JOURNAL OF SPATIAL INFORMATION SCIENCE
Number 14 (2017), pp. 1–29 doi:10.5311/JOSIS.2017.14.288

RESEARCH ARTICLE

Topological augmentation: A step forward for qualitative partition reasoning

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Received: May 25, 2016; returned: September 2, 2016; revised: October 26, 2016; accepted: December 14, 2016.

Abstract: The current state of the art for partition based qualitative spatial reasoning systems such as the 9-intersection, 9+-intersection, direction relation matrix, and peripheral direction relations is that of the binary set intersection—either empty or non-empty—conveying the intersection (or lack thereof) of an object in the sets deriving the partition. While such representations are sufficient for topological components of objects, these representations are not sufficient for various tasks in qualitative spatial reasoning (composition, representation transfer, converse, etc.) regarding partitions as tiles. Topological augmentation expands the current binary status quo into a system of assigning topological relations between objects and tiles. A case study is presented in the form of the direction relation matrix, demonstrating that an increased vocabulary has benefits for spatial information systems, providing localized context within a qualitative embedding.

Keywords: direction relation matrix, topological intersections, geographic information science, qualitative spatial reasoning, partitions

1 Introduction

From a cognitive perspective, qualitative spatial reasoning is a fundamentally important task that is broken apart in how it is initially learned developmentally [50]. Not surprisingly, the formalization of models for spatial intelligence for three decades produced a proliferation of formal models (see [12]) that have broken apart qualitative spatial information into three distinct components: *topology*, *direction*, and *distance* [36]. While these models are effective for various tasks in querying, the fact remains that human decisions are made by combining these three sources of spatial information into a unified view, despite that the

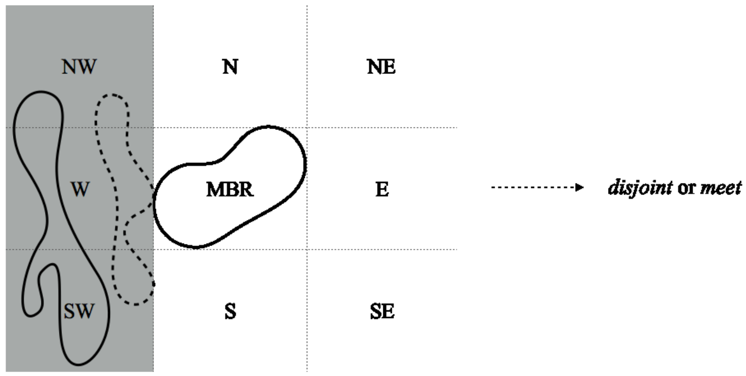

Figure 1: An example of converting a direction relation matrix symbol to a qualitative topological relation. In this case, the direction relation matrix symbol (NW,W,SW) is demonstrated to result in *disjoint* (the solid outline) or *meet* (the dashed outline), an uncertain result.

understanding of place cells and other brain structures has highlighted modularity in some aspects of the spatial process [34, 47, 57].

More recently, the literature in qualitative spatial reasoning over the past two decades has moved toward integrated forms of qualitative spatial reasoning, combining two or all of the separate spatial reasoning areas into productive and more verbose systems (e.g., [2, 10, 17, 41, 46, 59]). The combination of the topology, direction, and distance reasoning areas has helped to solve many spatial problems, but the realm of directions between areal objects has continued to provide a host of tasks where current methods of reasoning are inadequate or have not been applied substantively.

The source of direction reasoning issues is tied directly to how the current gold-standards are constructed. Two well-known approaches to the representation of areal direction [30, 49] focus on qualitative set intersections between an areal object (the figure object) and a set of partition tiles defined by the shape of a second areal object (the ground object). This qualitative set intersection approach has stark limitations linked to its missing information:

- Given the pervasiveness of geographic information systems (GISs), their usage across various areas of scientific endeavor, and the proliferation of volunteered geographic information [28], converting from one formal model to another formal model is often required to expand spatial knowledge within particular applications. One instance of this conversion is between the direction relation matrix and a 9-intersection topological relation [31]. This conversion currently cannot distinguish the topological relations *disjoint* and *meet* from one another in any circumstance (Figure 1). Adjacency is one of the primal concepts in the Natural Semantic Metalanguage [27], and thus can be considered essential to human understanding, though coarsening of topological information results in formalisms such as RCC-5 that neglect boundary contact [5].
- One key way to gather additional information from stored information is through the process of relation composition [60]. Composition fuses together relations over a common domain element. Composition can be classified in two forms: *strong* and *weak* [52]. The strong composition is the result that is applicable to any elements con-

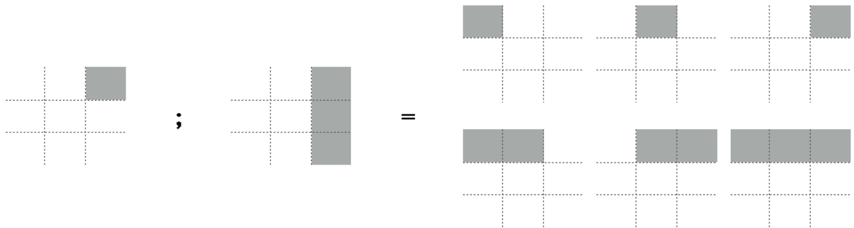

Figure 2: The weak composition of direction relation matrix symbols (NE) and (NE,E,SE). Not all members of these direction relation symbols can result in all of these resultant elements.

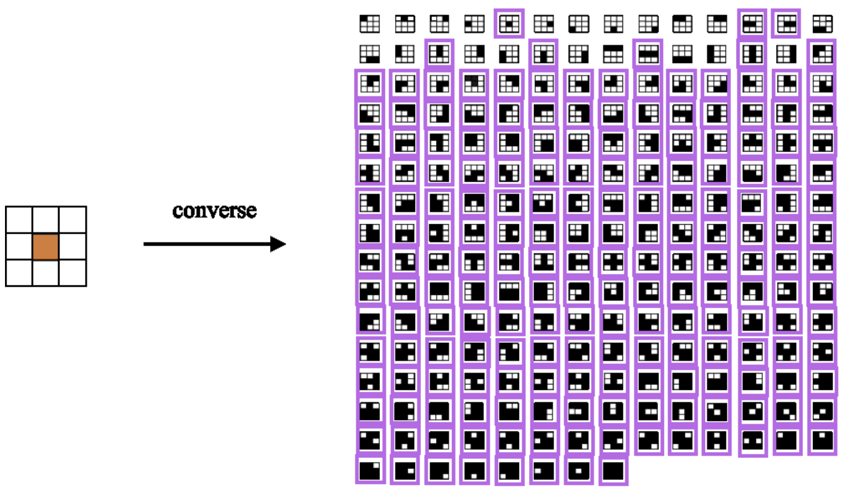

Figure 3: The converse relation of the minimum bounding rectangle tile in the direction relation matrix. The result has a high degree of uncertainty, causing potential errors in spatial reasoning applications where the direction relation is stored only in that order [61].

tributing to the relations in question, whereas the weak composition is an existential composition, claiming that at least one domain set can produce the result. Using the direction relation matrix, the weak composition has been computed [56]; however the strong composition of two specific instances has not been achieved in a verifiable form (Figure 2).

- The order that data/information are stored and communicated in a database is important. If the order of the relation in a database needs to be reversed, the converse relation must be considered. For simple topological relations, the converse relation is a one-to-one mapping (and thus invertible). Wang et al. [61] focused on this operation for the direction relation matrix, with some relations having as many as 198 converses (Figure 3)! Only four relations from the direction relation matrix have a unique converse that can also be undone uniquely.
- Given that both approaches employ areal partitions, consider the direction relation in either formalism between a linear feature that is a subset of the minimum bounding

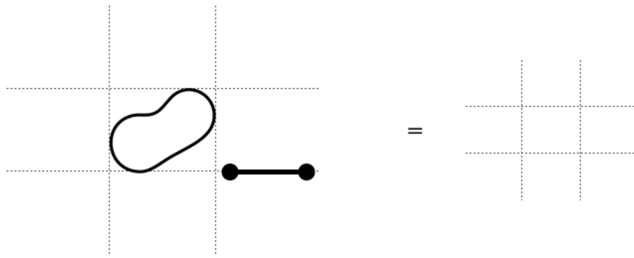

Figure 4: Application of the direction relation matrix to the relation between a line and a region. In this case, the line does not qualitatively intersect any of the tiles, therefore each tile is registered as empty, thus hiding the position of the line.

rectangle's boundary from an areal object [24]. No tile in this instance produces a non-empty qualitative intersection [29]. This result would be no different if the linear feature occupied the boundary between any two of the partition tiles (Figure 4). As such, these types of formalisms in their current form are ill-equipped for extension beyond areas.

A formal model that can answer the call for these tasks represents a substantial step forward for qualitative spatial reasoning with regard to directions. In this paper, it is argued that changing from a binary set intersection to a qualitative topological set intersection provides additional information in a large array of cases at the cost of inflating the set of realizable symbols. While this set of symbols will not always be needed at that granularity, storing data at a finer granularity and then deriving the coarsened view can provide future application benefit in tasks such as the ones listed above by implementing an approach that unifies two of the identified categories of qualitative spatial reasoning. Such extensions meet current needs in advanced cases, but may also one day meet currently unforeseen needs in the data science community, similar to the Google extension to the Brown Corpus and its potential effects on language analysis [32]. Notably, research has attempted to combine directional and topological reasoning [7, 9, 11, 35, 43, 44, 45, 54, 55], but none of those attempts have been employed to solve such existing challenges. In essence, this paper employs an approach akin to fuzzy sets [3, 6, 33, 64], however not in a numerical sense. The proposed theory assigns classes of membership to non-empty qualitative set intersections by infusing topological relations to provide additional context to an uncertain position [63]. In this light, not all members of the same current relation class are considered equal.

The rest of this paper is structured as follows. Section 2 highlights the spatial literature concerning the combination of spatial information types as well as the direction relation matrix and the standard topological relation formalisms based on intersections. Section 3 introduces topological augmentation and provides a proof that topological augmentation can provide the information available in the vanilla direction relation matrix in a simple and straightforward manner. Section 4 develops the integrity constraints for the various types of tiles that can arise in an arbitrary spatial partition of $\mathbb{R}^2$. Section 5 analyzes the results of topological augmentation on the symbol sets from the peripheral direction relations and the direction relation matrix. Section 6 revisits the examples from Section 1, showing that topological augmentation provides immediate impact for unsolved directional tasks. Finally, Section 7 provides conclusions and future work.

2 Motivating work in qualitative spatial reasoning

Qualitative spatial reasoning has been a fundamental factor in the spatial domain for the past fifty years of research. In this section, motivating literature within this corpus of research is reviewed, highlighting the need for a shift in how qualitative intersections are addressed. Section 2.1 addresses the field of topological relations. Section 2.2 addresses the direction relation matrix and its struggles regarding concepts in relation algebra and transfer of spatial representations. Section 2.3 addresses previous work in the arena of combining topological and directional qualitative spatial information.

2.1 Topological relations

One of the key areas that researchers have identified within spatial knowledge is that of topological spatial knowledge. Topological spatial knowledge reflects information pertaining to three specific concepts: *connectivity*, *containment*, and *intersection*. While the region connection calculus focuses on predominantly connectivity [51], the scope of this paper leans toward containment and intersection. Given the intended domain of this paper (simple regions), connectivity is functionally equivalent. One of the motivating tasks, however, benefits specifically from containment and intersection at the level of identification. These concepts generally form the backbone of spatial language [14, 38].

In qualitative topological spatial reasoning, containment and intersection are modelled through Boolean qualitative set intersections in three salient models: the 4-intersection [22], the 9-intersection [23], and the $9+$-intersection [37]. These three models have been developed using four key topological definitions: *interior*, *closure*, *boundary*, and *exterior* [1].

Definition 2.1. Let X be a set in a topological space T. Let O be the collection of all open sets O_n that are each a subset of X. The union of all members of O is called the **interior** of X, denoted as X^o.

Definition 2.2. Let X be a set in a topological space T. Let C be the collection of all closed sets C_n that are each a superset of X. The intersection of all members of C is called the **closure** of X, denoted as $\bar{X}$.

Definition 2.3. Let X be a set in a topological space T. The set $\bar{X} \setminus X^o$ is called the **boundary** of X, denoted as ∂X.

Definition 2.4. Let X be a set in a topological space T. Let O be the collection of all open sets O_n that do not intersect X. The union of all members of O is called the **exterior** of X, denoted as X^-.

Definitions 2.1–2.4 effectively create a partition of space based on a set. Each point in the space can be assigned to any of the three components *interior*, *boundary*, and *exterior*. This partition motivates the 4-intersection (Figure 5) and the 9-intersection (Figure 6), a combination of the topological components with respect to two objects. The $9+$-intersection extends this approach by considering a partition based on connected portions of these

Figure	$\partial A \cap \partial B$	$A^o \cap B^o$	$\partial A \cap B^o$	$A^o \cap \partial B$
(a)	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
(b)	$\neg\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$
(c)	$\varnothing$	$\neg\varnothing$	$\varnothing$	$\varnothing$
(d)	$\neg\varnothing$	$\neg\varnothing$	$\varnothing$	$\varnothing$
(e)	$\varnothing$	$\varnothing$	$\neg\varnothing$	$\varnothing$
(f)	$\neg\varnothing$	$\varnothing$	$\neg\varnothing$	$\varnothing$
(g)	$\varnothing$	$\neg\varnothing$	$\neg\varnothing$	$\varnothing$
(h)	$\neg\varnothing$	$\neg\varnothing$	$\neg\varnothing$	$\varnothing$
(i)	$\varnothing$	$\varnothing$	$\varnothing$	$\neg\varnothing$
(j)	$\neg\varnothing$	$\varnothing$	$\varnothing$	$\neg\varnothing$
(k)	$\varnothing$	$\neg\varnothing$	$\varnothing$	$\neg\varnothing$
(l)	$\neg\varnothing$	$\neg\varnothing$	$\varnothing$	$\neg\varnothing$
(m)	$\varnothing$	$\varnothing$	$\neg\varnothing$	$\neg\varnothing$
(n)	$\neg\varnothing$	$\varnothing$	$\neg\varnothing$	$\neg\varnothing$
(o)	$\varnothing$	$\neg\varnothing$	$\neg\varnothing$	$\neg\varnothing$
(p)	$\neg\varnothing$	$\neg\varnothing$	$\neg\varnothing$	$\neg\varnothing$

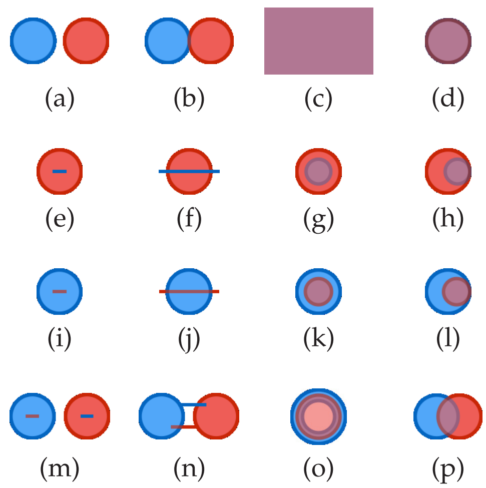

Figure 5: The sixteen different relations identified by the 4-intersection with a graphical instantiation of each. Wherever possible, the relation is expressed with a pair of simple regions [22].

topological components, generalizing to the 9-intersection for simple regions (that is, those bounded by a Jordan curve). Simple regions, given the Jordan curve property, are also *simply connected*, namely that each object can be continually collapsed to a point while still remaining within that domain [1]. Similarly, partition tiles within this paper are also simply connected as each tile does not contain a hole.

The 9-intersection has led to a proliferation of sets of relations based on specific domains of spatial interest, including line-line relations [18], line-region relations [24], simple region-region relations in $\mathbb{S}^2$ [20], digital relations [25], and complex region relations [42, 53]. While this paper focuses on those relations in Figure 6, the proposed methodology can be employed similarly in such other domains if the rules for assignment are established.

Topological relations (such as these) have been used in prior studies to help reconstruct spatial scenes [17, 21, 39] by storing the binary relationships between all objects in a space. The approach presented in this paper utilizes this approach in the context of a specifically structured partition of space.

2.2 Qualitative directions between areal objects

Qualitative topological spatial relations are just one perspective on a pair of objects and their association. Qualitative directional spatial relations are another key perspective, opening up a completely different level of spatial vocabulary [38]. While Euclidean spaces manage the direction between a pair of points through vectors, the task of assigning a direction between two areas of space is much more complicated. Approaches to this task range from using centroids of objects [46] to the qualitative partition of space into regions [8, 30, 45, 49].

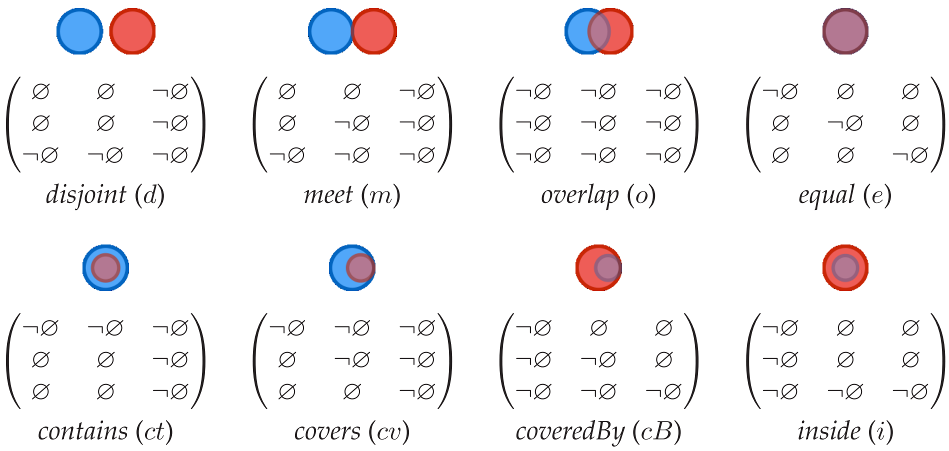

Figure 6: The eight topological relations identified by the 9-intersection between two simple regions [23].

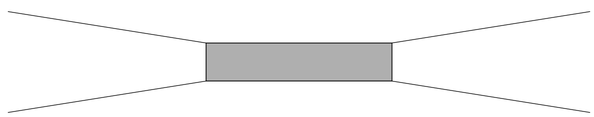

Figure 7: The peripheral direction relation embedding space [49], dropping a line from each corner of a rectangle toward the rest of the space at an arbitrary angle.

This paper focuses on two qualitative partitions of space designed to create a direction relation between two regions. The first approach is the peripheral direction relation [49], while the second is the direction relation matrix [30]. While the same methodology applies to the internal cardinal direction relations [45] and the objects interaction matrix [8], this pursuit is left to future work.

2.2.1 Peripheral direction relations

Motivated by cardinal direction relations and the way in which eyes are positioned in mammals [26], the peripheral direction relations [49] subdivide a space into partition tiles based on the minimum bounding rectangle (MBR) of the ground (or target) object. The vertices of the MBR serve as an anchor for a ray extending into infinity at a set direction angle, defining intuitive concepts for North (in front of), South (in back of), West (to the left of), and East (to the right of), as shown in Figure 7.

For simple regions, this method establishes a set of 29 distinct direction relations, one for each connected set of tiles. Only two sets of tiles: 1) West and East, and 2) North and South, do not satisfy this connected property.

2.2.2 Direction relation matrix

Though effective, the peripheral direction relations have a relatively small granularity and have difficulty in expressing ordinal directions. An answer to this shortcoming is found

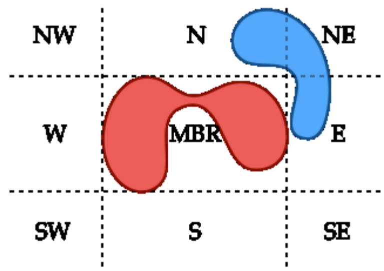

Figure 8: The direction relation matrix [30]. Using the object in the center of the figure, the minimum bounding rectangle (M) is constructed. The direction relation matrix then uses these tiles and assigns either a Boolean value or a proportion to represent the configuration.

with the direction relation matrix [30]. This approach uses the rectangular coordinates of the MBR and extends each line of the MBR in infinity. The intersection of these lines creates a 9-tile partition of space, one for each ordinal direction and the MBR itself (Figure 8).

The direction relation matrix has a much larger cardinality with respect to the peripheral direction relations. 218 distinct relations are available between simple regions, as demonstrated in Figure 9.

Numerous attempts have been made to exploit the direction relation matrix for additional types of information, including the determination of the converse relation [61], the determination of the topological relation [31], the composition of direction relations [56], and the use of the formalism to relate arbitrary objects [29]. In each case, specific deficiencies have been encountered as identified in the introduction.

2.3 Combinations of topology and direction

Though orthogonal, topology and direction play a pivotal role in the expression of spatial knowledge. Numerous research lines have been traversed in this area. Sharma [54] used direction relations and topological relations together in an effort to refine the composition of topological relations. Other formalisms have been developed that integrate the information together in varying capacities (e.g., [2, 7, 9, 11, 35, 41, 43, 44, 46]). None of these models, however, has been used to exploit properties in the composition of direction relations [56], nor applied to the general pursuit of knowledge regarding the properties of relation algebra within the direction relation matrix setting [60].

One interesting approach is the combination of topological and directional relations within the partitions themselves. Kor and Bennett [35] have employed this approach to expressing relations in maps using the region connection calculus and the direction relation matrix in a modified form. Their work does not, however, focus on the attainment of better contextual knowledge for the converse relation, composition result, and topological relations that are associated with the direction relation matrix itself. This work is intended to provide the bridge to filling those important gaps.

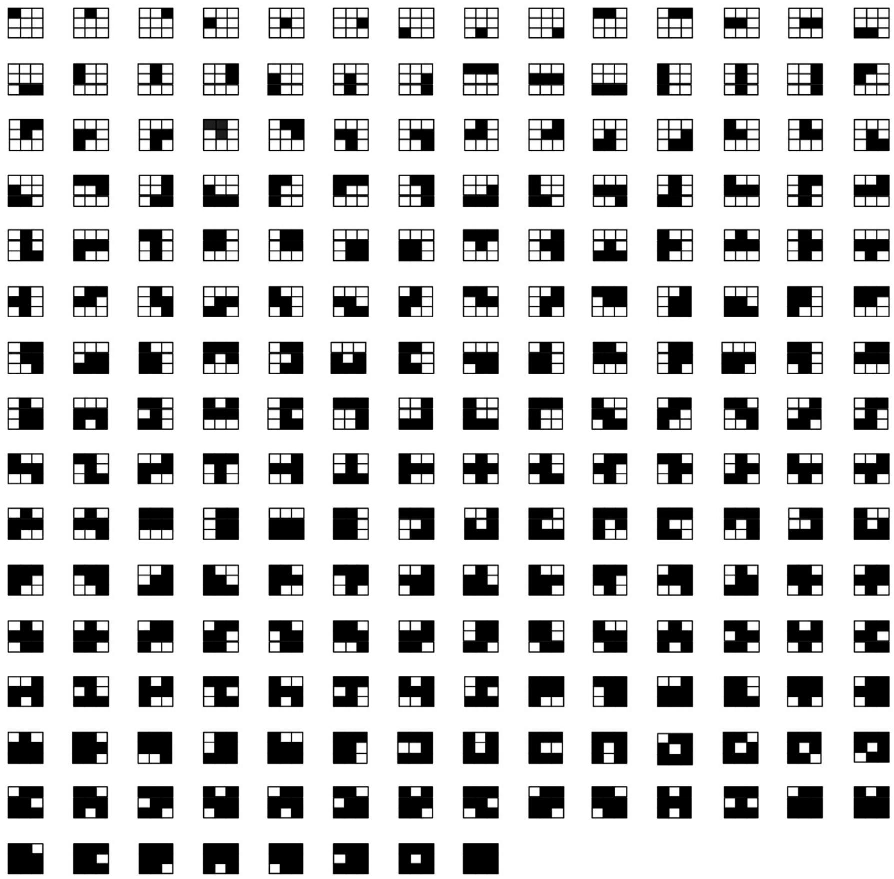

Figure 9: The 218 different symbols that can arise from the direction relation matrix with Boolean assignments with respect to two simply connected regions [29].

3 Topological augmentation

This paper defines the concept of *topological augmentation*. Topological augmentation is an approach to partition reasoning that assigns a topological relation between a figure object and a set of ground partitions to better manage the intricacies of the object within a partitioned space. This approach stands in contrast to Boolean set intersections, which provide only enough information to determine that the object is present or not. While Kor and Bennett [35] provide the first work in this area, the use of RCC-8 is not ideal when considering objects that can produce more rich topological relations achievable by other formalisms such as the 9-intersection. Topological augmentation is defined in Definition 3.1. For the remainder of this paper, the nomenclature of Talmy [58] is adopted in referring to objects as either *figure* (the first element of the spatial relation) or *ground* (the second element of the spatial relation). Additionally, the paper adopts the 9-intersection [23] as its method

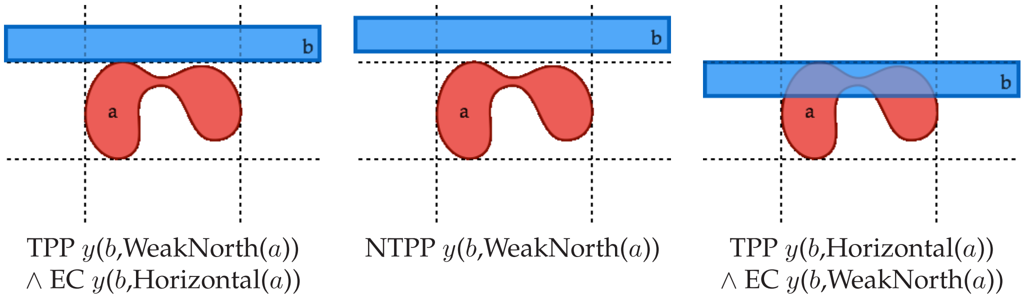

Figure 10: Direction relations with specified RCC-8 relations [35]. This approach provides a more flexible framework for describing further elements of object b's position relative to object a, but prior work does not focus on the gains that can be realized from a reasoning perspective by taking this approach.

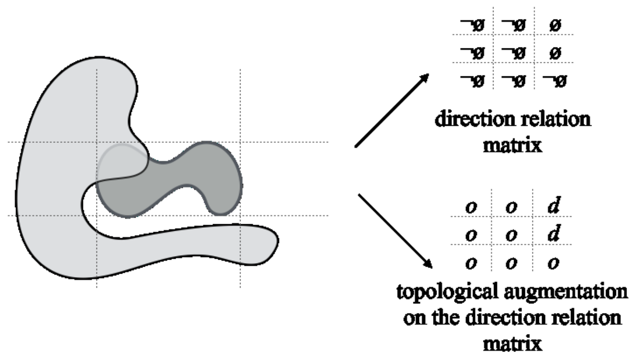

Figure 11: A comparison between the direction relation matrix of a pair of objects and the corresponding topological augmentation.

for assigning a qualitative topological spatial relation. The definition is flexible enough to account for other models of qualitative topological spatial relations.

Definition 3.1. Let X be a collection of sets partitioning a topological space T, and x_i a particular set from X called a **tile**. A **topological augmentation** is any method that assigns a binary qualitative topological spatial relation between a set Y and each individual x_i.

Topological augmentation can be seen as a refinement to a partition-based relation (such as the direction relation matrix). Rather than modelling just whether or not the set Y intersects each individual x_i, topological augmentation calculates the topological spatial relation and assigns that to each tile, rather than an empty or non-empty designation (as in the case of the direction relation matrix). As such, each tile is assigned information regarding the interior, boundary, and exterior of the figure object with regard to its own interior, boundary, and exterior. A topological augmentation of the 9-intersection within the direction relation matrix is shown in Figure 11.

The aim of topological augmentation is to produce the binary topological relation between the figure object and the ground tiles in such a manner that the original partition relation can be maintained in the standard Boolean form (empty or non-empty). Theorem 3.1 addresses this desideratum.

Theorem 3.1. *Topological augmentation maintains the partition relation in its Boolean form.*

Proof. Consider an arbitrary topological relation from an intersection-based method such as the 9-intersection [23], 4-intersection [22], or the $9+$-intersection [37]. These topological relations maintain information about the intersections of the topological components of their domain and co-domain by definition. If at least one point is shared between the two components, then the intersection is non-empty. If no points are shared, the intersection is empty.

Now, consider a qualitative partition relation in a Boolean form. Within this type of partition relation, the tile is registered as non-empty if the figure object shares a point with that tile, and empty otherwise.

Topological augmentation assigns a qualitative topological spatial relation between the figure object and each tile in the partition. Both the figure object and the tile have an interior, boundary, and exterior. If the interior or boundary of the figure object intersects the interior of the tile, then the figure object shares a point with the tile and thus has a non-empty intersection with it. Similarly, if only the exterior of the figure object intersects the interior of the tile, then the figure object does not share a point with the tile and thus has an empty intersection. Thus by observing only the $*$-interior intersections of the tile, the Boolean intersection is discernible for each individual tile, preserving the Boolean form of the partition-based relation. □

Given Theorem 3.1, we can also assert that a non-empty partition implies a topological relation that has a non-empty interior-interior or boundary-interior intersection, and that an empty partition implies that the topological relation does not have a non-empty interior-interior intersection. This information is imperative for the determination of the various combinations of topological augmentations that may mathematically exist within specific domains of object relations (region to region, line to region, etc.) within a partitioned environment.

In Section 4, the theory of topological augmentation is developed for the simple region-region relations identified in Figure 3.

4 Simple region-region relations

One formalism that can benefit from the use of topological augmentation is that of qualitative areal direction reasoning such as with the direction relation matrix [30] and the peripheral direction relations [49]. Each of these partition-based formalisms have simply connected tiles. This section provides the set of constraints that must be upheld for a particular topological augmentation within these partitions (and other arbitrary partitions with simply connected tiles) to be realizable given the constraint of simply connected figure objects. Each theorem details the conditions necessary for each of the eight topological relations between simple regions to be assigned to an individual tile based either on its qualitative set intersection (empty or non-empty) or a relation that is to be assigned to other specified tiles. This section is broken into three parts: Section 4.1 details empty tile intersections; Section 4.2 details general non-empty tiles; and Section 4.3 details bounded non-empty tiles.

Given that simple region-region relations are defined between objects bounded by Jordan curves, for tiles not bounded by a Jordan curve, we impose a Jordan curve beyond the scope of the figure object, a surrogate for an infinite extent. In so doing, particular relations are restricted from these cases: *contains*, *covers*, and *equal*, as each of these requires that the figure object is larger than its corresponding ground object. Since the tile is not bounded by a Jordan curve, this is intuitively not possible between any figure object and these particular tiles.

4.1 Empty tile intersections: *disjoint* and *meet*

From Theorem 3.1, given that a tile has an empty intersection, the corresponding relation in the tile must have an empty interior-interior intersection within topological augmentation. For the simple region-region relations in $\mathbb{R}^2$, this limits the choices of relations to *disjoint* and *meet*. The difference between these two relations is that *disjoint* shares no boundary points, whereas *meet* shares at least one boundary point.

Theorem 4.1. *Let tile x have an empty intersection with a figure object f. The relation between figure object f and tile x may be* disjoint *provided that none of x's neighbors have relation* equal*,* covers*, or* contains *with respect to f.*

Proof. The relation *disjoint* does not allow for a boundary intersection between f and x. To demonstrate Theorem 4.1, it must be shown that the relation *disjoint* can occur based on the composition of the relation between the figure object and the tile and the relation between two neighboring tiles (specifically, *meet*).

Consider the relation between f and an arbitrary neighbor of x, and the resultant compositions with *meet* [19]:

- If the relation is *disjoint*, *disjoint* ; *meet* = ***disjoint***, *meet*, *overlap*, *coveredBy*, *inside*.
- If the relation is *meet*, *meet* ; *meet* = ***disjoint***, *meet*, *overlap*, *coveredBy*, *covers*, *equal*.
- If the relation is *overlap*, *overlap* ; *meet* = ***disjoint***, *meet*, *overlap*, *covers*, *contains*.
- If the relation is *inside*, *inside* ; *meet* = ***disjoint***.
- If the relation is *coveredBy*, *coveredBy* ; *meet* = ***disjoint***, *meet*.

For each case, the composition includes the relation *disjoint*, therefore there is at least one instance where *disjoint* can occur between three objects/tiles satisfying these two relations. Since the 9-intersection produces a strong composition [19, 52], this result applies for arbitrary simply connected objects/tiles.

It must also be shown that the relations *contains*, *covers*, and *equal* do not allow for a *disjoint* neighbor. This can similarly be achieved through composition with *meet*.

- If the relation is *contains*, *contains* ; *meet* = *overlap*, *covers*, *contains*.
- If the relation is *covers*, *covers* ; *meet* = *meet*, *overlap*, *covers*, *contains*.
- If the relation is *equal*, *equal* ; *meet* = *meet*.

In each case, the relation *disjoint* is not a member of the composition result, therefore *disjoint* cannot exist in the presence of a neighboring tile with any of these relations. Therefore, a tile may have relation *disjoint* only in the case that its neighbors do not have relation *contains*, *covers*, or *equal* (Figure 12). ☐

Since *disjoint* and *meet* are the only two relations that can exist for empty tiles, any tile that may not be assigned *disjoint* must be assigned *meet* by default. Which cases allow for either to be assigned?

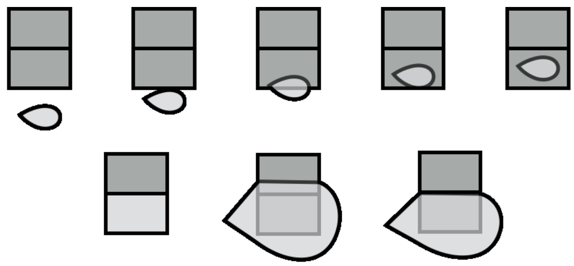

Figure 12: The set of tile relations for neighboring tiles that can produce *disjoint* (top), and the corresponding tile relations that cannot (bottom).

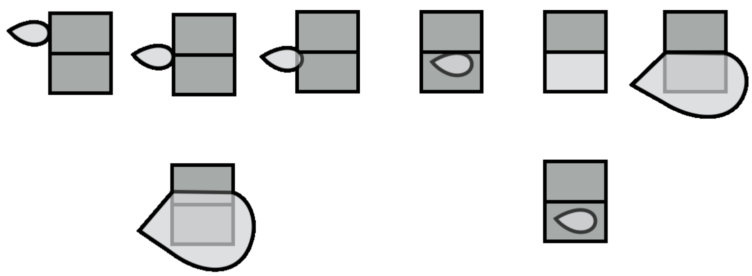

Figure 13: The set of tile relations that can produce *meet* (top) and those that cannot (bottom).

Theorem 4.2. *Let tile x have an empty intersection with a figure object f. For x to be assigned relation* meet *with respect to figure object f, a neighbor of x must be non-empty and must be assigned a relation other than* inside *or* contains.

Proof. Similar to Theorem 4.1, this proof is the result of the composition of the relation between the figure object and the tile and the relation between two neighboring tiles (specifically, *meet*). The compositions of *inside* ; *meet* and *contains* ; *meet* (from the proof of Theorem 4.1) do not allow for relation *meet*, but all of the others do (Figure 13)

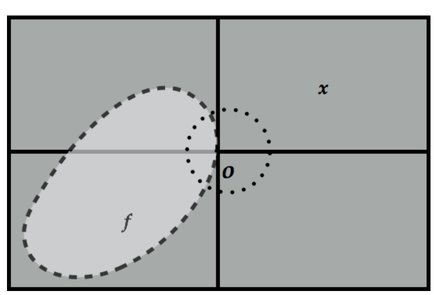

Figure 14: The visual argument to demonstrate that a neighboring tile to tile relation *meet* (in tile x) must be non-empty.

To demonstrate this theorem, it must also be shown that at least one neighboring tile is also non-empty. Since both the figure object f and the tile x are bound by a Jordan curve (or can be represented as such), one side of the boundary must consist entirely of interior points of f. Since x possesses an empty intersection with f, the side opposite x must be comprised solely of interior points of f. Since there is a boundary-boundary intersection between f and x, any open disc o containing that intersection point must include interior points of f and exterior points of f as well, independent of its radius (Figure 14). Since o is simply connected and x cannot contain interior points of f, these interior points of f must be in $o \setminus x$. Since the boundary of any object must be adjacent to its interior, at least one interior point of f must be adjacent to this boundary intersection between f and x. That point resides in a tile that neighbors x, therefore that tile's intersection with f is non-empty. □

There are circumstances, however, where the assignment of relation *meet* is dependent upon other tiles besides this non-empty one suggested in Theorem 4.2. Consider two tiles that intersect only at a point (such as direction relation matrix tiles M and SE). If M has a non-empty intersection and SE is to have relation *meet* with respect to a figure object f, dependencies must occur to validate the topological augmentation. These dependencies are covered in Theorem 4.3.

Theorem 4.3. *Let x be a tile with an empty intersection with f that is to be assigned* meet *based upon its point neighbor (as in the given example). For x to be assigned relation* meet*, all other tiles which share this point as boundary must have a relation that is not* disjoint, inside*, or* contains*.*

Proof. Since these tiles all have the point intersection as a boundary, the boundary-boundary intersection applied to x must also be shared with all of the specified tiles. This precludes relations *disjoint*, *inside*, and *contains*, as each relation has no boundary-boundary intersection available (Figure 6). □

Theorems 4.1–4.3 provide the basis for assigning topological relations to empty tiles in the prescribed environments with the specified domain of simply connected figure objects and tiles. Section 4.2 addresses generally non-empty tile intersections.

4.2 Generic non-empty tiles: *overlap*, *inside*, and *coveredBy*

Non-empty tiles must be subdivided into two classes because there are relations that are size dependent. Relations *equal*, *contains*, and *covers* require objects to be of identical or larger size to their co-domain. Since there are tiles where a simply connected figure object does not exist such that it can subsume the tile, these relations are not assignable to an arbitrary tile, specifically to those not bound by a Jordan curve. For relations *overlap*, *inside*, and *coveredBy*, any size constraints of these objects do not impact their general assignment, independent of the status of the boundary as a Jordan curve. These three relations also have very specific and straightforward parameters with which they can be assigned. These relations are addressed in Theorems 4.4–4.6.

Theorem 4.4. *Let tile x have a non-empty intersection with a figure object f. Tile x can be assigned the relation* overlap *only if a non-point neighbor tile of x is non-empty.*

Proof. The relation *overlap* implies an intersection between the figure object's interior and the tile's exterior, therefore some additional tile in the space must be non-empty. Since

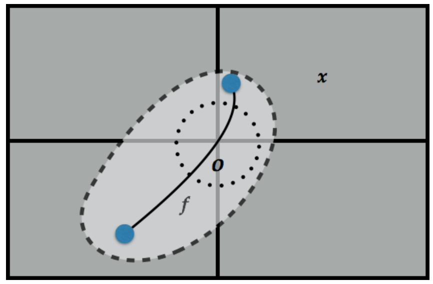

Figure 15: The visual argument for an edge-neighboring tile to tile x being non-empty when tile x has relation *overlap* with respect to object f.

the object must be simply connected, a path must exist that links any two such points that would generate those intersections. That path may, however, go from the tile being assigned to another point-adjacent tile, thus making those two tiles non-empty, using an argument similar to that of Theorem 4.2 for interior adjacency.

To demonstrate the Theorem, we must also show that some other tile that is an edge-adjacent neighbor of tile x also is non-empty. The interior adjacency argument from Theorem 4.2 can also be used to demonstrate this Theorem. Consider two cases. The first is the case where the edge-adjacent neighbor to x is also edge-adjacent to the point-neighbor to x. Since interior points are surrounded by interior points, there exists an open disc o around the point-adjacency that contains a set of only interior points of f (Figure 15). Since o has a non-zero radius, the edge-adjacent neighbor to both x and its point-neighbor has interior points of f and is thus non-empty.

The second case is the case where arbitrarily many tiles converge at that point-adjacency. The same argument for case one can be employed in this case as well as each point-neighboring tile will intersect o. One of these tiles must be edge-adjacent to x, satisfying the claim of the theorem. □

Theorem 4.5. *Let tile x have a non-empty intersection with a figure object f. Tile x can be assigned the relation* inside *only if it is the only non-empty tile in the partition and all other tiles have relation* disjoint*.*

Proof. This can easily be demonstrated by the composition of the relation between the figure object and the tile and the relation between the two tiles (*meet* for all adjacent tiles; *disjoint* for all others). *inside* ; *disjoint* = *disjoint* and *inside* ; *meet* = *disjoint*. Since *meet* and *disjoint* always occur in empty tiles, this verifies that the tile in question is the only non-empty tile in the partition. □

Theorem 4.6. *Let tile x have a non-empty intersection with a figure object f. Tile x can be assigned the relation* coveredBy *only if it is the only non-empty tile in the partition, and at least one other neighboring tile has the relation* meet*.*

Proof. This can also easily be demonstrated by the composition of the relation between the figure object and the tile and the relation between the two tiles (*meet* for all adjacent tiles; *disjoint* for all others). *coveredBy* ; *disjoint* = *disjoint* and *coveredBy* ; *meet* = *disjoint*, *meet*. Together, these make sure that all other tiles are empty. Since *coveredBy* implies a non-empty boundary-boundary intersection, any tile sharing that particular boundary-boundary intersection point(s) must produce relation *meet* by exhaustion, as it is the only available option with a boundary-boundary intersection per the result of Theorem 4.2. □

Theorems 4.4–4.6 can be applied to any tile, bounded by a Jordan curve or not. Since any size constraint implies smaller in these instances, there is no issue with these relations being applied in either context. Section 4.3 addresses the remaining cases that imply that the figure object is as large or larger than the tile.

4.3 Jordan-curve-bounded non-empty tiles: *contains*, *covers*, and *equal*

The relations in this section have a size dependency, and thus are only attainable when the tile in question is bounded by a Jordan curve. In both of the gold standard cases in the literature, only one tile may satisfy these demands (that of the minimum bounding rectangle), however for internal direction relations [45] or the objects-interaction matrix [8], many tiles can satisfy these demands. Theorems 4.7–4.9 address these relations.

Theorem 4.7. *Let tile x have a non-empty intersection with a figure object f. The relation* equal *can be assigned to x only if it is the only non-empty tile in the space, and additionally, all neighboring tiles have relation* meet *and all non-neighboring tiles have relation* disjoint*.*

Proof. This can be demonstrated through the composition of the relation between the figure object and the tile and the relation between the two tiles (*meet* for all adjacent tiles; *disjoint* for all others). *equal* ; *meet* = *meet* and *equal* ; *disjoint* = *disjoint*. As such, every neighbor must have relation *meet*, and every non-neighbor must have relation *disjoint*. Given that *meet* and *disjoint* are specifically empty tiles, the other requirement is satisfied. □

Theorem 4.8. *Let tile x have a non-empty intersection with a figure object f. The relation* contains *can be assigned to x only if all of its neighbors are non-empty.*

Proof. This can also be demonstrated through the composition of the relation between the figure object and the tile and the relation between the adjacent tiles (specifically *meet*). *contains* ; *meet* = *overlap*, *covers*, *contains*. All of these tiles have non-empty interior-interior intersections, and thus by Theorem 3.1 have non-empty tile intersections. □

Theorem 4.9. *Let tile x have a non-empty intersection with a figure object f. The relation* covers *can be assigned to x only if it is not the only non-empty tile in the space and none of its neighbors has relation* disjoint*.*

Proof. Similar to Theorem 4.8, consider the composition *covers* ; *meet* = *meet*, *overlap*, *covers*, *contains*. All neighboring tiles must have a relation from this set as the simple region composition is strong, thus no neighbor can have relation *disjoint*. Since *covers* has a non-empty interior-exterior intersection, the object must have a non-empty intersection with some other tile. □

Theorems 4.1–4.9 exhaustively cover the topological relations that can be assigned to tiles based on the properties of the tiles within the partition and within the Boolean setting. In Section 5, these methodologies are used to provide a large amount of diversity within the peripheral direction relations and the direction relation matrix.

5 Exposing diversity within qualitative spatial partitions

Using the foregoing Theorems from Section 4 and the sets of realizable relations between simply connected regions in both the peripheral direction relations [49] and the direction relation matrix [30], the diversity of realizable relations can be explored with topological augmentation. Figure 16 and Table 1 explore this diversity from the realm of *congruence classes*, relations that are identical under symmetry.

A *congruence class* refers to symbols that algebraically behave in similar ways to one another. In the presence of a congruence class, one exemplar need only be considered to provide information regarding all other members of that class. In the case of the peripheral direction relations and the direction relation matrix, congruence classes can be identified by the structure of the partitioning. There is a specific adjacency structure that remains consistent between the tile sets, independent of what orientation the partitioning is observed from. In the case of the direction relation matrix, when observing the space from the NE, NW, SE, or SW directions, the partitioning is structurally equivalent. When observing the space from the N, S, E, or W directions, the partitioning is structurally equivalent. In the case of the peripheral directions, observing the space from any direction produces a structurally equivalent result. As such, we can consider only a few cases to exhaustively cover all of these available symbols. These congruence classes are identified based on matrix transformations. Any matrix that can be produced by employing a horizontal, vertical, major diagonal, or minor diagonal transposition (or any combination thereof) to the same matrix are considered congruent to one another [15]. For the direction relation matrix, there are 45 different congruence classes, whereas for the peripheral direction relations, there are only 10.

Figure 16 and Table 2 determine the amount of topological augmentations that can exist within a particular congruence class for both qualitative areal direction formalisms. For the peripheral direction relations (Figure 16), these possibilities are enumerated (as the total set is more manageable in number). For the direction relation matrix (Table 2), the cardinality of each set is provided, allowing for the determination of the total number of relations available through topological augmentation over simple regions.

Each grouping in Figure 16 represents a different congruence class of peripheral direction relations that can be rotated. Topological augmentation thus produces a set of 180 different realizable symbols from the original set of 29, averaging just over six symbols from each peripheral direction relation.

Class (b) has the highest diversity amongst the relations. The one symbol belonging to class (b) produces 17 different topological augmentations: one for *inside*, one for *equal*, and 15 for *coveredBy*. The least diversity comes from class (h), with only two possible symbols. The impact of this diversity (or lack thereof) can be felt in many applications, as seen in the next section.

Similarly, each congruence class in Table 2 has a varying degree of flexibility with topological augmentation. Class 9 (the center stripe) has the most diversity with 65 relations, while class 44 (the empty center tile) has the least diversity. The most diverse single combination is that of *coveredBy* in class 5, producing 46 specific relations (with *inside* and *equal* producing the other two). Both class 44 and class 5 exhibit similar properties as exhibited in the peripheral direction relations. Topological augmentation thus produces a set of 3,084 different realizable symbols from the original set of 218, averaging just over 14 symbols

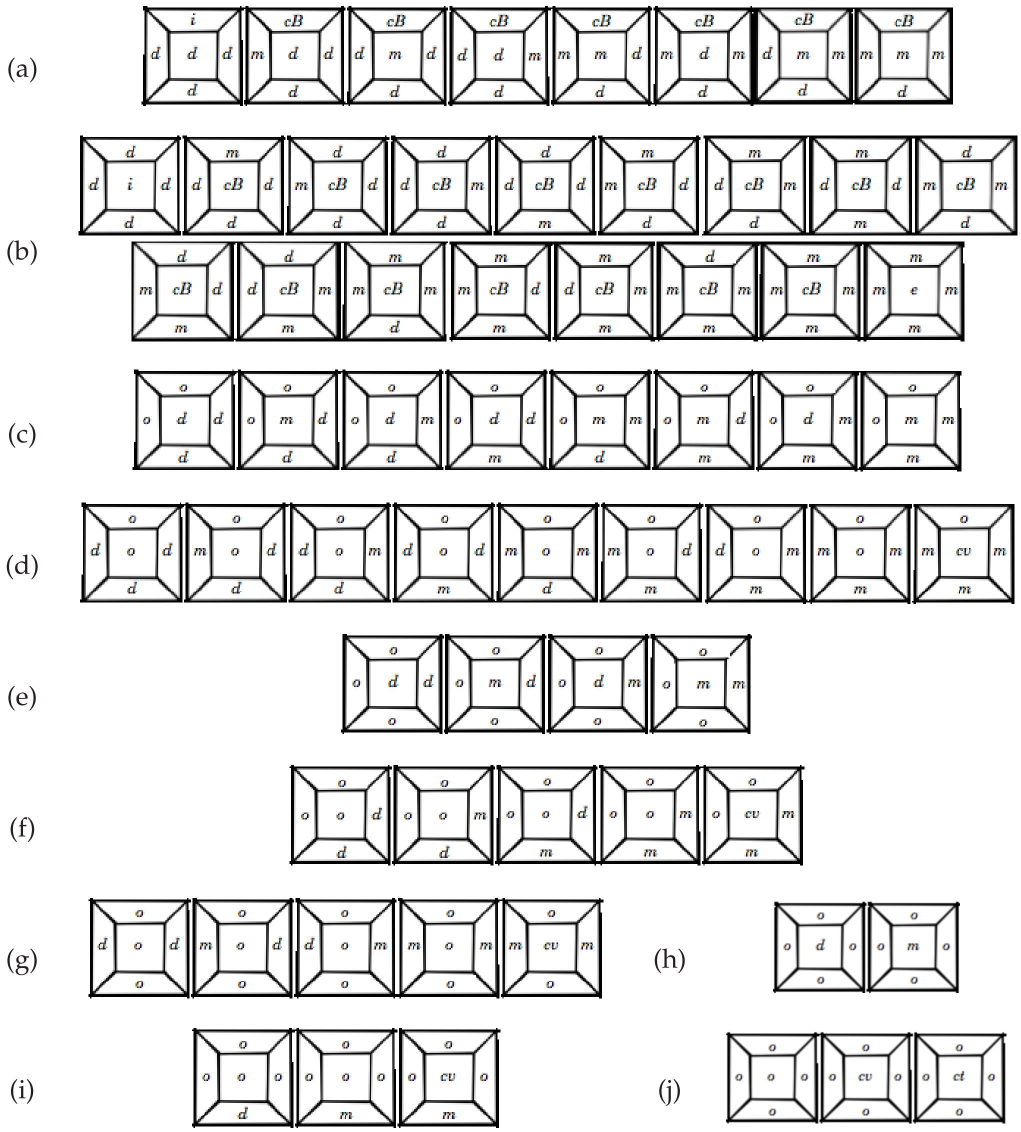

Figure 16: Topological augmentation applied to the peripheral direction relations. Each group (a–j) represents the members of an exemplar from the ten congruence classes. Each of these symbols (and rotated versions) can be used to provide additional context to peripheral direction relations. The symbols used to refer to each qualitative partition relation correspond to those abbreviations for the 9-intersection relations given in Figure 6.

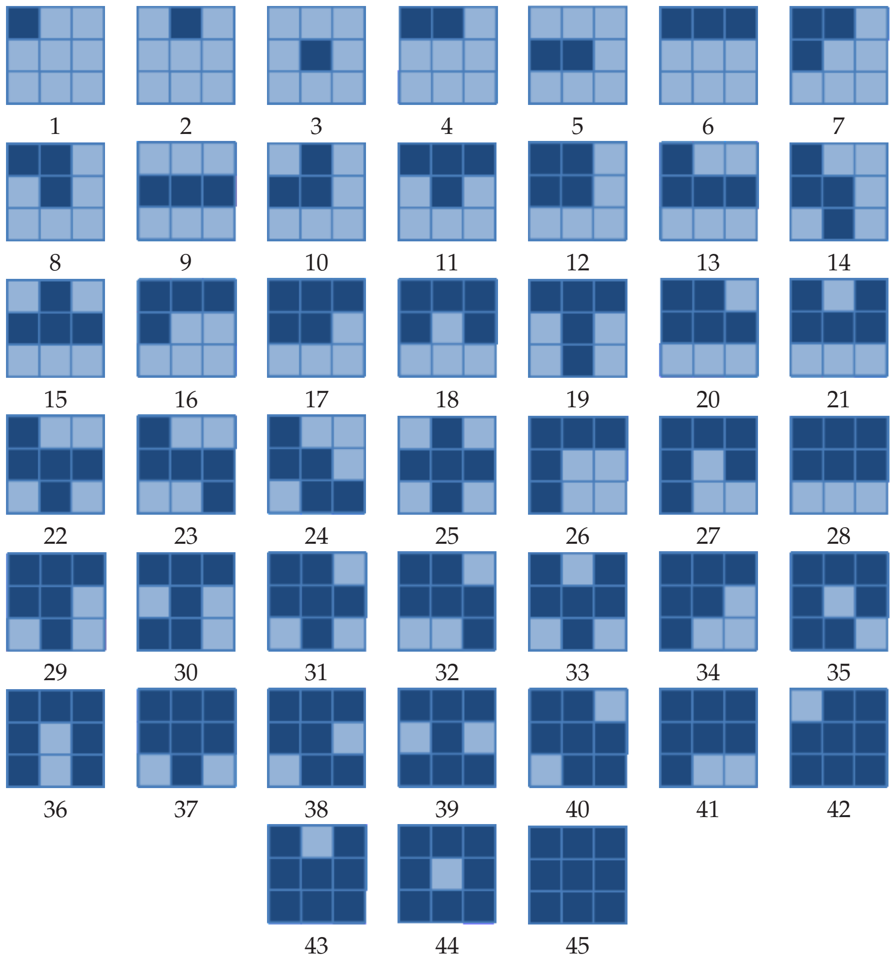

Table 1: Exemplars for each direction relation matrix class (reference for Table 2).

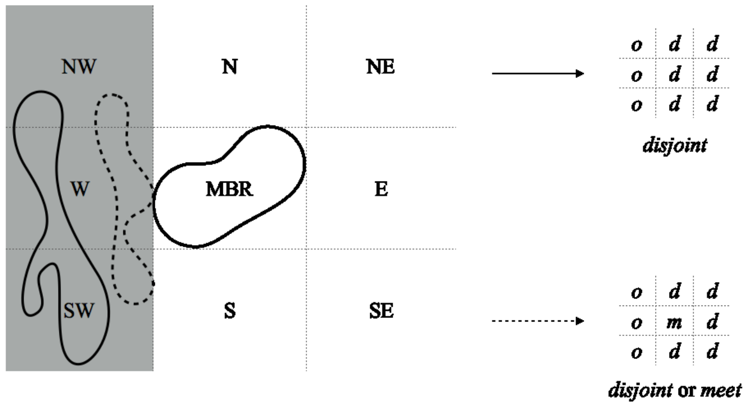

Figure 17: The impact of topological augmentation on the possible topological relations between a direction relation matrix symbol and a ground object. By observing the relation *meet* with the MBR tile, the relation *meet* with the object is possible, but not guaranteed. If the relation had been *disjoint* for this particular tile, the relation *meet* for the objects is not possible.

from each direction relation matrix. Similarly, the effects of this diversity can be felt in numerous applications as discussed in the conclusion.

While an increase in relational diversity seemingly violates the general premise of Occam's Razor, a computer's information systems or reasoning methods are not bound by this premise. What is bound, however, is the manner by which the computer conveys that knowledge back to users. In the next section, the benefits of this approach are displayed relative to important spatial reasoning tasks. Given that Theorem 3.1 ensures the maintenance of the original information contained in these qualitative partition relations, the clarification of context for the information system has no consequence on the given answers to the user other than creating a reduction of possibilities, a better source of information.

6 Perspectives on previous problems

At the outset of the paper, a set of four tasks were presented in the areal direction realm that provide substantial problems with respect to the advancement of partition-based direction reasoning. Topological augmentation as presented in this paper provides substantive advances for each.

Guo and Du [31] studied the transition from the direction relation matrix to topological relations. While their findings were accurate, they still could not distinguish *disjoint* from *meet* in any circumstance. With topological augmentation, this inability to distinguish topological relations is restricted only to cases where the boundary of a figure object intersects either the boundary of the MBR tile or the MBR tile itself. This result is demonstrated in Figure 17. Similar gains can be realized, distinguishing relations differing only by a boundary-boundary intersection (e.g., *contains* and *covers*, *inside* and *coveredBy*).

Skiadopoulos and Koubarakis [56] studied the composition of the direction relation matrix, but were only able to derive the weak composition [52]. Using topological aug-

Class	Symbols Augmentations	Topological Augmentations per Symbol	Total
1	4	5	20
2	4	13	52
3	1	48	48
4	8	10	80
5	4	53	212
6	4	8	32
7	4	13	52
8	8	27	216
9	2	65	130
10	4	41	164
11	4	14	56
12	4	21	84
13	8	33	264
14	8	21	168
15	4	33	132
16	8	10	80
17	8	11	88
18	4	11	44
19	4	17	68
20	8	17	136
21	4	17	68
22	8	17	136
23	4	17	68
24	4	11	44
25	1	17	17
26	4	8	32
27	8	8	64
28	4	9	36
29	8	9	72
30	8	9	72
31	4	9	36
32	8	9	72
33	4	9	36
34	4	6	24
35	4	4	16
36	4	4	16
37	4	5	20
38	8	5	40
39	2	5	10
40	2	5	10
41	8	5	40
42	4	3	12
43	4	3	12
44	1	2	2
45	1	3	3
	218		3,084

Table 2: Topological augmentation applied to the direction relation matrix (exemplar shown in Table 1).

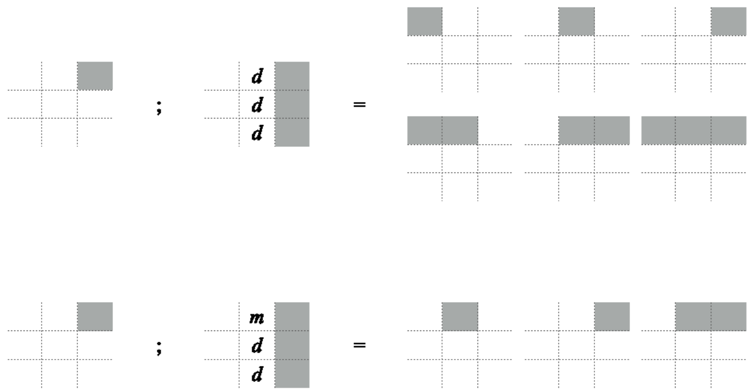

Figure 18: The strong composition of the direction relation matrix in the presence of topological augmentation. The distinction between *disjoint* and *meet* allows for distinguishing opportunities based on the maximum rectangle algebra relation that could be present between the objects [4, 13, 48].

mentation within the direction relation matrix setting allows for the calculation of a strong composition by utilizing information from the rectangle algebra. Revisiting the example from the introduction, by considering the topological augmentation of a given direction relation matrix, the size of the possible composition results are reduced. While Figure 18 shows a specific example, the partition carrying the *meet* relation in the topological augmentation could change (between North, MBR, South, or any combination thereof), but the same reduction would be achieved [13]. These gains are accomplished by restricting the extent rectangle of the figure object [4, 13, 48]

Wang et al. [61] studied the determination of the converse relation to a direction relation matrix. Their work considered rectangular sets of tiles (given the need to reduce to an MBR) as the vehicle for the computation of the converse relation. By employing topological augmentation, the computation of the converse can be related to the rectangle algebra by using the presence of the relation *meet*, thus reducing the cardinality of possible converse relations, and thus reducing uncertainty. By selecting only a single rectangle algebra relation, significant gains can be realized in the minimization of the converse set. While the maximum cardinality for a converse relation by the vanilla direction relation matrix is 198, the maximum for the topologically augmented direction relation matrix is 111. Wherever topological augmentation does not produce a unique converse direction relation matrix, it reduces the uncertainty present [13].

The fourth motivating example was the lack of support for all possible lines within the direction relation matrix framework [29]. By using line-region relations [24] as opposed to region-region relations [23], all lines can be represented within the direction relation matrix under topological augmentation (Figure 19). Similarly, other types of relation domains can be applied under this philosophy. With respect to this line-region context, such an approach can be used to better describe the relationship between a line and direction partition tiles. This approach could be used to sequentially address the interplay of a line with each topological tile, providing valuable information to scene reconstructive processes.

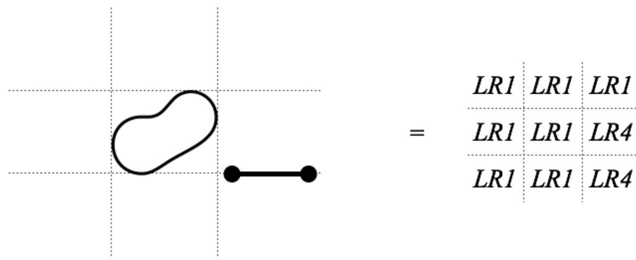

Figure 19: An example of topological augmentation for line-region relations, exemplifying the presence of a linear feature along the boundary of the E and SE tiles of the direction relation matrix.

7 Conclusions and future work

In this paper, a new formalism for partition-based qualitative spatial reasoning systems called *topological augmentation* was presented. This formalism augments the Boolean set intersection common to the direction relation matrix [30] and the peripheral direction relations [49] with a topological relation, providing additional granularity that has previously been unavailable to users of these spatial representations (180 for the peripheral direction relations, and 3,084 for the direction relation matrix). While similar to the work of Kor and Bennett [35], this work extends beyond the region connection calculus [51], allowing for entrance into such domains as line-region relations [24] and complex region-region relations [42, 53].

This work deviates substantially from the philosophical appeal of Occam's Razor insofar as it advocates for an inflation of an already sizable set of base relations that are not necessarily needed in a fully refined sense by humans. The examples provided in Section 6, however, demonstrate that reasoning systems can create meaningful refinements to current tasks in spatial artificial intelligence. This approach is presented not as a means by which to present results to users, but rather as an approach to help directional information systems founded upon these types of formalisms reduce the amount of uncertainty presented to their users.

There are many avenues for future work within this formalism. Currently, the direction relation matrix is the state of the art for areal-based direction relations. Defining and examining the potential for the relation algebra upon these relations has been a task wrought with difficulties with both the converse operation [61] and the composition operation [56]. Given the properties of the rectangle algebra [4, 48], it is hypothesized that topological augmentation can provide meaningful insights into both of these arenas, making the quest for the ascertainment of a relation algebra, or a systematic characterization of its missing components, a more realizable task. More specifically, one task that has been particularly difficult is the determination of an involution [60] over these two operations.

While the examples provided in this paper are in relation to direction tasks, topological augmentation can also be employed in topological constructions as well. Previous work regarding *surrounds* relations uses relations between partition tiles as a means for isolating surrounding configurations in maps [16]. By applying topological augmentation, this approach can be expanded for more complex configurations. Specifically, this approach allows for the determination of similar relations in cases where the whole partition tile need not be exhausted by some object of importance.

Topological augmentation is also a move toward a scene level view of space, rather than a binary view of space. Approaches such as the o-notation [39], i-notation [40], Swiss canton region [17], *surrounds* [16], MapTree [62], and the combination of this work and that of Kor and Bennett [35] highlight that scene level information such as with partitions (a restricted view of scene level representations) and the more diverse arbitrary scene provide benefits that binary relations do not. Many decisions are made based on more than just a binary relation. These approaches provide the foundation for the future of this field.

Acknowledgments

The author wishes to acknowledge sources of funding for this project, including NSF grants NMA 201-01-1-2003 (PI: Max Egenhofer), DGE 0504494 (PI: Kate Beard), and the Maine Economic Investment Fund Michael J. Eckardt Dissertation Fellowship. The author also gratefully acknowledges contributions from dissertation advisory committee Max Egenhofer, Kate Beard, Torsten Hahmann, Andre Khalil, and Reinhard Moratz. The author also wishes to acknowledge the valuable contributions gleaned from the review process.

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