon graphical representations of sine and cosine functions to illustrate the change in the angle as a mathematical shift. The rote-computational reasoning resulting in the change in the trigonometric expression substituted for a connection to the projection, as it did for Rachel and Silas.

Notably, even in the interviews in which students treated the differential as a product of lengths, mathematical formalism appeared before geometric reasoning: groups CD and EF first see the shift as mathematical transformation *before* identifying the geometric motivation. Students here engaged with the "doing" of mathematics first and sense-making second. Furthermore, geometric reasoning was not easily accessed, even though the task involved quantities directly related to coordinate system geometry. This shows that students do not necessarily have a strong conceptual understanding of the relationship between coordinate system geometry and differential element construction.

6.7 Conclusions

The construction of and ability to reason about non-Cartesian differential length and volume elements are keys to many of the concepts in E&M that make use of vector calculus. Addressing students' conceptual understanding of the differential elements and how they are constructed in non-Cartesian coordinates, this work shows that students do not necessarily have a strong understanding of the geometrical aspects of three dimensional polar coordinate systems that are important to the invocation or construction of these differential elements in physics contexts in particular.

Our results suggest that students struggle to think critically about the aspects that go into the construction of differential elements, but that some are able to check the validity of their expressions and make appropriate adjustments when prompted. Following construction of a differential length vector in an unconventional spherical coordinate system, analysis of differential volume construction showed three approaches taken by students: multiplication of length components, determination of the sides of a differential cube, or recall and transliteration from a spherical differential volume element. The group initially using the second approach did not include a trigonometric term and subsequently switched to recall and transliteration after not being able to determine the justification for inclusion of the term. In general, recall and transliteration was used in groups that had greater difficulty with construction of the differential lengths. These groups either had difficulty recognizing the need to account for multiple components, suggesting that the task did not evoke the *component and direction* aspect of the differential vector concept image, or instead constructed a differential length vector with Cartesian unit vectors. Additionally, these groups did not try to connect the expressions for the differential length vector and differential volume element.

Furthermore, the construction and checking aspects of these tasks provide stark contrast between those groups who could connect the necessary geometric ideas to the differential volume and those who could not. The checking process only cued projection to students who were already performing more strongly on the task and had accessed *arc length* or *projection* during length construction (see section 5.1), while others only saw the use of cosine as the result of a variable change from theta to alpha into the sine term in the spherical differential volume. Thus some students have an incomplete understanding of the coordinate systems due to misapplication of particular ideas, while for other students the prerequisite ideas are sometimes present but not accessed or activated in this particular context.

Regardless of variations in students' geometric reasoning ability, the differential volume element appeared more accessible to students than the differential length vector, CD, GH, PQ, RS, and T were all able to recall the spherical differential volume element, but only CD was able to reconstruct the differential length components from the volume. The disconnect between the differential length and volume elements for students made it difficult for students to construct or correct their length elements accordingly. CD and GH, in particular, explicitly attempted to use the spherical differential volume element to make sense of their schmerical length vector after failing to directly recall a spherical length vector. Students' difficulty reconstructing a spherical differential length from these terms, as well as a blanket difficulty recognizing the need for a trigonometric projection, further supports earlier work reporting student difficulty accessing necessary aspects for the construction of a differential length vector (see section 5.1).

Lastly, overreliance on spherical coordinates and attempts to map trigonometric functions directly are findings reminiscent of *x,y syndrome* [88], in which a particular process is remembered in terms of symbols rather than underlying relationships between quantities. Likewise, the symbols and trigonometric functions of the differential volume element are remembered in the way they are first taught and lose particular meaning over continued use. This is consistent with the formation of a restricted concept image [38]: prolonged use of a particular idea in a formulaic context or limited range of situations can obscure underlying understanding. Thus, when students meet a broader context, they struggle with the application of fundamental ideas. Bollen and colleagues similarly report that students are able to perform calculations with differential vector operators but struggle to interpret the conceptual meaning [37]. In our work, students' mostly computational use of spherical volume and area elements earlier in the semester appears to obscure the underlying understanding of how these elements are constructed. Meeting the broader context of the unconventional system, students then struggle to apply appropriate concepts.

This accessibility of the differential volume elements, and students' failure in connecting mathematical aspects to geometric aspects, imply that in order to improve instruction of non-Cartesian differential elements in E&M, more focus should be given to how length, area, and volume elements are constructed and determined when problem solving, with explicit emphasis on building the requisite ideas by connecting them to geometric aspects and motions within the space of the coordinate systems.

In order to address these concerns, results of this study have been used to develop preliminary instructional materials in the style of *Tutorials in Introductory Physics* [65] to be used at the beginning of E&M or in a mathematical methods for physics course (Appendix C). These activities structure students' construction of a differential length element in schmerical coordinates in order to engage them with the act of element construction within a non-Cartesian system, and additionally use 3D physical manipulatives to allow students to construct the elements within a physical space in order to elicit geometric reasoning. Based on the pedagogical value of the differential volume construction and checking tasks in helping students recognize issues with the differential length expressions in the interviews, these tasks are included in the materials. Preliminary results of the implementation are promising; the materials seem to generate discussions similar to those in the interviews but allow students to harness an understanding of the physical space, to realize the geometric features of the differential length elements, and to connect those features to properties of the differential volume element. Ongoing testing and development are occurring.

CHAPTER 7

STUDENT CONSTRUCTION AND DETERMINATION OF DIFFERENTIAL AREA ELEMENTS

"Great things are done by a series of small things being brought together."

-Vincent Van Gogh

The differential area is one of the more ubiquitous differential quantities, especially in the electrostatics portion of E&M. While much of the literature has addressed student understanding in various areas of E&M (see Chapter 2), little of this research has given specific attention to differential areas [12,34]. Nguyen and Rebello [34] have also shown cases in which students were unable to interpret the meaning of da in integration. As part of a project to determine student understanding of differential elements used in vector calculus, this chapter addresses students' conceptual understanding of the differential area element and the construction or determination of the differential area.

The differential area is commonly used as both a vector and a scalar quantity throughout E&M. When applying Coulomb's Law to a surface charge distribution the integral takes the form,

$$\vec{E} = k\int \frac{dq}{(\vec{r}-\vec{r}')^{3/2}}(\vec{r}-\vec{r}') = k\int \frac{\sigma da}{(\vec{r}-\vec{r}')^{3/2^2}}(\vec{r}-\vec{r}')$$

Here, students solve for the electric field by accumulating the effects of infinitesimal charges expressed in terms of a scalar differential area. The differential area, *da*, is represented based on the coordinate symmetry of the charge distribution (*i.e.*, $r^2 \sin\theta\, d\theta d\phi$ for a spherical shell of charge). Conversely, the vectors $\vec{r}$ and $\vec{r}'$, which represent vectors from the origin to the location of the differential charge and from the differential charge to the point of interest, are then constructed in terms of their Cartesian elements.

The differential area also appears when calculating electric flux, $\Phi_E = \iint \vec{E} \cdot d\vec{a}$, or magnetic flux, $\Phi_B = \iint \vec{B} \cdot d\vec{a}$, due to varying electric and magnetic fields respectively. The dot product isolates the amount of field passing through differential portions of area, and the integral then accumulates these effects over the whole of the defined surface. To account for relative vector directions, the differential area is a vector but still takes the shape of the chosen coordinate system.

This chapter explores students' treatment of differential area elements, as vectors and scalars, with specific attention to how these elements are built or determined in multivariable coordinate systems. The first sections addresses data collected over the breadth of this project as a depiction of students' treatment and understanding of the differential area, including students' use of Gauss's Law to find the electric field of the point charge during the spiral task (Section 7.1.1), and interviews in which students' were asked to check an imaginary student's solution containing an incorrect differential area (section 7.1.2). Then I contrast two sets of pair interviews in which students were asked to construct a generic differential area vector for spherical and cylindrical coordinates (Section 7.2). This chapter then reports on students' understanding of differential areas, categorizing the various evoked concept images [38] as students construct differential areas in two physics contexts (Section 7.3). This set of tasks address student understanding of the differential area as used in a common equation, the relation of differential areas in terms of coordinate geometry, and the construction of differential areas in specific physics contexts.

7.1 Gauss's Law and the hidden differential area

The work presented in this section addresses student understanding and use of the differential area as part of Gauss's law, one of the most common instantiation of a flux integral. The full expression is given as,

$$\Phi_E = \oint \vec{E} \cdot d\vec{a} = \frac{q_{enclosed}}{\varepsilon_o},$$

where the flux through a defined closed surface is proportional to the charge enclosed by that surface. As an "inverse problem" [15], the use of Gauss's Law involves highly symmetric argumentation to isolate the electric field as the target quantity. This involves defining a Gaussian surface where the electric field is perpendicular to the surface at all points (resolves the dot product) and has a constant magnitude over the whole of the surface (allows the electric field to be pulled out of the integral as a constant). Common Gaussian surfaces include cylinders and spheres, where the surface area is a known quantity. In these case, as the penultimate mathematical step, $\oint da$ can be replaced with the appropriate surface area of the given shape.

The complete bypassing of the writing of the differential area can potentially be obscuring students' understanding in problems where da construction is necessitated. Preliminary classroom observations and review of student work show that students are much less successful with constructing cylindrical differential area elements on course assignments and employ Gauss's Law in cases where the underlying symmetry does not dictate Gauss's law as an appropriate solution pathway (see Chapter 4). The current literature has shown the latter is a common difficulty for students [12,24,32,33].

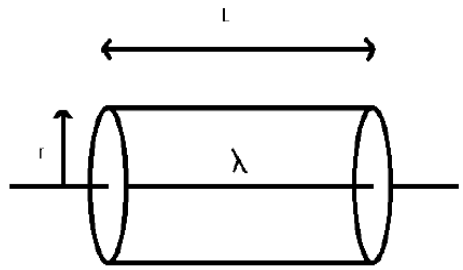
In the main task described here, students are given an imaginary student's solution employing Gauss's law (section 7.1.1). This task provides insight into students' attention to differential areas as part of a common solution pathway where the writing of the differential area can be bypassed in favor of expressing the final area of the surface. In the remaining subsection, there is discussion of two students who employed Gauss's law during the spiral task (section 5.2) to provide additional insight into students' use of Gauss's law as a solution.

7.1.1 Check solution of Gauss's law task

7.1.1.1 Research design and methodology

As part of an attempt to assess students' attention to the differential area used in Gauss's Law, a task was designed in which students were asked to check the solution of an imaginary student trying to find the change in potential between two points at different distances from a line charge (Appendix B.5; Fig. 7.1). To solve for the change in potential, the imaginary student first uses Gauss's Law with a cylindrical Gaussian surface of radius, r, and length, l (depicted in task). The student then uses Gauss's Law to solve for the electric field, with some mistakes. Using this incorrect electric field, the student then derives an expression for the change in potential, to which the student incorrectly attributes a unit vector. This type of task assesses students' ability to follow and critically reason about a given solution as well as gauges students' attention to the differential area as part of a solution method where it is commonly ignored.

The actual focus of this task was the solution for the electric field prior to the calculation of change in potential, given that the writing of the differential area for



$$\oint \vec{E} \cdot \overrightarrow{da} = \int_0^{2\pi} \int_0^{\pi} (E\hat{r}) \cdot (rLsin\theta d\theta d\varphi \hat{r}) = E(4\pi r L)$$

$$\oint \vec{E} \cdot \overrightarrow{da} = \frac{Q}{\epsilon_o} = \frac{1}{\epsilon_o} \lambda L$$

$$\text{Thus } \vec{E} = \frac{\lambda}{4\pi\epsilon_o} \frac{1}{r} \hat{r}$$

$$\text{And } \text{V(f)-V(e)} = -\int_e^f \vec{E} \cdot \overrightarrow{dl}$$

$$= -\int_e^f \left(\frac{\lambda}{4\pi\epsilon_o} \frac{1}{r} \hat{r} \right) \cdot (dr\hat{r}) = \frac{-\lambda}{4\pi\epsilon_o} \int_e^f \frac{dr}{r} \hat{r}$$

$$\text{V(f)} - \text{V(e)} = \frac{-\lambda}{4\pi\epsilon_o} \ln \left(\frac{f}{e} \right) \hat{r}$$

Figure 7.1. Figure provided for check solution task. Students were given the image of a long wire with cylindrical Gaussian surface and imaginary student's solution.

Gauss's Law can be bypassed in favor of plugging in the full area of the targeted Gaussian surface. As such, the writing of the differential area was added as a step in the process, but replaced the correct differential area, $sd\theta dz$, with one used by an actual student on a homework assignment during course observations, $rL \sin\theta \, d\theta d\phi$ (note that r and s may be used interchangeably as the radius). The incorrect differential area includes both spherical and cylindrical components but is suited to neither system. Purposefully, this area also yields a result that is close enough the actual surface area of the outer cylindrical shell, $4\pi rl$ (rather than $2\pi rl$) so that students could decide that the final area was correct and overlook the differential area. This would support the idea that the emphasis on the final area obscures the understanding or attention to the underlying coordinate representation.

The incorrect unit vector was added as a mistake so that students could be satisfied if they felt they had to find an error in the students' solution. If they could then not determine whether the area was incorrect, they could claim to have completed the task.

Clinical think-aloud interviews were conducted with pairs of students (N=8) at University A at the end of the first semester of a two-semester junior-level E&M course sequence. Pair interviews facilitated more authentic student discussion and allowed them access to each other's conceptual understanding, thus minimizing the input and influence of the interviewer. Groups are identified as AB, CD, EF, and GH, to label pairings of students (given pseudonyms) Adam and Bart, Carol and Dan, etc. Adam and Bart were graduate students enrolled in the course for credit. The four pairs identified here are the same pairs interviewed at University A on the Schmerical Task.

As with other tasks, interviews were videotaped and transcribed. Both the transcripts and video data were analyzed to isolate which aspect of the imaginary solution students attended to as well as the understanding of the differential area.

7.1.1.2 Results and Discussion

Three of the four pairs recognized that differential area was wrong, while the fourth accepted the final answer as correct. However, two of the pairs claimed the differential was correct and attributed it either to spherical coordinates or a second cylindrical differential area. This shows that students don't necessarily recognize the appropriateness of differential areas. Furthermore, as part of the derivation of the correct response, each of the three pairs restarted the task.

Students in groups EF and CD immediately identified the differential area as incorrect. Elliot and Frank made this realization while writing out the students' work, but Frank attempted to correct the students' response by replacing the differential area with the cylindrical volume element, which Elliot immediately corrected.

Elliot: So we're integrating the electric field dotted with the area element, which is, I don't think that is right.

Frank: Yeah, that is wrong... So this should just be $rdrd\theta dz$.

Elliot: ...dr though? Because you're not going to do a dr. You're not going to increase the size of the cylinder. You're staying at r.

Notably, after supplementing the correct differential area element, Elliot continued to analyze the incorrect differential area by identifying it as the wrong coordinate system. Then Elliot momentarily suggested the final area was still correct before recognizing it should be $2\pi rL$.

Elliot: There is only a change in $d\phi$, this is, no. He is using the wrong coordinates. First of all, if you use cylindrical coordinates, there is not going to be a θ in there.

Frank: Yeah, so that is problem number one, that probably the main problem.

Elliot: But he still gets the right area.

a) b) c)

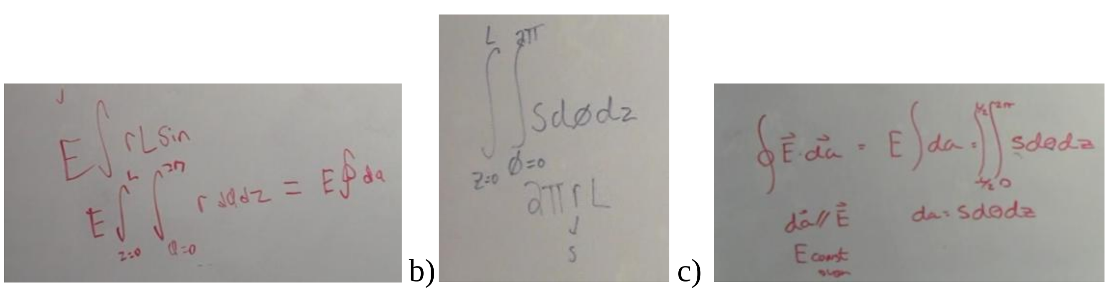


Figure 7.2. Students work showing restarting of Gauss's law for the check solution task. a) Elliot and Frank. b) Carol and Dan. c) Adam and Bart.

Carol and Dan also immediately recognized the differential area as incorrect and attempted to correct the differential area but could not immediately recall the correct element. Dan then tried to reason about the final area so they could correct their element.

Dan: Do they have their $\overrightarrow{da}$ right?

Carol: Um, no.

Dan: It's gross, who the [heck] did that?

Carol: A cylindrical area should be... so they should be taking a cylindrical area

Dan: Of a side, so it should be a circumference, so $2\pi rL$ sounds right to me. I don't know where they're getting $4\pi rL$. That's the part that disturbs me.

Carol: I think it's their sine.

Dan: Well that gave them an extra 2, so, but I mean, I know the circumference of a circle is whatever it is.

The two students eventually restarted the calculation so that they could determine the correct differential area (Fig. 7.2b). After arriving at the correct answer, they sought to identify the source of the imaginary student's mistake. Dan claimed the student (which he engenders as male) incorrectly used the differential area for the end cap because the imaginary student had not been paying attention to the directionality of the electric field.

Dan: Right, so the way he is looking at it is he's taking them to be perpendicular [gestures E parallel to line charge] which would make the dot product 0.

Interviewer: So what is he doing wrong?

Dan: I think the student is looking at the end caps of the cylinder.

Interviewer: Okay.

Dan: When he is doing his da integral, which is incorrect if we have a line charge, and the electric field we assume is pointing straight up.

Interviewer: So the $rL \sin\theta \, d\theta d\phi$ is, that's from the end cap?

Dan: Yes.

Dan here did not attend to the incorrect use of two angles for cylindrical coordinates in the way that Elliot did and had little qualm claiming it to be the differential area for the end cap.

When asked to explain where the terms of the correct differential element came from, Dan's attempt to unroll the cylinder into a sheet shows he had a less robust understanding of cylindrical coordinates. In doing so, he labeled the radius as what would actually be half of the circumference (Fig. 7.3). Carol interjected and offered an alternative (and correct) explanation despite not being able to quickly recall the element earlier.

Dan: So, our Gaussian surface, we want it to be perpendicular to the electric field. We want it to be perpendicular to this chunk, which we can unroll the cylinder=

Carol: Oh, yeah.

Dan: =and get a square, a square with some radius, s, because squares have radii, or rectangles sorry, has some radius, s. If we're still picturing this like a circle, this side goes from 0 to 2π and θ, as we know, goes from o to π.

Interviewer: So how does the s end up in there?

Carol: Or you can do what I do, which is just treat this like a circle. So, you have your radius s and $d\theta$ all around and then you push it through the cylinder dz

Dan: $d\phi$ all the way around and dz

Carol: That's how I, that's how I think of it because you want just what is on the outside of the cylinder, which is like the edges of the circle all the way through.

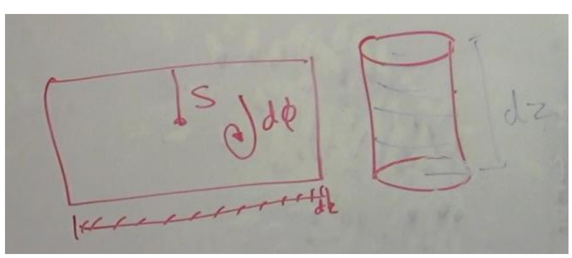


Figure 7.3. Figure drawn by Carol and Dan in order to explain expression for the differential area.

Despite being unable to produce the element without restarting the problem, Carol's explanation depicts a richer understanding of the underlying coordinate systems, showing that she would be able to reconstruct the elements to describe the given area.

As Adam and Bart approach the problem, Adam suggested there is something wrong with the imaginary student's work. He attributed this to the differential area, despite having arrived at a final answer he "would expect." Following Adam's lead, Bart identified the differential area as incorrect but claims the final area is what they should be getting.

> Adam: There is something I don't like about this. Look at, look at their limits, the area they've chosen. They're using cylin/, er, spherical coordinates for a cylindrical symmetry.... I don't think E is okay though. It looks, like that is kind of the answer I would expect. Okay, I'm going to write it out just to check it with myself. [starts recalculating Gauss's Law]
>
> Bart: B: This is correct [points to $4\pi rl$] and this is wrong [da].

Adam restarted the problem for himself and determined the correct differential area (Fig. 7.2c). After calculating the integrals, he then also determined the final area as incorrect. It isn't until the end of the interview that Adam recognized the correct surface area as $2\pi rL$ and suggested the da came from another coordinate system.

> Adam: I don't, I don't think this is right, though. Shouldn't it be 2π, because you were going 0 to 2π. So they were off by a factor of two. That is a part of them using the wrong coordinate system.

In contrast to the other three groups, Greg immediately determined the differential area to be correct after reading off of the final result times the area.

> Greg: $rL \sin\theta \, d\theta d\phi$, so we have our $4\pi rL$, so that's correct.

Harold also tried to restart the problem, but drew two Gaussian surfaces and quickly became confused. GH eventually recognized the incorrect unit vector and engaged in a discussion about whether the potential integral needed another negative sign, but at no point returned to the Gauss's Law calculation.

Overall, students were generally able to recognize that the differential area was inappropriate for the particular task. However, the focus of some groups on the final area and the claim of the differential area belonging to other surfaces show that students possessed an incomplete concept image of the differential area element in terms of coordinate geometry.

7.1.2 Student use of differential areas for Gauss's law during the spiral task

During student interviews of the spiral task (see section 5.2 for task and methodology), two students in separate interviews attempted to derive the electric field due to a point charge. These students then intended to use the electric field to solve for the change in potential. Most students in the remaining interviews simply recalled $\vec{E} = \frac{q}{4\pi\varepsilon_o r^2}\hat{r}$. As the focus of the interview was students' construction of a differential length element while determining a change in electric potential (described previously in section 5.2), their determination of the electric field was not subject to scrutiny. Lenny and Nate however, could not recall the formula, and attempted to use Gauss's law to rederive the expression. While a point charge has sufficient symmetry for this solution pathway, neither student in this case applied the right area element.

Both students, being in the second semester of E&M, struggled with the use of Gauss's law. After first being provided with the correct expression for the electric field, Lenny dismissed it as being the result of Coulomb's law (an equally valid solution pathway) and instead began to employ Gauss's law.

Lenny: I guess I would start by using Gauss's law to find the electric field that we got I guess.

Interviewer: Which I gave you.

Lenny: That's Coulomb's Law.

Interviewer: What do you mean?

Lenny: Uh, [starts writing Gauss's law].

Interviewer: Alright, go ahead then.

As Lenny continued to work out the electric field, he made the requisite symmetry arguments for a Gaussian surface, until he has isolated the integral of the differential area (Fig. 7.4). At this point, he wrote a separate integral where he used rdθdr for his differential area, yielding πr^2 as his surface area. Without completely defining his Gaussian surface, Lenny arrived at the differential area for a circular sheet, rather than the spherical shell necessary to enclose the point charge with sufficient symmetry. He questioned the absence of the "4" that would appear with the correct surface area, but reasoned it away, apparently associating it with the $4\pi\varepsilon_o$ coefficient term.

> Lenny: Uh, so, I guess I got to draw it out [draws a dotted circle around a point charge]. So my Gaussian surface, I call that r, or big R, so it doesn't get too confusing. E is constant as we go furt/ or, uh, it's parallel to the area so that is just E da is equal to q enclosed, which q is just equal to big Q, over epsilon not. uh, yeah, so E is constant over the Gaussian surface, so it will just be E closed integral da, q over ε_0, where da is just equal to θ from 0 to 2π, big R from 0 to r, rdθdr. So it'd just be 2πr squared over 2, so it'll just be πr^2 which we know is the area and I guess that's just the same thing. Oh, am I forgetting the $4\pi\varepsilon_0$? No that's different, okay.

Despite making the appropriate symmetry arguments and being able to recall a differential area, Lenny did not recall the correct differential area or seek to rectify his use of Gauss's law with appropriate coordinate symmetry. Having arrived at what he

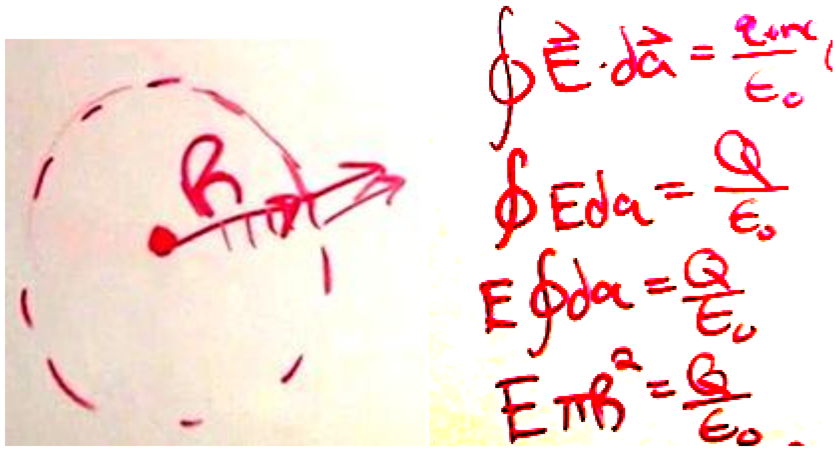


Figure 7.4. Lenny's incorrect use of Gauss's law. Lenny uses symmetry but includes the incorrect surface area.

considered the right area ("it'll just be πr^2 which we know is the area"), he was satisfied with the difference from the correct provided formula.

Nate, also enrolled in the second semester of E&M, at first did not recognize that he could use Gauss's law for a point charge. After not being able to recall the electric field, the interviewer asked how he would go about getting the electric field if he couldn't remember, then offered the idea of Gauss's law.

> Interviewer: How would you go about it [solving for the electric field] if you couldn't remember it?
>
> Nate: I would look it up in a book.
>
> Interviewer: Uh, so like Gauss's law then.
>
> Nate: Hmm?
>
> Interviewer: Do you think you could use Gauss's law?
>
> Nate: For this?
>
> Interviewer: For a point charge.
>
> Nate: For a single point charge? To my knowledge, the way I learned Gauss's Law is that if you have an object that is symmetrical, you can draw a Gaussian surface around it and solve for that electric field at that Gaussian surface but I don't think we ever did that for a point charge. We did it for a sphere. We did it for a cylinder, for a plane.
>
> Interviewer: What is a sphere but a really big point charge?

Nate: I, I see what you're trying to say.

In order to see how Nate would go about using Gauss's Law, I gave him the equation after he spent some time trying to remember it himself, and only being able to recall pieces of the finally result which he attempted to attribute to the original expression for Gauss's Law. Once I wrote out the flux integral part of the expression, he finished the equation. Without reasoning through any of the steps as Lenny did, he immediately wrote "$E2\pi r$" (Fig. 7.5).

Nate: [Writes E2πr] um, so when we do q enclosed that's when we have to, um. God it's been a long time, um, so that like. I guess I'm confused when using a point charge because my instinct says it will also be, um, q times $2\pi r$ over ϵ_o because when we do the/ When we do the um/ When we integrate over/... I mean q enclosed is just going to be Q times some area, so it would be, because its dq. um/

[...]

Interviewer: I: So what is your $2\pi r$ here?

Nate: N: It's the point.

Interviewer: I: The $2\pi r$ is the point?

Nate: N: But, well no it's the Gaussian surface around the point, but r/ So this is where I could use some assistance when talking about a point charge, when we're doing the Gaussian surface around the point we can make r really, really, tiny to the point where it is just infinitesimal.

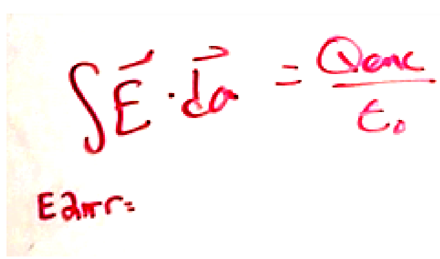


Figure 7.5. Nate's incorrect use of Gauss's law. Nate doesn't make symmetry arguments, but instead just includes an incorrect surface area.

Nate struggled to define this aspect of Gauss's law and at several instances attempted to use portions of partially remembered equations. This, followed by the forceful insertion of an incorrect surface area, which also lacked the proper dimensionality, shows Nate sufficiently struggled with the use of a fundamental E&M equation or the implementation of the appropriate coordinate system to this task. While Lenny explicitly attended to the differential area element, he did not appropriately attend to the underlying symmetry that went into the construction of their differential area, showing that even after two semesters of E&M students struggle to account for the underlying symmetry.

7.1.3 Conclusions

Findings from this section focus on students' attention to differential areas within a solution employing Gauss's law, a high symmetry technique that typically bypasses the writing of the differential area. In the first task, students were provided students with a mock Gauss's law solution with an incorrect differential area. Within four interviews, students generally recognized that the differential area was incorrect. Only one group failed to recognize the mistake in the solution after accepting the final expression for area as correct, despite it being a factor of two off. When solving the spiral task, two students attempted solving for the electric field using Gauss's law but ended up with incorrect areas.

In order to verify the differential area was incorrect, each group restarted the problem from Gauss's Law and rederived the differential element in order to determine the correct expression. Additionally, students in two pairs wrongly attributed the incorrect differential area as being part of another coordinate system or as the end cap of the cylinder. Another student attempted to use the differential volume, a common mistake seen during course observations. In the spiral task, Lenny and Nate both incorporate areas that are inappropriate for describing the type of spherical Gaussian surface needed for a point charge. Instead, they introduced an area for the end cap of a cylinder and a circumference of a circle as stand-ins. These aspects of the interview findings suggest that not all students have a completely robust understanding of coordinate system geometry and how the geometry connects to the representation of the differential area element.

However, when asked to explain where the terms in the correct differential area originated, several students were able to do so. Elliot attended to the ideas of arc length as he did in the previous schmerical task (section 5.1). Carol was able to describe the construction of a differential area using the differentials to define a circle via a radius and differential angle which was then added up over the length of the cylinder. Notably, she was now able to access the ideas of arc length, which the group was unable to attribute the unfamiliar schmerical system.

Lastly, while students were generally able to recognize the incorrect differential area expression, they initially accepted $4\pi rL$ as the surface area for a cylindrical shell, focusing on the final area result as correct. Upon deciding the final area was correct, Greg accepted the incorrect differential area despite it being inappropriate for any coordinate system. Similarly, Adam and Bart did not recognize the final area was wrong until the end of the task, as it looked like what they "would expect." For the spiral task, where students actually calculated Gauss's law, Nate, who struggles to remember Gauss's Law at all, did not attempt to construct a differential area and shifts from Gauss's law to the expression of electric field multiplied with his determined area. Distrusting the correct electric field expression obtained via Coulomb's law, Lenny went through the application of Gauss's law. During this analysis, he stated that the electric field is parallel with the area vectors and constant over his surface, but he does not connect these back to the actual coordinate system and thus doesn't recognize the incorrect area element. These instances suggest that students do not necessarily recognize the final expressions for surface area despite the common use of Gauss's-law-type problems.

7.2 Generic differential area element task

7.2.1 Research design and methodology

In an effort to see if students could spontaneously construct the three differential area elements in both spherical and cylindrical coordinates, clinical think-aloud interviews were conducted at the end of the second semester of junior-level E&M at University A. These interviews were conducted with two pairs of students: Bart and Harold, and Dan and Victor. Given the limitation in the number of interviews conducted, the purpose of this section is to add to the current presentation of student understanding of construction of differential elements within specific coordinate geometry.

Designed to be similar to the schmerical length construction task, students were first asked to construct a generic differential area vector for spherical coordinates. A correct response would include three components, one for each pairing of differential lengths, as derived in Appendix A. Once students were satisfied with their response, they were asked to construct a generic differential area vector for cylindrical coordinates. This sequence of questions on the generic differential area followed three other tasks; two of which were the flux task (see section 7.3; Appendix B.3), and third task involving current density through a section of circular wire (not described in this work). Purposefully, the flux and current density tasks involved a different differential area vector from cylindrical coordinates so that they could be used as references for students.

The protocol was designed to allow the interviewer to ask how this differential area compared to either task, should the student only construct one component. Should the students recognize the existence of multiple differential area vectors for a cylindrical coordinate system but not for spherical coordinates, then they could be asked to compare the two systems.

The interviews were videotaped and transcripts were written to account for student dialogue, writing and drawing. As only two pairs of students were interviewed, this set of data was not collected to make broad claims about student understanding, but to pilot a possible instructional activity building conceptual understanding of differential area elements. This line of questioning represented an early attempt at eliciting and building student understanding of differential areas. As the results stand now, I use the data to contribute to the discussion in this chapter on students' understanding of differential area elements in curvilinear coordinates. A more robust data collection is presented in section 7.3, which in turn guided the actual development of instructional materials described in section 9.5 and Appendix D.

7.2.2 Results and Discussion

Victor immediately constructed the differential area for the surface of a spherical shell, which he drew in order to get a sense of the shape (Fig. 7.6). After drawing the differential area on the shell, Victor next determined one contributing differential length component accounting for the changing theta component, then tried to reason about how the side lengths would change as it was moved to higher position on the sphere. Dan interrupted at this point, suggesting the addition of a dz, which Victor dismissed as it is not a component of this coordinate system. Victor then added terms for the second differential length: an arc length of $rd\phi$ to which he then added a $\sin\theta$ based on the geometry of the sphere.

> Victor: Assuming this is a small angle, which it is because it's $d\theta$. So this is $rd\theta$. Then this part could be at any height.
>
> Dan: $rd\theta dz$.
>
> Victor: Well we don't have z...then this bit would be $rd\phi$ but then we need a $\sin\theta$ because this area could also be higher.

After constructing the surface differential area, Victor immediately stated that others could be constructed based on combinations of different coordinate variations. While he initially listed off an area and direction that were incompatible, he was able to construct the component through attention to the geometry of the coordinate system. Dan mentioned that he was able to integrate to an area, showing a focus on getting to the final resulting area. He attempted to follow this reasoning and ended up rederiving the differential area for the outer shell.

> Victor: But you could also do, like the area, depending on you could pick any two of the varying variables. We could do like a $drd\theta$ area if we needed to, which would point, uh, it's be like a square in that direction, in the $\hat{\theta}$-direction.
>
> Interviewer: What are you thinking Dan?
>
> Dan: That if I was to integrate that I'd get exactly what I'd want. You'd get an area...I agree with what you said, that we could alter it based on how things change... based on what variable we're looking at there are different ways to rewrite things that give you like an area product, if like dr was, if like we could fix r then you get the $r^2\sin\theta$.

At this point, Victor stepped in and constructed one of the remaining differential areas by drawing the coordinate changes on his diagram and reasoning through labeling each side of the constructed area (Fig. 7.6).

> Victor: I'm just working in the plane of $\theta = 0$. We'd have some flat plane. This would be dr, this would be $rd\phi$. That would be a legitimate area vector too, in the theta direction and if we move this up to some different theta location, this would get small with the $\sin\theta$...That would be a legitimate area vector too if we wanted to integrate over an area slicing into a sphere.

Dan then attempted to construct the last differential area in the $\hat{\phi}$-direction. He initially included a $\sin\theta$, showing further difficulty with the construction process, as these types of areas are not ones he would have used in calculation or been directed to think about. Victor questioned the inclusion of the trigonometric function, at which point Dan assumed he needed a sine of the other angle. The difficulties centered around the $\sin\theta$ term and its specific role in differential area construction in these interviews support previous findings where students were unsure of the origin of trigonometric functions when constructing differential length vectors (section 5.1).

Victor attempted to help Dan by suggesting he needs to figure out what the sides of the differential area are, at which point Dan mentioned the motion of ϕ. Knowing ϕ was

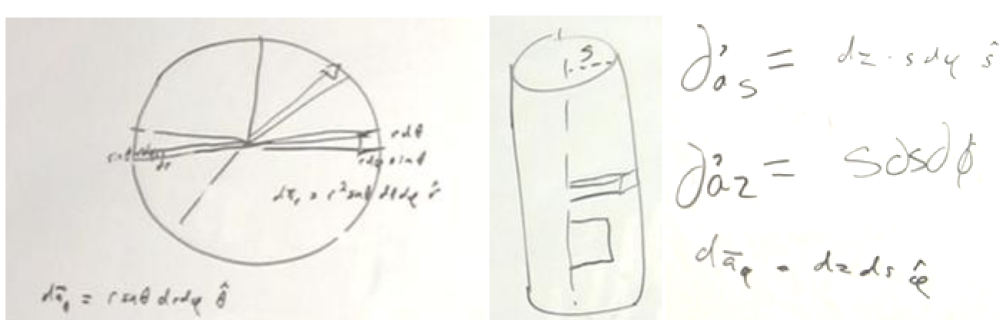


Figure 7.6. Dan's and Victor's construction of differential areas in both coordinate systems.

being kept constant, Victor then constructed the differential area by tracing out the sides corresponding to the changing variables.

Victor: Why do you have a $\sin\theta$?

Dan: Is it $\sin\phi$?

Victor: Well, it was helping me to draw it, because it'd be like a little square, and this one would be in the r direction and the θ direction. Just have to figure out what these side length are.

Dan: So ϕ is going around.

Victor: It's theta that is varying. We're keeping it at a constant ϕ [construct with dr and $rd\theta$]. The question is, does it need another angle. If I put this at a different ϕ, does it look the same...where I had the other one before if I took it and moved it up to a different θ, it would decrease in size because the angle goes up.

While considering the construction of the term and how the differential area would look at different measurements of ϕ, Victor reengaged with his earlier decision to include a $\sin\theta$ in da_θ. Thus, while Victor proved fairly adept at geometric reasoning within the coordinate system, he wasn't explicitly connecting the differential areas to the differential length components. As he sought to justify the inclusion of the trigonometric function beyond his assumed geometric conceptions, he connected the differential areas and volumes, recognizing that multiplying any area by the missing change yields the differential volume element. This solidified all the differential elements for the pair.

Victor: I'm not sure. Maybe this doesn't need a $\sin\theta$ either

Dan: I think you're right. If you were like at $\pi/2$, that is where sine is at its largest, then it gets smaller as you go you, and I agree with [da_ϕ]

Victor: I mean, I guess you can kind of check because if you multiply [surface da] by dr, you get your volume element, and if you multiple [da_θ] by $rd\theta$ which is the missing thing, you get the volume thing...

The construction of terms in cylindrical coordinates was more successful for Dan and Victor as they were able to construct each differential area component without difficulty.

Victor was successful in constructing the multiple differential areas of spherical coordinates due to a specific attention to the geometry and changes in variables. However, despite his success, Victor still struggled with the inclusion of the sine function, until he was able to connect the area and volume elements. Dan was less adept at construction and struggled to construct either of the two unfamiliar components, suggesting that Victor's recognition and ease with construction of the other two differential elements is not common among most students.

Bart and Harold had significantly more difficulty being able to construct any differential areas in either coordinate system. When first asked to construct a differential area vector in spherical coordinates, each student simultaneously wrote a different incorrect element.

Bart:	[writes $sdsd\phi$]
Harold:	That's going to be, spherical, $\rho^2 \sin\theta \, d\rho d\theta$.
Bart:	Spherical?
Interviewer:	Spherical.
Harold:	Yes, spherical. No, that's cylindrical.
Bart:	Sorry. I forget.
Harold:	Unless I made a mistake. Spherical or cylindrical?
Interviewer:	Spherical.
Bart:	Okay. Let me try something. The area is $4\pi r^2$. So if I take the integral. This is correct.
Harold:	Oh. I did a volumetric sphere.

After verifying that they should be doing spherical coordinates, Bart suggested his answer was correct because it would integrate to the correct area. Harold, accepting this, claimed his response was a volume element. They each simultaneously calculated an integral of Bart's differential area element but didn't get the correct area. In order to fix this, Bart added a $d\theta$ to his differential area, which still does not yield the desired $4\pi r^2$.

After Bart changed the s to s^2, and still failed to get the correct area, the interviewer asked them to construct a differential area vector in cylindrical coordinates, hoping that this would be an easier system for them to work in. Harold wrote a correct element, $rdrd\theta$, and Bart returned to an $s^2 d\theta d\phi$ deciding that now the final area needed to be $\pi s^2 L$.

Harold then incorrectly included an l in his differential area so that it would integrate to what he deemed the correct term. Notably, this shows an inattention to dimensionality of terms, as the expression was already in units of area. When then asked for the direction of their differential area, Bart initially suggested the $\hat{\phi}$ direction, then replaced that with an $\hat{n}$ to show that it depends on which surface of the cylinder is chosen. The "$\hat{n}$" would then be replaced with the unit vector for that surface. Notably, in his depiction, $\hat{\phi}$ is incorrectly depicted as radially outward (Fig. 7.7). The correct unit vector for a differential area on this surface is $\hat{s}$, which is written but not attributed to any surface on the diagram.

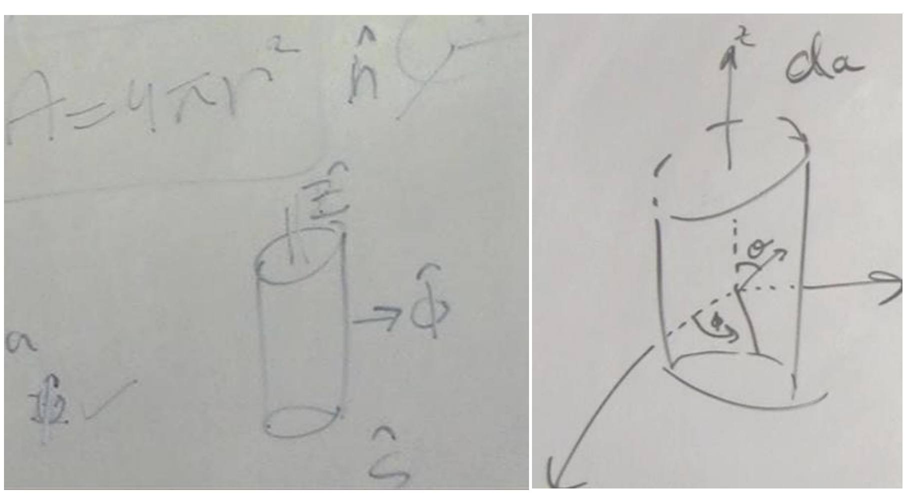


Figure 7.7. Attempts to depict cylindrical coordinates by Bart and Harold.

Interviewer: So what would the direction be then?
Bart: ϕ.
Interviewer: Okay. Why ϕ?
Bart: No, n, $\vec{n}$. So when/ It depends.

At this point Harold also constructs a cylindrical coordinate system with the angles θ and ϕ (Fig. 7.7). Harold then articulated that he couldn't decide between the $\hat{\theta}$ or $\hat{\phi}$ as the unit vector for his differential area. Due to time constraints and the difficulties encountered in both either coordinate systems, the pair was not asked to return to spherical coordinates.

Compared to Dan and Victor, Bart and Harold struggled immensely and were unable to settle on one correct differential area. The pair spent most of each task tacking elements onto a differential area so that it yielded the final area upon integration. While they recognized the correct surface area of a sphere, they used the volume of the cylinder instead of either area. In order to arrive at this post-integration, Harold unknowingly altered his correct differential area into a differential volume element, failing to recognize the incorrect dimensionality of this expression. The group then struggled to determine the unit vector for the area, revealing some underlying misunderstandings about basic properties of cylindrical coordinates, notably Harold's use of two angles to describe the coordinate system. Bart insisted that the area is sufficient for whichever direction is needed by including an $\hat{n}$. This shows that even after two semesters of E&M, some students struggle with basic properties of coordinate system representation and connecting those properties to differential element construction.

7.3 Construction of differential areas in physics contexts

Extending the investigations of student understanding of the construction of differential areas, two tasks were designed involving integration over a given area. The main purpose of constructing differentials in E&M is for use in integration to find physical quantities. Both tasks were adapted from standard problems in the widely used course text[14]. This examination allowed for identification of students' conceptual understanding of differential areas in terms of students attention to geometric representation and aspects of the physical system.

7.3.1 Research design and methodology

In the first of the two tasks, students were given the expression for the magnitude of the magnetic field induced by a long straight current-carrying wire and asked the find the magnetic flux through a square loop (Fig. 7.8). The task as it was presented to students is included in Appendix, B.3. The varying magnetic field requires an integral expression for flux, $\Phi_B = \iint \vec{B} \cdot d\vec{A}$. This leads students to consider the differential area as a vector

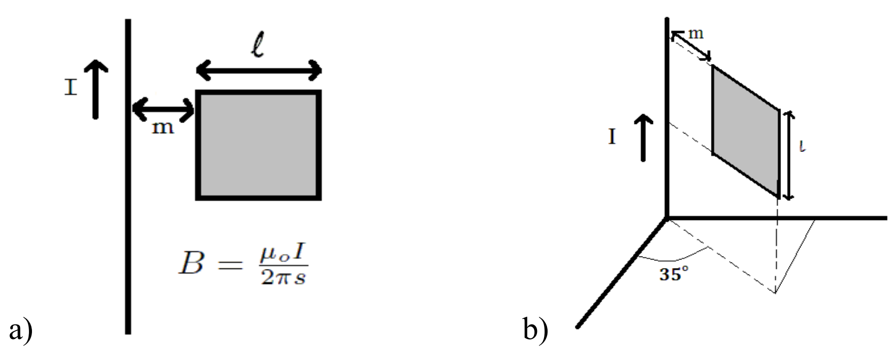


Figure 7.8. Figures provided for the flux task. (a) Depiction of a square loop (shaded) of side length l at a distance m from a current-carrying wire. (b) Figure showing a rotated loop given to students that worked only in Cartesian coordinates.

quantity. Given the curling nature of the magnetic field, cylindrical coordinates are optimal, but Cartesian coordinates can be used if students rewrite the magnetic field with the appropriate variable. The magnetic field was purposefully written as a magnitude so that the unit vector, $\hat{\phi}$, did not influence student choice of coordinate system. Students invoking a Cartesian differential element were asked how their answer would change if the square were rotated out of the board by some angle; the students were given a second figure to illustrate this (Fig 7.8b).

In the second task, students were asked to construct an integral to solve for the electric field a distance x from a circular sheet of constant charge density, σ (Fig. 7.9). The full task as it was presented to students is provided in Appendix, A5. The typically approach for this problem, given the distance between where the field is being measured and the charges, involves using Coulomb's Law, $\vec{E} = k\int dq(\vec{r} - \vec{r}')/(\vec{r} - \vec{r}')^{3/2}$, where dq is a differential charge and $(\vec{r} - \vec{r}')$ is a displacement vector from the location of dq to the electric field measurement. Since the charge is distributed over a circular sheet, dq can be expressed as the product of the surface charge density and a differential area representing the charged surface.

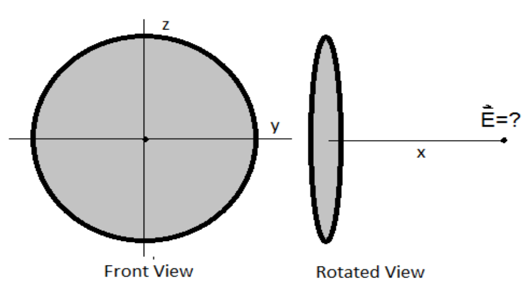


Figure 7.9. Figure provided for the charged sheet task. Depiction of a charged sheet (shaded), with front and rotated view.

These tasks were administered as parts of multi-task interviews to students in the second semester of junior-level E&M at two universities. Two pair interviews (student designations B&H and D&V) were conducted at University A, followed by six other individual interviews (J, K, L, M, N, and O) with a different set of tasks the subsequent year. Interviews at University B involved two pairs and one individual student (P&Q, R&S, and T). Students in pair interviews were only given the flux task. Individual interviews featured both of the described tasks, separated by a line integral task. Pseudonyms are provided for students corresponding to their identifying letter (i.e., Jake for J).

As part of both interview questions, after completing the task students were asked to elaborate on their choices of differential areas in terms of how they was chosen or why they contained particular components. Interviews were videotaped and later transcribed. Transcriptions and video data were analyzed to seek commonalities in students' treatment of differential areas, as well as related difficulties, using a *concept image* [38] framework from mathematics education. A student's concept image is a multifaceted and dynamic construct, including any ideas, processes and figures the student associates with a topic. The particular aspect(s) called forth, referred to as the *evoked concept image(s),* depends on the task and context. Our analysis sought to identify evoked concept images of differential areas elicited during integration in E&M tasks. This categorizing of students' treatment and invocation of the quantity provides insight into students' use of differential area quantities as part of problem solving in class.

7.3.2 Results

From students' progression through the interviews, we identified several particular concept images of the differential area evoked across students' integral construction. In approximately a third of interviews, students treated the differential area as a small quantity, which is a common treatment of differential quantities by physics students [49,56,81]. Students commonly treated the differential area as constructed of differential lengths, which was largely productive. Due to the focus of students' attention, the specific nature of the concept image ranged from a product of differential lengths to an incorrect sum of differential lengths to the product of a constant length with a differential in one direction. Other representations of the differential area included the derivative of the expression for the given area and the full area itself. Ideas related to using the full area to construct *dA* were a hindrance to students in the absence of high symmetry. These five processes for constructing the differential area encapsulate all interviewed students' choices for these two specific tasks. Additionally, several students' evoked concept images varied over the course of the interview task, reflecting a multifaceted concept image.

7.3.2.1 Small portion of area constructed from differential lengths

Students commonly associated the differential area as a small quantity. However, due to the focus of students' attention, the specific construction of terms ranged from a product of differential lengths to an incorrect sum of differential lengths to the product of a constant length with a differential in one direction.

7.3.2.1.1 Product of differential lengths

Treatment of the differential area as a product of differential lengths was productive for students and most typically led to the correct expression. This entailed students recognizing a differential area on a particular surface as a product of two small changes in two given directions, respective to the needed area and the given coordinate system.

Despite recognizing the curling magnetic field, students typically approached the first task with a Cartesian coordinate system, attending more to the square shape of the loop.

> Molly: Since it's a square, Cartesian coordinates would just be the easiest to integrate over it, so that would just be like a little bit, like the differential area is just a little bit in the x and then a little bit in the y.

Thus the two differentials here were a combination of a dx and dy, or dy and dz, depending on how students placed their Cartesian axes. Three other students, Kyle, Oliver, and Tyler, expressed similar reasoning with their choices of differentials, using either the idea of little changes in the necessary variables or referring to specific Cartesian axes.

While the use of Cartesian coordinates are sufficient for solving the flux task and otherwise appropriate, cylindrical coordinates are more appropriate given the curling nature of the magnetic field, the direction of which is defined with $\hat{\phi}$. Molly displayed no difficulty in solving the task in cylindrical coordinates as opposed to her earlier solution using Cartesian. However, when asked how their answer would change if the square were rotated out of their Cartesian plane, the three other interviewees responded that the differential area would now include a trigonometric function to account for the decrease in flux. Students in two of these cases indicated that the magnetic field would still be directed perpendicular to the board, despite not being in the plane of the board, which is physically incorrect.

Kyle: So if we do it like that...where the angle relative to the $\hat{z}$ direction is 30^o so it ends up being a, you get a dz where we only want the component in the $\hat{z}$-direction, so that's going to be *mumbles* the cosine...yeah, so what it would end up being is $da \cos 30^o$, where da is just our magnitude, still in the $\hat{z}$ direction.

Interviewer: Okay, so the magnetic field is still in the $\hat{z}$ direction when we're rotated our plane out?

Kyle: Yeah, the magnetic field should still be in that [$\hat{z}$] direction since it's just induced by the wire.

Oliver, while reasoning about this portion of the task, defined the magnetic field with the unit vector, $\hat{\phi}$, and still insisted that the amount of flux through the rotated loop would be less.

Oliver: So it adds a sine or cosine component because you're changing the amount of field lines by like θ.

...

Oliver: Yeah, I would need the equation that relates B and I to do that. ...I mean, it's a curl. I'm pretty sure it's a curl, so if I is in the direction, I'm pretty sure it would be around the wire in the $\hat{\phi}$ direction.[rewrites given magnetic field with $\hat{\phi}$]

...

Interviewer: Yes, talk more about the $\hat{\phi}$ and does that change anything for you.

Oliver: Does it change anything for me.

Interviewer: It may not. That's just the only way I can phrase it.

Oliver: No, it really doesn't. So it means that I'm thinking that/ So like this is what I mean by the ϕ, B in the $\hat{\phi}$ equals. that is equal to the/ ...And so it's I were to draw the magnetic field this would be curling around to go through this loop and when you change it, the amount of them would change.

Interviewer: Okay, so you're saying when you rotate, you're still going to have that trig function there.

Oliver: Yes, yeah, because it doesn't change that.

Rather than thinking in terms of cylindrical coordinates or arguing that a rotation of the plane preserves the parallel nature of the magnetic field and area vector, these students continue to express their differential area in Cartesian components with the addition of the trigonometric function of the given angle.

When solving the circular charged sheet task, where students more easily associated the task with polar coordinates, the *product of differential length* concept image was equally productive for students in defining a differential area. Because the differential area in polar coordinates is not exactly a simple square, students needed to include the necessary scaling factors.

> Molly: ...to create a differential area on this circle we have we'd move a little bit ds and then we'd move a little $d\phi$, which is, well, a little bit in the $\hat{\phi}$-direction. Which is $sd\phi$ because of the arc length formula.

Only two other students were able to correctly include the radius in the expression for arc length. Kyle specifically wrote out the differential length for spherical coordinates, from which he'd chosen the two appropriate lengths, explaining dA as "length times length" (Fig. 7.10). A fourth student recognized the need for two lengths but used the full radius of the circle for his arc length, which he treated as a constant during his integration (Fig. 7.11). Thus while he demonstrated an understanding of how to construct a differential area, he was unable to arrive at the appropriate expression.

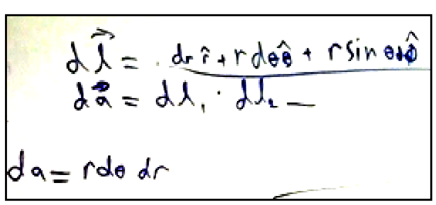


Figure 7.10. Kyle's explanation for his choice of dA in the charged sheet task, where he selects appropriate differential length elements from the generic length vector.

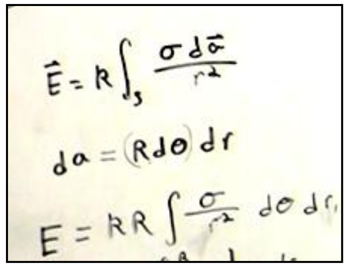


Figure 7.11. Oliver's solution for the charged sheet task, where he treats R in $Rd\theta$ as constant.

7.3.2.1.2 Rectangle with constant height and differential width

The last categorization, where students treated the differential area as a strip of height, l, and width, ds, is specific to the flux task. This is an appropriate solution as the magnetic field only changed in the direction of increasing distance from the wire. In two interviews, students reasoned about the physical symmetry and implicitly integrated in the direction parallel to the wire, producing an l in the equation. While Dan and Victor quickly asserted this solution, Lenny struggled with his solution, first attempting to define the current direction as the vector representing the magnetic field. After further analysis of the task he decides upon ldl, noting that the differential area has the proper dimensionality.

> Lenny: … ldl in the $\hat{n}$-direction… dl being the length to integrate the field over… that l I'd assume to be this one l right here, which would make the area, but I wouldn't feel like I'd have to integrate because the field is constant on that portion... If $[s]$ was the distance away, so that would be like ds maybe. (Fig. 7.12)

In effect, this method adds up the magnetic flux through rectangular strips of height l and width ds. Students reasoning this way used the physical geometry to obtain the right solution but bypassed a choice of a coordinate system.

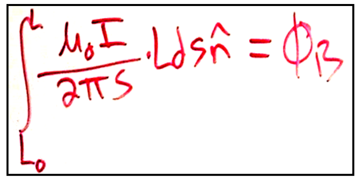


Figure 7.12. Lenny's second attempt on the flux task, where he reasons about only adding up the magnetic field in one direction.

7.3.2.1.3 Sum of differential lengths

Jake expressed *dA* as a sum of lengths rather than as a product for both tasks for reasons expressed in the charged sheet task:

> Jake: Actually no, it will be $drd\theta$ because it's a surface area so I'll need two dimensions that my dθ is probably going to come in from my dq. Because I should have a differential area shouldn't I, and a differential area should be $drd\theta$ [writes $dr + rd\theta$]. (Fig. 7.13)

Jake's can be interpreted as a symbolic forms error [47]. He clearly stated a need to include two dimensions for an area but instead of representing this as a product, he invoked an additive template, such as parts-of-a-whole. Similarly in the flux task, Jake represents his differential area as $ds\hat{s} + sdz\hat{z}$, using an incorrect differential length.

The representation of an area as a sum of lengths appears also in Lenny's initial approach to the flux task, which involved attempting to skip integration by multiplying the field and the area of the shape. He also failed to account for the changing magnetic

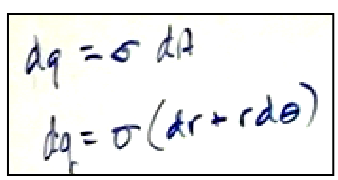


Figure 7.13. Jake's second attempt to express dq for the charged sheet task. To account for the need to integrate over "theta," expressed "da" as the sum of two differential lengths.

field over the square, which he mentions earlier, and uses just the value of the magnetic field at the first side of the shape. Yet, rather than l^2, his depiction of the whole area is represented as $2l$, corresponding to an addition of the two sides of the square rather than a multiplication (Fig. 7.14).

Lenny: I guess $\mu_o I \;\; 2l$ if that was the area. s would be m because that's the distance away is equal to Φ_B.

In Lenny's solution, he skips the dot product and integration aspects to arrive at a final expression of the magnetic field times an area element. This is reminiscent of students' treatment of Gauss's law problems where the symmetry aspects can be reasoned away. Here, however, Lenny's final area is incorrect. He then returned his attention to the s dependence on the field and decided upon the ldl expression above.

7.3.2.2 Derivative of the area expression

Students attempted to functionalize the given area and take a derivative to gain an expression for the differential area across three interviews. This is consistent with students' treatment of the derivative as a machine[8] that acts on a function: students

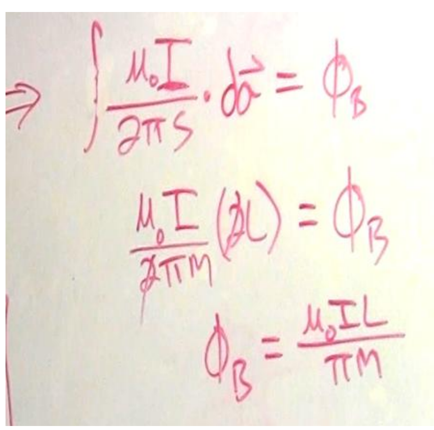


Figure 7.14. Lenny's initial solution for the flux task, where he expresses the area of the square as 2l.

interpret the d in dA as a cue to differentiate the function represented by the second variable. For Jake and Tyler, the ensuing difficulty was which variable to take their derivative with respect to. Both decide to integrate with respect to r (Fig. 7.15), which neglected the integrand's dependence on θ. This caused Jake to switch back to his *sum of differential lengths* concept image.

A pair of students employed this idea for the flux task.

> Percy: You still need… d something. I mean, what is your area? The area equals l^2 so da equals… $2ldl$… What we would do is say: "Oh look at this, what I have is: integral of some dA. Well, what is the area of this? Oh, that's l^2"… We would just recognize the fact that it's an integral of… an area element, so we take the area of the object and we'd do it easy.

Here, Percy reasons about the differential area represented in their flux equation as just the derivative of the area in an attempt to justify his final answer as just the multiplication of the magnetic field with the area of the square. Neither student attends to the fact that the magnetic field is changing in one direction or would need to be constant to bypass the use of dA. This particular reasoning speaks more to the treatment of dA as something that gets replaced with the expression for area after integration rather than a geometrical object accounting for integration of a quantity in two different coordinate directions.

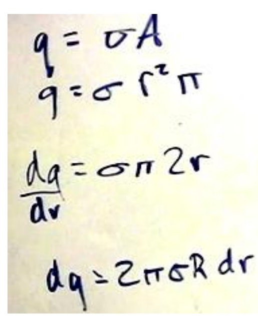


Figure 7.15. Jake's first attempt to express dq for the charged sheet task.

7.3.2.3 The area of the region itself

A third overall approach was to insert a functional form of the area for the whole region as dA. This was often the result of inattention to differentials and/or students' perceived need to plug in the area.

> Bart: The dA is the area of the square…you want just the square loop. I mean, there is flux everywhere but you want just the square loop. This is B [gestures to summing of fields at each edge of the loop] and [dA] is l^2. (Fig. 7.16)

Throughout the interview, Bart was persistent about plugging in the area, much in the way Percy was above. However for Bart, the area being l^2 was subsumed into the integral, which then resulted in a multiplication of his (incorrect) magnetic field and the full area. This was not something on which these students sought consistency.

Nate applied this reasoning to both tasks, replacing dA with the perceived area of the given space.

> Nate: ...but with da, when we're talking about this, we're talking about the area inside, so you'd think it'd be l^2 but I'm never confident in my ability to figure out what da is...It makes sense to me that it would be l^2

Nate included these differentials in his integrals in an attempt to identify what quantities needed to be integrated over (Fig. 7.17). Nate's explanation later in the interview of the flux task illustrated an understanding of the physical nature of dA as a

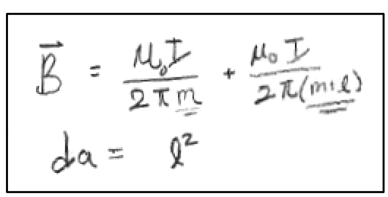


Figure 7.16. Bart's and Harold's expressions of magnetic field and da for the flux task. Bart explicitly writes an incorrect "B" and "da" before taking the product of the terms for the purposes of integration.

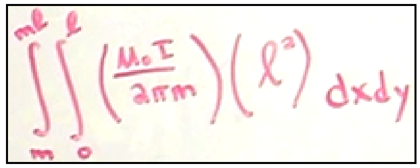


Figure 7.17. Nate's solution for the flux task. He explains his choice of dA as l^2 and the inclusion of $dxdy$ due to the need to integrate over the given boundaries.

"little chunk of area," an idea that Nate failed to connect to his earlier representation or to his addition of differentials. Nate's treatment of the differentials dx and dy is consistent with the differential as a nonphysical quantity, or just a variable of integration [8]. These conceptions of both dA and differentials persisted into the charged sheet task, where Nate described the area of a circle as $\cos^2\theta$, which would be multiplied by σ to express the differential charge dq.

As depicted, students attempting to express the differential area with an equation for the area of the full region have additional trouble with other parts of the tasks. This type of solution appears on a similar order as students who are taking a derivative in order to arrive at the final area, but represents a higher level of student difficulty, as the differential aspect remains unused.

7.3.3 Conclusions

Analysis of student interviews about differential area in the context of typical E&M tasks allowed us to identify several evoked concept images and to gauge student understanding of differential quantities as they are used in typical E&M problem solving. As part of a larger integration task, the differential area was commonly treated as a small portion of area constructed from differential lengths, as the derivative of the given area, or as the given area itself. Notably, the particular solution method employed was independent of coordinate system, suggesting students' methods for determining differential areas are detached from students' choice of coordinate systems.

The most productive instantiation of students' concept images was to express the differential area in terms of a product of differential lengths. This was especially productive for students working in polar coordinates, where they were not able to use aspects of the physical system to bypass defining a differential area. Other students possessed correct ideas pertaining to differential area but either had difficulty with the correct expression of individual differential lengths or displayed confusion with the overall symbolic template of the expression (e.g., added lengths).

All students using the product of differential lengths concept image for the flux task expressed their area in Cartesian coordinates, despite the curling nature of the field. While this is a reasonable solution pathway, when asked how their response would change for the square being rotated out of the plane of the board, three students failed to recognize the magnetic field still remained entirely parallel to the area vector for the square, even as one student explicitly labeled the magnetic field with a cylindrical unit vector. This suggests that cylindrical coordinates are not as readily accessed by students, as they still show preference to a Cartesian system and incorrectly adjust their expressions because of that preference. This connects to work in both physics and mathematics education research where students show preference for Cartesian systems over polar ones [10,44] and also have difficulty employing the various resources of the systems relating to unit vectors [10,43].

Students incorrectly expressing differential areas most commonly focused on the final area of the given region, whether attempting to take a derivative to account for the need to integrate or by forcibly inserting a function for the full area into the integral. Emphasis on plugging in the area is most likely an artifact of generalizing common textbook problems that are highly symmetric, such as Gauss's law, where they can "do it easy," as Percy states, and neglect the dot product and vector nature of the dA. At this point, they can simply express the integral as a product of the field and the area of a Gaussian surface. Lenny attempted to treat the flux task as a Gauss's Law problem and ended up using an incorrect final area. While in very specific cases inserting a given area after integration or taking a derivative of the area to use in the integrand may produce a correct result (*e.g.,* Jake's *derivative of area* response for the charged sheet task, where symmetry eliminates the need to integrate over θ), these methods are not as universal as students perceive them to be. Students' use of area in this way is another example of overuse of symmetry arguments in problems where symmetry is not present [12,32,89].

Results suggest that an explicit instructional focus on the construction of differential areas as the product of differential lengths in specific coordinate systems, even in high-symmetry situations, may help dissuade students' overemphasis on a "plugging in the area" approach. Preliminary versions of instructional materials were developed in the style of *Tutorials in Introductory Physics* [65] to build the understanding of differential areas in Cartesian, cylindrical, and spherical coordinates as a product of associated differential length components (see section 9.5 and Appendix D for details).

7.4 Summary of findings on student understanding of differential areas in non-Cartesian coordinates

This chapter presents findings from targeted research tasks evoking students' conceptual understanding of differential area elements in non-Cartesian coordinate systems. Little prior research on student understanding in E&M has addressed these quantities [34,90]. Interview tasks were designed as part of a larger project to investigate the extent to which students understand the construction of the differential area element in terms of non-Cartesian coordinates.

Findings from various tasks involving students reasoning about or constructing differential area elements show that students struggle connecting differential areas to the underlying geometry of a particular coordinate system. During the check solution task (7.1.1) some pairs incorrectly identified the nonsensical differential area, $rL \sin\theta \, d\theta d\phi$, as belonging to spherical coordinates or another cylindrical surface. In other cases, such as the spiral task (7.1.2) or the generic differential area construction task (7.2), students struggled to construct an appropriate differential area. When constructing generic differential areas, students were still seen to have difficulty including or accounting for the trigonometric function in spherical coordinates, which verifies earlier difficulty in differential length construction (see section 5.1).

Across multiple interviews and tasks, several students placed emphasis on expressing the final area rather than interpreting the geometry of a given coordinate system or physical scenario. In both the flux task and the charged sheet task (7.3), several students attempted to represent the differential area as a derivative or as the full surface area, rather than reasoning about the geometric motions within the targeted surface. This idea echoes student responses in earlier tasks. In the check solution task, one group accepted the nonsensical differential area because they incorrectly acknowledged the final area, $4\pi rL$, as correct for the curved side of a cylinder. In the generic differential area construction, Bart and Harold invoked incorrect surfaces areas in an attempt to construct differential areas. They added or subtracted terms from a given differential area based on whether integration of the term was giving the targeted result.

This emphasis on the final area is most likely connected to the invocation of Gauss's law,

$$\Phi_E = \oint \vec{E} \cdot d\vec{a} = \frac{q_{enclosed}}{\varepsilon_o}.$$

This expression is taught early in E&M as a method to solve for the electric field due to a charge distribution requires a high degree of symmetry. Furthermore, due to the high symmetry, students can bypass the writing of the differential area in favor of replacing the integral with a product of the electric field and given surface area. Research has shown students often use this solution pathway in cases where the symmetry is inappropriate [12,24,32,33]. Students emphasis on the final area is likely a manifestation of a familiarity with this high-symmetry type of problem solving that, as shown by the findings in the last section, hampers students' problem solving in tasks where the explicit writing of the differential area as part of the integrand is necessary (*e.g.,* the flux task has a magnetic field which decreases over the width of the square loop, requiring integration to be carried out in this direction). This is consistent with the formation of a restricted concept image [38], where students have worked within the context of high symmetry for such a long period of time that they experience difficulty in contexts where such symmetry is absent.

In contrast, a number of students were able to invoke a product understanding to connect the differential area to differential length and volume quantities. This was productive in the flux and charged sheet tasks for students who constructed the differential area in terms of coordinate system geometry, as it allowed students to expediently carry out calculation. The product understanding was also productive for Victor and Dan, allowing them to more easily determine all three differential areas in spherical coordinates. Beyond this, the pair checked the correctness of these elements by multiplying each by the missing component to verify that it gave them the volume element.

Following these findings, a tutorial [65] was developed to place more emphasis on the construction of differential areas as a product of differential lengths and foster further understanding of the construction of differential elements in terms of coordinate system geometry (Appendix D). This activity was specifically made as part of a sequence with a prior tutorial on differential length construction (Appendix C) and also includes the use of three-dimensional manipulatives to connect motions in three-dimensional space to the expressions for differential elements.

CHAPTER 8

INCORPORATING SYMBOLIC FORMS IN CONCEPTUAL BLENDING TO INTERPRET STUDENT MATHEMATIZATION: CONSTRUCTING EXPRESSIONS FOR DIFFERENTIAL ELEMENTS IN VECTOR CALCULUS

"My goal is simple. It is a complete understanding of the universe, why it is as it is and why it exists at all."

– Stephen Hawking

Application of symbolic forms [47] and concept image [38] frameworks to students' construction of differential length vectors in schmerical coordinates (see section 5.1, [91,92]) provided two complementary analyses of students' structural understanding of the expressions and conceptual understanding of the differential element. Findings from these analyses showed that students generally understood the structure of the differential length vector but not the expression of terms based on coordinate system geometry.

In order to better describe the way in which students connected the structural representation and conceptual understanding, the conceptual blending framework [76] was applied. From this, a model was developed to described students construction and interpretation of equations. This incorporation of conceptual blending provides contextual understanding to a symbolic forms analysis, while the incorporation of the conceptual schema from symbolic forms provides an underlying structure previously absent from literature describing the blending of mathematics and physics [39,81,93] and further addresses the research question concerning the way in which students' conceptual understanding and knowledge of symbolic expressions impact differential element construction. This chapter outlines the model that connects these frameworks and the particular affordances of such a model in the analysis of students' work with equations.

This chapter is in preparation for submission for journal publication.

8.1 Introduction

One of the fundamental drives of physics education research has been in interpreting the way students use and understand the mathematics used in physics. There is great purpose in this venture as mathematics forms the underlying foundation for representation of physics content. We use mathematics to construct expressions that allow us to relay information, manipulate expressions to further advance this understanding, and interpret derivations to gain new insight into physical systems. From kinematic equations like $v_f = v_o + at$, to divergence of an electric field in electricity and magnetism (E&M), to Dirac notation and linear algebra in quantum mechanics, mathematics provide us fundamental language for physics.

Researchers in physics education have previously described mathematics as the language of physics [29] and developed theoretical models to frame the ways in which mathematics and physics interact in problem solving [29–31]. A common feature of these diagrams is mathematical modeling or "mathematization," in which a physical system is abstracted, often into a mathematical expression.

The theoretical framework of symbolic forms was developed specifically to address how students construct and understand the mathematical underpinnings that provide the structure to equations [47]. Building off of a *knowledge-in-pieces* approach [72], symbolic forms account for what Sherin saw as students writing an equation from a “sense of what they wanted to express” [47]. The purpose of identifying the underlying mathematical-based structures through which students understand equations speaks to the larger goal of how mathematics is used by students and ties to their understanding of mathematization in physics. Symbolic forms, however, were designed as acontextual constructs with explicit focus on the mathematical justifications for equations, and therefore were not intended to address students’ conceptual understanding of the associated physics.

Other researchers have incorporated conceptual blending [76], a theoretical framework from linguistics that describes the connection and combination of elements from separate domains of knowledge (referred to as mental spaces) into a blended domain. Conceptual blending has served as a means to describe the ways in which mathematics and physics are woven together, both at the introductory [94,95] and upper levels [39,81]. Previous adaptations of conceptual blending to discuss the interaction of mathematics and physics have generally not included a *generic space*, which serves as an underlying structure for each of the two input domains and determines which pieces combine to form a new blended concept.

Extending from the depth of the theoretical work and its applications in physics and mathematics education literature, the concept of an equation emerges as a statement of a physical-mathematical language where meaning is embedded (or modeled) in the way variables and procedures are embedded into specific forms. Much in the way that the rules of writing a sentence govern structure, punctuation, and clauses, and thus put forth a certain meaning, the way an equation is written conveys a very specific message of meaning and of how the quantities relate.

As such, we present a model for analysis of students' construction and interpretation of equations by connecting students' use of symbolic forms [47] with their physics conceptual understanding through the use of formal conceptual blending theory [76]. In this model, aspects of symbolic forms serve as the underlying structure for the blending of mathematics and physics, while the incorporation of symbolic forms brings conceptual understanding to an acontextual symbolic forms analysis. To fully explore this theoretical model, we use data from our research in upper-division E&M, where we asked students to construct a differential length vector for an unconventional spherical coordinate system (see section 5.1, [91,92]). However, this model can be extended to analyze students' connection of structural/mathematical understanding to any physics context.

In this chapter, we first review the development of previous models for mathematization in physics to situate our work within the realm of physics education research on students understanding of mathematics. As a continuation of a review of relevant literature, we include detailed overviews of the symbolic forms and conceptual blending frameworks and discuss each of the instantiations of these frameworks within the physics and mathematics education research. We then introduce and critique previous work, which attempted to connect symbolic forms and conceptual blending theories [93].

In section 8.3, we present the proposed model for students' construction of equations. We argue that the combination of the aforementioned frameworks is complementary in that we can use the aspects of each framework to fill missing analytical aspects within the other. Extending this, we present the affordances of our model by further connecting various analytical pieces of each framework as a means to show the scope and reach of the model. Lastly, we summarize the model and discuss future work, specifically in line with Sherin's suggestions for extending symbolic forms literature to account for further physics contexts, as well as other kinds of mathematical representation.

8.2 Review of relevant theoretical literature

The following section presents an overview of the relevant theoretical lens for interpreting students' use and understanding of mathematics in physics as background for the development of the theoretical model described in section 8.3. The first subsection describes the large-scale models that have been developed to describe student work at the mathematics-physics interface. Section 8.2.2 introduces the specific perspective of symbolic forms framework [47] as it has been used to describe students' construction of equations as mathematical objects. Section 8.2.3 introduces the conceptual blending framework [38] as an additional means to describe the interaction between physics and mathematics. Lastly, we draw attention to previous work within the literature that has used a conceptual blending framework to describe students use of symbolic forms in physics.

8.2.1 Review of models for students' mathematization within physics

The incorporation of mathematics in physics goes beyond calculation, as mathematics plays a role in reasoning about relationships between physical quantities or state of the system, as well as conveying these relationships with graphs or equations. Several physics education researchers have sought to describe and represent the way students incorporate mathematical concepts throughout physics (Fig. 8.1). Notably, these models involve a number of common elements, suggesting key areas of mathematics

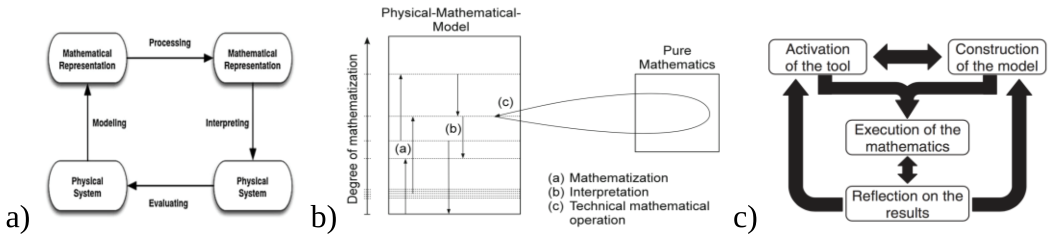


Figure 8.1. Models of mathematization. (a) Model from Redish and Kuo [29]. (b) Model from Uhden and colleagues [30]. (c) Model from Wilcox and colleagues [31].

understanding necessary for physics. An early instantiation separated the mathematics and physics domains into two distinct spaces that students cycled between: the physical system and mathematical representation [29]. Within this framing, modeling appears as the action that moves students from the physical system into a mathematical representation space (e.g., setting up an integral). This representation is then processed within the mathematical domain (e.g., calculating an integral). Interpretation of this new representation brings one out of the physical system and back into the physics domain.

Uhden and colleagues developed a more sophisticated representation that considers a blended space of mathematics and physics [30]. Each level in this portrayal of the mathematics-physics interface represents a degree of mathematical modeling, which has also been referred to as mathematization. The closer to the bottom of the vertical axis, the more grounded in the physical system. As students model the physical system by defining proportionalities, writing equations to connect variables, or using various laws, theorems, or physics relationships, the level of mathematization increases. Interpretation of these results corresponds to a lesser degree of mathematization.

A third model of students' use of mathematics resulted from work in upper-division E&M [31]. The ACER framework designed a more student-centered script in which the arrows in the previous two diagrams are now where steps in problem solving occur. This framework designates spaces for the "activation of a tool" (tool referring to the choice of an equation), "construction of the [mathematical] model," "execution of mathematics," and "reflection on the results." While each diagram represents students' use of mathematics in a different way, they all include features to account for modeling, calculation, and interpretation.

The idea that physics is a combination of these two spaces is not isolated to the physical-mathematical-model of Uhden and colleagues. Other researchers have used ideas related to conceptual blending [76] to depict the interaction of the physical world with conceptual understanding of mathematical operation (or "mathematical machinery") within introductory physics [94,95]. This work has spilled over to the upper division, specifically in research into student understanding of the mathematics in E&M, where both the mathematics knowledge and physics knowledge required of students becomes more sophisticated. Use of mathematics at this level has led researchers to identify broad student difficulties related to interpretation of underlying physical symmetry, connecting mathematical calculation to physics ideas in terms of setting up representations and interpreting results, and recognizing the appropriate method of solution or "mathematical tool" [12]. The plethora of models suggests that identifying students' interaction with mathematics in physics in non-trivial. However, the presence of common features (modeling, processing, interpretation) suggests these are several key aspects to student understanding and use of mathematics in physics. The work presented in this paper deals with the idea of modeling as a means of creating mathematical representation, specifically during the process of equation construction. We further use the analysis from the construction of equations to describe students' interpretation of equations as they read information out from these abstracted representations.

8.2.2 Development and use of symbolic forms to address students' understanding of physics equations in terms of mathematical structures

Analysis using symbolic forms [47] provides a means to address student understanding of the mathematical representation used in equations. In this section, we provide an overview of symbolic forms and describe its use in the literature. Lastly, an overview of the use of symbolic forms within our work is provided to lay the groundwork for the presentation of the model.

8.2.2.1 Overview of symbolic forms

In an effort to explore the mathematical structures in equations students use to construct and interpret equations, Sherin [47] asked junior physics majors several introductory physics problems. Sherin found that rather than trying to derive an expression or manipulating known equations, students built or attempted to build an equation from a sense of what they wanted it to express. Motivated by this analysis, Sherin developed an analytical tool for interpreting symbolic forms to provide a critical lens for the investigation of students' construction and sense-making of equations in terms of mathematical understanding.

A symbolic form, in line with a *knowledge-in-pieces* model [72], is an element of a mathematical expression defined in a pairing of two parts. The main element of a symbolic form is the symbol template, the externalized structure of the equation. For example ($\square + \square + \square$) would be a template in which the students would place terms/numbers/variables to add them. The particular associations underlying or motivating the template are what Sherin refers to as the conceptual schema. For $\square + \square + \square$, the associated schema is identified by Sherin as "amounts of a generic substance contributing to a whole." Sherin identifies this symbolic form as *parts-of-a-whole*.

The conceptual schema comes from the idea that students learn to associate meanings with structures in equations. Thus the conceptual schemata are acontextual, meaning that they don't rely on a particular physics context, but on an underlying mathematical understanding of how the equation is written. *Parts-of-a-whole* could be seen in a student's writing of an expression for the total energy of a system in terms of kinetic and potential energy, $\frac{1}{2}mv^2 + mgh$, or in an attempt to express the surface area for a cylinder of radius, r, and length, l, as a sum of the end caps and shell, $2\pi r^2 + 2\pi rl$. While these equations contain drastically different variables and physical meanings, they share the symbolic structure of *parts-of-a-whole*. Sherin illustrates parts-of-a-whole through students' construction of an equation around an incorrect idea of the coefficient of friction.

> *Karl:* *...the frictional force as having two components. One that goes to zero and the other one that's constant.* [47] (Fig. 8.2)

$$\mu = \mu_1 + C\frac{\mu_2}{m}$$

Figure 8.2. Mike and Karl's final equation depicting the invocations of the *parts-of-a-whole* symbol template. Image reproduced from [47].

It is important to note here that symbolic forms can be used correctly even when students have incorrect conceptual ideas of the associated physics. In this example, students' invoke parts-of-a-whole because it is consistent with their underlying idea that two quantities need to be added.

Sherin identified the *base-plus-change* symbolic form in a one pair of students' expression of a kinematics equation, $v_f = v_o + \frac{1}{2}at^2$, despite the equation not having any physical meaning.

> *Mark:* *'Cause we have initial velocity [circles v_o] plus if you have an acceleration over a certain time [circles $\frac{1}{2}at^2$]. Yeah I think that is right.* [47]

As before, students' conceptual schema is illustrated during the construction and connected explicitly to the associated structures in the base-plus-change template.

It is important to note that *parts-of-a-whole* and *base-plus-change* both describe an identical mathematical operation: addition. While parts-of-a-whole describes addition of independent quantities, base plus change, $\square + \Delta$, is a specific case where the first term is a fixed quantity augmented by a variable second term. While this may seem to be cued primarily by a physics understanding as seen in kinematics equations, it is also the form for the equation of a line ($mx + b$) and thus can be imagined to appear in many other physics equations, connected explicitly to graphical representations.

Returning to the coefficient of friction example, Sherin describes the conceptual schema for the parts-of-a-whole template as "seen behind Karl's statement that the coefficient of friction consists of two components" [47]. This further supports the idea that despite an incorrect physics understanding, students can show correct use of a symbolic form and that the symbolic forms are divorced from physics understanding. For

Sherin, the conceptual schemata are simple acontexual structures similar to diSessa's phenomonological primitives (p-prims). P-prims are intuitive knowledge elements that aren't learned but intrinsically held by individuals, such as the idea that larger objects are heavier [72]. While addition is certainly a learned mathematical skill, the idea of it is built up by years and years of association to the operation, so that students arguably develop an intrinsic sense of what it means to express two quantities contributing to a whole. In this sense, the conceptual schemata of symbolic forms can be thought of as the intuitive knowledge elements through which students intrinsically understand the written structures in an equation.

Beyond this, it is important to note that equation construction on the whole involves the invocation of several symbolic forms, which when used together carry the associated meaning of the symbols. Students' construction of an expression to describe the coefficient of friction invoked the *prop-* ($\frac{\ldots\square\ldots}{\ldots}$), *coefficient* ($\square$ ...), and *no dependence* ([...]) symbolic forms [47] to express the full mathematical meaning students attached to the variables in the equation. Symbolic forms can thus be nested within each other in whatever manner is deemed necessary to convey the full meaning of the equation. In order to interpret or convey this meaning beyond reading the mathematical structures, we must bring in another piece, the conceptual understanding, which is the aim of section 8.3.

8.2.2.2 Previous application of symbolic forms in related literature

Meredith and Marrongelle [48] adapted the conceptual schemata of symbolic forms to account for the features of electrostatics problems that cued integration among students.

They found students invoking the conceptual schema of the *dependence* form, a symbolic form that establishes the need for a particular variable that an expression "depends on." Students invoked this conceptual schema when eliciting the reliance of an integral on a particular variable. Students invoked the *parts-of-a-whole* form when acknowledging the need to sum up multiple small charges along a charged rod. While this study does not identify invocation of the accompanying symbol templates for these schemata, the underlying ideas of parts-of-a-whole and dependence were revealed as aspects driving students choices to integrate.

Attempting to expand symbolic forms to the realm of integration, the ideas of symbolic forms were additionally used to analyze calculus students' ideas when making sense of integrals [50]. Jones identified variation in students' conceptual understanding when interpreting the various structures associated with (mostly definite) integrals given as part of the tasks. This led to the creation of several distinct symbolic forms, some of which possessed the same template to distinguish between *Riemann sum, area and perimeter,* and *function matching* interpretations. Some of these forms were duplicated to account for an integral expression without limits on the integrand, while others had more varied templates to account for types of integration: area between two functions or integration over a physical shape. Notably, students' exposed conceptual understandings often led to depictions graphical representations of the given functions and use of the depictions to explain the integration.

The symbolic forms framework has been further extended to analysis of physical chemistry students' use of partial derivatives in a thermodynamics context [78]. This work illustrated the ways in which students understood and applied symbolic forms reasoning when working with common mathematical expressions in physical chemistry. In several cases students recalled specific processes, such as that of taking the total derivative, or concepts, such as $dx = 0$ when x is constant. This showed the specific role of recall in mediating student construction of and reasoning about expressions when working with upper-division content, consistent with findings of analyses of student construction of differential vector elements (see section 5.1).

8.2.2.3 Symbolic forms analysis of differential length vector construction

As part of work looking at students' understanding of mathematics and mathematical methods in upper-division physics, we identified symbolic forms appearing in students' construction of differential length vectors for an unconventional spherical coordinate system we called "schmerical coordinates" (see section 5.1, [91,92]). Differential length and area elements, the latter constructed as products of the former, appear in vector and scalar integration involving electric and magnetic fields. Due to the symmetry of physical situations, much of vector calculus in physics uses non-Cartesian coordinate systems, such as spherical and cylindrical coordinates. The development of schmerical coordinates allowed us to assess students' underlying understanding in terms of arc lengths and differential changes without allowing them to explicitly recall the differential length vector for spherical coordinates.

Pair interviews were conducted at two universities. Pair interviews facilitated more student-driven interaction with less input or influence from the interviewer. Interviews were videotaped and later transcribed for analysis.

Interviewed pairs were asked to construct a differential length vector in schmerical coordinates. Preliminary analysis (see section 5.1, [92]) identified student's concept images [38] associated with the differential length vector as a means to identify the specific ideas or properties that students' associated with such elements. The concept image analysis is born from mathematics education research and is similar in many aspects to resources [42] or *knowledge-in-pieces* [72]. While students focused on several key aspects, such as a need for appropriate dimensions or for multiple components, other aspects were not employed by students (see section 5.1, [91,92]). With further desire to understand the construction process and the terms with which students wrote their expressions, secondary analysis [91] involved identification of symbolic forms [47] by attending to the structures students expressed in equations and their understanding of that structure. We incorporated this analysis into the upper division to investigate students' structural understanding of differential length vectors as they constructed a generic differential length vector for a non-Cartesian coordinate system.

Our analysis identified several symbolic forms from the original literature (*parts-of-a-whole, coefficient, no-dependence*), as well as new symbolic forms that emerged due to the increased sophistication of the mathematics in upper-division physics (*differential, magnitude-direction*) (see section 5.1.5, [91]). With the concept image analysis in mind, we noticed that students' inclusion of specific structures in their expressions sometimes resulted from differing conceptual ideas that were not accounted for in a strict symbolic forms analysis.

For example, Carol and Dan often motivated the inclusion of a differential as a change in a particular quantity, without reference to size.

Carol: ...you have a change in your $\widehat{M}$ is going to be your dM, it's your change in your M.

Elliot and Frank, however, emphasized the infinitesimal aspect of the differential, often articulating it as a “little” amount of a given quantity.

Here there are two differing ideas contributing to the same symbolic structure. Varying conceptual representations of a differential make sense, given that literature has identified several ways in which students use and understand differential quantities [25,28,49,51,52]. Our interpretation of symbolic forms as acontextual constructs does not account for these varying conceptual understandings that lead to students expression of terms in equations, only the recognition of the need for structures to express specific mathematical ideas, such as a vector being composed of distinct magnitude and direction terms. The *why* of writing the components this way is not addressed. If, indeed, symbolic forms accounted for contextual analysis it would then have to describe symbolic forms in a way that distinguishes variability between physics contexts, which would inevitably confound analysis and obscures the understanding of the underlying mathematical reasoning for symbol arrangement and representation.

A more stark depiction of how varying conceptual understandings can motivate the same symbolic structure can be seen when looking at students’ reasoning about the inclusion of the scaling factors. Given the curvature of non-Cartesian coordinate systems, the differential length components in the angular directions are arc lengths. For spherical coordinates this yields $rd\theta$ for the $\hat{\theta}$-direction and $r\sin\theta\, d\phi$ for the $\hat{\phi}$-direction. As students constructed differential length vectors, one pair of students recognized the nature of the component as an arc length using geometrical reasoning, while others often only reasoned about the inclusion of the radius terms as necessary to give the appropriate dimensions.

> Adam: This doesn't have any units of length, so it needs to have some M term. (Fig. 8.3)

Here, Adam recognizes that the differential angle component doesn't have the units of length and thus fills the blank space in front of $d\alpha$ with an M.

Others still, engaged in a third line of reasoning, recognized that the coefficient box needed to be filled; but as the groups lacked the requisite knowledge to derive the terms via conceptual understanding, these students used a process of recall to a more familiar spherical coordinate system and mapped quantities to schmerical coordinates.

> *Bart: so now we have just to compare* so we have r it is M, θ is α, ϕ is β

Students in each of these groups recognized that an extra term was needed in their expressions. We identify their treatment of this space before the differential angle terms as *coefficient*, in line with Sherin's form, (□ ...). The associated conceptual schema describes the coefficient form as a factor or constant multiplied on the left of an expression that attenuates the value of the quantities. In the case of Sherin's coefficient of friction task, the constant, C, was added "almost as an afterthought" [47]. In our case, students reasoning geometrically can easily see how increasing the radius would attenuate the value of the arc length, while those using dimensionality express the inclusion of M as just a factor that contributes needed units to the term without explicitly

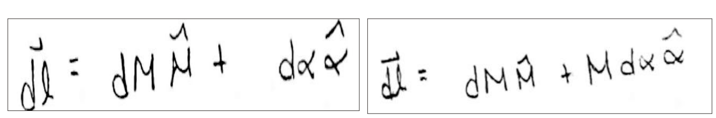


Figure 8.3. Adam's inclusion of "M" based on dimensional reasoning.

accessing the underlying idea. Students using recall display little underlying conceptual reasoning, only arguing that some term needs to fill the spot because it needs to bearresemblance to an earlier problem. While each of these cases invoke the coefficient symbolic form, the particular reasoning for the invocation is distinct and not addressed with attention to the underlying mathematical schema.

Recall, specifically, presents an interesting mechanism for the invocation of symbolic forms, as it sidesteps attention to the underlying conceptual schema. Yet previous literature has shown that recall of specific ideas is relevant to equation construction at the upper-level [78]. Utilizing a *conceptual blending* framework [76], we later address the role of recall as it is connected to the students' construction of expressions or equations.

8.2.3 Connection of mathematics and physics through Conceptual Blending analyses

As a means to address the integration and networking of conceptual ideas with students' understanding of the symbolic structures in an equation, we draw on the theory of *conceptual blending* [76].

8.2.3.1 Overview of conceptual blending

Conceptual blending originated from the study of linguistics as a way to discuss the interaction of form and meaning in the development of language and human understanding. At its most basic, a conceptual blend describes the compression of ideas from two distinct *mental spaces,* often containing information connected to one's previous experiences. The result is a *blended space* where new meaning/understanding emerges.

One of the more accessible examples involves two rival CEOs in a business competition:

> *We say that one CEO landed a blow but the other recovered, one of them tripped and the other took advantage, one of them knocked the other out cold.* [76]

This example represents a compression of two input spaces: the business space, which contains the CEOs and market strategies; and the boxing space, which contains two competitors engaging in fisticuffs. Each input space represents a collection of individual ideas that do not inherently belong to one narrative. It isn't until we connect a CEO to a boxer or a blow to an effective business strategy within the blended space that we can make sense of "one knocked the other out cold," as the CEOs are not engaged in actual physical combat or being rendered unconscious by shifts in the market.

The typical figure presented to illustrate blending shows the compression of these spaces into the blend, as well as a *generic space* (Fig. 8.4). The generic space is a fourth space used in conceptual blending to provide the underlying structure to the two input spaces, identifying the commonalities within each space and allowing one to see which element in each space is being mapped to an element in a second space. This often drives blending as an active process of compression of elements into the emergent blended space (solid line). Using this representation, we can develop a conceptual blending diagram for the boxers/CEO blend (Fig. 8.4). Here we see the connections laid out as the conception of boxing CEOs emerges as an amalgamation of the two different spaces.

The boxing CEOs example represents a specific type of blending network identified as a single-scope blend. In such a blend, the frame of one space (boxing) provides the

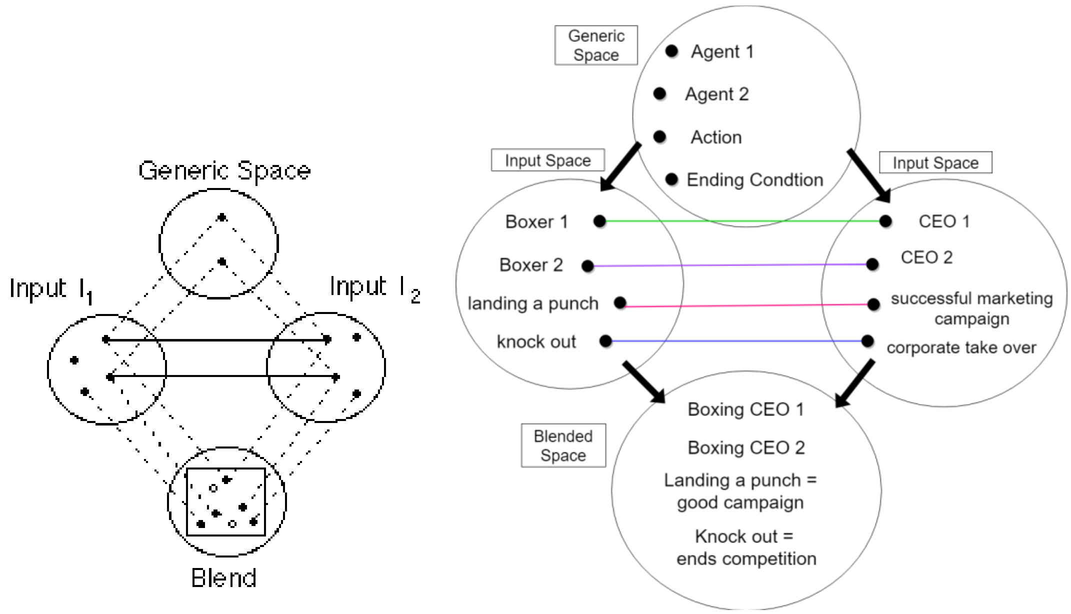


Figure 8.4. Basic diagrams depicting conceptual blending. (a) Generic model of a conceptual blend. Image reproduced from [76]. (b) Model for the boxing CEO blend. Adapted from [76].

organization of the blend, bringing the two CEOs into spatial and temporal proximity. The boxing input space is mapped entirely onto the business frame to provide a lens of physical combat onto business adversaries. As such, single scope blending provides the prototypical network for most conventional metaphors [76].

The other commonly cited type of network is identified as a double-scope blend. In this type of blend, the organizing frame of the blended space is integrated from both spaces. Drawing on conceptual blending literature [76], when one describes your foolish investments as “digging your own grave,” there is a conceptual blend of grave digging and foolish actions. While the grave digging provides most of the framing, presenting you as the grave digger and your actions as the “*che che*” of a shovel sinking into the earth, the causality is projected from the foolish action space, since the completion of one’s grave plot does not immediately imply death within the space of grave digging. Yet the implication is emergent in the blended space, as the causality of foolish action leads to failure is brought into the blend. Whereas in a single-scope blend one input space contributes the entirety of the organizing frame, in a double-scope blend the other input space provides more beyond the elements it contains. Double-scope blends incorporate aspects of structure as well, such as causation, and time- and space-compressions as well [76].

In some cases, with either conceptual blending network, backward projection can occur, in which the blended space provides guiding information back to an initial input space. For example, the blending of mathematics and physics ideas may provide insight into the meaning of a particular mathematics operation or physics concept [39]. While reasoning about the curl of a given field, a student had difficulty connecting the symbolic interpretation of Maxwell-Ampère's law to the graphical representation of the field. Bollen and colleagues [39] describe that a fluent calculation allowed the student to reinterpret the curl (how much the field rotates) at a given location without needing further intervention.

> *This makes sense, if the changing electric field vanishes, the curl of the magnetic field should vanish as well. However, the magnetic field itself is non-zero. [...] the drawing confused me at first, but now I can see that a paddle wheel would not spin here.* [39]

In this case, the students' calculation and subsequent interpretation of the equation leads them to reevaluate the nature of the physical system and arrive at the correct expression. The student then recognizes the curl is 0, by invoking the imagery of a paddle wheel spinning in the field (a common visual test used for quickly determining curl at a point). The backward projection is the use of the blended mathematics-physics space to make sense of one of the input spaces, in this case the physics input space.

8.2.3.2 Previous application of conceptual blending in related literature

Given the tilt of conceptual blending toward providing a lens for understanding how ideas are connected and combined in the learning process, conceptual blending has been specifically adapted to physics education research to explain how students connect mathematics and physics [39,81,93–95], and to explain the interplay of various physics principles in terms of wave mechanics [5] and energy [96].

At the introductory level, Bing and Redish [94] have adapted the language of conceptual blending to discuss the ways in which students combine mathematical and physics knowledge using two examples of air drag and kinematics. In these examples, the two input spaces are defined as "mathematical machinery" and "physics world." An example of a blend here takes "positive and negative quantities" as mathematical machinery and maps it with "up and down directions" to arrive at a typically defined one-dimensional coordinate system with "+" meaning up. In the single-scope example, students map a mathematical template for equating two fractions onto the numeric values of a given velocity and distance, without regard to the physical meaning or units of the quantities (Fig. 8.5). Since students focus on the mathematical process without attention to units, Bing and Redish identified this as a single-scope blend. Furthermore, the researchers distinguish this from double-scope blending, in which students use the signs as algebraic rules that encode the physical direction of the forces.

Researchers have adapted conceptual blending to upper-division physics in order to explain how students connected concepts in electricity and magnetism to the mathematical ideas of integration [81] and vector differential operators [39]. The blending at this level takes a similar form to the work at the introductory level, separating

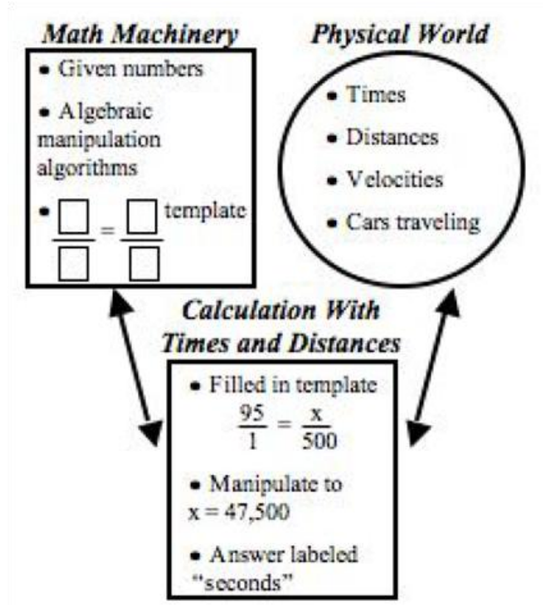


Figure 8.5. Math-physics blending diagram from Bing & Redish [94].

out three spaces as "math notation space," "symbolic space," and "physics space" (Fig. 8.6). Across the conceptual blends at this level, the physics space and symbolic space remain uniform lists of quantities (electric field, charge density, etc., for the physics space) or equations (e.g., $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_o$ in symbolic space) [39]. The blended spaces, then, are dictated by changes in the mathematics notation space, or the ways in which students understand or express concepts of integration, differentials, or divergence of a vector field. By separating out various realms that function together to establish a students' conceptual understanding, the results of this work establish several cases where students' conceptual understanding of an equation or mathematical idea leads to an incorrect response.

Wittmann adapted conceptual blending to explain the origin and intricacy of students' emergent conceptualizations of wave pulses with intuitive ideas related to throwing a ball

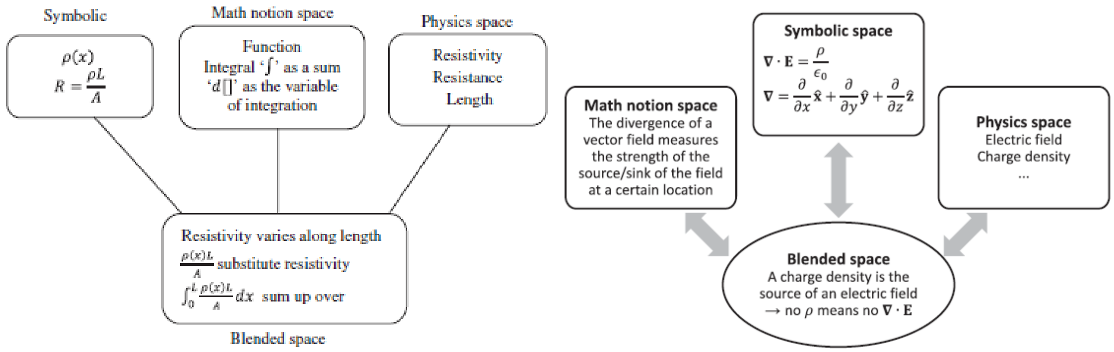


Figure 8.6. Math-physics blending diagrams from Hu & Rebello [81] and Bollen and colleagues [39], respectively.

(Fig. 8.7) [5]. Depending on the aspect of the physical system that students attend to, he identified a "wave-ball blend," where a faster flick corresponds to faster movement in the way a harder throw means a faster ball, and a "beaded-string blend," where the nearest-neighbor interactions are responsible for pulse speed. The blends here are depicted with concise compressions by connecting elements directly between input spaces and then subsequently to an element representing the blend. This representation is similar to that in work depicting integration of location and substance metaphors for energy into a coherent picture of "absorbing energy makes things go up" (Fig. 8.8) [96].

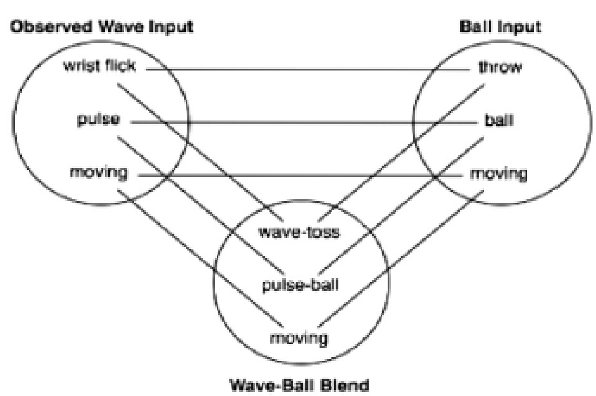


Figure 8.7. Wave-ball blending diagram from Wittmann [5].

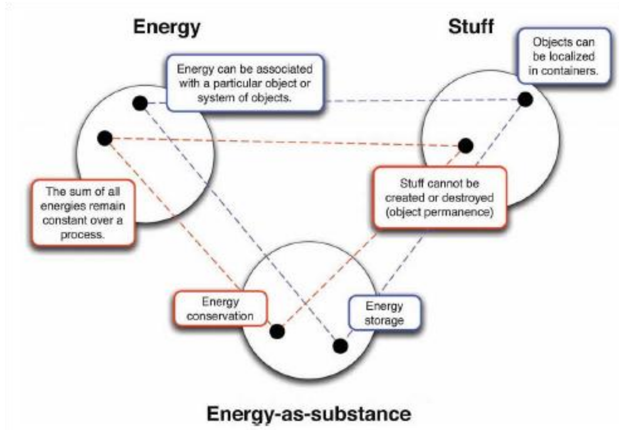


Figure 8.8. Energy-stuff blending diagram from Dreyfus and colleagues [96].

It is notable that none of the examples presented here make use of the generic space within conceptual blending literature. For the latter two examples, this space is arguably tacit and redundant (as in the boxing example): the compressions of the two input spaces are concise in that elements that share analogous aspects in other input spaces are explicitly connected by a dotted line (representing a compression in the original blending literature [76]). In the examples connecting mathematics realms to physics realms, the input spaces represent three distinct spaces from which students draw knowledge, without structure or connection among the input spaces (Fig. 8.6). As such, the active nature of a students' blending process is obfuscated. We argue that the generic space, or depiction of compression, is necessary to the invocation of blending, especially in cases for which the blending is not so clear cut and students' combination of ideas is unclear from a conceptual standpoint, in order to highlight underlying ideas that drive the compression of two elements.

8.2.4 Previous attempts to argue symbolic forms as elements of a conceptual blend

Recognizing the role of symbolic forms in the constructing of equations within physics, Kuo and colleagues framed symbolic forms as a conceptual blend of the symbol template and conceptual schema [93]. They addressed students' qualitative reasoning or "processing" of equations by presenting two contrasting case studies in which students interpret the kinematics equation ($v = v_o + at$). Here, v is a function of time, t. Additionally, v_o is the velocity of an object at $t = 0$, and represents an acceleration, the rate at which a velocity changes in a given time. While one student reasons formulaically, the other is said to engage in a blended process of mathematics and physics as they interpret the mathematical structure of the equation in terms of the physical situation.

The authors then discuss students' reading or failure to read out a *base+change* symbol template, $\square + \Delta$, from the given equation, and connect this to students' responses to the second prompt.

> *Pat: Because I mean, if you look at it from the unit side, it's clear that acceleration times time is a velocity, but it might be easier if you think about, you start from an initial velocity and then the acceleration for a certain period of time increases or decreases that velocity.* [93]

Pat's attention to the "at" component as changing the velocity is the key aspect of the reasoning that evokes the *base+change* formalism.

The authors identify this as conceptual blending of the symbol template and conceptual schema of the *base+change* symbolic form. However, symbolic forms are acontextual: ideas of velocity and acceleration are not included in the definition of *base + change*. A *base+change* symbolic form only accounts for the summation of terms in which "one is the base value; the other is a change to that base" [47]. It is only through an understanding of physics principles that we recognize that acceleration is related to a change in velocity, which shares the same underlying conceptual schema as the $\square + \Delta$ template.

Whereas in introductory physics, a symbolic form's conceptual schema and students' conceptual understanding are closely related, the conceptual schema is not the content idea itself, but the underlying mathematical essence of the idea. The parts-of-a-whole template appears in equations when there is a need to add aspects of a substance together. As an argument in semantics, this does not stipulate why such quantities need to be added. Kuo and colleagues present the conceptual schemata of parts-of-a-whole with an example of how guests at a wedding belong to multiple groups: close relatives, close friends, business contacts, and others [93]. The idea that wedding guests can be split into various groups that can be summed to give the guest list is a property of the wedding in the same way vectors can be represented as a sum of components. In both cases, the conceptual schema appears buried within the property of the target quantity, but is defined by neither, as "substances contributing to a whole" maintains it acontextual nature so it may be applied across multiple physical laws. The representation of vector quantities using equations, while guest lists for weddings are often devoid of mathematical symbology, is related to the mental integration of the properties of vector quantities with the appropriate mathematical template. This becomes the essence of what has driven the theoretical lens that we later describe.

Notably, while the work speaks of symbolic forms as an act of conceptual blending, there is very little attention to the actual blending process or the associated formalism, as this is not the focus of their work. As such, an underlying structure to the blend is not addressed. Blending is adopted as a broader process within this model, leaving room for deeper interpretation and further efforts connecting students' conceptual understanding to symbolic forms in general.

In the next section, we present an argument as to why symbolic forms is not a full blend in and of itself. We address the missing analytical aspects in previous literature, such as the underlying generic space, and provide theoretical structure for how blending occurs when constructing equations. In particular, we argue that students' interpretation of equations, such as in the task presented by Kuo and colleagues, is actually an act of backward projection rather than of forward blending.

8.3 Blending forms: Structuring students' use of symbolic forms as a conceptual blend

In the same way conceptual blending was used to attach meaning to form in the development of language, our goal for analysis of differential length vector construction has been to connect conceptual meaning (understanding) to symbolic forms as students develop equations. The writing of an equation in physics serves as the creation of a mathematical representation of the relationships between measurable or quantifiable entities. As such, there is need of an understanding of the physical system or variables, and of the mathematical representations. In analysis of student work, these mathematical relationships appear as symbolic forms.

While a strict symbolic forms analysis reveals students' structural understanding and associations related to the mathematics context, it does little to draw out or assess the students' conceptual understanding that dictates the need for the specific form. That is, the content basis for choices made as to the symbolic arrangement of expressions is neglected within the formal theory. As discussed previously, the literature utilizing symbolic forms often bypasses this by equating the student's the mathematical understanding of the expression with the understanding of the physics content, such as the ideas of velocity and acceleration describing the *base+change* symbolic form in the previous section.

This model proposes the two aspects of symbolic forms as spaces within a conceptual blend. This combination gives a focus on content knowledge to extend symbolic forms, in a way that students' varied conceptual understanding can be tied to explicit representations in their equations. This allows us to look at the physics justification for the representation of terms, which is irrelevant to the structural focus of a symbolic forms approach.

Furthermore, the generic space that structures the blending of elements within the input spaces has typically been absent in previous analyses of students blending of mathematics and physics. The incorporation of symbolic forms establishes this underlying structure for the blending of mathematics and physics in terms of constructing and interpreting equations.

We present this model in the context of earlier work investigating students construction of differential vector elements in upper-division E&M (see section 5.1, [91,92]). Upper-division physics provides several boons in regard to parsing students' conceptual understanding and expression of equations. By the time a student has entered upper-division physics, they have encountered and used symbols for addition, notation for vectors, and calculated numerous integrals and derivatives in both mathematics and physics courses. Therefore, we expect the symbol templates used during construction are often fairly ingrained in what we could call a students' conceptual toolbox. Thus, we can think of this process as a blending of these template understandings with physics understanding rather than a spontaneous creative process.

In this section, we present the model of equation construction by interpreting symbolic forms in terms of conceptual blending. We further show the affordances of this model in terms of other analyses, both from our own work and in previous literature. Here, we further elucidate the importance of the generic space in conceptual blending in terms of accounting for variation in conceptual understanding. We also show how such a model can account for how variations in representation can account for the same conceptual information. Next, discussion focuses on the role of recall and backward projection in construction in terms of such a model with heavy focus on conceptual understanding. Lastly we elaborate on the utility of this particular model in interpreting students' errors while constructing equations as belonging to either structural or conceptual understanding.

8.3.1 Proposal of the model of conceptual blending and symbolic forms

Armed with some level of conceptual understanding, students can condense their understanding of a physical situation into an equation, choosing from various symbolic representations, such as choosing to add when it is dictated by the relationship between physical quantities. Additionally, keeping symbolic forms in mind to account for mathematical understanding, two large input spaces appear. One of these spaces includes a selection of the mathematical representations, which we identify as the symbol template piece of Sherin's symbolic forms. The remaining input space contains the sum of students' conceptual understanding regarding a specific topic, including associated variable representations.† As students combine aspects of these input spaces, the equation is constructed: a sentence in a physics-mathematics language, given form by the understanding of mathematical relationships but meaning because of the physics conceptual understanding. This leads to a final representation or emergence of an equation within the blended space.

Further still, the conceptual schema of symbolic forms, which describes the justification for the mathematical structures of an equation, serves as the underlying generic space in a conceptual blending framing of students' construction of equations. As such, the conceptual schema is preserved as the underlying mathematical schema of a template but now also appears as the underlying understanding of students' ideas. With the conceptual schema appearing as the underlying understanding, it drives the blend of two input spaces. We discuss the deeper role of the generic space in the next section.

By sufficiently mapping symbolic forms and conceptual understanding onto conceptual blending, we can create a blanket blending diagram that can later be used to parse students' construction of equations (Fig. 8.9). Blending of this sort, involving the connection of physics and mathematics ideas, can take either a single- or double-scope form. The distinctions are discussed by Bing and Redish [94], who present two cases

† This allows a smoother depiction of physics ideas and equation construction and detracts little from the construction process as most students who, at this level, are now more expert-like physicists and have much familiarity with treating a concept and the variable used to represent it as one in the same.

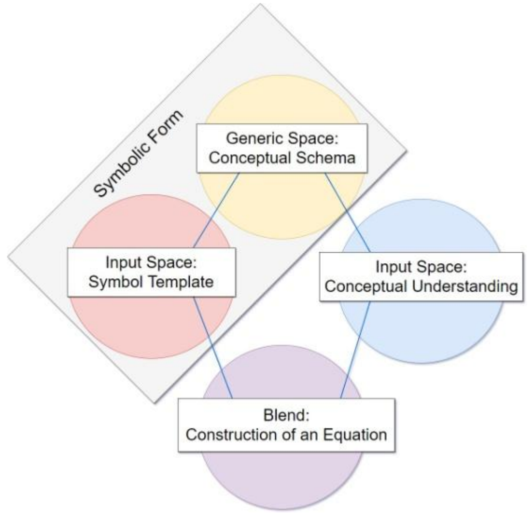


Figure 8.9. Diagram of conceptual blending for the modeling equation construction,

discussed in a previous section, one in which the mathematics structures the physics and another where mathematical and physical statements interact (e.g., +/- signs behave given algebraic rules but also convey physical meaning). Interpreting this model into work with symbolic forms means in some cases the conceptual understanding may entirely drive the construction of an equation (single-scope), while in others the external template may have more emphasis on guiding students conceptual physics ideas (double-scope).

As an example of how students blend conceptual information with symbolic representation, consider a pair of students, Eliot and Frank, as they constructed a differential length vector for schmerical coordinates.

Frank: Yeah, so like there, dl, there are three different dl's. There is dl with respect to M, dl with respect to α, and dl with respect to [β]

[construct each component individually]

Elliot: You sum them, so it is those added together [Fig. 8.10]

Looking at the conceptual ideas here, there is focus on the component nature of a vector; specifically, these two students focus on the idea that a differential length vector

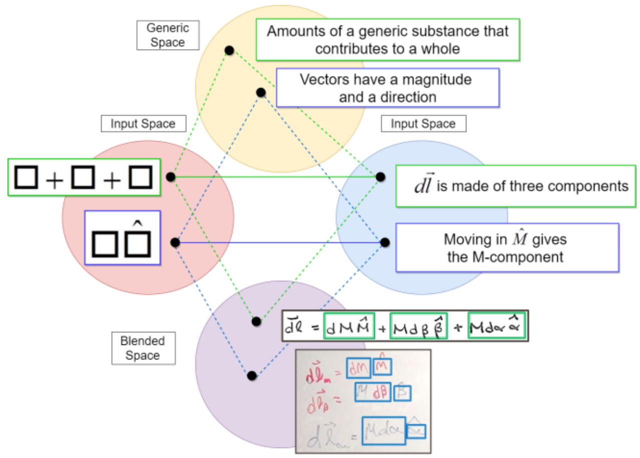


Figure 8.10. Blending of symbol template and conceptual understanding for Elliot and Frank

has three components for each of the three directions of motion. The idea of three components is a property belonging to the essence of a differential length vector, which students understand as three components (or parts) being summed to define the differential length vector. Likewise students associate each component as being taken "with respect to" a given variable direction, which is expressed in the final magnitude-direction pairing of a vector. Elliot and Frank articulate the "with respect to" later as they specifically express things like "now you're going to have a length component in the beta-hat direction." With a symbolic forms perspective, observations of students' written work and discussion of the expression reveal two main structures: *parts-of-a-whole* [47] to account for students' addition of the multiple components and the newly defined *magnitude-direction* symbolic form to account for the specific instantiation of the vector notation (see section 5.1.5, [91]).

We argue that these specific combinations of students' conceptual knowledge and symbolic representation can be treated as a conceptual blend of the two understandings as it results in the construction of complete or partial expressions, which only have meaning when understood through both of the initial input spaces.

The generic space then consists of the conceptual schema of symbolic forms. In the symbolic forms literature, behind the template [$\square + \square + \square$] is this conceptual schema of amounts of a substance contribute to a whole. Of course we want to remember here that the conceptual schemata of symbolic forms are the underlying *mathematical* understandings of those external structures. Bringing in the conceptual understanding side of this, we can also see that essence behind the understanding of the vector component property of a differential length vector. This symbol template and the conceptual understanding of three-dimensional vectors are then compressed in a conceptual blend into the final result of the equation, which depicts the summation of individual components of the differential length vector. Put another way, combining the knowledge that a vector in three dimensions can be represented as three magnitude-direction pairings pursuant to the coordinate system (in schmerical coordinates these being $\hat{\alpha}$, $\hat{\beta}$, and $\widehat{M}$) with the understanding of the template for addition of substances that contribute to a whole results in an final equation that is the sum of vector components.

The final equation is a product of the blend. Similar to the earlier statement "the CEO knocked out his competition," which only makes sense in a space where business and boxing are blended, an equation only has interpretable meaning when there are symbolic and contextual spaces from which to draw information.

While the previous example depicts Elliot and Frank's broader characterizations of the differential length vector, this model for conceptual blending can be mapped onto students' processes of construction, connecting the pieces of the template to the physical reasoning and discussion as the template is filled out.

Carol and Dan begin their interview by calling forth the need to have the three unit vectors of each component, leaving space between each to fill in the magnitudes.

Carol: So we're going to have, um, we're going to have [$\hat{\alpha}$], [$\hat{\beta}$], and some $\widehat{M}$. That's what we usually do and then they each need to be a length (boxes each component with hands). You need a length vector…This is, there is going to be a plus here.

Dan: Dan: (writes M with the $\widehat{M}$)

Carol: Carol: Okay, yup, so some M in the $\widehat{M}$. Isn't this dM?

Dan: Dan: Yeah, because it is dl, yup.

Carol: Carol: Right. So you have a change in your $\widehat{M}$ is going to be your dM, it's your change in your M.

While Carol and Dan do not elaborate on the specific underlying reasoning as they hybridize the parts-of-a-whole and magnitude-direction symbolic forms, the statement "that's what we usually do" suggests a level of recall moderating the construction. Notably, invoking forms together, rather than each independently, is not unexpected for upper-division students [25]. Using this perspective, it then also makes sense that Carol's and Dan's dual invocation was accompanied by a level of recall. The students have become familiar with these quantities and representations to a specific extent and they believe they recognize how the differential length vector needs to be structured. Here, Carol and Dan are correct with the structures that they have carved out from memory. While recall has been shown to mediate students' construction of equations and use of symbolic forms [78], here recall plays a role in the conceptual input space (Fig. 8.11). Students access the underlying mathematical understanding of the need for vectors of multiple components through this recall and blend the requisite elements of the

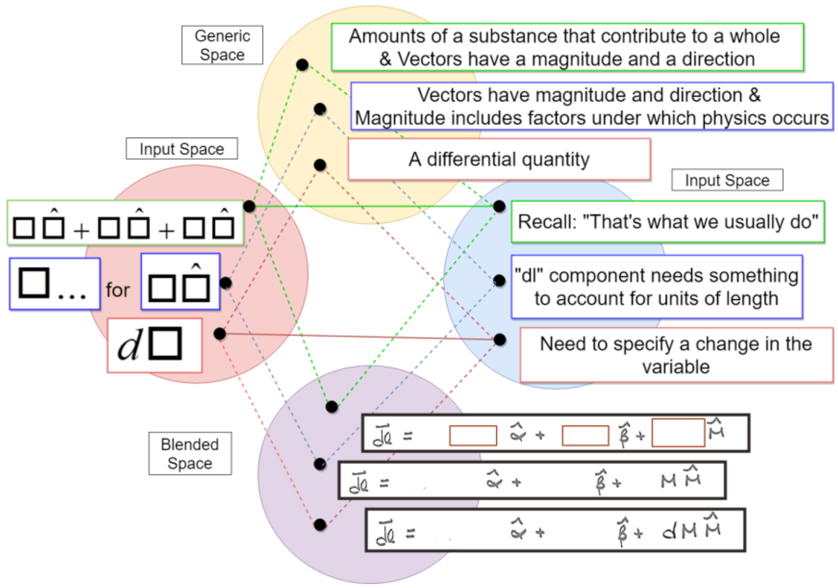


Figure 8.11. Blending diagram for Carol and Dan as they begin construction.

coordinate system with the symbol template. Had the students been asked to elaborate on why they had written the trappings of this expression in such a way, we can imagine, they would say something similar to that of Elliot and Frank above. The further role of recall in this type of model will be discussed later.

Following the structuring of their differential length vector, Carol articulates that each component needs to be a length and then curves her hands into a parenthetical shape and isolates each magnitude and unit vector pairing. This statement then cues Dan to write an M in the space before the $\widehat{M}$. In terms of symbolic forms, they're attending to the magnitude direction template nestled in the *parts-of-a-whole* structure and identifying that each needs to contain an element of length. Carol emphasizes the existence of structure of this template at this moment by articulating "yup, some M in the $\widehat{M}$."

Students' emphasis on dimensionality in other places in the interview appeared as an invocation of the coefficient symbolic form (see section 5.1.5, [91]). In these cases, groups of students (e.g., Carol and Dan) were building angular components and recognized that a differential angle did not carry the needed units of length,

> Adam: …This doesn't have any units of length, so it needs to have some M term.

These represented manifestations of the coefficient symbolic form, because students explicitly argued that something else needed to be included just to account for the units of length. With the coefficient symbolic form representing a constant or static factor that "defines the circumstances under which physics is occurring," [47] we can see the placement and treatment of M within this light. Our blending diagram for Adam and Bart in this moment of the interview accounts for this treatment. In the case of the angular components, M is a constant radius at which the differential length would be traced out in an angular direction.

When considering motions in the $\widehat{M}$-direction, the variable M is no longer static but needs to account for variation in the length of the coordinate vector. Carol and Dan invoke a new symbolic form representation upon recognizing this. They represent this as a dM, as the differential length vector component in the $\widehat{M}$ direction needs to account for the change in M. The differential concept image aspect and differential symbolic form identified in previous work (see section 5.1, [91,92]) go hand in hand, as students' invocation of the differential symbolic form is easily related to differential ideas. The conceptual blending template now allows the connection of these two ideas from different theoretical lenses, and dually allows on to model variations in students' conceptual ideas related to the differential. For example, Elliot and Frank invoke the differential symbolic form, but do so by attending to the infinitesimal nature of the differential.

> Elliot: So it's M times some Δ. I think it's M times $\Delta\beta$, a small β. (Fig. 8.12)

The pairs CD and EF both invoke the differential with "change in quantity" and "small quantity," respectively. While both conceptual understandings are appropriate in the given context, we consider these distinct evoked concept images. The connection of multiple conceptual ideas to the same symbol template highlights the importance of including the generic space, which is discussed in greater detail as part of section 8.3.2.1.

The last of the symbolic forms identified in this study was the *no-dependence* form, which accounts for the absence of a variable or quantity in an expression after a student explicitly dictates that the expression is independent of said quantity. This appeared in two sets of interviews, where students attended to components in the angular directions. When constructing the $\hat{\alpha}$ component, Adam and Bart correctly decide that the term

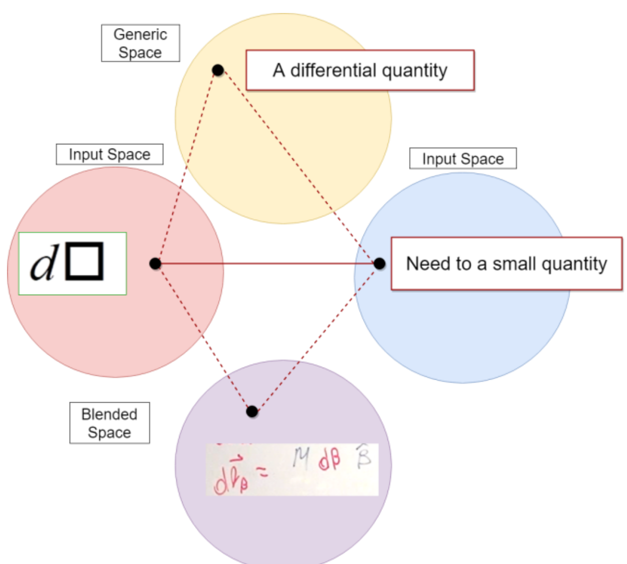


Figure 8.12. Blending diagram for differential template with varied conceptual understanding.

should not include any aspect of the other angle. This *no-dependence* form appears because of a comparison to the $\hat{\beta}$-component, which does scale with the placement of the azimuthal angle.

Adam: (sweeps arm vertically) For [motion in] α, it doesn't have any dependence on this other angle.

As with the other symbolic forms, we can now elaborate upon students' use of the no dependence form and connect it explicitly to students reasoning about the geometric motions by using conceptual blending (Fig. 8.13). Again, Sherin's conceptual schema "a whole does not depend on a quantity" takes the role of the generic space. Then Adam's explicit exclusion of a β-term in the α-component can be compressed with the symbol template that shows the absence.

By importing a conceptual blending framework, we gain a sense of the mechanism through which symbolic forms are activated as students make sense of the mathematics used in physics. As such, a depiction of deeper conceptual physics and mathematics understanding emerges, one that is needed by students in upper-division physics.

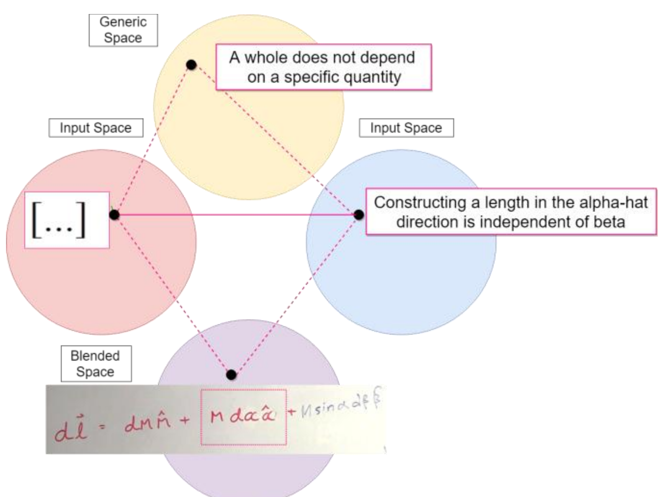


Figure 8.13. Blending diagram including no dependence symbolic form.

The introductory kinematics context involved connecting acceleration to changing velocity, which is a portion of the way in which the concept of acceleration is defined in kinematics. As such, the line between the conceptual schema of “change in base quantity” and the contextual understanding of “acceleration as a change in an object’s initial velocity” is difficult to distinguish. The conflation of the conceptual schema and contextual understanding by Kuo and colleagues [93] indicates that their suggestion of a model of blending between the two components of symbolic forms (conceptual schema and symbol template) was, in essence, a blend of contextual understanding and symbolic expression. In this section, we have fully articulated such a model by representing the conceptual schema as the generic space in a blend of contextual understanding and symbol template.

In upper-division physics, the expression of an equation often involves a substantial background of conceptual understanding in terms of physics concepts. Expressions of vector calculus connect to various coordinate systems, vectors fields, and charge/current distributions, which are built into students’ expressing of equations and in turn can be interpreted from the expressions. As shown above, variations of students’ conceptual understanding of quantities, such as the differential are now present. The presented model accounts for such variation by separating the conceptual schema and conceptual understanding in the analysis of students’ in-the-moment construction of equations, which becomes increasingly important to developing an understanding of students’ work as they move beyond algebraic contexts to include ideas such as those that involve vector calculus.

8.3.2 Affordances of the model

8.3.2.1 Connecting the underlying generic space and variations in conceptual understanding

In conceptual blending, the generic space does the work of providing the underlying connections between two distinct input spaces. These underlying connections drive the compression of these ideas and the emergence of the blend. To analyze how students engage in the construction of equations, we have equate the generic space as the conceptual schema of symbolic forms. Just as before, it is important to note that Sherin's conceptual schema is not a stand-in for physics conceptual understanding. This is even more true in upper-division work, where students' conceptual understanding pertains to more complex and intricate mathematical and physical ideas.

We have argued that the conceptual schema that underlies a symbol template also underlies the student's contextual knowledge or understanding. In line with Sherin's depiction of the underlying conceptual schema as consistent with phenomenological primitives [72], we see the conceptual schema as the fundamental "behind-the-scenes" [47] understanding of the conceptual input of the blend. We elaborate upon this by returning to the discussion of varying conceptual ideas being attached to the representation of a differential element $d\square$. By the time students make it to upper-division physics, the ideas related to vector and differentials have been largely ingrained, in that the structures are generally identifiable and understood by many students. The differential has become a fundamental quantity involved in everyday calculation, but the meaning of the quantity can vary. As Carol and Dan worked on constructing their differential length vector, they only referenced the differential as a change in a quantity, while Elliot and Frank were mostly focused on the size of the quantity, invoking the differential as part of a need for a small bit of a variable. Other research in E&M has identified other ways in which students treat or conceptualize the differential: as a small amount, a dimensionless point, a cue to differentiate, and an identification of what to integrate with respect to [25]. Investigations of calculus students' interpretation of integrals revealed interpretations of the differential related to the width of a Riemann rectangle, shape in space, and "way to obtain the original function" [50]. Small quantities or changes are often the more prevalent understanding of the quantity for students using mathematics in physics problem solving [25,27,49], but that does not prevent the other ideas from appearing in physics students' problem solving.

In a symbolic forms sense, the box of the template for the differential is not large enough to encapsulate the entirety of those ideas. Instead, we put forth that there is some underlying conceptual schema, a fundamental essence of what is a differential, that exists beneath these ideas. This idea is consistent with Sherin's association of the conceptual schema with phenomenological primitives. However this becomes difficult to define, given the difference in conventions and pedagogical emphases between disciplines. For the sake of our work, we retain the conceptual schema as "a differential quantity," in order to maintain that such an idea can extend to the various conceptualizations depending on the given context.

Isolating Sherin's conceptual schema in such a way now allows a reengagement with prior literature utilizing symbolic forms, specifically work with integration, without detracting from the value of that work. Meredith and Marrongelle [48] originally identified the conceptual schema of parts of a whole and dependence as cues for integration. Our model of conceptual blending identifies these cues as the underlying mathematical understanding of the generic space connected to students' conceptual understanding, not necessarily the conceptual schema given that students would invoke different symbol templates.

Separating the conceptual and symbolic input spaces, we allow a different categorization of Jones's integration symbolic forms [50]. Now, rather than having multiple symbolic forms tied to the same symbol template, we can see each template as the manifestation of one symbolic form with a single conceptual schema tied to the use of each box in the template (Fig. 8.14). Much like the conceptual ideas associated with the differential, the ideas of adding up pieces, adding up the integrand, perimeter and area, and function matching, which all utilize the same template are now multiple departures from a more representational understanding of what the arrangement of symbols within the integration means. It is further likely that these templates for integrals may exists as an amalgamation of smaller units of symbolic forms, in the way that students often

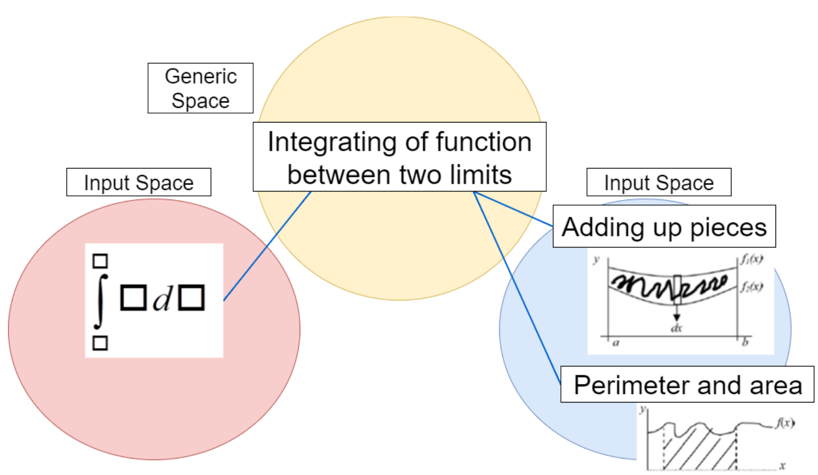


Figure 8.14. Interpreting symbolic forms for integration using conceptual blending.

combine multiple templates to express more complex physical relationships among numerous quantities. However, by utilizing the generic space, what was originally identified as a conceptual schema takes the place of the conceptual understanding input space, separating out students' conceptual ideas from the more fundamental template understanding as done in the original symbolic forms literature.

8.3.2.2 Recall, backward projection, and reading-out

While Carol and Dan were able to produce the appropriate structural representations from repeated use and teachings within the classroom, students across several interviews experienced difficulty in generating or applying the correct conceptual ideas as they constructed the beta-hat component. In order to fill in the template, students recall the similar spherical coordinates in order to make sense of the unfamiliar system.

Bart: You can, you can check from [spherical dl], um

Adam: For α it doesn't have any dependence on this other angle over here, but when you're talking about β, um/

Bart: So this is dl (g. to spherical $\overrightarrow{dl}$ he wrote), okay, $dr\ r$ [hat], $r\ d\theta\theta$ [hat],=...= $r \sin\theta\ d\phi\phi$ [hat], so now we have just to compare so we have r it is M, θ is α... ϕ is β. Go ahead [Adam]

...

Adam: Yeah I can see now, this α here is independent of whatever β is, yeah, so $M \sin\alpha\ d\beta$.

Here we see Adam working within the coordinate system to construct the differential length vector. In contrast, Bart immediately begins to map spherical coordinates, drawing on the spherical differential to finish the construction. After an attempt to redraw the coordinate system, and some confusion between the mathematics and physics representations of spherical coordinates, Adam finally settles on the mapping of $\sin\alpha$

into the $\hat{\beta}$-component. For Adam and Bart, the recall of a spherical differential takes the place of conceptual understanding and neither student draws back on the conceptual understanding that went into the construction of the spherical differential length element (Fig. 8.15). Within conceptual blending, we would here only insert the recalled element into the conceptual input space regardless of its correctness.

In contrast, other groups attempt to use recall as a sensemaking tool. Carol and Dan recall the spherical volume element as well as the Cartesian coordinate transformations to, as Carol states, "make sense of the new coordinate system." However, the group struggles to find anything to dissuade them from a direct mapping of variables and thus settles on the $\sin\alpha$ as part of the beta component. In contrast, Elliot and Frank acknowledge the differences between the two coordinate systems. Frank correctly dictates the comparable spherical component as, but unable to discover conceptual basis for the inclusion of a trigonometric function, Elliot was hesitant to use recall as a justification.

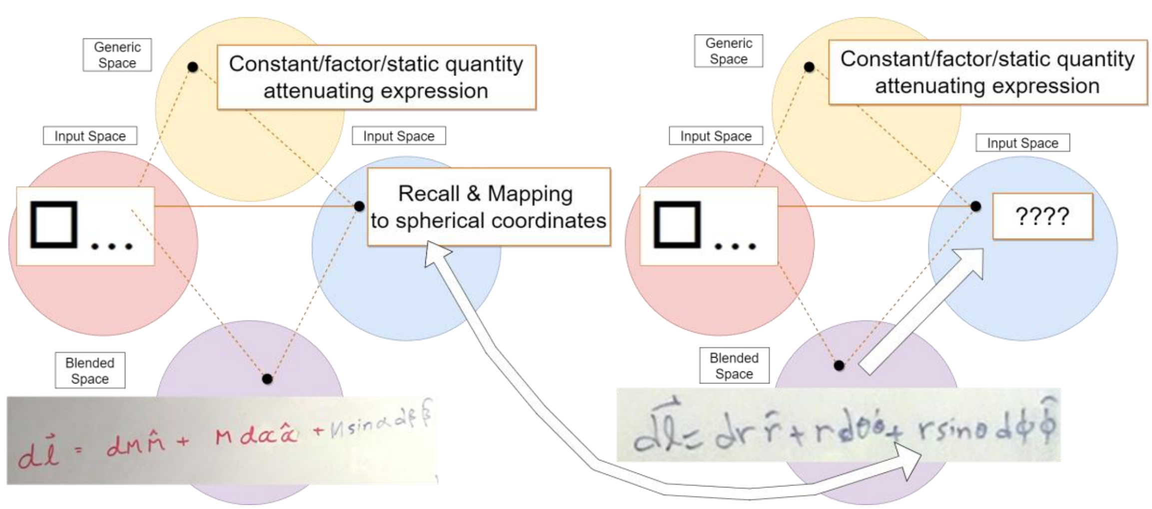


Figure 8.15. Students' conceptual blending involving recall and backward projection.

Elliot: Yeah, because if it were spherical coordinates, you'd have a $\sin\theta$ somewhere in there, you know...which it's very similar, I agree, but I feel like we should just work only by what we see here and try not to fog our mind with preconceived notions of how this should work.

At this point the group settles on $Md\beta$, relying on their conceptual understanding of arc length but still missing the necessary projection aspect that explains the trigonometric function. Later the group returns to this idea, as Frank feels the need to have their differential length resemble the one in spherical coordinates absent of the conceptual idea with this space.

Frank: I mean, uh, spherical coordinates don't look like that. They have sines in there and I agree but if I can't find a reason to put it in there, you know, and there must be something wrong with the way I'm thinking. If that's true but I just don't, I don't see it yet, so why do you have $r\sin\theta$?

This statement of "I can't find a reason" marks a backward projection in the blending literature [76]. A backward projection describes the use of the blended space to interpret or look back at one of the input spaces. We identify the use of the spherical differential elements within the latter of the two groups as an attempt to use spherical coordinates to draw out the associated conceptual understanding attached to the angular components. With neither group recognizing the need for a $\cos\alpha$ in the $\hat{\beta}$-term, they each take different paths: Carol and Dan directly mapping the elements into the schmerical coordinate elements, and Elliot and Frank choosing to stick to the elements constructed within the realm of what they understand. This shows students experiencing difficulty with contextual knowledge, despite having the correct structural understanding of the template. Therefore, it extends the explanatory power when compared to the individual theoretical frameworks.

Students' maneuvering within the blending diagram in order to ascertain the relevant conceptual information from a previously constructed equation further connects conceptual blending to symbolic forms. Sherin not only identified symbolic forms as a way to analyze students' abilities to construct equations, but as a means to address their abilities to "extract implications from a derived expression," thus students' abilities to read out information from an equation based on the given structures [47]. While we see an aspect of this in attempts to isolate the coefficient template of a spherical differential, we suggest this reading out more explicitly draws on the backward projection. Drawing again on *parts-of-a-whole*, a student seeing an equation in which multiple things were being added together could recognize the parts-of-a-whole template and then infer a conceptual understanding of the nature of the relationship between the added quantities. In essence, the equation then carries this information, which is then projected into the larger conceptual space. This is seen in the earlier example presented by Bollen and colleagues [39] in which interpretation of a calculation led a student to correct aspects of the physical system.

8.3.2.3 Interpreting template errors in equation construction

One of the benefits of applying conceptual blending in any context is the ability to isolate particular realms of ideas. In research on the use of mathematics in upper-division physics, this has manifested as the ability to isolate particular errors to difficulties with mathematics or physics ideas [39,81]. While this model has given us a means to assess errors in a final expression that can be attributed to missing or unaccessed conceptual understanding, the benefits of this model extend to analyzing students' mistakes in their symbolic forms understanding, meaning insight can be gained about students' mistakes with the representational mathematics.

In a different study, we conducted individual interviews to investigate students' understanding and construction of differential area elements within common E&M contexts (see section 7.3, [97]). One question in particular required students to construct a scalar differential area to solve for the electric field above a circular sheet with constant charge density (Fig. 8.16). In this task, a student seemingly displays the correct conceptual information but invokes the incorrect symbol template. After first attempting to ascertain the differential area by taking the derivative of (πr^2) with respect to r, Jake then recognizes he can build a differential area from differential length components.

> Jake: Actually no, it will be $drd\theta$ because it's a surface area so I'll need two dimensions... that my dθ is probably going to come in from my dq. Because I should have a differential area shouldn't I, and a differential area should be $drd\theta$... [writes $dr + rd\theta$].

Despite recognizing the need for two dimensions, which would imply multiplication between the two length components, Jake's "$drd\theta$" evidently contains an implicit

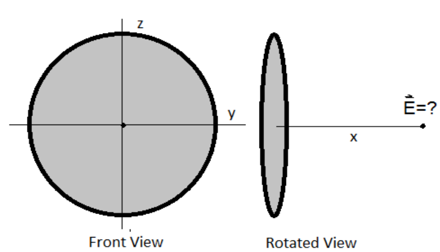


Figure 8.16. Figure provided for the charged sheet task. Full details of the task are presented in section 8.3

addition symbol, as well as a radius term. Jake makes this error on an earlier task as well, despite having an otherwise appropriate concept image of a differential area as a small portion of area (see section 7.3.2.1.3, [97]).

Within our proposed model for equation construction, Jake's conceptual understanding input space for differential area contains the correct information, yet it is blended with an inappropriate parts-of-a-whole template (Fig. 8.17). Using this symbolic forms understanding, we can hypothesize that Jake's underlying conceptual schema was skewed to that of parts-of-a-whole. He thus could be seen approaching the idea of area as being made up of two lengths and used the incorrect template during the compression of ideas. As such, he wrote the terms as a sum rather than a product. Much later in the interview, Jake was able to correct his differential area by reasoning about dimensionality, which shifted the representational form to the correct multiplication of lengths.

Likewise, Sherin noted instances of students accessing the requisite conceptual information but applying the incorrect template [47]. Within our work analyzing

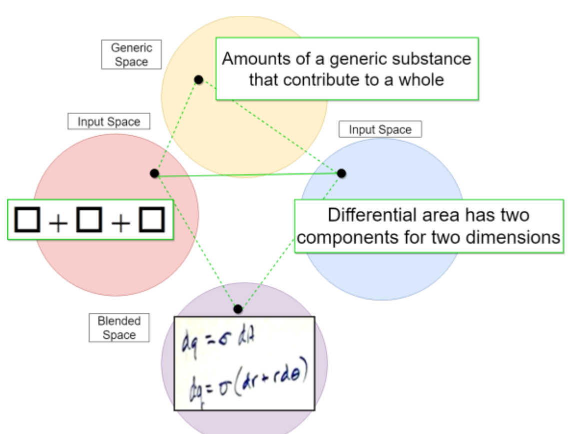


Figure 8.17. Conceptual blending where Jake invoked the incorrect template.

students' differential length elements, we noted that students had a general understanding of the symbol template in terms of the structural representation of the differential length vector, but had more specific difficulty with understanding the geometry of the coordinate system and expressing it appropriately.

In a further study, students constructed differential length vectors during a calculation of change in electric potential around a curved path (see section 5.2, [98]). During these interviews we noted an incorrect encoding of vector notation which has been seen commonly in students' work from course observations. The correct expression involves a differential length with two components to represent each polar direction of motion, as Molly easily demonstrates.

> Molly: So first I travel in the r-direction so I go dr in the $\hat{r}$ and then I travel in the $\hat{\theta}$-direction and the arc length of a circle is the radius times the angle that you move so that is $rd\theta$, here in the $\hat{\theta}$. (Fig. 8.18)

Here, we see her emphasis on the unit vectors and associated components, which she deftly represents using the magnitude-direction template.

In contrast, Lenny only constructs a component in the theta direction. Despite similar conceptual understanding, Lenny expresses his differential component as $rd\vec{\theta}$. When asked to describe why he wrote the term in such a way, his reasoning was absent of magnitude-direction reasoning.

> Interviewer: What do you mean by $d\vec{\theta}$ there?
>
> Lenny: So I guess, any differential shift in θ…because that's just the direction of the change in θ. (Fig. 8.18)

Mathematically speaking, the use of $d\vec{\theta}$ makes the expression incorrect. While $d\vec{x}$ would make sense for a differential shift in the x-direction, polar unit vectors are not static

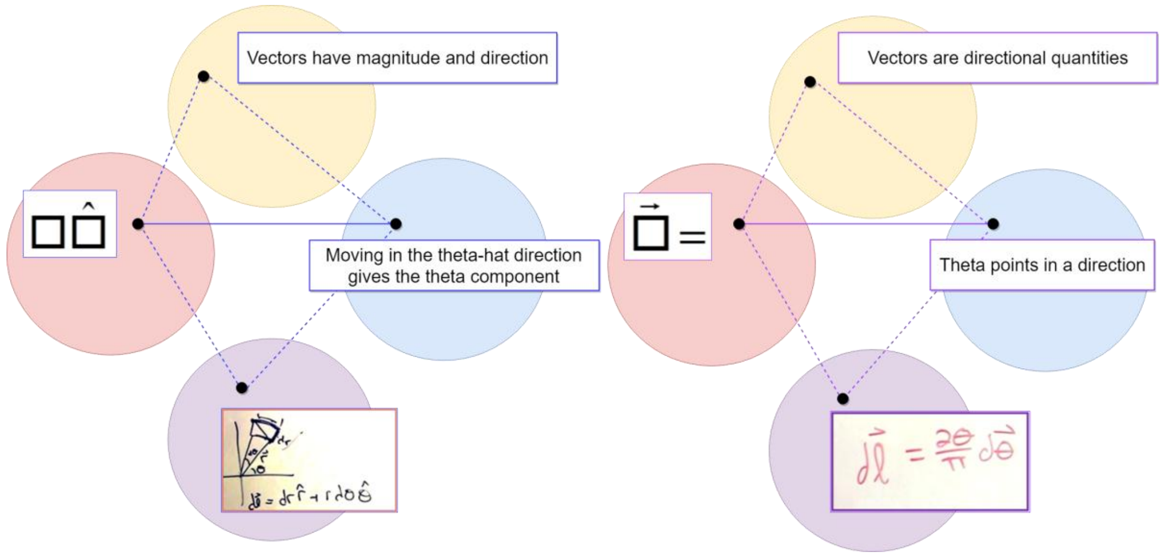


Figure 8.18. Comparison of students' blending diagrams for expression differential vector elements

quantities and vary based on position in space. In our analysis, Lenny's idea of representing a vector within this space is reduced to a representation of "the direction of change in theta." His emphasis on directionality without a separation of magnitude and unit vector leads to his encoding of this expression with a vector arrow template, [$\vec{\square}$], rather than the magnitude-direction template, and thus makes sense within the presented model of conceptual blending and equation construction.

8.4 Summary and Conclusions

In this paper, we have used conceptual blending to analyze students' mathematical sense-making when constructing equations in upper-division physics. As part of previous work, we analyzed data on students' construction of differential length elements in an unfamiliar spherical coordinate systems using two different approaches: concept image [38] and symbolic forms [47]. Analysis involved the use of a concept image framework to identify specific properties students associate with a differential length vector in a non-Cartesian coordinate system, as well as a symbolic forms approach to investigate students structural understanding during equation construction (see 5.1.5, [91,92]). As symbolic forms were designed to assess the mathematical understandings of the structures within an equation, and not the physics conceptual understanding, we recognized these as naturally compatible to give a picture of both sides of the equation; yet they still remained independent analyses without a cohesive tie. By incorporating a conceptual blending lens [76], originally designed to describe the connection of meaning to form in the use of language, we have developed a model with the means to analyze students' construction of equations as an expression of a mathematical-physical language in which they connect conceptual understanding and structural expression.

This approach to analysis of equation construction uses the aspects of one theoretical framework to complement missing analytical aspects of the other. Use of conceptual blending adds a component of conceptual understanding to a symbolic forms analysis, which becomes increasingly important within upper-division physics where concepts connected to equations become more rigorous. Likewise, incorporating symbolic forms into a conceptual blend provides a guiding generic space to analyze student understanding and use of mathematics in physics contexts. To represent the union of these frameworks and illustrate the model, we designed a blending diagram that represents the conceptual blending generic space as the symbolic forms conceptual schema and depicts the compression of conceptual and representational understanding into the final construction of an equation.

This work has presented a number of examples in which our model is employed within the context of the differential length vector study, as well as several other instances in our own work. This serves to illustrate the model as well as to show the utility of bringing conceptual blending to the construction of equations and symbolic forms. We have also provided discussion as to how this model is consistent with and reinterprets the use of symbolic forms within the current literature base [48,50,93], where the conceptual schema of symbolic forms has equated with the conceptual understanding of the contextual content. Similarly it shows how use of the generic space, which is generally absent from conceptual blending analysis of mathematics in physics [39,81,94], can provide deeper explanation of students' conceptual and representational choices when constructing equations.

Lastly, we have outlined several benefits of such a model as well as the full scope of its explanatory power. The incorporation of the generic space as the underlying mathematical meaning or idea has provided the ability to connect diverse student conceptual understanding to similar template use. This model also isolates the specific structures of an equation so as to connect student difficulty to either template understanding or incorrect/incomplete conceptual understanding. This model also supports the backwards projection of the conceptual blending model, by connecting it to the reading of information out of an equation to gain conceptual understanding. We also showed how backward projection was useful in describing errors in students' recall in which they use previous ideas to make sense of new contexts.

The presented model provides the opportunity for obtaining a deeper and more complete understanding of students' construction of equations in situations that draw on sophisticated mathematical and/or physical understanding. The connection of aspects across these theoretical frameworks allows for analysis on both the level of conceptual understanding and of structural representation.

8.5 Future Work

With the understanding of the affordances of such a model to the analysis of student construction of equations in terms of conceptual and representational understanding, we envision further applications of the model.

Just as Sherin suggests the symbolic forms framework could be extended into other domains of physics, we believe that our model presents as a key analytical tool to the study of mathematics used in physics problem solving, especially in an upper-division context where, throughout the course of their academic track, students connect physics to concepts of vector calculus, partial derivatives, and linear algebra.

Sherin also suggests that “stretching farther still,” symbolic forms could be generalized to discuss other representational forms that contain sets of meaningful structures. We hypothesize that the incorporation of conceptual blending takes a step in that direction by providing the generic space as a means to connect ideas by their underlying similarities. As such, we can extend the template space to a representational space and connect students’ conceptual understanding of linear relationships and graphing knowledge to graphical representations, and additionally with conceptual ideas of wave vectors, wave functions, or probability density graphs. Researchers have recently begun to address students’ understanding of the various representations of Dirac notation, wave function notation, and matrix notation [17]. Other researchers have explored students' metarepresentational understanding of these notations, finding when students make judgments about which notation is easier or better suited to a task [99]. More broadly, a model of conceptual blending as we have presented could be extended to analyze student work as they translate between various representations that effectively convey the same conceptual understanding.

CHAPTER 9

DISCUSSION AND CONCLUSIONS: CROSSCUTTING CONCEPTS AND COORDINATE SYSTEMS

"It is good to have an end to journey toward,
but it is the journey that matters in the end"

–Ursula K. Le Guin

The work presented in this dissertation is the result of several years of investigation into students' understanding of one aspect of the vector calculus concepts that are ubiquitous junior-level electricity and magnetism. Specifically, this investigation has explored students' conceptual and structural understanding of differential length vectors, differential area elements (scalar and vector), and differential volume elements, as these elements are constructed and determined in a given coordinate system. This is a continuation of a recent focus of physics education research both in the emphasis on the application and understanding of mathematics and as an inquiry into student understanding of upper-division content. While previous work has involved exploration of mathematics in E&M, little work has previously addressed construction of differential elements in the non-Cartesian symmetries used throughout the course. This study contributes empirical research that addresses student understanding and informs instruction of these quantities.

Interviews were designed using tasks similar to those presented in course instruction as well as a task using an unfamiliar, unconventional coordinate system. Data from interviews using the tasks within a physics context provided insight into the connection between contextual features and students' construction of differential elements. Data generated from the task based in the unconventional coordinate system provided insight into the particular ideas associated with a generic differential length vector in non-Cartesian coordinates. Analysis focused on identifying student difficulties [57], aspects of students' concept image [38], and students' understanding of equations in terms of symbolic forms [47]. The instantiation of these frameworks focused investigation on students' understanding of symbolic expressions and conceptual aspects and how these impact construction. In chapter 8, we combined the concept image and symbolic forms frameworks using conceptual blending [76] as a theoretical model to depict how students' contextual knowledge and representational understanding are combined in the construction of equations. We further extend this model as a means to address students' mathematization.

In this chapter, we present the conclusions as a discussion of common threads woven across the previous chapters. Initial attention is given to the extent to which coordinate system understanding influenced determination or construction of differential elements. Secondly, focus is turned to common concept image elements as they were or were not evoked across the interview tasks. Given the analytical focus on student understanding and invocation of symbolic forms and emphasis of multiple tasks on construction, further discussion highlights the common representational understandings in terms of how students encoded information in equations across chapters. Following this, I discuss the extent to which students recognized or utilized the relationships among differential lengths, areas, and volumes. Finally, there is a summary of instructional implications and suggestions for future works.

9.1 Overview of findings: Coordinate system choice and geometric reasoning in curved spatial coordinates

The choice of coordinate system due to field symmetry and charge/current distribution is generally the first step in the mathematization of a physical situation in E&M. This choice impacts the expression of differential elements, fields, and vector operators. While the use of Cartesian coordinates dominates much of both mathematics and physics instruction, the physical symmetries of E&M dictates the use of other coordinate systems as a means to simplify calculation. Use of non-Cartesian systems, however, requires an understanding of how the curvature affects the geometry and expression for the differential elements.

Results presented as part of this research project corroborate findings in the literature regarding student overuse of Cartesian coordinate systems for situations in which a curvilinear coordinate system would ease the calculational burden [10,44]. In some cases, the use of Cartesian coordinates can be equally productive, such as the flux task, on which a number of students used Cartesian to express the differential area for a square loop (see section 7.3.2.1.1, [97]). However, Oliver's attempt to use Cartesian coordinates for the spiral task offers an example of when use of Cartesian coordinates leads to unwieldy and calculationally inefficient expressions (see section 5.2, [98]).

Students' construction of differential length vectors in schmerical coordinates (section 5.1, [91,92]) also revealed the predominance of expressions related to Cartesian coordinates. Pairs GH and PQ constructed schmerical differential length expressions that were rooted in the Cartesian system. Rather than associating a differential length vector with a sum of components resulting from motions of the coordinate variables, these pairs isolated components of the $\vec{M}$ in each of the Cartesian directions. CD engaged in a similar activity as they try and find the $\hat{\beta}$ and $\hat{\alpha}$ components. This originally led to a cosine term in one component and a sine term in the other, before a comparison to spherical guided the remainder of the construction. Notably, a decomposition of a vector into Cartesian coordinates in terms of spherical components is an important problem-solving step when applying Coulomb's Law, since this generic brute force approach often utilizes both Cartesian and non-Cartesian representations.

Students' responses in the generic differential length construction echo those found in the classroom: students attempted to construct generic differential length expressions in spherical and cylindrical coordinates, and even included inappropriate trigonometric functions (see section 4.3, Table 4.1). However, even students who constructed a differential length vector utilizing the elements of schmerical coordinates had significant difficulties reasoning about the geometry of the system.

Generally, this overuse of Cartesian in any case speaks to a difficulty connecting to the underlying symmetry of the physical situation [12], a difficulty that leads to larger issues of determining appropriate coordinate systems. Analysis across several interviews shows that students struggle to connect the symmetries of the vector fields to the coordinate system of choice, and thus to the choice of differential elements. While working through the flux task (see section 7.3, [97]), students attended more to the shape of the given area (square loop - Cartesian) rather than to the curling magnetic field (circular symmetry - cylindrical). When asked how their response would change if the square loop was rotated out of the plane, three of the four students using Cartesian coordinates did not recognize the field was still perpendicular to the loop and suggested the dot product of field and differential area would yield a trigonometric function. At the time students were enrolled in E&M II, which commonly involves curling magnetic fields and cylindrical symmetry. Oliver specifically added a $\hat{\phi}$ to express the curling field, but demonstrated a strong preference for use of Cartesian coordinates.

Analysis of the spiral task (see section 5.2.2, [98]) further shows student emphasis on the given shape of the path with little attention to the contextual physics. While students in the interviews more often utilized curvilinear symmetry, there was emphasis on the rotational aspect of the spiral path and little attention to the radial direction of the electric field.

In a small number of cases, some students never explicitly chose a coordinate system when problem solving or showed a limited understanding of coordinate systems. Lenny, in particular, never defined a coordinate system when approaching the flux task and only stumbled upon the correct solution after spending some time attempting to ascertain the direction of the magnetic field. Similarly, Kyle incorrectly associated the circular charged sheet (see section 7.3.2.1.1, [97]) with spherical symmetry, rather than cylindrical. Bart and Harold both displayed difficulty with determining directions of cylindrical unit vectors, and even drew cylindrical coordinates as having two angles (see section 7.2.2). In the checking solution task (section 7.1), pairs were able to recognize that the differential area was inappropriate for the given task but some went further to incorrectly attribute the element to spherical coordinates or another surface within spherical coordinates.

Students' difficulties recognizing the scaling factors in the checking solution task as inappropriate for any coordinate system speaks to larger difficulties for students in regards to geometric reasoning. Only a small number of students explicitly attend to arc length across the body of interviews (i.e., EF and RS during construction of the schmerical differential length, and Molly for the spiral and charged sheet task). This does not mean that other students do not have an understanding of arc length, but that it was not evoked in the given contexts. This suggests that students have a limited understanding of the construction of these terms, as arc length is monumentally important to the construction and understanding of differential elements in curvilinear coordinates (see Appendix A).

Notably, for both EF and RS, who explicitly discussed the need for arc length in the schmerical length and schmerical volume constructions, respectively, the trigonometric function needed to account for projection was absent from their final expressions. Thus while arc length was accessible for these students, it was not tied to other aspects of the coordinate system geometry. The understanding of projection that results in the trigonometric function in spherical-like coordinate systems (see Appendix A) was difficult for all groups in the schmerical task. Only three groups in the seven interviews were able to connect the trigonometric function to projection, and this only occurred after students checked their differential volume element and calculated an incorrect volume.

Results have also shown a number of instances in which students have trouble reconstructing the differential area elements in regards to the scaling factors that needed to be expressed. While Bart and Harold have significant difficulty constructing generic differential areas, even Dan and Victor, who are successful with the task, question the inclusion of the trigonometric term (See chapter 7.2.2). Analysis of area element construction in other contexts suggests that the instantiation of high symmetry tasks obscures the origin of differential terms. The large number of problems in E&M that involve bypassing the writing of the differential element or that consistently only use one component (such as the radius) could result in a restricted concept image of differential elements where the reason for the trigonometric function or other scaling terms is lost.

In conclusion, students appear to struggle with determining appropriate coordinate systems, often relying on Cartesian coordinates. Further investigation on construction of a generic differential length element within an unconventional system revealed student difficulty with recognizing the affordance of leveraging the geometry of a system to determine the expressions for the differential components. Unsurprisingly, students with a higher tendency to connect vector fields and charge/current distribution to coordinate systems and expression of differential elements performed better on these tasks. This leads to suggestions for instruction, which are further discussed later in this chapter.

9.2 Overview of findings: Ubiquity of concept image aspects in differential element construction

This section gives explicit attention to prominent concept image aspects identified in the schmerical differential length vector construction (see section 5.1.4, [92]) and their influence on construction of differential elements as a whole. These include students' attention to aspects such as *dimensionality* and *differential*. These aspects pervade construction of differential elements, as lengths, areas, and volumes all need to express appropriate dimensions. Furthermore, differential elements are differential quantities. Thus we can compare students' treatment of these quantities (which are sometimes vectors) to previous literature looking at the differential in other contexts. Lastly I discuss attention to *component & direction.* I omit discussion of the *projection* aspect here, due to its connection to the discussion of geometric reasoning in the previous section.

9.2.1 Role of dimensionality

Attention to dimensionality was noticeably constructive for students during the schmerical differential length task. Students in pairs AB, CD, and EF regularly attended to dimensionality, making sure each component expressed units of length. On the extreme end, the radius term was sometimes only included following argumentation that the term needed to include lengths, such as for Adam in the $\hat{\alpha}$ component and for Carol when constructing the $\hat{\beta}$ component, saying "sine of something isn't a length, so we need something else in there" (see section 5.1, [91,92]). In these cases the overt attention to dimensionality overshadowed the geometric reasoning related to arc lengths. Carol and Dan gave explicit focus to each term being a differential length and at one point questioned whether the differential angles or unit vectors also carried units of length. For other students in the task, there was not discussion of dimensionality, which may have resulted in the length components that contained both an x and a dx. It is likely that in these cases, students did not recognize differentials as quantities that have dimension, which is a finding common with other studies of differentials [25,52].

EF used dimensionality to reason about the correctness of their differential volume element later during the schmerical task, claiming it was likely correct, as it would integrate to an M^3 (see section 6.6.2, [92]). When Elliot acknowledged that integration over the angles could yield any coefficients beyond the $4/3\pi$ that were needed for the volume of a sphere, the pair carried out the necessary integration.

When determining differential area elements, several students also explicitly addressed dimensionality. Jake, having first incorrectly reduced the flux integral to a dot product with a length, recognized he needed an element which expressed two dimensions (see section 7.3.2.1.3, [97]). However, he incorrectly represented this as a sum rather than a product, which we discuss later in this chapter.

Overall, dimensional consistency of differential lengths, areas, and volumes is important to construction. While some students attend to this explicitly, in other cases not associating units to the differential elements contributed to their incorrect representations of terms.

9.2.2 Student understanding of differentials

Interviews during which students were asked to construct differential lengths, areas, and volumes, revealed myriad understandings of the differential quantity consistent with previous literature.

As part of the schmerical differential length task (see section 5.1, [91,92]), students commonly discussed needing small amounts of motion or changes in a given quantity. These concept image aspects were helpful for students building the components rather than using recall. The treatment of the differential in this way is common to physics instruction [25,27,49,56] and productive for students making sense of integration [28,48,52,69].

This particular concept image also appeared in students' construction and determination of differential area elements (see section 7.3, [97]). In these tasks, rather than constructing a generic expression for a differential element, students were constructing an expression explicitly for the purpose of integration. Here, thinking about the differential area as a small portion of the surface in question, specifically as a product of differential lengths, was productive for students.

Students also associated the differential as a cue to take a derivative of another quantity [25]. This is most prominent in the differential area context, where Jake attempted to take a derivative of the area of a circle but struggled to determine what the derivative was with respect to. The idea also appeared in the schmerical length construction when Tyler began with an incorrect expression for the vector and attempted to take derivatives to find the differential length vector. In the spiral task, Oliver started with a dx and dy and attempted to take the derivative of the Cartesian transformations to convert the expressions into terms of theta. This type of representation and transitioning between understanding of the differential as an object and an understanding of the associated process to differentiate can be productive in some physics contexts when used appropriately. Only Jake would have been able to arrive at a correct response using this method, but only due to the given symmetry of this task. Other students struggled with this due to other difficulties.

Lastly, results showed at least one student routinely approached differentials as identification of the variable of integration [25,52]. In this representation the differential has no physical meaning. In both differential area tasks, Nate added differentials to indicate the variables over which integration occurred. Notably, the equations he used included a differential area, which he replaced with an expression for the full area of the surface and didn't attend to as a differential quantity.

Attention to students' treatment of differential quantities spans the space of understanding detailed in the literature. As such, this means there is no single understanding students have of differential lengths, areas, or volumes when associated with the context of vector calculus. However, association of the differential as a change in a direction or as a small portion of a line or surface remain the most productive representations for this context.

9.2.3 Recognition of component and direction

In the construction of the schmerical differential length element, students in all but one interview eventually recognized the need to express multiple components. Transitions to a more contextual task, which included a spiral path (where the differential still included two components), involved more students only expressing a single term for a differential length vector, in line with highly-symmetric situations seen in class and on homework assignments.

9.3 Overview of findings: How students encode information: Symbolic forms understanding

Analysis of the schmerical differential length construction in terms of invoked symbolic forms [47] revealed students had a general understanding of the structures in the equation. The difficulty appeared in determining the quantities or variables that filled the structure. For example, students recognized where a coefficient was needed and often left space to write terms, but did not access the ideas of arc length or projection that would have yielded the appropriate terms. In many cases, the filling of the associated symbol template was mediated by recall to spherical coordinates. In other cases, the correct symbol template was cued with two different and equally valid conceptual understandings. Both CD and EF correctly expressed differentials, but CD continually used the concept of change in a variable while EF focused on needing a little amount.

Complimentary results from concept image [38] and symbolic forms analyses led to the use of conceptual blending [76] to account for the types of variation in students' construction of equations described above (see Chapter 8). Importing conceptual blending provided a way to account for variation in conceptual understanding when using a symbolic forms analysis. Likewise, importing the underlying conceptual schema from symbolic forms provided a necessary structure missing in previous literature on students' blending of mathematics and physics. As described in the previous chapter, this work extends beyond the schmerical differential length to other contexts in our study where students construct and interpret expressions.

Students' success with structural representation and understanding extended to construction of differential area elements. Students generally were able to invoke requisite templates and in some places articulate the differential area as a product of differential lengths. However, to some extent, a structural analysis is obscured in this context due to the "plugging in area" mentality cued with the instantiation of high symmetry in physics contexts. This results in fewer students constructing the differential area element outright as an infinitesimal.

Over the course of the study, a fair number of students have shortcut the magnitude-direction representation of a differential element by writing the differential as a vector (e.g., $d\vec{\theta}$ in place of $d\theta\hat{\theta}$) on homework, quizzes, and interviews. Both Lenny and Oliver utilized this representation during the spiral task (see section 5.2.2). Students articulated that it represents the direction in which the change in taken. Notably, course observations show that this representation is not introduced by the instructor. While not mathematically correct, students' specific encoding is suggestive of expert-like behavior in that the expression in shortened using the introduction of specific notation. This goes further to show that students' structural understanding of vectors and some calculus concepts are fairly ingrained and understood by the time they enter upper-division E&M.

Building on this structural understanding, instructional materials were developed in which the equations' structures were isolated and students built the associated concepts (see sections 5.1.4, 7.3). Based on the productivity of this line of reasoning for students in the interviews, this approach should help students build the necessary connections between coordinate system geometry and the expression of differential elements.

9.4 Overview of findings: Students understanding of connections between differential lengths, areas, and volumes

Over the course of interviews, recognition of the interconnectedness of the differential elements was a tool that allowed students to be more productive. Students who had a stronger connection between the differential length vector and the differential volume were able to easily construct the differential volume element as a product of lengths. Furthermore, students who were most productive in the differential area construction were those with the concept image of the differential area as a product of differential length components that describe the surface. When constructing generic differential area elements in spherical coordinates, Victor attended to the multiplication of different pairs of differential lengths to construct different differential areas. Then when checking his responses, he multiplied his conjectured differential area by the third length component to verify whether or not he arrived at the volume, as a means to validate the correctness of his differential areas and justify the inclusion of a $\sin\theta$ in the $\hat{\theta}$ term. Jake fixed his representation of differential area as a sum by recognizing that a Cartesian differential volume was a product of lengths.

Granted, any differential element could be determined from scratch with sufficient geometric reasoning (RS attempt fail to construct a volume element in this way because of a missing trigonometric function; Lenny interpreted the geometry of the flux task to construct a differential area), but a more fundamental understanding of constructing differential lengths and an infusion of product understanding allow students to efficiently determine subsequent differential elements.

Notably, it was much more difficult for students to deconstruct a non-Cartesian differential volume element into associated length terms. Both pairs CD and GH experienced difficulty determining a spherical differential length vector from the more easily recalled spherical volume element. The terms were entirely estranged for PQ, RS, and T, who experienced the most difficulty with schmerical differential length construction; they were easily able to recall the spherical volume element but did not connect the terms within the volume as components of a differential length vector. Students AB, CD, and EF were able to use the differential volume to correct their length terms but it was only these three groups that built the volume element as a product of differential lengths.

Therefore, a product understanding is useful for the construction of differential volume and area elements, as long as students possess sufficient understanding of how differential length terms are constructed within a given geometry.

9.5 Implications for instruction

Results suggest that instruction should give greater emphasis to the way the underlying coordinate system geometry connects to the construction of the differential elements. Students with stronger geometrical reasoning were better able to construct differential elements both as generic expressions and within specific contexts. Further emphasis should connect differential area and volume elements more explicitly to the origin of differential lengths. The connection of these differential elements to differential length terms was significantly productive for students, whereas the absence or inattention to these connections resulted in greater difficulty.

These instructional implications have already led to the purposeful design of instruction tasks in the spirit of previously developed physics tutorials [6,66]. The first portion of the developed tutorial sequence builds the geometrical understanding of a spherical-type (schmerical) coordinate system while using a rubber ball to leverage the three-dimensional space the coordinate system represents (Appendix C). This tutorial activity structures the building of each length component by connecting the ideas arc length and projection to the expression of the differential length vector through attention to geometric motions on the surface of the ball.

After connecting the first tutorial to differential length construction in the more common Cartesian, cylindrical, and spherical coordinates, the second portion of the tutorial sequence leverages the understanding of differential lengths to construct differential areas in each of the coordinate systems (also with 3D examples) (Appendix D).

The tutorial pair includes pre-tutorial homework, a tutorial designed for small-group work, and post-tutorial homework. The inclusion of pre-tutorial homework is consistent with previous upper-division tutorials [66] to situate and prepare students to engage with the tutorial. Each tutorial sequence was test-run with physics faculty and graduate students with experience in physics education research. This provided input to further design and modifications. The tutorials were implemented in E&M I near the third week of the course, in subsequent classes. Observations suggest tutorial implementation is promising: the materials seem to generate discussions similar to those in the interviews but allow students to harness an understanding of the physical space, connecting length components to geometric motions. Likewise, implementation of the area tutorial showed it was helpful for students in connecting differential length components in a given coordinate system to a needed differential area element describing a surface. Future implementation of these tutorials should include more discussion about how these ideas appear when problem solving in E&M. These materials will continue to be developed, tested in-house and at external pilot sites, and eventually disseminated more widely.

9.6 Suggestions for future work

This dissertation adds to the growing body of literature on student understanding of mathematics in E&M. While prior studies have explored E&M students' understanding of differentials [25], cues for integration [14,48], understanding of physical symmetry [12,16,24,33], and understanding of vector fields and vector differential operators, little previous work has addressed the construction of differential lengths, areas, and volumes as they connect to vector calculus in non-Cartesian coordinate systems [12]. As such, there is room for further investigation, specifically on the emphasis of physical context on choice of differential elements. This includes how variation in particular features of charge/current distribution and vector fields cue the implementation of different coordinate systems and the associated differential elements.

The theoretical development derived from this study has far reaching implications and thus more work could be done extending this model to other physics contexts outside of E&M as well as other mathematical representations (i.e., graphs, matrix notation) beyond equations. (See Chapter 8 for more discussion.)

9.7 Summary

In conclusion, the work in this dissertation has explored student conceptual understanding of differential vector elements in non-Cartesian coordinate systems. Results document that even after explicit instruction and application of different lengths, areas, and volumes, students in E&M had difficulty with the geometric reasoning related to constructing non-Cartesian differential elements or connecting differential areas and volumes to the components of the differential length vector. Students successfully attending to these ideas were more proficient with problem solving in physics contexts; thus, instructional materials have been designed to guide students to explicitly attend to the development of these ideas.

Furthermore, specific attention to how students connected representation and contextual understanding has led to the development of a model for students' construction and interpretation of equations, by combining complementary theoretical frameworks of symbolic forms and conceptual blending. The theoretical frameworks are complementary in that missing analytical aspects of one are supplemented by the other. This combination provides affordances in regards to previous analyses and can provide deeper insight into how students connect representation mathematics understanding to other physics contexts at the physics-mathematics interface.

REFERENCES

[1] L. C. McDermott and E. F. Redish, "Resource Letter: PER-1: Physics Education Research," Am. J. Phys. **67**, 755 (1999).

[2] R. A. Lawson and L. C. McDermott, "Student understanding of the work-energy and impulse-momentum theorems," Am. J. Phys. **55**, 811 (1987).

[3] H. G. Close and P. R. L. Heron, "Student understanding of the angular momentum of classical particles," Am. J. Phys. **79**, 1068 (2011).

[4] M. C. Wittmann, E. F. Redish, and R. N. Steinberg, "Understanding and addressing student reasoning about sound waves," Int. J. Sci. Educ. **25**, 991 (2003).

[5] M. C. Wittmann, "Using conceptual blending to describe emergent wave propagation," Proc. 9th Int. Conf. Learn. Sci. **1**, 659 (2010).

[6] L. C. McDermott and P. S. Shaffer, "Research as a guide for curriculum development: An example from introductory electricity. Part I: Investigation of student understanding," Am. J. Physics1 **60**, 994 (1992).

[7] J. Li and C. Singh, "Investigating and improving introductory physics students' understanding of the electric field and superposition principle," Eur. J. Phys. **38**, (2017).

[8] President's Council of Advisors on Science and Technology, "Engage to Excel: Producing One Million Additional College Graduates with Degrees in Science, Technology, Engineering, and Mathematics," *Engage to Excel: Producing One Million Additional College Graduates with Degrees in Science, Technology, Engineering, and Mathematics* (2012).

[9] S. R. Singer, N. R. Nielson, and H. A. Schweingruber, editors , "Discipline-Based Education Research: Understanding and Improving Learning in Undergraduate Science and Engineering," *Discipline-Based Education Research: Understanding and Improving Learning in Undergraduate Science and Engineering* (National Academies Press, Washington, D. C., 2012).

[10] E. Sayre and M. Wittmann, "Plasticity of intermediate mechanics students' coordinate system choice," Phys. Rev. Spec. Top. - Phys. Educ. Res. **4**, 020105 (2008).

[11] M. D. Caballero, L. Doughty, A. M. Turnbull, R. E. Pepper, and S. J. Pollock, "Assessing learning outcomes in middle-division classical mechanics: The Colorado classical mechanics and math methods instrument," Phys. Rev. Phys. Educ. Res. **13**, 010118 (2017).

[12] R. Pepper, S. Chasteen, S. Pollock, and K. Perkins, "Observations on student difficulties with mathematics in upper-division electricity and magnetism," Phys. Rev. Spec. Top. - Phys. Educ. Res. **8**, 010111 (2012).

[13] C. Singh and E. Marshman, “Student difficulties with determining expectation values in quantum mechanics,” in *2016 Phys. Educ. Res. Conf. Proc.*, edited by D. Jones, L. Ding, and A. Traxler (AIP Conference Proceedings, 2016), pp. 320–323.

[14] L. Doughty, E. McLoughlin, and P. van Kampen, “What integration cues, and what cues integration in electromagnetism,” Am. J. Phys. **82**, 1093 (2014).

[15] C. Manogue, K. Browne, T. Dray, and B. Edwards, “Why is Ampère’s law so hard? A look at middle-division physics,” Am. J. Phys. **74**, 344 (2006).

[16] C. Wallace and S. Chasteen, “Upper-division students’ difficulties with ampère's law,” Phys. Rev. Spec. Top. - Phys. Educ. Res. **6**, 020115 (2010).

[17] E. Gire and E. Price, “Structural features of algebraic quantum notations,” Phys. Rev. Spec. Top. - Phys. Educ. Res. **11**, 020109 (2015).

[18] G. Passante, “Energy measurement resources in spins-first and position-first quantum mechanics,” in *2016 Phys. Educ. Res. Conf. Proc.*, edited by D. Jones, L. Ding, and A. Traxler (AIP Conference Proceedings, 2016), pp. 236–239.

[19] H. R. Sadaghiani, “Conceptual and mathematical barriers to students learning quantum mechanics,” Unpubl. Diss. (Physics), Ohio State Univ. (2005).

[20] E. Marshman and C. Singh, “Student difficulties with representations of quantum operators corresponding to observables,” in *2016 Phys. Educ. Res. Conf. Proc.*, edited by D. Jones, L. Ding, and A. Traxler (2016), pp. 216–219.

[21] T. I. Smith, D. B. Mountcastle, and J. R. Thompson, “Identifying student difficulties with entropy, heat engines, and the Carnot cycle,” Phys. Rev. Spec. Top. - Phys. Educ. Res. **11**, 020116 (2015).

[22] D. E. Meltzer, “Investigation of students’ reasoning regarding heat, work, and the first law of thermodynamics in an introductory calculus-based general physics course,” Am. J. Phys. **72**, 1432 (2004).

[23] J. R. Thompson, C. A. Manogue, and D. Roundy, “Representations of partial derivatives in thermodynamics,” in *2011 Phys. Educ. Res. Conf. Proc.*, edited by C. Henderson, M. Sabella, and C. Singh (AIP Conference Proceedings 1413, 2012), pp. 85–88.

[24] J. Guisasola, J. Almudí, J. Salinas, K. Zuza, and M. Ceberio, “The Gauss and Ampere laws: different laws but similar difficulties for student learning,” Eur. J. Phys. **29**, 1005 (2008).

[25] D. Hu and N. S. Rebello, “Understanding student use of differentials in physics integration problems,” Phys. Rev. Spec. Top. - Phys. Educ. Res. **9**, 020108 (2013).

[26] T. I. Smith, J. R. Thompson, and D. B. Mountcastle, “Student understanding of Taylor series expansions in statistical mechanics,” Phys. Rev. Spec. Top. - Phys. Educ. Res. **9**, 020110 (2013).

[27] J. Von Korff and N. S. Rebello, “Teaching integration with layers and representations: A case study,” Phys. Rev. Spec. Top. - Phys. Educ. Res. **8**, 010125 (2012).

[28] J. Wagner, “Students’ obstacles and resisitance to Riemann sum interpretations of the definite integral,” in *Proc. 19th Annu. Conf. Res. Undergrad. Math. Educ.*, edited by T. Fukawa-Connelly, N. Engelke Infante, M. Wawro, and S. Brown (MAA, Pittsburgh, PA, 2016), pp. 1385–1392.

[29] E. F. Redish and E. Kuo, “Language of physics, language of math: Disciplinary culture and dynamic epistemology,” Sci. Educ. **24**, (2015).

[30] O. Uhden, R. Karam, M. Pietrocola, and G. Pospiech, “Modelling Mathematical Reasoning in Physics Education,” Sci. Educ. **21**, 485 (2012).

[31] B. Wilcox, M. Caballero, D. Rehn, and S. Pollock, “Analytic framework for students’ use of mathematics in upper-division physics,” Phys. Rev. Spec. Top. - Phys. Educ. Res. **9**, 020119 (2013).

[32] A. L. Traxler, K. E. Black, and J. R. Thompson, “Student’s use of symmetry with Gauss's Law,” in *2006 Phys. Educ. Res. Conf. Proc.* (AIP Conference Proceedings 883, 2007), pp. 173–176.

[33] C. Singh, “Student understanding of symmetry and Gauss’ law of electricity,” Am. J. Phys. **74**, 923 (2006).

[34] D. Nguyen and N. S. Rebello, “Students’ difficulties with integration in electricity,” Phys. Rev. Spec. Top. - Phys. Educ. Res. **7**, 010113 (2011).

[35] L. Bollen, P. van Kampen, and M. de Cock, “Students’ difficulties with vector calculus in electrodynamics,” **11**, (2015).

[36] C. Astolfi and C. Baily, “Student reasoning about the divergence of a vector field,” 2014 Phys. Educ. Res. Conf. Proc. **11**, 31 (2014).

[37] L. Bollen, P. van Kampen, and M. De Cock, “Students’ difficulties with vector calculus in electrodynamics,” Phys. Rev. Spec. Top. - Phys. Educ. Res. **11**, 020129 (2015).

[38] D. Tall and S. Vinner, “Concept image and concept definition in mathematics with particular refernce to limits and continuity,” Educ. Stud. Math. **12**, 151 (1981).

[39] L. Bollen, P. Van Kampen, C. Baily, and M. De Cock, “Qualitative investigation into students’ use of divergence and curl in electromagnetism,” Phys. Rev. Phys. Educ. Res. **12**, 020134 (2016).

[40] T. Dray and C. Manogue, “Miscellanea spherical coordinates,” Coll. Math. J. **34**, 168 (2003).

[41] B. Hinrichs, “Writing position vectors in 3-d space: A student difficulty with spherical unit vectors in intermediate E&M,” 2010 Phys. Educ. Res. Conf. Proc. **1289**, 173 (2010).

[42] D. Hammer, “Student resources for learning introductory physics,” Am. J. Phys. **68**, 52 (2000).

[43] M. Vega, W. M. Christensen, B. Farlow, G. Passante, and M. E. Loverude, "Student understanding of unit vectors and coordinate systems beyond cartesian coordinates in upper division physics courses," in *2016 Phys. Educ. Res. Conf. Proc.*, edited by D. Jones, L. Ding, and A. Traxler (AIP Conference Proceedings, 2016), pp. 364–367.

[44] T. Paoletti, K. C. Moore, J. Gammaro, and S. Musgrave, "Students' emerging understanding of the polar coordinate system," in *Proc. 16th Annu. Conf. Res. Undergrad. Math. Educ.* (2013).

[45] K. C. Moore, T. Paoletti, and S. Musgrave, "Covariational reasoning and invariance among coordinate systems," J. Math. Behav. **32**, 461 (2013).

[46] K. C. Moore, T. Paoletti, and S. Musgrave, "Complexities in students' construction of the polar coordinate system," J. Math. Behav. **36**, 135 (2014).

[47] B. Sherin, "How students understand physics equations," Cogn. Instr. **19**, 479 (2001).

[48] D. Meredith and K. Marrongelle, "How students use mathematical resources in an electrostatics context," Am. J. Phys. **76**, 570 (2008).

[49] M. Artigue, J. Menigaux, and L. Viennot, "Some aspects of students' conceptions and difficulties about differentials," Eur. J. Phys. **11**, 262 (1990).

[50] S. Jones, "Understanding the integral: Students' symbolic forms," J. Math. Behav. **32**, 122 (2013).

[51] V. Sealey and J. R. Thompson, "Students' interpretation of and justification of 'backward' definite integrals," Proc. 19th Annu. Conf. Res. Undergrad. Math. Educ. 410 (2016).

[52] S. Jones, "Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts," J. Math. Behav. **38**, 9 (2015).

[53] S. Jones and A. Dorko, "Students' generalizations of single-variable conceptions of the definite integral to multivariate conceptions," Proc. the18th Annu. Conf. Res. Undergrad. Math. Educ. **1**, (2015).

[54] V. Sealey, "A framework for characterizing student understanding of Riemann sums and definite integrals," J. Math. Behav. **33**, 230 (2014).

[55] M. Zandieh, "A Theoretical Framework for Analyzing Student Understanding of the Concept of Derivative," CBMS Math. Educ. 103 (2000).

[56] D. Roundy, C. Manogue, J. Wagner, E. Weber, and T. Dray, "An extended theoretical framework for the concept of derivative," in *Proc. 18th Annu. Conf. Res. Undergrad. Math. Educ.* (2015), pp. 919–924.

[57] P. Heron, "Empirical investigations of learning and teaching, part I: Examing and interpreting student thinking," in *Enrico Fermi Summer Sch. Phys. Educ. Res.*, edited by E. F. Redish and M. Vincentini (Italian Physical Society, Varenna, Italy, 2003), pp. 341–351.

[58] D. Griffiths, “Introduction to Electrodynamics,” *Introduction to Electrodynamics*, 4th ed. (Pearson Education, New York, 2013).

[59] R. P. Hunting, “Clinical interview methods in mathematics education research and practice,” J. Math1 **16**, 145 (1997).

[60] K. A. Ericsson and H. A. Simon, “How to Study Thinking in Everyday Life: Contrasting Think-Aloud Protocols With Descriptions and Explanations of Thinking.,” Mind, Cult. Act. **5**, 178 (1998).

[61] K. A. Ericsson and H. A. Simon, “Protocol analysis: Verbal reports as data, Rev. ed.,” *Protocol Analysis: Verbal Reports as Data, Rev. Ed.* (MIT Press, Cambridge, MA, 1993).

[62] M. W. Van Someren, Y. F. Barnard, and J. A. C. Sandberg, “The Think Aloud Method: A Practical Guide to Modeling Cognitive Processes,” *The Think Aloud Method: A Practical Guide to Modeling Cognitive Processes* (Academic Press, London, 1994).

[63] L. S. Vygotsky, “Tool and symbol in child development,” in *Mind Soc. Dev. High. Psychol. Process.*, edited by M. Cole, V. John-Steiner, and S. Scribner (Harvard University Press, Cambridge, MA, 1978).

[64] M. C. Wittmann and K. E. Black, “Mathematical actions as procedural resources: An example from the separation of variables,” Phys. Rev. Spec. Top. - Phys. Educ. Res. **11**, 020114 (2015).

[65] L. C. McDermott, P. S. Shaffer, and the P. E. Group, “Tutorials in Introductory Physics,” *Tutorials in Introductory Physics* (Prentice Hall, 2002).

[66] T. I. Smith, “Student understanding of the Boltzmann factor,” Phys. Rev. Spec. Top. - Phys. Educ. Res. **11**, 020123 (2015).

[67] J. Piaget, “Structuralism,” *Structuralism* (Basic Books, New York, 1970).

[68] J. Piaget, “The equilibration of cognitive structures,” *The Equilibration of Cognitive Structures* (The University of Chicago Press, Chicago, 1975).

[69] V. Sealey, “Definite integrals, riemann sums, and area under a curve: What is necessary and sufficient,” PME-NA Proc. **2**, 46 (2006).

[70] B. G. Glaser and A. L. Strauss, “The discovery of grounded theory,” Int. J. Qual. Methods **5**, (1967).

[71] A. Bryant and K. Charmaz, “The SAGE handbook of grounded theory,” *The SAGE Handbook of Grounded Theory* (SAGE Publication Ltd, London, 2007).

[72] A. A. diSessa, “Toward an Epistemology of Physics,” Cogn. Instr. **10**, 105 (1993).

[73] T. Dray and C. Manogue, “Bridging the gap between mathematics and physics,” APS Forum Educ. (2004).

[74] T. Dray and C. Manogue, “The vector calculus gap: Mathematics ≠ physics,” Primus **9**, 21 (1999).

[75] M. Boas, “Mathematical Methods in the Physical Sciences,” *Mathematical Methods in the Physical Sciences* (Wiley, New York, 2006).

[76] G. Fauconnier and M. Turner, “The Way We Think: Conceptual Blending and the Mind’s Hidden Complexities,” *The Way We Think: Conceptual Blending and the Mind’s Hidden Complexities* (Basic Books, New York, 2002).

[77] B. E. Hinrichs, “Writing position vectors in 3-d space: A student difficulty with spherical unit vectors in intermediate E&M,” in *2010 Phys. Educ. Res. Conf. Proc.*, edited by M. Sabella, C. Singh, and N. S. Rebello (AIP Conference Proceedings 1289, 2010), pp. 173–176.

[78] N. Becker and M. Towns, “Students’ understanding of mathematical expressions in physical chemistry contexts: An analysis using Sherin's symbolic forms,” Chem. Educ. Res. Pr. Chem. Educ. Res. Pr. **13**, 209 (2012).

[79] T. Dray and C. A. Manogue, “Using differentials to bridge the vector calculus gap,” Coll. Math. J. **34**, 283 (2003).

[80] A. F. Heckler, “The ubiquitous patterns of incorrect answers to questions: The role of automatic bottom-up processes,” in *Psychol. Learn. Motiv.*, edited by J. P. Mestre and B. H. Ross (Elsevier Inc., Waltham, MA, 2011), pp. 227–266.

[81] D. Hu and N. S. Rebello, “Using conceptual blending to describe how students use mathematical integrals in physics,” Phys. Rev. Spec. Top. - Phys. Educ. Res. **9**, 020118 (2013).

[82] C. Baily, L. Bollen, A. Pattie, P. Van Kampen, and M. De Cock, “Student thinking about the divergence and curl in mathematics and physics contexts,” PERC Proc. (2015).

[83] M. Montiel, M. R. Wilhelmi, D. Vidakovic, and I. Elstak, “Using the onto-semiotic approach to identify and analyze mathematical meaning when transiting between different coordinate systems in a multivariate context,” Educ. Stud. Math. **72**, (2009).

[84] B. Parzysz, “Representation of Space and Students ’ Conceptions at High School Level,” Educ. Stud. Math. **22**, 575 (1991).

[85] B. Parzysz, “‘ Knowing ’ vs ‘ Seeing ’. Problems of the Plane Representation of Space Geometry Figures,” Educ. Stud. Math. **19**, 79 (1988).

[86] J. Tuminaro and E. F. Redish, “Elements of a cognitive model of physics problem solving: Epistemic games,” Phys. Rev. Spec. Top. - Phys. Educ. Res. **3**, 1 (2007).

[87] T. J. Bing and E. F. Redish, “Analyzing problem solving using math in physics: Epistemological framing via warrants,” Phys. Rev. Spec. Top. - Phys. Educ. Res. **5**, 020108 (2009).

[88] P. White and M. Mitchelmore, “Conceptual knowledge in introductory calculus,” J. Res. Math. Educ. **27**, 79 (1996).

[89] C. Singh, “Student understanding of symmetry and Gauss’s law,” in *2004 Phys. Educ. Res. Conf. Proc.*, edited by P. Heron, S. Franklin, and J. Marx (AIP Conference Proceedings 790, 2005), pp. 65–68.

[90] R. E. Pepper, S. V. Chasteen, S. J. Pollock, and K. K. Perkins, “Our best juniors still struggle with Gauss’s law: Characterizing their difficulties,” in *2010 Phys. Educ. Res. Conf. Proc.*, edited by M. Sabella, C. Singh, and N. S. Rebello (AIP Conference Proceedings 1289, 2010), pp. 245–248.

[91] B. P. Schermerhorn and J. R. Thompson, “Students’ use of symbolic forms when constructing differential length elements,” in *2016 Phys. Educ. Res. Conf. Proc.*, edited by D. Jones, L. Ding, and A. Traxler (AIP Conference Proceedings, 2016), pp. 312–315.

[92] B. P. Schermerhorn and J. R. Thompson, “Physics students construction of differential elements in an unconventional spherical coordinate system,” Proc. 19th Annu. Conf. Res. Undergrad. Math. Educ. (2016).

[93] E. Kuo, M. M. Hull, A. Gupta, and A. Elby, “How students blend conceptual and formal mathematical reasoning in solving physics problems,” Sci. Educ. **97**, 32 (2013).

[94] T. J. Bing and E. F. Redish, “The Cognitive Blending of Mathematics and Physics Knowledge,” in *2007 Phys. Educ. Res. Conf. Proc.*, edited by L. McCullough, L. Hsu, and C. Henderson (AIP Conference Proceedings 951, 2007), pp. 26–29.

[95] S. Brahmia, “Mathematization in Introductory Physics,” Unpubl. Diss. (Physics), Rutgers Univ. (2014).

[96] B. W. Dreyfus, A. Gupta, and E. F. Redish, “Applying conceptual blending to model coordinated use of multiple ontological metaphors,” Int. J. Sci. Educ. **37**, (2015).

[97] B. P. Schermerhorn and J. R. Thompson, “Students ’ determination of differential area elements in upper -division physics,” in *2017 Phys. Educ. Res. Conf. Proc.*, edited by L. Ding, A. Traxler, and Y. Cao (AIP Conference Proceedings, 2017).

[98] B. P. Schermerhorn and J. R. Thompson, “Physics students’ construction of differential length vectors for a spiral path,” Proc. 21st Annu. Conf. Res. Undergrad. Math. Educ. (accepted) (2018).

[99] M. Wawro, K. Watson, and W. M. Christensen, “Meta-representational competence with linear algebra in quantum mechanics,” in *Proc. 20th Annu. Conf. Res. Undergrad. Math. Educ.*, edited by A. Weinberg, J. Rasmussen, J. Rabin, M. Wawro, and S. Brown (MAA, San Diego, CA, 2017), pp. 326–337.

[100] B. P. Schermerhorn and J. R. Thompson, “Physics students’ use of symbolic forms when construction differential elements in multivariable coordinate systems,” Proc. 20th Annu. Conf. Res. Undergrad. Math. Educ. (2017).

APPENDIX A – MATHEMATICAL BACKGROUND

DIFFERENTIAL ELEMENTS IN NON-CARTESIAN COORDINATE SYSTEMS

The use of coordinate symmetry in physics largely eases the calculational burden. Just as Dirac notation is an elegant expression of vectors and matrices in quantum mechanics, the expressions of these natural physical symmetries (e.g., a point charge with a radial electric field or a long straight wire with a curling magnetic field) in terms of coordinate systems that leverage said symmetry is a matter of elegance. The caveat now comes in understanding that transitions from the more familiar rectangular coordinates to systems involving curved surfaces means one must interpret and keep track of how these new lines and areas are described.

The purpose of the following sections is to give the reader enough background information to understand the differences between particular coordinate systems and how one goes about constructing differential elements for the purposes of vector calculus in E&M. This appendix may also serve as a reference for later chapters discussing student work in this area. Section A.1 first explains the nature Cartesian coordinates and develops background for how one may approach thinking about differential line, area, and volume elements. Sections A.2 and A.3 then go into detail about spherical and cylindrical coordinates and what use that particular coordinate system is to E&M. Since differential elements in spherical coordinates represent a greater deviation from Cartesian coordinates, more time is spent here to illuminate the differences between these two systems. As cylindrical coordinates draw on ideas from both systems, this will be developed more quickly.

A.1 Cartesian coordinates and Cartesian differential elements

Cartesian, or rectangular, coordinates are the most commonly used coordinate systems for problem solving. Used almost exclusively mathematics taught vector calculus courses [74,73], Cartesian coordinates are also used as the predominant coordinate system in the first few years of physics courses up to post-introductory mechanics and electricity and magnetism. The coordinate system is defined using three perpendicular axes denoted, *x*, *y*, and *z*, and therefore allow one to describe a coordinate point in three dimensional space using up to three straight perpendicular lines, each corresponding to an change along only one axis. This representation of vectors is how students commonly work with vectors in introductory physics courses.

Representing a vector drawn to any point in three-dimensional space can be done by decomposing it into three vectors along the three coordinate directions (Fig. 2.1a). The particular length of a component is specified by the magnitude of the vector while the direction is given by a unit vector that points in the direction of a positive increase along a specific axis. Unit vectors are designated as $\hat{x}, \hat{y},$ and $\hat{z}$ or $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ for the x-, y-, and z-axes, respectively (Fig 2.1b). Unit vectors in Cartesian coordinates are static, meaning that they always point in the directions defined by the Cartesian axes for any vector three-dimensional space. A generic vector, $\vec{C}$, in Cartesian coordinates can then be given as

$$\vec{C} = x\hat{x} + y\hat{y} + z\hat{z}.$$

This becomes the given form for any vector in this coordinate system, regardless of whether it is defined from the origin or another point in space.

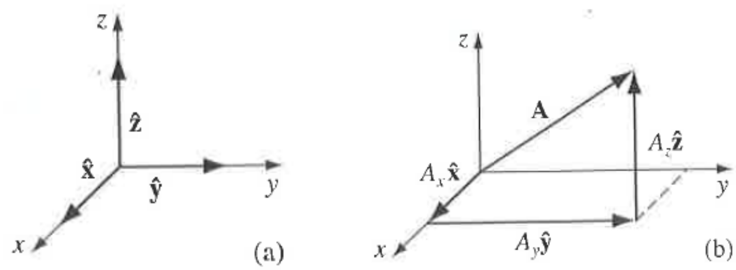


Figure A.1. Cartesian vector notation. (a) Unit vectors for each of the Cartesian axes. Also commonly expressed as $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$ for the x-, y-, and z-axes, respectively. (b) A generic vector $\vec{A}$, or $\boldsymbol{A}$, represented in Cartesian coordinates. Images reproduced from E&M course text [58].

E&M then deals with vector fields produced by distributions of charges or currents. A vector field is a set of position-dependent vector quantities. (Fig. A.2). E&M courses typically deal with electric and magnetic fields that establish symmetric patterns that students can interpret. Calculation involving these fields, however, must also account for the direction of the fields at points of interest. This involves employing vector calculus to account for the specific effects of fields along lines and through surfaces.

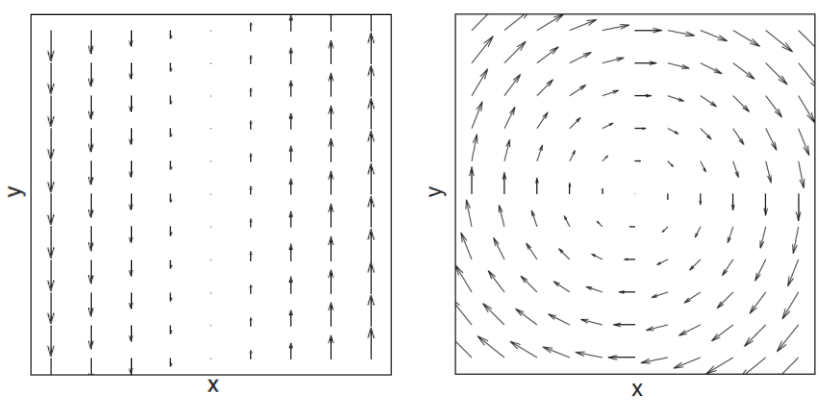


Figure A.2. Two examples of vector fields, showing position dependent vectors. Assuming an origin in the center of the image, the field on the left is expressed by $\vec{F}(x,y) = x\hat{y}$ and the field on the right is expressed by $\vec{F}(x,y) = y\hat{x} - x\hat{y}$. Images reproduced from work by Bollen and colleagues [37].

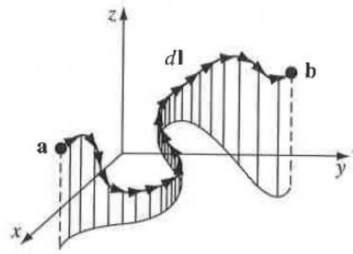


Figure A.3. Multiple differential lengths along a curve. The differential lengths here represent infinitely small vectors used to accumulate the effects of a field along a line segment. Image reproduced from E&M course text [58].

A differential length vector, $d\vec{l}$, is an infinitesimal segment of length along a curve represented by a vector tangent to this curve (Fig. A.3). A $d\vec{l}$ is typically used in vector calculus to sum up the effects of a particular vector field over a given curve or path. Working in Cartesian coordinates, this is easily represented by

$$d\vec{l} = dx\hat{x} + dy\hat{y} + dz\hat{z},$$

where dx, dy, and dz represent infinitesimal lengths in each Cartesian direction.

Similarly, differential area vectors can be created to represent infinitesimal portions of planes. These are typically used in vector calculus to calculate the amount of flux, or field passing through a given area. The differential unit vector for any given planar area is perpendicular to that area. Thus, an area represented in the $\hat{x}$-direction is given by y- and z-length components. Mathematically this corresponds to a cross product of the two differential length vectors in the $\hat{y}$- and $\hat{z}$- directions, where the magnitude is the area of the resulting parallelogram (here a rectangle), and the direction is perpendicular to the plane spanned by the original vectors (Fig 2.4a).

$$d\vec{a}_x = d\vec{l}_y \times d\vec{l}_z = dydz(\hat{y} \times \hat{z}) = dydz\hat{x}$$

This follows for each of the Cartesian directions, giving a completed differential area vector as follows

$$d\vec{a} = dydz\hat{x} + dxdz\hat{y} + dxdy\hat{z}.$$

Just as with the curve, components are selected based upon what is needed to represent the given area. In many cases, textbooks develop the differential area as a scalar quantity and use a unit vector $\hat{n}$ to describe the surface, which is developed later in the context of the problem [75].

Differential areas have a particular importance when working with flux. The vector field will have more effect when acting perpendicular to a surface area than when acting parallel with it; this will specifically appear as a dot product with the differential area vector within integration. The differential area describing a surface is co-opted as a vector quantity in order to account for the amount of field perpendicular to a surface (parallel with the unit vector that describes a surface).

Volume elements, typically used in vector calculus integration to describe sources of vector fields, are then given using each of the Cartesian differential lengths and the equation for the volume of a parallelepiped. By taking an area given by a cross product of

(a) (b)

Figure A.4. Cartesian area and volume elements. (a) A differential area element made from differential lengths in the y- and z-directions. The unit vector of the area, $\hat{x}$, is perpendicular to the given area. (b) A differential volume where the sides are given by differential length components along each axis. The sides of the differential volume element represent the different possible differential areas. The differential area vector is created via a cross product of the two lengths.

two vectors and performing a dot product with a third vector, one can find the volume of a parallelepiped with sides defined by the three vectors [Fig 2.4b].

$$d\tau = d\vec{l}_x \cdot \left(d\vec{l}_y \times d\vec{l}_z\right) = dxdydz(\hat{x} \cdot \hat{x})\text{=dxdydz}$$

Given that unit vectors for any coordinate system are defined to be perpendicular to each other, the differential volume is commonly used and taught as a multiplication of each of the three differential lengths, bypassing the vector nature of the construction. The resulting volume is the same for any combination of (right-handed) cyclic combination of components. Additionally the differential volume element is a scalar quantity and does not have three independent parts in the way that the volume and area elements do.

A2. Spherical coordinates and spherical differential elements

Spherical coordinates are often invoked in the analysis of physical systems with spherically symmetric fields. Typical systems include a single point charge, sphere of charge, or shell of charge where the amount of charge at any distance r is the same as any other point given at the same distance (*e.g.*, (0,1,1) and (1,0,-1) have the same value of r but different Cartesian coordinates). In these cases any non-zero resulting electric fields at any given point are directed along a line between the center of the charge source and the given point.

To this effect, spherical coordinates utilize a vector, r, measured from the origin to the point of interest (Fig. A.5). The coordinate system is then mapped by the length of

(a) (b)

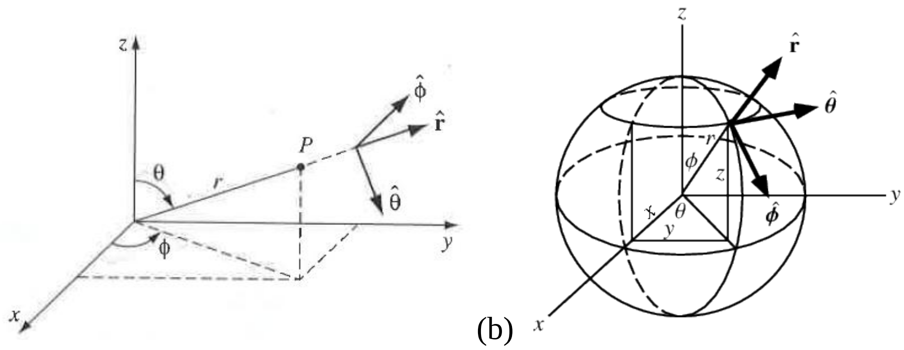


Figure A.5. Notation for spherical coordinates. (a) Standard physics conventions for spherical coordinates. Image reproduced from E&M course text [58]. (b) Standard mathematics conventions for spherical coordinates. Image reproduced from http://mathworld.wolfram.com/SphericalCoordinates.html

the vector, r, and two angles. In physics, θ is the polar angle, meaning it is measured between the radial vector and the z-axis. In terms of an Earth-like coordinate system, this measures the particular co-latitude of a point starting with zero at the northern pole (positive z-axis), measuring $\pi/2$ at the equator and ending with π at the southern pole (negative z-axis). The second angle, ϕ, is called the azimuthal angle. It measures the rotational distance of the radial vector in the xy plane. This can range from 0 to 2π. In mathematics, the assignment of these variables is reversed, with ϕ being the polar angle and θ the azimuthal. The distinction in convention between the two disciplines has previously been proposed as a potential area of confusion for students [40]. For the purposes of this work, I will continue to use the physics definitions for particular coordinate systems. Despite the disciplinary discrepancy, in either representation, spherical coordinates allow us to adequately describe any point in space with a single ordered triplet of variables in this domain.

Establishing the conventions of the coordinate system, one can write $\vec{r}$ terms of a Cartesian coordinates system.

$$\vec{r} = r \sin \theta \cos \phi \, \hat{x} + r \sin \theta \sin \phi \, \hat{y} + r \cos \theta \, \hat{z}$$

Associated with this is a radial unit vector $\hat{r}$, which points directly away from the origin in the direction of increasing coordinate (Fig. 2.5). Thus within spherical coordinates,

$$\vec{r} = r\hat{r}$$

maps to any point in space by defining a set of concentric spherical shells. To define any single point in particular, one must explicitly account for the measurements of the two angles used to define $\vec{r}$: θ and ϕ. Similarly $\hat{r}$, which defines the direction of increasing radius, is dependent upon location of the vector. Therefore, this unit vector is not static in the way Cartesian unit vectors were defined.

Just as with the unit vector in the radial direction, two additional unit vectors, $\hat{\theta}$ and $\hat{\phi}$, define the directions of increasing θ and ϕ, respectively. Given our condition of orthogonality of unit vectors, these vectors are tangent to a spherical shell but will also change direction whenever $\vec{r}$ is placed at different values for the angles. This dependence is made apparent when examining the relation between the spherical unit vectors and unit vectors along the original Cartesian axes we use to describe this system.

$$\hat{r} = \sin \theta \cos \phi \, \hat{x} + \sin \theta \sin \phi \, \hat{y} + \cos \theta \, \hat{z}$$

$$\hat{\theta} = \cos \theta \cos \phi \, \hat{x} + \cos \theta \sin \phi \, \hat{y} - \sin \theta \, \hat{z}$$

$$\hat{\phi} = -\sin \phi \, \hat{x} + \cos \phi \, \hat{y}$$

While construction of a differential length vector in Cartesian coordinates involves tracing out lengths in completely independent directions, a cursory observation reveals that lines traced out by changing either variable angle in spherical coordinates creates circular arcs. A change of θ maps out a circumference of the sphere (also known as a great circle) – a circle of a particular longitude, to return to our geographical analogy. The length of this arc is given by the formula,

$$l = r\theta$$

where l is arc length. Changes of the azimuthal angle ϕ yield small circles, traced out on latitudinal rings. Further observation of the coordinate representation yields that circles traced out by changes of ϕ are smaller closer to the z-axis. This is because the radius measured to the z-axis is amended to $r \sin\theta$, rather than the full radius r used before hand. This gives the following expression for arc length for any value of ϕ:

$$l = (r \sin\theta)\, \phi$$

These expressions for arc length for one fixed angle become relevant when we consider the effects of differential changes in angles [Fig A.6]. While in Cartesian coordinates, one was able to consider a small change in a variable and equate it to differential length, spherical coordinates does not trace out rectangular-like coordinates. However, the differential length does remain a straight line due only to the infinitesimal nature of the change. Engaging in a limiting process, one can determine expressions for differential changes in variables as defining differential lengths.

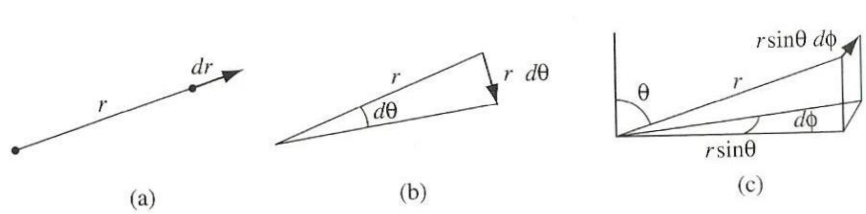


Figure A.6. Construction of differential length components in spherical coordinates. A differential change in each variable produces a differential length component traced by the vector, $\vec{r}$. Image reproduced from E&M course text [58].

Accounting for a small change in the radial direction yields a simple dr. For arc lengths, differential shifts in the angle yield differential lengths in those directions. Thus, one can construct the following differential length vector:

$$d\vec{l} = dr\hat{r} + rd\theta\hat{\theta} + r\sin\theta\, d\phi\hat{\phi}$$

The differences resulting from a comparison to Cartesian coordinates are again a result of the need to consider infinitesimal arc lengths. Construction of further differential elements, however, retains the same procedural aspect and only requires attention to the inclusion of the spherical scaling factors.

The cross product of the two differential lengths in the $\hat{\theta}$- and $\hat{\phi}$-directions results in an infinitesimal portion of the surface area of a sphere. This differential area vector points in the $\hat{r}$-direction and has a magnitude $da = (rd\theta)(r\sin\theta\, d\phi)$ [Fig A.7a]. This area is most commonly used in E&M when considering spherical charge distributions, which produce radial electric fields. Doing this requires recognizing that a centered spherical shell will mean that the radial field is perpendicular to the surface at all points, then recognizing which differential lengths describe that surface.

However, just as in Cartesian coordinates, we can continue the combination of differential length elements to describe differential areas in the two other directions, resulting in the following generalized expression for a differential area vector:

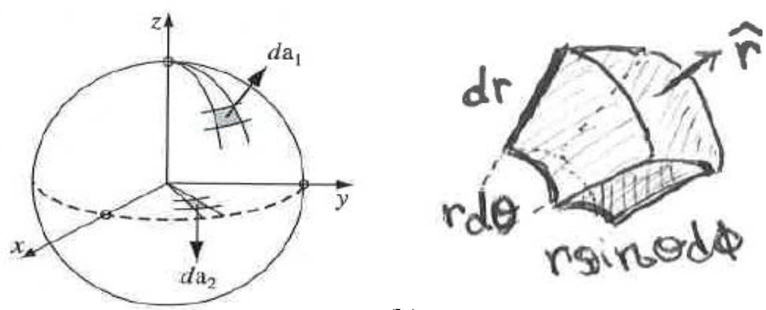
$$d\vec{a} = r^2\sin\theta\, d\theta d\phi\hat{r} + r\sin\theta\, drd\phi\hat{\theta} + rdrd\theta\hat{\phi}$$

While differential areas in the $\hat{\theta}$- and $\hat{\phi}$-directions are not commonly used when problem solving in physics, the recognition of how to derive them is pertinent to da construction. This derivation is more relevant for differential areas in cylindrical coordinates, where each of the three possible das is used in various situations.

A spherical differential volume is then found by taking the volume of a parallelepiped, as shown in Cartesian coordinates. A physical representation is illustrated in Figure A.7b. A simple multiplication of the three length components yields the same differential volume element:

$$d\tau = r^2 \sin\theta \, dr d\theta d\phi$$

Notably, the representations of the differential area and volume elements typically depict the scaling factors written to the left of the expression followed by the differential variables in coordinate order. While this represents a simplified mathematical form, it hides the origins of the particular length terms.



(a) (b)

Figure A.7. Spherical differential area and volume elements. (a) Examples of differential areas in spherical coordinates. da_1 depicts the differential areas for the surface of a sphere and is constructed as a product of two differential length components representing changes in each of the angles. Image reproduced from E&M course text [58]. (b) A differential volume in spherical coordinates constructed as a product of each differential length component.[3] The sides of the differential volume element represent the different possible differential areas. The differential area vector is created via a cross product of the two lengths.

[3] Note that here the curving nature of the sides is exaggerated to depict the need to consider a differential arc length. The $\hat{r}$ is shown to establish the outward direction. While it represents the unit vector for the differential area of a spherical shell, the differential volume element in a scalar quantity.

A3. Cylindrical coordinates and cylindrical differential elements

Cylindrical coordinates are another of the common coordinate systems used to describe physical systems in E&M, used to analyze line charges and a wealth of current-carrying wires in magnetostatics. These systems contain two-dimensional radial electric fields and curling magnetic fields, respectively. Cylindrical coordinates become useful in these cases as they leverage two dimensional polar coordinates and extends three dimensionally using a Cartesian axis, typically considered, but not limited to, the z-axis (Fig. A.8). Just as with spherical coordinates, typical mathematics convention differs from that of physics. While mathematics conventions make use of variable notation for two-dimensional polar coordinates (where disciplines commonly agree on r and θ), for the purposes of this work, the author will use Griffiths's notation [58], where s gives the radius into the xy-plane and φ measures the polar angle. Using this coordinate system, one can represent any point in space in terms of Cartesian unit vectors as

$$\vec{r} = s\cos\varphi\,\hat{x} + s\sin\varphi\,\hat{y} + z\hat{z}.$$

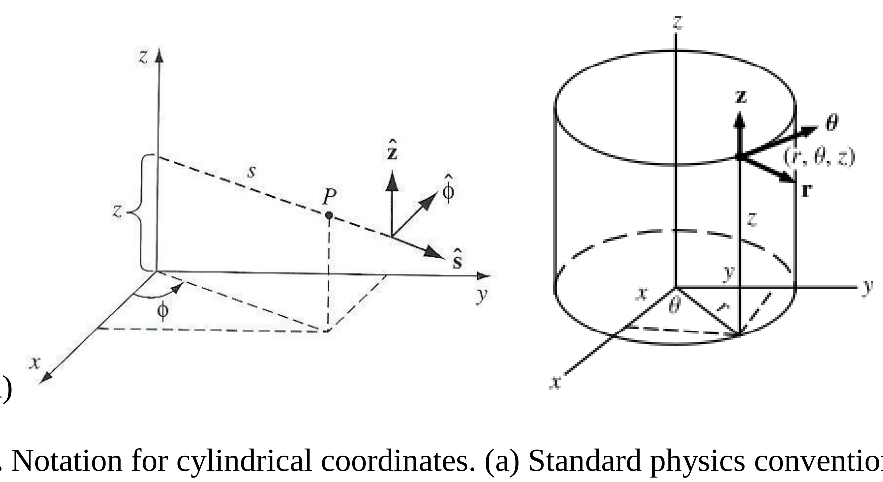
(a) 

Figure A.8. Notation for cylindrical coordinates. (a) Standard physics conventions for cylindrical coordinates. Image reproduced from E&M course text [58]. (b) Standard mathematics conventions for cylindrical coordinates. Image reproduced from http://mathworld.wolfram.com/CylindricalCoordinates.html

Within this coordinate system, the same vector is expressed as

$$\vec{r} = s\hat{s} + z\hat{z},$$

accounting for a position along the z-axis coupled with a circle traced out at that radius. Just as spherical coordinates allowed the definition of concentric spherical shells, defining cylindrical coordinates allows one to think about either circles or cylindrical shells centered on an axis.

Further analysis reveals that while $\hat{z}$ is now a static unit vector, always pointing in the direction parallel to the z-axis, $\hat{s}$ and $\hat{\phi}$ are both dynamic in that they are dependent on the measurement of ϕ. The specific relationship is drawn out when decomposing the unit vectors into the Cartesian axes:

$$\hat{s} = \cos\varphi\,\hat{x} + \sin\phi\,\hat{y}$$

$$\hat{\phi} = -\sin\varphi\,\hat{x} + \cos\phi\,\hat{y}$$

The complete vector form of the differential length element can be arrived at by again considering lengths traced out by differential changes in each of the three variables. This is now a simpler process than in spherical coordinates in that it only needs to account for one arc length when a change is made in the $\hat{\phi}$-direction:

$$d\vec{l} = ds\hat{s} + sd\phi\hat{\phi} + dz\hat{z}.$$

The differential areas are constructed as before and can again be compiled into a larger vector:

$$d\vec{a} = sd\phi dz\hat{s} + dsdz\hat{\phi} + sdsd\phi\hat{z}.$$

What differs here from spherical coordinates is that each of these differential area components is eventually used individually in E&M [see Chapter 7 for description of tasks using various differential areas]. Whereas in spherical coordinates it may be easier to recall the $\hat{r}$-component of the differential area for problem solving, a problem making use of cylindrical coordinates requires students to understand which component is relevant given the physical systems (i.e., what differential lengths account for the surface they need to describe).

Lastly, combining all of the differential length elements, the differential volume element takes the form (Fig. A.9):

$$d\tau = sdsd\phi dz$$

Just as with spherical coordinates, the typical expression of the volume element separates the scaling factors, obscuring the original expression of the differential lengths.

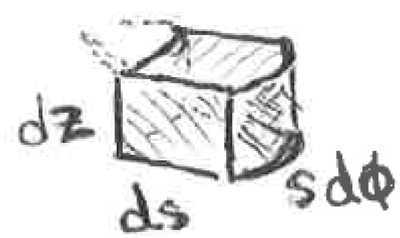


Figure A.9. Cylindrical differential volume element. The sides of the differential volume element represent the different possible differential areas. The differential area vector is created via a cross product of the two lengths.

APPENDIX B - INTERVIEW TASKS

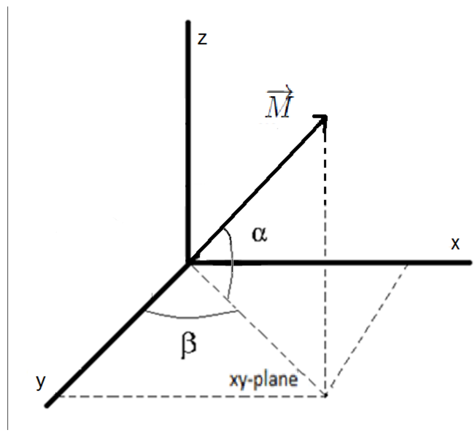
B1: Schmerical task

Consider the following coordinate system measured using the following variables:

$$M: 0 \rightarrow \infty$$

$$\alpha: -\pi/2 \rightarrow \pi/2$$

$$\beta: 0 \rightarrow 2\pi$$



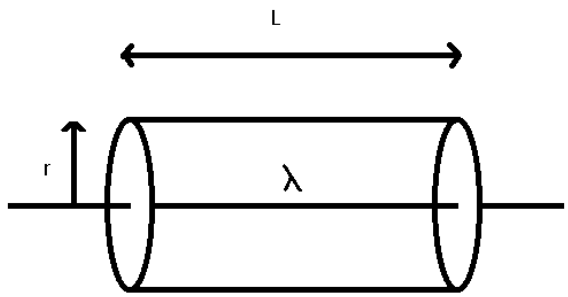
i) Does this depict a feasible coordinate system and if it is valid what type of situations (kinds of problems) would it be appropriate for?
ii) Construct a generic differential length element for this system.
iii) Construct a differential volume element for this coordinate system.
iv) Check that the volume element is correct.

B2: Check solution task

Consider an infinite line of charge with a constant linear charge density, λ . Student B is working a homework problem to find the change in potential from radius *e* to a radius *f*>*e*. Find any errors that exist in Student B's reasoning.

Student B's Solution:

To solve for the electric field, imagine a Gaussian surface a radius r from the surface.



$$\oint \vec{E} \cdot \overrightarrow{da} = \int_0^{2\pi} \int_0^{\pi} (E\hat{r}) \cdot (rLsin\theta d\theta d\varphi \hat{r}) = E(4\pi rL)$$

$$\oint \vec{E} \cdot \overrightarrow{da} = \frac{Q}{\epsilon_o} = \frac{1}{\epsilon_o} \lambda L$$

Thus $\vec{E} = \frac{\lambda}{4\pi\epsilon_o} \frac{1}{r} \hat{r}$

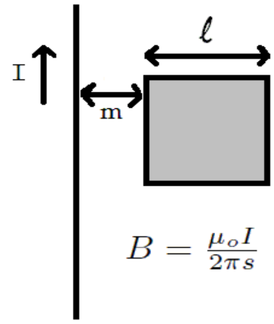
And V(f)-V(e) $= -\int_e^f \vec{E} \cdot \overrightarrow{dl}$

$$= -\int_e^f \left(\frac{\lambda}{4\pi\epsilon_o} \frac{1}{r} \hat{r} \right) \cdot (dr\hat{r}) = \frac{-\lambda}{4\pi\epsilon_o} \int_e^f \frac{dr}{r} \hat{r}$$

$$\mathrm{V(f)} - \mathrm{V(e)} = \frac{-\lambda}{4\pi\epsilon_o} \ln \left(\frac{f}{e} \right) \hat{r}$$

B3: Flux task

Consider a wire lying along the z-axis with constant current, I, in the direction indicated in the figure.
The magnitude of the magnetic field is $|\vec{B}| = \frac{\mu_o I}{2\pi s}$, where μ_o is a constant and s is the distance from the wire.

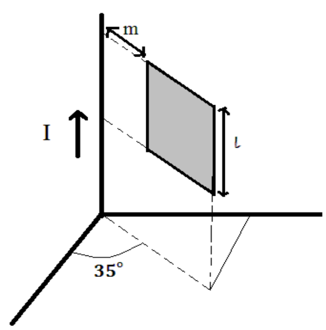


a) b)

What is the magnetic flux through a square loop (side length *l*), if the first side is a distance *m* from the wire?

[If student's use Cartesian coordinates]

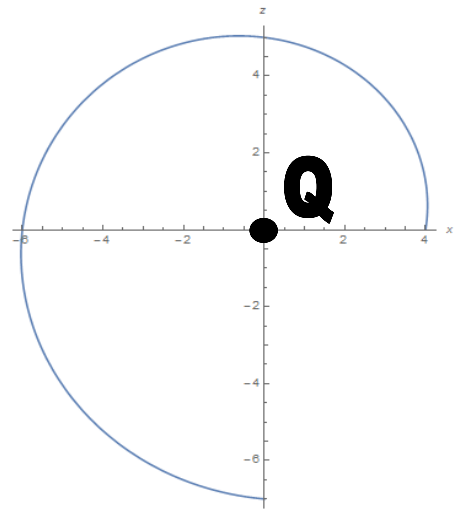
How would your answer change if the loop was rotated out of the plane by some angle?



B4: Spiral task

Consider a charge, Q, located at the origin.

A test charge is moved along the following path given by $r = 2\theta/\pi$ as shown in the following diagram from (4,0,0) to (0,0,-7).

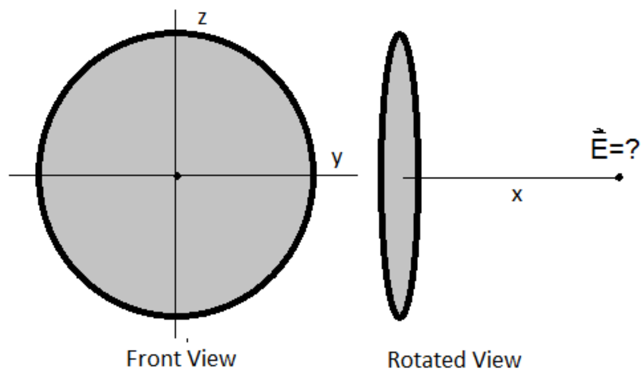


i) What is the differential length, $\overrightarrow{dl}$, for the path along which the charge is moved?

ii) What is the change in electric potential experienced by the test charge?

B5: Charged sheet task

You have a circular sheet in the *yz*-plane with a constant surface charge density, σ, and radius R.



Set up an expression to solve for the electric field a distance, x, far from the center of the sheet.

APPENDIX C – UPDATED LENGTH TUTORIAL SEQUENCE

For the first portion of the tutorial sequence focusing on differential length construction, there is an attached pre-tutorial homework (Appendix C1). This assignment presents schmerical coordinates and asks students to reason about the feasibility of the system as was done in the first part of the interview task (section 5.1). This is designed to prepare students for working within the unfamiliar schmerical coordinate system. A second task was added to the pre-tutorial homework asking students to derive an expression for the distance traveled by two cars around a circular track at different radii. The purpose of this task is to refamiliarize students with the ideas of arc length before they applied it in such an unfamiliar context.

The length tutorial (Appendix C2) was also greatly augmented to provide a more structured differential length construction in the second iteration. The largest difference was the inclusion of a physical manipulative, motivated by research showing student difficulties reasoning about 3D objects from 2D images [84,85] and in part by observing students in interviews and in our previous tutorial implementation be challenged by considering motion in 3D space from 2D images. Each group is now provided with a rubber ball ($r{\sim}11$ cm) that could be drawn on with erasable markers. Students are instructed to draw latitude and longitude lines, which are explicitly connected to measurements of alpha and beta in the schmerical coordinate system. This change allows students to actually consider and interact with motions at a fixed radius along the surface of the ball. Additionally, a small task is added to have students compare unit vectors at two different locations, as students have been shown to struggle with defining unit vectors in two dimensional coordinates [43].

The length task then asks students to describe changes in each variable direction and construct length components as before. In line with work on conceptual blending [76] (Chapter 8), this tutorial sequence attends to the specific structural components of the differential length vector and attempts to have students build the associated contextual information related to arc length and projection. For the angular components, this updated tutorial includes a discussion that compared lines of longitude for the $\hat{\alpha}$-component and latitude for the $\hat{\beta}$-component. The result of such a task shows how, for fixed changes in alpha, longitude lines remain the same at different locations, but that lines of latitude (changes in the β-direction) are dependent on the value of alpha at which the change is measured. This leads students toward the inclusion of the trigonometric function as a scale factor for the beta-hat component. For each angular component, students are asked to express a large change on the physical surface of the ball, then find an expression for a differential change in the same direction.

After constructing the three components, students are asked to express the total differential length vector and compare this to that for spherical coordinates. At this point, a student who has correctly expressed the schmerical element would say the trigonometric function had changed, but to a student who has used spherical coordinates as a means to construct components they are the same. The purpose of this step is to allow students to engage in a sense-making task by employing a coordinate system with which they were more familiar.

The post-tutorial homework (Appendix C3) asks students to construct a differential area element for the surface of a sphere in schmerical coordinates, then asks how this related to the terms in the total differential length vector. The purpose of this is to help students’ recognize that the area elements can be constructed from length elements. A second task was added to the post-tutorial homework in which students are explicitly asked to use ideas from the tutorial to construct the length elements for spherical, cylindrical, and Cartesian coordinates as a way to cement ideas within the more familiar coordinate systems, but also to prepare students for the area tutorial designed to be implemented the following class.

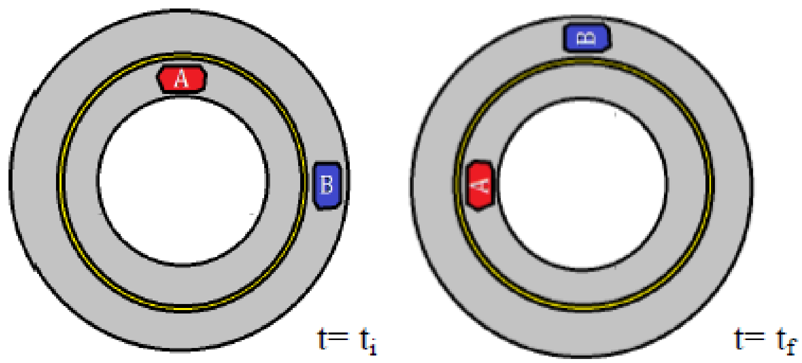
C1: Pre-tutorial HW

DIFFERENTIAL VECTOR ELEMENS IN NON-CARTESIAN COORDINATES: LENGTHS

Pre-tutorial Homework

I. Motion in a circular path

Two cars are racing on a circular race track a time t_i. At time t_f, it appears as if each car has traveled $\pi/2$ radians.



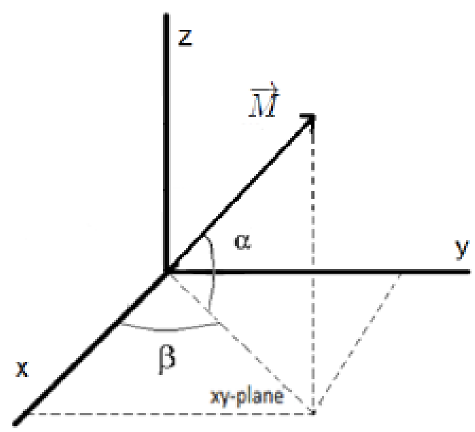
Is the distance traveled by car B in this time *greater than, equal to*, or *less than* the distance traveled by car A? Generalize an expression for the distance traveled by each car using the amount of angle covered. What would be the result for motion through an angle θ?

II. Coordinate system validity

Consider the following coordinate system, "schmerical coordinates," consisting of the following variables with the domains given:

$$M \in [0, \infty)$$
$$\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$
$$\beta \in [0, 2\pi)$$

Is this a valid coordinate system? Explain why or why not.



C2: Differential length vector tutorial

DIFFERENTIAL VECTOR ELEMENTS IN NON-CARTESIAN COORDINATES: LENGTHS

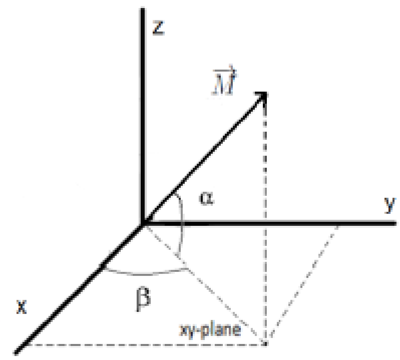
I. Determining the validity of an unconventional coordinate system

In Electricity and Magnetism courses, you will often exploit the symmetry of a problem and employ a particular coordinate system to simplify the mathematics of your calculations.

A. Consider the following coordinate system, "schmerical coordinates," consisting of the following variables with the domains given:

$$M \in [0, \infty)$$
$$\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$
$$\beta \in [0, 2\pi)$$

Discuss the validity of the coordinate system as a group.



B. Considering the various problems and coordinate systems you've encountered in class, what type of problem symmetry would this be useful for? (What coordinate system is this like?)

II. Modeling Schmerical coordinates in three dimensions

Obtain a rubber ball and white board marker from your instructor.

Assume the center of the ball is the origin of schmerical coordinates, so that the surface of the ball represents a constant radius M. Start by drawing an equator ($\alpha = 0$) around the surface of the ball. Choosing one hemisphere, draw several circles parallel to your equator. These horizontal lines will map out latitudes for values of **fixed alpha**. Next, mark out longitudinal lines to represent values of **fixed β**. Draw these so they are equally spaced at the equator and cross lines of latitude at perpendicular angles. Cover at least half of the hemisphere.

Pick *two* separate locations where a latitude and longitude intersect. Sketch/determine unit vectors corresponding to the variables of this coordinate system. How do the directions of the unit vectors compare at these locations?

- ➢ Check your answers with your instructor.

III. Constructing differential length elements in schmerical coordinates

In order to construct a differential length, we isolate each variable direction, introduce a small change in that variable, and account for the length which $\vec{M}$ has traced out.

A. Considering one of the points on the ball, describe how applying a change in the $\hat{M}$ direction affects your position vector, $\vec{M}$.

$\vec{M}$ before and after change in the $\hat{M}$ direction

In the space provided, sketch $\vec{M}$, before and after the prescribed change. Make sure you use the same origin.

If we consider a differential change in this direction, what is the produced differential length component in terms of the given coordinate system?

B. Consider a change in the $\hat{\alpha}$ direction.

1. Mark a segment along the surface of your ball that corresponds to a change in α. Have it cross several lines of latitude (i.e. lines measuring constant α).

   For a different value of β, sketch a segment on the ball corresponding to the same change in the α (same number of crossed latitudes). How does the length of this segment compare to the original segment?

   $\vec{M}$ before and after change in the $\hat{\alpha}$ direction

   To the right, sketch the vector $\vec{M}$ before and after a change in α from one origin. Include the path that is traced out.

2. Write an expression for this path in terms of $\Delta\alpha$ and other relevant variables.

How would your answer change when considering a differential change in α? Write an expression in terms of relevant variables. What is the associated unit vector?

Imagine inflating the ball to a larger radius. What happens to the lines measuring α, and β. How does this inflation affect the lengths of the segment you traced out and the differential length component? Is this consistent with both your expressions above?

C. Consider a change in the $\hat{\beta}$ direction.

1. Mark a segment along the surface of your ball that corresponds to a change in β. Have it cross through several longitude measurements.

   For a different value of α, sketch a segment on the ball corresponding to the same change in the variable β (same number of crossed longitudes). How does the length of this segment compare to the original segment?

   To the right, sketch the vector $\vec{M}$ before and after a change in β from one origin. Include the path that is traced out.

$\vec{M}$ before and after change in the $\hat{\beta}$ direction

2. Where on the ball would your length segment due to $\Delta\beta$ be a maximum? A minimum? What are the corresponding values of α in each case? What are the values/expressions for the length component in each case?

3. Derive an expression for the value of this length in terms of $\Delta\beta$ and relevant variables.

   How would your answer change when considering a differential change in β? Write an expression in terms of relevant variables. What is the associated unit vector? Considering this as an arc length, what represents the "radius" portion? How does this compare to the "radius" for the α component? Does this surprise you?

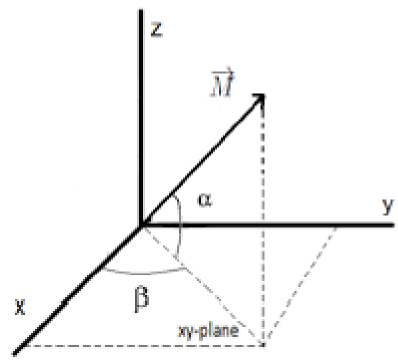
Imagine inflating the ball to a larger radius as earlier. How does this inflation affect the lengths of the segment you traced out and the differential length component? Is this consistent with both your expressions above?

D. Express the total differential length vector for this coordinate system.

- Check your answers with your instructor.

III. Constructing and checking a differential volume element

A. Determine the differential volume element for this system. Explain how you arrived at your answer.



B. How does your volume element for schmerical coordinates compare to the differential volume element for spherical coordinates?

C. Consider the following student's statement.

"The differential volume should just be a really tiny cube. It should just be $dMd\alpha d\beta$."

Do you *agree*, *partially agree*, or *disagree*? Explain.

D. If you integrated the volume element to a constant radius M, what total volume you would expect to calculate for this system?

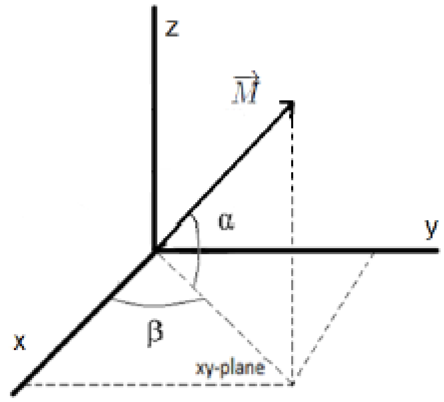
What is the result of such a calculation for the differential volume element you determined? Resolve any inconsistencies.

C3: Mid-tutorial HW

DIFFERENTIAL VECTOR ELEMENTS IN NON_CARTESIAN COORDINATES: LENGTHS

Mid-Tutorial Homework

I. Constructing differential area elements



A. Holding the length of $\vec{M}$ constant and changing the angles α and β, you can describe a surface area.

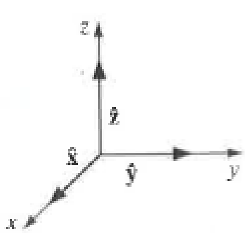
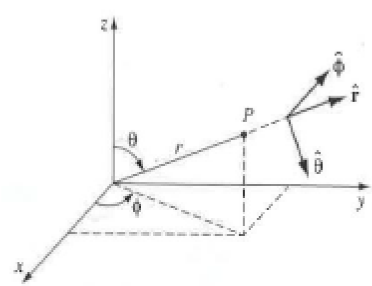
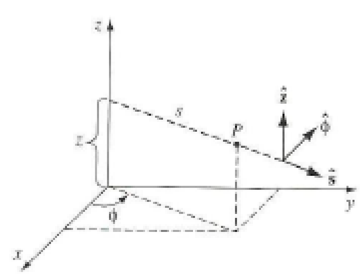
Determine the differential area element on this surface. Is this a vector or a scalar?

How does the terms of your differential area element compare to those in the differential length vector you've already determined in the tutorial?

B. Can you construct any other differential area elements for this system (magnitudes and directions)?

II. Differential lengths in typical multivariable coordinate systems

Only using methods developed in the *Differential Lengths* activity, construct a general differential length vector for each of the following three coordinate systems.

Cartesian	Spherical	Cylindrical
 $d\vec{l} =$	 $d\vec{l} =$	 $d\vec{l} =$

APPENDIX D – AREA TUTORIAL

Following the results of the interviews dealing with differential areas within the context of physics, a second tutorial activity as a companion to the schmerical length tutorial (Appendix C). This tutorial (Appendix D1) seeks to guide students to explicitly connect differential area elements to the product of associated length elements, which several students productively employed in interviews.

This activity begins by having students define an area vector for a flat plane using a grid-marked sheet of paper at the end of the packet. This portion of the tutorial is adapted from the beginning of the "Electric Field and Flux" tutorial which builds students' understanding of a differential area vector [65]. Students then define a differential area for a gridded region, using the appropriate coordinate system (Cartesian). At this point, the mathematical relationship for the area between two vectors is given, $\left|\vec{A} \times \vec{B}\right| = AB \sin \theta$, and students are asked to interpret what these vectors would be for the previously determined differential area. After doing this for a Cartesian coordinate system, students are given a polar coordinate grid and again asked to determine the differential area and to connect that expression to the equation for the area between two vectors. This shows that a polar differential area can be constructed using an arc length as one of the differential vector components.

Expanding this into three dimensions, this tutorial makes further use of physical manipulatives. Students are instructed to take the sheet of paper and roll it into a cylinder in order to discuss the differential area that would be created for this surface. Likewise, the ball from the schmerical tutorial is used for the construction of a spherical differential area element.

This tutorial also addresses the disconnect between vectors having to represent straight lines and flat planes. As these elements are differential quantities, they can be treated as straight lines and flat planes even though they represent curved dimensions. Then as they are accumulated over a surface, we arrive at the curved shapes dictated by the symmetry of E&M.

The last section of the tutorial addresses the idea of a coordinate system having multiple differential area elements by eliciting students' construction of the less commonly used differentials by having them multiply other length components as a way of cementing the construction of area vectors.

D1: Differential area vector tutorial

DIFFERENTIAL VECTOR ELEMENTS IN NON-CARTESIAN COORDINATES: AREAS

I. Area for a flat surfaces

With a developed understanding of differential length elements at our back, we now trudge on to develop a proper sense of differential area elements.

A. Detach the last page of this handout: a sheet of paper with a grid on each side. Leave one side blank to use in section II.

1. Often in E&M we treat area as a vector quantity, where the magnitude is the value of the area and the direction is normal to the surface. Why does a perpendicular vector better to define an area vector, rather than an area in the plane?

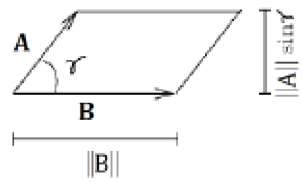
2. As differential areas are infinitesimally small, the grid region would contain *a lot* of them. However, in order to develop an understanding of how they are constructed, let us consider each box cut out by the grid as a differential area. Choosing just one of these boxes, how would you describe the magnitude and direction of a differential area vector for this box *in words*?

   Using an appropriate coordinate system, *write an expression*, for the differential area vector (magnitude and direction).

3. The area of a parallelogram made by two vectors can be calculated using a cross product.

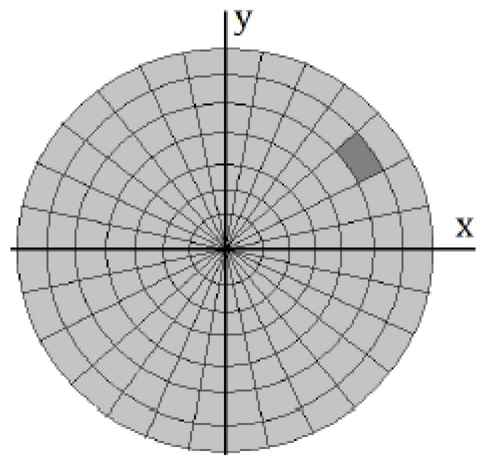
   Example: The area between $\vec{A}$ and $\vec{B}$ is given by: $|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin \gamma$ where γ is the angle between the two vectors.

   Can you interpret your differential area in this way? If so, define vectors $\vec{A}$ and $\vec{B}$ for your differential area for the grid?



B. Consider a polar coordinate grid in the xy-plane, with the z-axis coming out of the page.

1. Is it possible to use cross-product formalism to define this differential area? If so, define vectors $\vec{A}$, and $\vec{B}$. If not, construct a differential area by other means.



2. For what type of problems might the polar representation of da be more appropriate than the one for the earlier grid?

3. Consider the following student statement.

Ava: Wait. Vectors are supposed to be straight lines, so you can't have a differential length vector around a circle. Circles have curves. An arc through 35° is clearly a curve, so you're still going to have the same shape when you zoom in to a $d\theta$ change.

With what aspects of this student's statement do you *agree*? *disagree*?

➢ Check your answers with your instructor.

II. Area for curved surfaces

In electricity and magnetism, it is likely you will come across surfaces that span three-dimensional space. We now seek to construct differential areas for these surfaces.

A. Taking the sheet of paper we detached in section I, roll it along the long axis of the paper to form a tubular surface (you can wrap it around a water bottle for a harder surface), with the unused grid facing outward.

1. Is the expression for the differential area from the flat grid sufficient to describe this surface? Why or why not?

2. Using an appropriate coordinate system, *write an expression* for the differential area vector (magnitude and direction). If a cross-product formalism is applicable, define vectors $\vec{A}$, $\vec{B}$.

B. Obtain a ball like that used in the *Differential Lengths* activity. If the latitude and longitude lines have been erased, draw them back onto a section of the ball.

1. Using an appropriate coordinate system, *write an expression* for the differential area vector (magnitude and direction) on this surface. If cross-product formalism is applicable, define vectors $\vec{A}$, and $\vec{B}$.

2. Consider the following student discussion. How does this relate to your thinking?

Nate: I thought you could only define area vectors for completely flat surfaces. How can you have a unit vector to describe the surface of a sphere?

Esme: It's okay because we're defining such a small area. A football field looks flat to us, but it's actually on the curved surface of the earth. So 100 yds is actually an arc length, but we don't see it that way. It's *very* small compared to Earth.

III. Multiple differential areas

A. In section II, you constructed $\overrightarrow{da}$ for a polar coordinate system. This differential area can also describe an area in cylindrical coordinates. What area does this represent geometrically?

Recall the rolled sheet of paper in section II. This also represents an area for cylindrical coordinates. Is there a third possible differential area vector for cylindrical coordinates? What region of space does it describe? What would be the unit vector?

Considering the use of vector calculus, what properties would a vector field have that this differential area vector be useful?

B. Consider the following student's statement.

Isaiah: There is only one differential area for spherical. You only have one surface, just the outer part of the sphere. It's the only differential area we ever use.

Do you *agree* or *disagree*? If you *disagree*, what are the remaining differential lengths.

BIOGRAPHY OF THE AUTHOR

Raised on the outskirts of Gouverneur, New York, Benjamin Paul Schermerhorn spent most of his early life throwing hay bales around to feed 20-25 horses at the farm he shared with two loving parents and a sister. When he wasn't mucking out horse stalls or shoveling lake-effect snowfall, he would often be climbing trees and rocks down near the beaver pond. At 18, he graduated from Gouverneur High School, where he developed a passion for science, mathematics, and teaching, and often spent his free class periods helping his physics teacher in physics labs or in the 6th grade science classroom. Benjamin attended the State University of NY at Potsdam where he received two Bachelor of Arts degrees, one in Physics and one in Mathematics, as well as a minor in Creative Writing. In addition to making it home on the weekends to help on the farm, by his senior year, Benjamin was a resident assistant, President of the LGBTAA, Vice President of Sign Language Club, a Physics tutor, and a teaching assistant in the physics department. Shifting his sights to teaching at the collegiate level, he worked as an adjunct professor for a year at SUNY Potsdam as he researched graduate programs.

Upon discovering physics education research, Benjamin left the idyllic farm life behind and subsequently began his Ph.D. work on student understanding at the interface between math and physics. Having experienced the disciplinary disconnect first hand, he had a strong desire to "figure some of this nonsense out." Benjamin as a member of the American Association of Physics Teachers (AAPT) and has attended numerous national conferences, such as AAPT/Physics Education Research Conference, Conference on Research on Undergraduate Mathematics Education, and the conference on Transforming Research on Undergraduate STEM Education, where his has given a plethora of talks and posters, many of which became conference proceedings and longer papers [91,92,97,98,100]. At the University of Maine, Benjamin has worked as a teaching assistant and eventually was promoted to be the first head graduate teaching assistant, where he interfaced between the lecturer and the other teaching assistants to prepare for recitations.

In his spare time, Benjamin still enjoys creative writing and has successfully participated in two National Novel Writing Months where he has written over 134,000 words over two Novembers. Passions for writing, having a good laugh, teaching, and researching student understanding have helped him through harder times and carry him forward to a post-doctoral fellowship in California researching in and developing curricula for a spins-first quantum mechanics course. Benjamin strives to make the classroom a place where all students to feel accepted, enjoy themselves, and become more well-rounded individuals.

Benjamin Schermerhorn is a candidate for the Doctor of Philosophy degree in Physics from the University of Maine in May 2018.