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RESEARCH ARTICLE

Spatial refinement as collection order relations

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Abstract: An abstract examination of refinement (and conversely, coarsening) with respect to the involved spatial relations gives rise to formulated order relations between spatial coverings, which are defined as complete-coverage representations composed of regional granules. Coverings, which generalize partitions by allowing granules to overlap, enhance hierarchical geocomputations in several ways. Refinement between spatial coverings has underlying patterns with respect to inclusion—formalized as binary topological relations—between their granules. The patterns are captured by collection relations of inclusion, which are obtained by constraining relevant topological relations with cardinality properties such as uniqueness and totality. Conjoining relevant collection relations of equality and proper inclusion with the overlappedness (non-overlapped or overlapped) of the refining and the refined covering yields collection order relations, which serve as specific types of refinement between spatial coverings. The examination results in 75 collection order relations including seven types of equality and 34 pairs of strict or non-strict types of refinement and coarsening, out of which 19 pairs form partial collection orders.

Keywords: refinement, partition, covering, spatial granule, class relation, collection relation, order relation

1 Introduction

Refinement (and conversely, *coarsening*) [16] plays an important role in analyzing and reasoning with spatial hierarchies. The hierarchical nature of spatial knowledge [17] has been studied via multiple approaches and modeled with various spatial data structures. Nevertheless, an abstract examination of refinement between complete-coverage representations, such as partitions and coverings, reveals the underlying patterns of the involved spatial relations, which leads to formal interpretations of spatial refinement with a series of order relations, and thereby potentially offers new insights into hierarchical geocomputations.

1.1 Background

Structured in many different ways, spatial hierarchies underlie and facilitate *hierarchical spatial reasoning* [5, 39], analyzing, and modeling, for example, in handling complexities via multi-level abstractions [38], switching between different *resolutions* [42], and representing space as nested *containers* [22]. A current computational abstraction of space and its regions is *granules* [18, 32, 47], which will be used hereinafter in this paper. A spatial hierarchy is either *functional*—based on task decomposition, or *structural*—based on space subdivision [5, 21].

A *spatial inclusion hierarchy* [39], a typical form of structural hierarchies, is a set of regional spatial granules partially ordered on *inclusion* [14, 34]. With hierarchies building on *levels* [13, 31, 35, 38, 44, 46], granules ordered by inclusion are intuitively situated at different levels, where an *including* granule carries higher coarseness [39] at a higher level, while an *included* one lies at a lower level exposing more detail. Though a level may form with granules that make up only a partial coverage of the hierarchized space, which may be as large as the surface of the Earth, or as small as a city block, levels here refer to those having complete coverages only.

A spatial inclusion hierarchy consisting purely of complete-coverage levels is commonly induced by one or more series of partitions. The partitions, each populating a distinct level with its granules, have a partial order on refinement such that each granule in the refining partition is *included* or *contained* by some granule in the refined one [13, 14, 16]. A refinement is *proper* if it divides at least one granule of the refined partition, or *non-proper* between equal partitions [14, 16].

The above conceptualizations of the refinement–inclusion association suggest formal interpretations of spatial refinement with specific types, which may be revealed by two ordered partitions' granule participation and granule correspondence in relevant spatial relations. Granule participation refers to whether one partition has each of its granules dividing or divided by some granule in the other partition, and vice versa. Granule correspondence refers to patterns captured by the relations' *cardinality properties* such as uniqueness and totality, which are formally defined for *class relations* [27, 28], yet may set-theoretically be reformulated for partitions as sets—instead of classes—of granules.

1.2 Overlapped representations

While partitions serve as a prevalent type of representations for spatial levels, typically as planar map layers in current GIS, their advantage is also their limitation: the granules of a partition do not overlap. A spatial level, however, is more general than a map layer and thus may arise in more-complex manners than from a partition. The following are some cases where the granules populating a level form a *covering* [24, 43], a generalization of a partition that allows granules to overlap.

Case 1 The modifiable areal unit problem (MAUP) [30] occurs when a coarsening operation scales a lower-level, fine-grained representation up to a higher-level, coarse-grained one regarding the complete-coverage distribution of a phenomenon (e.g., poverty, unemployment, job opportunities, real estate price, health services accessibility, public opinion on same-sex marriage, and support for a political candidate). In dealing with the MAUP, multiple datasets are incorporated into a single hierarchy. Then, for each region (i.e., granule) in the fine-grained representation, one out of a set of candidate regions—each from a different dataset—is returned at the higher level, dynamically generating the coarse-grained representation as an optimal zoning for the variable(s) (e.g., unemployment rate) being analyzed. Both the representation and the collection of all the candidate regions at the higher level form coverings by possibly having overlapping granules from different datasets. Similar coarsening operations may be used in hierarchical spatial reasoning and applications such as improving location privacy by returning the least revealing candidate region at a higher level.

Case 2 On the occurrence of an event such as an industrial pollution or a natural disaster, spatial analyses and reasoning are performed at the same time for multiple government agencies that use respectively different datasets (e.g., administrative areas versus physiographic/topographic/geologic regions). Hierarchical computations within each dataset (vertical) as well as cross analyses between them (horizontal) are made within an integrated spatial hierarchy, where geographic categories with commensurate coarseness are modeled at the same level, resulting in coverings as in Case 1.

Case 3 Multiple complete-coverage levels are combined into a single level, or a level is itself a nested series of *sub-levels* forming a sub-hierarchy [46], which may further be finitely recursive. Such a *nested level*, however complex its internal structure is, can be represented by a single covering. Nested levels are useful in simplifying the level structure of a complex inclusion hierarchy, which is typically a result of merging multiple hierarchies.

Case 4 Most real-world geographic features such as urban and rural regions adjoin through gradual transition zones, which are either arbitrarily divided by crisp boundaries in current GIS or treated as broad boundaries of fuzzy or vague regions. It is, however, realistic and intuitively reasonable to model these features with their full extents such that they overlap in the otherwise transition zones (e.g., forest–grassland ecotones). Such overlapped representations by coverings reflect *near-decomposability* [35] between weakly interacting granules within any hierarchical level in the real world, and as well reveal—through intersections of overlapping features—the extents of the transition zones, where ecological *edge effects* occur. Besides, some transition zones are big enough and have their unique characteristics distinguishing them from the regular categorical features. Then, a more comprehensive view of the variations within a landscape is usefully obtained by adding such transition features to a partition (or covering) in relevant categorical features, which results in an overlapped representation.

The above cases present evidence that overlapping granules not only are inherent in applications dealing with integrated hierarchies based on multiple spatial datasets, but also bring about a sensible approach to cognizing and modeling real-world geographic features. As such, an examination of spatial refinement between coverings will enhance hierarchical geocomputations by probing the complexities of the overlapped scenario.

1.3 Approach

This paper develops a formal model of spatial refinement between coverings in the following four steps.

- Introduction of *collection relations* based on *set cardinality properties*, which are formulated by extending the cardinality properties of class relations [27, 28] to relations between sets (i.e., collections).
- Conceptualization of 2D point-set *spatial granule*, formulation of a *spatial covering* of a decomposable space as a complete-coverage collection of spatial granules, and categorization of coverings into partitions (non-overlapped) and *o*-coverings (overlapped) through *overlappedness*, as interpreted by coarse topological relations [9].
- Formulation of spatial refinement as *collection order relations*. Applying certain combinations of set cardinality properties to relevant coarse topological relations [9] results in such collection relations of equality and proper inclusion that underlie refinement between spatial coverings. Then conjoining the underlying collection relation(s) with optionally the overlappedness of the refining and the refined covering yields each specific type of refinement as a collection order relation. Coarsening is derived as the converse relations of refinement.
- A further examination that reveals which of these collection order relations form *partial collection orders* between spatial coverings.

The remainder of the paper is organized as follows. Section 2 reviews related work on granulation, topological relations, and class relations. Section 3 presents collection relations as an extension to class relations. Section 4 formulates refinement and coarsening as collection order relations. Section 5 examines whether the collection order relations form partial collection orders. Section 6 draws conclusions and future work.

2 Related work

Levels in spatial hierarchies commonly arise from *multiple representations* [1, 3] such as land *subdivisions* [20]. A computational abstraction regarding such notions as representation and subdivision is *granulation*, lying in the center of *granular computing* [25, 48], in which a hierarchy consists of *families of granules* [45].

2.1 Covering versus partition

In current computing theories, a *granule* is a clump or cluster of objects drawn together by indistinguishability, similarity, proximity, functionality, or indiscernibility [18, 32, 47], or more generally, a binary neighborhood [23]. Richly modeled with rough set [32] as well as fuzzy set [47] approaches, a *granulation* is a collection of granules that result from decomposing a finite universe U or clumping the objects in it, which in the crisp mode has two basic mathematical forms—partition and covering [24, 43].

A *partition*, π, is a collection of non-empty subsets of U such that (1) the non-empty subsets are *pairwise disjoint* granules, therefore also called *blocks*, and (2) π is *complete*, that is, the union of its blocks is U [12, 24, 43].

Addressing also the *non-partition cases* [25] by allowing granules to overlap, a *covering*, τ, is a collection of non-empty subsets of U with their union being U, therefore partitions are a special type of coverings [24, 43].

Granule in the spatial context has different conceptualizations. Some examples are: a *cell* of a space used possibly with the neighboring cells to describe the location of a region [36]; a regular closed subset of a spatial domain, which may have holes and appear in disconnected pieces [2]; and a subset of a spatial domain regarding a granularity such as *kilometers* or *provinces* [4].

Geospatial granulations are consistently modeled as partitions mainly because of their simple non-redundant way of representing complete coverages rather than out of a cognitive consideration. A *spatial partition*, or a *planar subdivision*, typically divides a space or a map's plane into *areas* or *regions* [12, 13, 14], which may form a *categorical coverage* [1] based on attribute values, or be modeled as spatial objects linked to an identifying property [14].

Considering the regularity of the shapes of geographic regions and the common boundaries shared by bordering regions, spatial partitions are different from *set partitions* that are composed of pairwise disjoint point sets [12, 29]. A database-oriented approach formalized spatial partitions as a spatial data type by mapping each point in $\mathbb{R}^2$ to one or more label values. A partition thus obtained is based on *interior regions* and *borders* as regular open point sets, each forming a block with the points in it being labeled by an identical set of values [12]. A graph-theoretic model further presents *spatial partition graphs* as discrete, implementable representations of map geometry [29].

Spatial coverings, in spite of a lack of conceptualizations, may arise from complete-coverage collections of *delimited spatial entities* [15], which are allowed to overlap. In addition, *multi-valued vector maps* [7], where points, lines, and regions of multiple themes may overlap, provide implementation options that facilitate topological queries across map layers (e.g., What provinces does the Loess Plateau extend over?).

2.2 Refinement, inclusion, and class relations

Given partitions π_1 and π_2 of a finite universe U, π_2 is a *refinement* of π_1 if each block of π_2 is a part of some block of π_1, and conversely π_1 is a *coarsening* of π_2; a *proper refinement* of π_1 has at least one block of π_1 split, with π_1 being a *non-proper refinement* of itself [14, 16, 43]. Refinement, which applies to coverings in a similar way [43], induces orders on coverings as well as partitions [14, 43], though the possibility for granules to overlap leads to higher complexities.

Refinement and coarsening between spatial coverings are embodied by *inclusion* or *containment* [14, 34, 41] between each other's granules, which is formalized as binary topological relations by either the 9-intersection model [11] or the RCC theory [33]. Both models yield *eight base* region–region relations in $\mathbb{R}^2$, among which three pairs are distinguished by whether the regions' boundaries intersect (9-intersection model) or whether a subset relation is tangential (RCC), resulting in *five coarse* relations [9], referred to as DE5 here, and RCC-5 [6], respectively. The coarse topological relations further reduce to more-general *spatial order relations* [19] accounting for region–region comparability regarding inclusion (Table 1), with $\{\mathrm{OV}, \mathrm{OUT}\}$ in DE5 and {PO, DR} in RCC-5 both generalized to *incomparable*.

A sensible way to interpret refinement between spatial coverings is by describing their granule participation and correspondence in the embodying, inclusion-conveying topolog-

RCC-5	DE5	Spatial order relations
EQ (EQual)	EQ $=$ *equal*	$=$
PP (Proper Part)	IN $=$ *coveredBy* or *inside*	$\leq$
PPI (Proper Part Inverse)	IN^{-1} $=$ *covers* or *contains*	$\geq$
PO (Partially Overlapping)	OV $=$ *overlap*	incomparable
DR (Distinct Regions)	OUT $=$ *disjoint* or *meet*	incomparable

Table 1: The corresponding RCC-5 [6] and DE5 [9] coarse topological relations, and spatial order relations [19].

ical relations (e.g., EQ and IN in DE5). A current approach to a similar issue is *class relations* [8, 26], where *individuals* are treated as *instances* of *classes* through *instantiation* such that $Inst(a, A)$ means *individual a is an instance of class A*. A class relation R from class A to class B is denoted by $R_{CP}(A, B)$, where CP specifies the *cardinality properties* [27, 28] involving *uniqueness* and *totality* with respect to the corresponding *instance relation* $r(a, b)$ from a to b, taking all a and b for $Inst(a, A)$ and $Inst(b, B)$ into account.

Donnelly and Bittner [8] first presented a model of five class relations by defining their CPs: *some*, *all-1*, *all-2*, *all-12*, and *all-all*. As an extension to it, the Mäs [26, 27, 28] framework is based on four *basic cardinality properties*, LT, RT, LD, and RD, corresponding to *left-total*, *right-total* (*surjective*), *left-definite* (*left-unique* or *injective*), and *right-definite* (*right-unique* or *functional*) in binary relations, respectively. The framework presents a set of 17 *abstract class relations* [26], which are *jointly exhaustive and pairwise disjoint* (JEPD). Among the 17 JEPD relations, 15 are yielded by conjoining at least one of the four basic cardinality properties with a negation of the others [27]. Two examples are the definitions of $R_{RT}(A, B)$ and $R_{RD\,LT}(A, B)$ as follows.

$$R_{RT}(A, B) := RT(A, B, r) \wedge \neg LD(A, B, r) \wedge \neg RD(A, B, r) \wedge \neg LT(A, B, r)$$
$$R_{RD\,LT}(A, B) := RD(A, B, r) \wedge LT(A, B, r) \wedge \neg LD(A, B, r) \wedge \neg RT(A, B, r)$$

Being two special cases, $R_{LT\,RT-all}(A, B)$ means every instance of A has an instance relation r to every instance of B; $R_{some}(A, B)$ means some instance of A has relation r to some instance of B excluding LT, RT, LD, and RD, such that it is JEPD with the other 16 relations and thereby more restrictive than the Donnelly and Bittner [8] definition.

The Tarquini and Clementini [37] model is another extension to the Donnelly and Bittner [8] approach, which, by treating classes as sets, links individuals to classes through set membership instead of instantiation. By forcing spatial relations to hold between distinct individuals only, the model does not apply to reflexive topological relations such as EQ and *equal*.

3 Collection relations

Uniqueness and totality are cardinality properties of binary relations, therefore binary topological relations between coverings, like any binary relations between classes, have corresponding patterns conveyed by them. However, applying the cardinality properties formally defined for class relations would cause spatial coverings, which are sets, to be treated as classes.

Classes in class relations are types or kinds (e.g., geographic categories) abstracted from reality and particularized by their instances, therefore a class (e.g., *country*) remains to be what it is no matter how many instances it has, where the instances are located, or whether the number of its instances changes [8]. Sets, in contrast, are identified by their members. A covering as a set of granules turns into a different set if any change to the memberships of its granules happens. Specifically, treating coverings as classes has the following problems, which do not exist, or are conveniently dealt with, if coverings are treated as sets.

First, although the instances of some geographic categories form complete coverages, for example, when restricted within a country, complete coverage is not a property of classes and thus is not guaranteed as by spatial coverings. Some classes are in certain hierarchical orders with respect to inclusion between their instances, but such orders do not necessarily coincide with refinement. For example, *towns* and *counties* are ordered, with *towns* being contained by, and therefore subordinated to, *counties*. However, the proposition "*counties* are refined by *towns*" is generally false, as a county is typically divided into not just *towns*, but also *cities* and possibly other categories of town-level entities. Even *counties* may form only a partial coverage of a country, where there might be *boroughs*, *districts*, or other categories of county-level entities as well. Since many simple classes such as *county*, *borough*, *city*, and *town* have complex relations among them, they are not directly applicable for interpreting refinement.

While simple classes may be combined into complex, abstract classes to form complete coverages that are ordered on refinement, such relations are in essence between the sets of the involved instances rather than between the classes. For example, the *states* are divided into *counties* and their *statistical equivalents* in the United States, where the *statistical equivalents* include at least *parishes*, *boroughs*, *census areas*, and *independent cities* [40]. By taking *counties* and *statistical equivalents* as one complex class of entities, the proposition "*counties and statistical equivalents* are a refinement of *states*" is true if the classes are confined within the United States such that their involved instances form exact complete coverages of the country. It implies, however, that the refinement holds between *the set of counties and statistical equivalents* and *the set of states* of the United States rather than between the classes, which also have instances in other countries. In fact, a refinement does not hold from *county and statistical equivalent* to *state*, as their instances form different coverages of the world.

Alternatively, refinement may be enforced to hold between such specially defined classes, which are possibly complex and abstract, and have the same geographic restriction, such that each class has all of its instances forming an exact complete coverage of the same space. Examples of such classes are in the ordered series of {*US state, US county and statistical equivalent, US county and statistical equivalent subdivision*}, and {*New England state, New England county, New England county subdivision*}. A series of new classes thus have to be defined whenever a different hierarchized space (e.g., the coastal interstate region *Southern Maine* – *Southeastern New Hampshire* – *Northeastern Massachusetts*) is selected for analysis. These geographically determined classes correspond to, and almost serve as names of, the relevant sets. At this point, interpreting refinement as relations between sets is much more flexible and convenient.

Moreover, a complete coverage by granules that belong to a single class is usually a partition (e.g., a partition of a country in *states* or *provinces*), which does not help analyzing and reasoning with overlapped representations. For the classes that may have overlapping instances due to factors such as different criteria (e.g., *cultural region*), whether a coverage is complete or overlapped is often unpredictable and variable as the instances are subject to change. This makes it difficult to distinguish between different types of coverages, which is crucial in examining refinement relations.

Finally, there are numerous ways of forming a covering with granules from different classes. For example, a covering that combines features of economy and culture may be generated by merging *economic regions* and *cultural regions* of the same state/province or country, or by augmenting a partition by either of the two classes with at least one instance of the other. As such, spatial coverings greatly outnumber classes. Since it would not be sensible to define a distinct class for each possible covering, treating coverings as sets is the necessary option for examining spatial refinement.

Therefore, instead of using LT, RT, LD, and RD [27, 28], here referred to as *class cardinality properties*, this paper formulates four corresponding *set cardinality properties*—Lt and Rt for totality, and Ld and Rd for uniqueness—by replacing instantiation with set membership. In the following definitions regarding sets X and Y, a relation R from individual x to individual y is denoted by $R(x, y)$.

$$\begin{aligned}
Lt(X, Y, R) &:= \forall x \in X\ \exists y \in Y : R(x, y) \\
Rt(X, Y, R) &:= \forall y \in Y\ \exists x \in X : R(x, y) \\
Ld(X, Y, R) &:= (\forall x, z \in X\ \forall y \in Y : R(x, y) \wedge R(z, y) \rightarrow x = z) \wedge Exist(X, Y, R) \\
Rd(X, Y, R) &:= (\forall x \in X\ \forall y, z \in Y : R(x, y) \wedge R(x, z) \rightarrow y = z) \wedge Exist(X, Y, R) \\
Exist(X, Y, R) &:= \exists x \in X\ \exists y \in Y : R(x, y)
\end{aligned}$$

Applying set cardinality properties to individual relations gives rise to similar relations between sets. Such set-cardinality-constrained relations are referred to as *collection relations*, which are thereby distinguished from the relations such as set inclusion. Conjunctions and negations of Lt, Rt, Ld, and Rd result in 15 pairwise disjoint collection relations in the same way the corresponding class relations [26] are yielded. For example, $R_{Rd\ Lt\ Rt}(X, Y)$ is defined as the conjunction of Rd, Lt, and Rt with the negation $\neg Ld$ as follows.

$$R_{Rd\ Lt\ Rt}(X, Y) := Rd(X, Y, R) \wedge Lt(X, Y, R) \wedge Rt(X, Y, R) \wedge \neg Ld(X, Y, R)$$

Also, the two corresponding special cases are defined as follows.

$$R_{Lt\ Rt-all}(X, Y) := \forall x \in X\ \forall y \in Y : R(x, y)$$

$$\begin{aligned}
R_{Some}(X, Y) := Exist(X, Y, R) \wedge \neg Ld(X, Y, R) \wedge \neg Rd(X, Y, R) \\
\wedge \neg Lt(X, Y, R) \wedge \neg Rt(X, Y, R)
\end{aligned}$$

As an extension to the Mäs [26, 27] framework, the Lt–Rt–Ld–Rd set cardinality constraint system yields a corresponding JEPD set of 17 *abstract collection relations*, which are applicable to any topological relation between two sets such as coverings.

4 Collection order relations

As shown in Table 1, two regions are comparable if they are related by equality (e.g., EQ) or a converse pair (e.g., IN and IN^{-1}) of coarse, inclusion-conveying topological relations [6, 9]. Such two regions thus have *inclusion comparability*, with the coarse relations EQ, IN, and IN^{-1} turning into *individual order relations*. Similarly, two coverings have *refinement comparability* if they are equal or one is a refinement (or coarsening) of the other, giving rise to *collection order relations*. Due to the refinement–inclusion association (Section 1.1), the essence of refinement, then, is that inclusion comparability (between regions) collectively turns into refinement comparability (between coverings). As such, relevant collection relations of the coarse, inclusion-comparable topological relations (e.g., EQ, IN, and IN^{-1}) appropriately serve as building blocks for constructing refinement-comparable collection order relations.

4.1 Spatial covering

The eight base and five coarse topological relations are restricted to *disc-like regions*, or *regions without holes*, each formalized as a point set in $\mathbb{R}^2$ with a connected non-empty interior, a connected exterior, and a connected boundary [9, 10, 11]. Correspondingly, a 2D point-set *spatial granule*, or simply a *granule*, is here defined as a disc-like region in $\mathbb{R}^2$. Topological relations between granules are hereinafter interpreted by the DE5 coarse relations [9].

Geospatial granulations provide representations of the real world, where a decomposable space such as a country often comprises multiple disconnected regions. Thus for the remainder of this paper, a decomposable space S is a non-empty point set as the union of a finite number of disc-like regions that are mutually disconnected. A *spatial covering*, or simply a *covering*, of S is a set of spatial granules with their union being S. In the following equation that defines V as a spatial covering of S, denoted by $\mathrm{Covering}(V, S)$, $\mathrm{S\text{-}granule}(v)$ restricts that v is a spatial granule, and no two different granules of V are equal (EQ).

$$\begin{aligned}\mathrm{Covering}(V, S) := \forall v \in V : \mathrm{S\text{-}granule}(v) \wedge \bigcup V = S \\ \wedge\, \forall v_1, v_2 \in V : \mathrm{EQ}(v_1, v_2) \rightarrow v_1 = v_2\end{aligned}$$

A covering is characterized by whether its granules overlap, or its *overlappedness*. Yet "two granules overlap" may be interpreted in three different ways with decreasing extensions: a) by having common points, thus bordering granules overlap by sharing boundary points, from the set partition point of view, b) by having common interiors [33], or c) by having common interiors excluding the cases of equality and proper inclusion [11]. For the overlappedness of spatial coverings, b) is the appropriate interpretation and is used in this paper. Then, in terms of the DE5 coarse relations [9], a covering, V, is either *non-overlapped* if its granules have pairwise OUT relations, or *overlapped* if at least two of its granules are related by IN, IN^{-1}, or OV, defined as follows.

$$\begin{aligned}\mathrm{Non\text{-}overlapped}(V) &:= \forall v_1, v_2 \in V : v_1 \neq v_2 \rightarrow \mathrm{OUT}(v_1, v_2) \\ \mathrm{Overlapped}(V) &:= \exists v_1, v_2 \in V : \mathrm{IN}(v_1, v_2) \vee \mathrm{IN}^{-1}(v_1, v_2) \vee \mathrm{OV}(v_1, v_2)\end{aligned}$$

Then, rather than defining with a separate predicate, this paper implies the type of a covering (e.g., partition) with a predicate conjunction, where its overlappedness is explicitly specified. In consequence, a *spatial partition*, or simply a *partition*, X, of S is a *non-overlapped covering* of S, implied by $\mathrm{Covering}(X, S) \wedge \mathrm{Non\text{-}overlapped}(X)$. Since the granules in X share common boundaries with their adjoining ones, they are not referred to as blocks.

An *overlapped covering*, or an *o-covering*, O, of S is implied by $\mathrm{Covering}(O, S) \wedge \mathrm{Overlapped}(O)$. An *o*-covering is either *non-redundant* [43] if every granule has its own unique subset and therefore a complete coverage of S can only be obtained through the union of all its granules, or *redundant* such that the arbitrary union of a non-empty proper subset of it equals S. Such redundancy concepts apply to coverings in general, and therefore partitions are non-redundant. However, the work presented in this paper is based on not distinguishing these cases.

For better indication and readability, this paper hereafter uses V, X, and O, typically with a subscript, to denote the same covering in general, implied to be a partition, or an *o*-covering, respectively. For example, covering V_i is replaced where applicable by either partition X_i or *o*-covering O_i if its overlappedness is known or specified.

Let V_i, V_j, X_i, X_j, O_i, and O_j denote two coverings of S in general and with their overlappedness being explicit as non-overlapped or overlapped, respectively. A collection relation from V_i to V_j with respect to a DE5 coarse relation has the form $\mathrm{DE5}_{Cp}(V_\mathrm{i}, V_\mathrm{j})$, where Cp specifies the Lt–Rt–Ld–Rd set cardinality properties.

4.2 Refinement

Combining the definitions of refinement between partitions [14, 16] and between coverings [43], refinement intuitively falls into three broad, mutually exclusive types, given V_i as a refinement of V_j:

- Type A: a covering is an improper (in lieu of "non-proper") refinement of itself;
- Type B: V_i is a proper refinement of V_j such that each granule of V_i is a proper subset of some granule of V_j, and for each granule of V_j some granule of V_i is a proper subset of it; and
- Type C: V_i is a proper refinement of V_j such that a proper subset of V_i forms a Type B refinement of a non-empty proper subset of V_j, while the remaining granules of V_i equal the remaining granules of V_j in a bijective way.

The following subsections explore these broad types and their combinations.

4.2.1 Type A: Improper refinement

Type A refinement implies equality between two coverings in terms of both geometry and position, which holds if and only if they have exactly the same granules. The underlying collection relation based on the DE5 coarse relations is $\mathrm{EQ}_{Ld\ Rd\ Lt\ Rt}$, a bijection of relation EQ, giving rise to a *general improper refinement* from V_i to V_j, denoted by $V_\mathrm{i} =_S V_\mathrm{j}$ (equation 1).

$$V_\mathrm{i} =_S V_\mathrm{j} := \mathrm{Covering}(V_\mathrm{i}, S) \wedge \mathrm{EQ}_{Ld\ Rd\ Lt\ Rt}(V_\mathrm{i}, V_\mathrm{j}) \wedge \mathrm{Covering}(V_\mathrm{j}, S) \quad (1)$$

Such equality implies that the two coverings have the same overlappedness, therefore V_i and V_j are either partitions X_i and X_j or *o*-coverings O_i and O_j, captured by two cases of $\mathrm{EQ}_{Ld\ Rd\ Lt\ Rt}$, respectively. Figure 1 illustrates the two cases with geometric abstractions of their real-world counterparts. The decomposable space S is simplified to one disc-like region, represented by the two unlabeled bounding boxes on the left of each case. The granules composing the related partitions (Figure 1a) and *o*-coverings (Figure 1b) are drawn as the labeled boxes, which are translucent such that in O_i (similar in O_j) the overlaps are revealed by the gray boundary segments as if granule i_0 partially covers i_1, i_2, and i_3. On the right of each case is an illustration of the cardinality restriction in $\mathrm{EQ}_{Ld\ Rd\ Lt\ Rt}$ by a mapping of EQ between the related partitions or *o*-coverings.

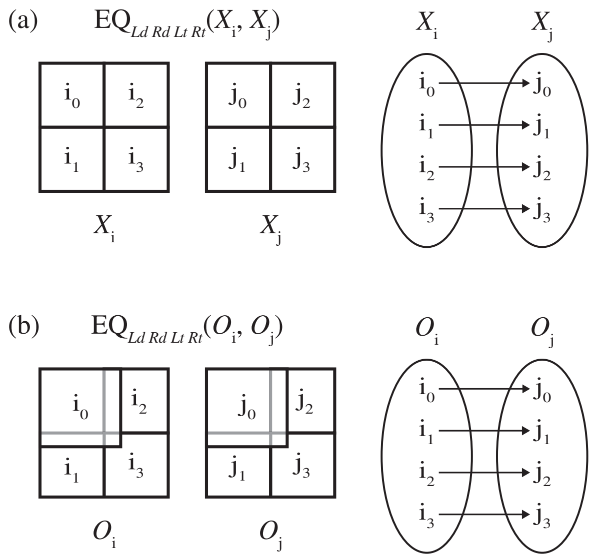

Figure 1: The two cases of $\mathrm{EQ}_{Ld\ Rd\ Lt\ Rt}$: (a) from a partition ($X_\mathrm{i}$) to a partition ($X_\mathrm{j}$); (b) from an *o*-covering (O_i) to an *o*-covering (O_j).

More specifically, the overlappedness of the *refining* covering (the refinement) is referred to as *left overlappedness*, denoted by Lo, and the overlappedness of the *refined* covering is referred to as *right overlappedness*, denoted by Ro, corresponding to the positions of V_i and V_j (equation 1), respectively. With each of Lo and Ro being either specified with a predicate or unspecified, and thus each having three possible settings, combinations of these settings distinguish refinement into specific types.

Then, any collection order relation of improper refinement from V_i to V_j is obtained as $V_\mathrm{i} =_S^{\text{[L-][-R]}} V_\mathrm{j}$ from a *tripartite formula* that has the central component being $V_\mathrm{i} =_S V_\mathrm{j}$ (equation 2). The value of L or R is "N" for *non-overlapped* or "O" for *overlapped* indicating the particular specified Lo or Ro respectively, with L-, -R, $Lo\wedge$, and $\wedge Ro$ enclosed in [ ] being optional.

$$V_\mathrm{i} =_S^{\text{[L-][-R]}} V_\mathrm{j} := [Lo\wedge]\ V_\mathrm{i} =_S V_\mathrm{j}\ [\wedge Ro] \tag{2}$$

Out of the nine (3^2) combined settings of Lo and Ro, seven are valid, each resulting in a relation with its Lo and Ro being *unopposite*, that is, either both specified as non-overlapped or overlapped, or at least one unspecified, such that the equality is possible. Table 2 lists these seven relations, with COR indicating collection order relation.

The superscript [L-][-R] for describing the specific type of a relation indicates whether overlappedness is specified on neither side (e1), on either the right side (e2, e3) or the left side (e4, e5), or on both sides (e6, e7). The descriptive terminology for specified overlappedness is *non-overlapped* or *overlapped* for Lo serving as an adjective modifying *improper refinement*, which correspond to *partition* or *o-covering* for Ro indicating the type of the refined, respectively. Two examples, e4 ($=_S^{\text{N-}}$) from a partition (X_i) to a covering (V_j), and e7 ($=_S^{\text{O--O}}$) from an *o*-covering (O_i) to an *o*-covering (O_j), are yielded as follows.

COR	Specific type	Denotation	Terminology
e1	general improper	$=_S$	*general improper refinement*
e2	-N improper	$=_S^{\text{-N}}$	*improper refinement of a partition*
e3	-O improper	$=_S^{\text{-O}}$	*improper refinement of an o-covering*
e4	N- improper	$=_S^{\text{N-}}$	*non-overlapped improper refinement*
e5	O- improper	$=_S^{\text{O-}}$	*overlapped improper refinement*
e6	N--N improper	$=_S^{\text{N--N}}$	*non-overlapped improper refinement of a partition*
e7	O--O improper	$=_S^{\text{O--O}}$	*overlapped improper refinement of an o-covering*

Table 2: The collection order relations of improper refinement.

$$X_{\mathrm{i}} =_S^{\text{N-}} V_{\mathrm{j}} := \text{Non-overlapped}(X_{\mathrm{i}}) \wedge \text{Covering}(X_{\mathrm{i}}, S) \wedge \text{EQ}_{Ld\ Rd\ Lt\ Rt}(X_{\mathrm{i}}, V_{\mathrm{j}}) \wedge \text{Covering}(V_{\mathrm{j}}, S)$$

$$O_{\mathrm{i}} =_S^{\text{O--O}} O_{\mathrm{j}} := \text{Overlapped}(O_{\mathrm{i}}) \wedge \text{Covering}(O_{\mathrm{i}}, S) \wedge \text{EQ}_{Ld\ Rd\ Lt\ Rt}(O_{\mathrm{i}}, O_{\mathrm{j}}) \wedge \text{Covering}(O_{\mathrm{j}}, S) \wedge \text{Overlapped}(O_{\mathrm{j}})$$

An identity relation as is EQ, $\text{EQ}_{Ld\ Rd\ Lt\ Rt}$ is reflexive, symmetric, and transitive, with its converse collection relation having the same cardinality restriction, that is, $(\text{EQ}_{Ld\ Rd\ Lt\ Rt}(V_{\mathrm{i}}, V_{\mathrm{j}}))^{-1} = \text{EQ}_{Ld\ Rd\ Lt\ Rt}(V_{\mathrm{j}}, V_{\mathrm{i}})$. In consequence, e1–e7 serve as the types of both improper refinement and its converse, *improper coarsening*.

4.2.2 Type B: Total refinement

Type B refinement is *total* as it is both left-total (Lt) and right-total (Rt) of IN, the embodying proper subset relation in DE5, not involving the cases of $\text{IN}_{Lt\ Rt-all}$. A total (Type B) refinement of a partition, X_{j}, implies additionally right uniqueness (Rd) since the granules of X_{j} have pairwise OUT relations. The refinement is non-overlapped if it is also a partition, X_{i}. The collection relation from X_{i} to X_{j} is $\text{IN}_{Rd\ Lt\ Rt}(X_{\mathrm{i}}, X_{\mathrm{j}})$, which holds if each granule of X_{i} has an IN relation to exactly one granule of X_{j}, and for each granule of X_{j} some granule of X_{i} has an IN relation to it. The same cardinality restriction applies if the refinement is overlapped being an *o*-covering, O_{i}, resulting in $\text{IN}_{Rd\ Lt\ Rt}(O_{\mathrm{i}}, X_{\mathrm{j}})$. Figure 2 illustrates such two cases of $\text{IN}_{Rd\ Lt\ Rt}$, where the granules of O_{i} (Figure 2b) are equal to their correspondents in X_{i} (Figure 2a) except that i_0 in O_{i} extends downwards to overlap i_1, which is indicated by the gray top border of i_1 as if i_0 partially covers i_1.

Combining these two cases yields a *T-N refinement* (*total refinement of a partition*) from V_{i} to X_{j}, denoted by $V_{\mathrm{i}} <_S^{\text{T-N}} X_{\mathrm{j}}$, where "T" indicates total, and "N" indicates *non-overlapped* regarding X_{j} (equation 3).

$$V_{\mathrm{i}} <_S^{\text{T-N}} X_{\mathrm{j}} := \text{Covering}(V_{\mathrm{i}}, S) \wedge \text{IN}_{Rd\ Lt\ Rt}(V_{\mathrm{i}}, X_{\mathrm{j}}) \wedge \text{Covering}(X_{\mathrm{j}}, S) \wedge \text{Non-overlapped}(X_{\mathrm{j}}) \quad (3)$$

If a total refinement is of an *o*-covering, O_{j}, then its underlying collection relation is one of the pairwise disjoint collection relations of IN with both Lt and Rt (excluding $\text{IN}_{Lt\ Rt-all}$). There are four such relations, which are $\text{IN}_{Ld\ Rd\ Lt\ Rt}$ (Figure 3), $\text{IN}_{Ld\ Lt\ Rt}$

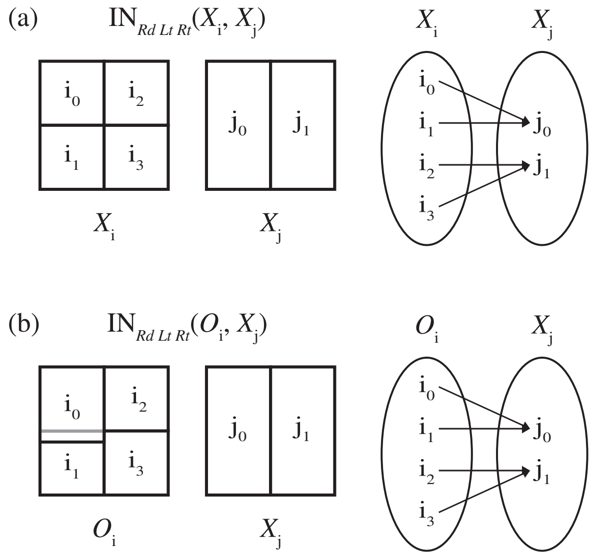

Figure 2: The two cases of $\mathrm{IN}_{Rd\ Lt\ Rt}$ to a partition (X_j): (a) from a partition (X_i); (b) from an *o*-covering (O_i).

(Figure 4), $\mathrm{IN}_{Rd\ Lt\ Rt}$ (Figure 5), and $\mathrm{IN}_{Lt\ Rt}$ (Figure 6), each forming a *candidate collection relation* to O_j with two cases, from a partition (X_i), or from an *o*-covering (O_i). In Figure 3a, the granules of O_j are the four labeled regions (j_0–j_3) having the same shape and size but different orientations, with j_3 being partially covered by j_1 and j_2, which are both partially covered by j_0. In Figure 3b, i_1 partially covers i_0 in O_i, while in O_j both j_1 and j_2 partially cover j_0, with j_1 being partially covered by j_2. The relevant granules in Figures 4–6 overlap in similar ways.

Then, combining the cases in Figures 3–6, a *T-O refinement* (*total refinement of an o-covering*) from V_i to O_j, denoted by $V_\mathrm{i} <_S^{\text{T-O}} O_\mathrm{j}$, where "O" indicates *overlapped* regarding O_j, is defined based on the disjunction of the four candidate collection relations (equation 4).

$$\begin{aligned} V_\mathrm{i} <_S^{\text{T-O}} O_\mathrm{j} := \mathrm{Covering}(V_\mathrm{i}, S) \wedge (\mathrm{IN}_{Ld\ Rd\ Lt\ Rt}(V_\mathrm{i}, O_\mathrm{j}) \\ \vee \mathrm{IN}_{Ld\ Lt\ Rt}(V_\mathrm{i}, O_\mathrm{j}) \vee \mathrm{IN}_{Rd\ Lt\ Rt}(V_\mathrm{i}, O_\mathrm{j}) \vee \mathrm{IN}_{Lt\ Rt}(V_\mathrm{i}, O_\mathrm{j})) \\ \wedge \mathrm{Covering}(O_\mathrm{j}, S) \wedge \mathrm{Overlapped}(O_\mathrm{j}) \end{aligned} \quad (4)$$

Not distinguishing the overlappedness of either covering, a *T refinement* (*general total refinement*) from V_i to V_j, denoted by $V_\mathrm{i} <_S^{\mathrm{T}} V_\mathrm{j}$, is obtained through the disjunction of T-N and T-O refinements (equation 5).

$$\begin{aligned} V_\mathrm{i} <_S^{\mathrm{T}} V_\mathrm{j} := \mathrm{Covering}(V_\mathrm{i}, S) \wedge (\mathrm{IN}_{Ld\ Rd\ Lt\ Rt}(V_\mathrm{i}, V_\mathrm{j}) \vee \mathrm{IN}_{Ld\ Lt\ Rt}(V_\mathrm{i}, V_\mathrm{j}) \\ \vee \mathrm{IN}_{Rd\ Lt\ Rt}(V_\mathrm{i}, V_\mathrm{j}) \vee \mathrm{IN}_{Lt\ Rt}(V_\mathrm{i}, V_\mathrm{j})) \wedge \mathrm{Covering}(V_\mathrm{j}, S) \end{aligned} \quad (5)$$

Then any collection order relation of total refinement is obtained as $V_\mathrm{i} <_S^{\text{[L-]T[-R]}} V_\mathrm{j}$ from the second tripartite formula (equation 6), where the central component is $V_\mathrm{i} <_S^{\mathrm{T}} V_\mathrm{j}$ (T refinement).

$$V_\mathrm{i} <_S^{\text{[L-]T[-R]}} V_\mathrm{j} := [Lo\wedge]\ V_\mathrm{i} <_S^{\mathrm{T}} V_\mathrm{j}\ [\wedge Ro] \quad (6)$$

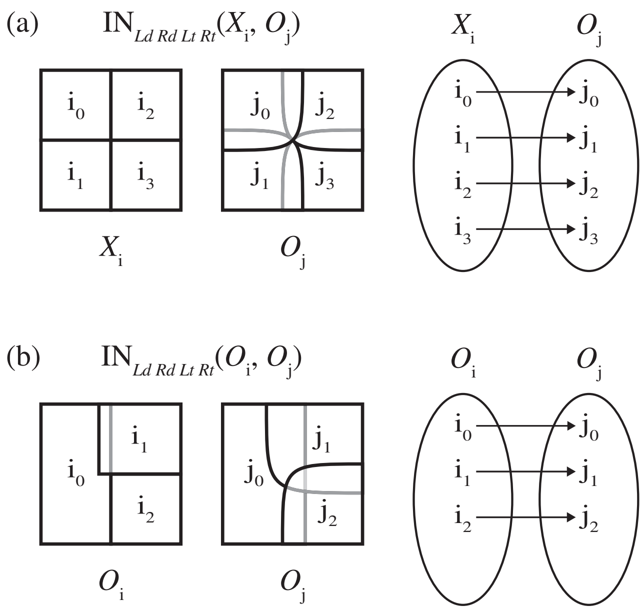

Figure 3: The two cases of $\mathrm{IN}_{Ld\ Rd\ Lt\ Rt}$ to an *o*-covering (O_j): (a) from a partition (X_i); (b) from an *o*-covering (O_i).

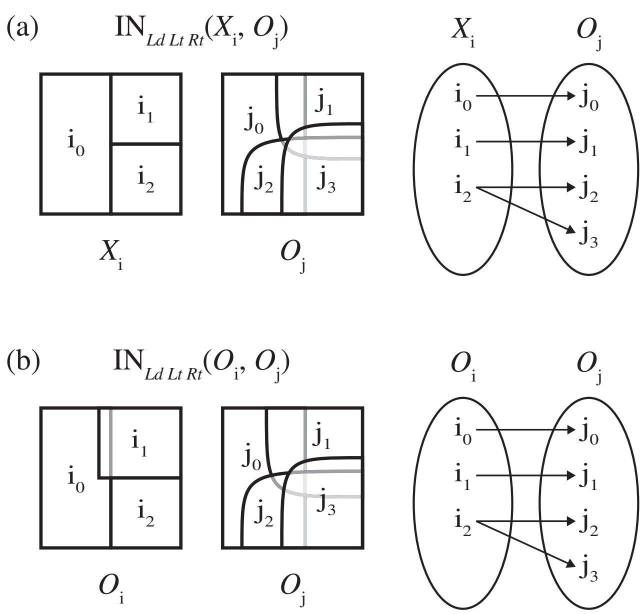

Figure 4: The two cases of $\mathrm{IN}_{Ld\ Lt\ Rt}$ to an *o*-covering (O_j): (a) from a partition (X_i); (b) from an *o*-covering (O_i).

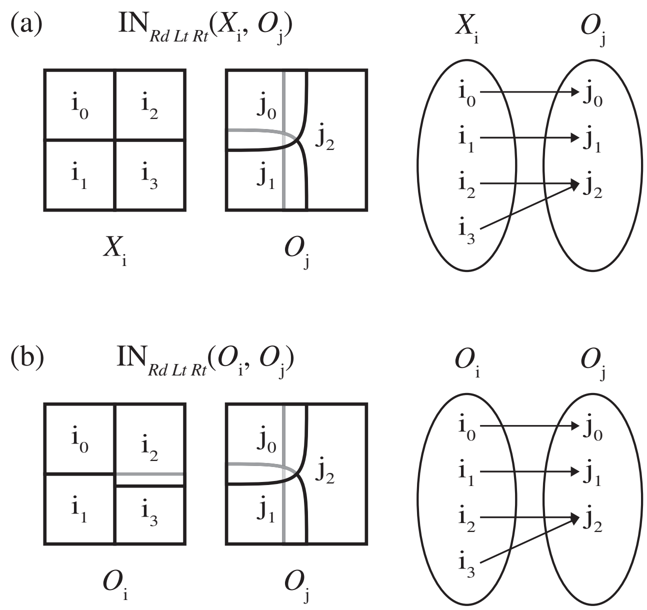

Figure 5: The two cases of $\mathrm{IN}_{Rd\ Lt\ Rt}$ to an o-covering (O_j): (a) from a partition (X_i); (b) from an o-covering (O_i).

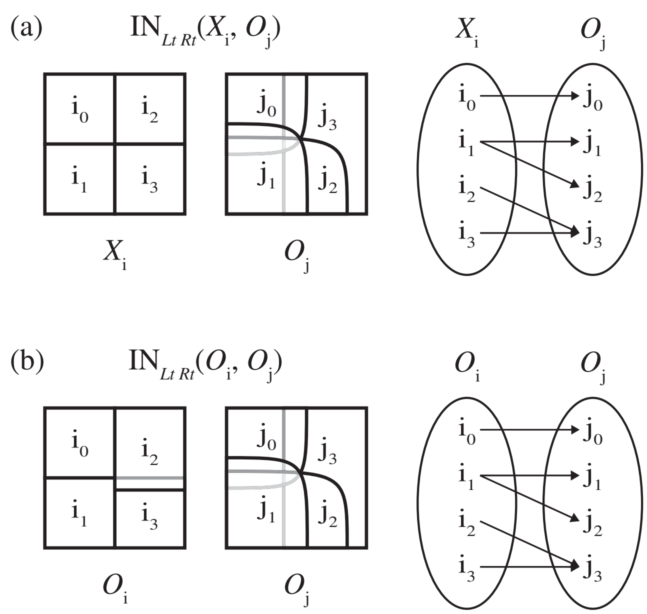

Figure 6: The two cases of $\mathrm{IN}_{Lt\ Rt}$ to an o-covering (O_j): (a) from a partition (X_i); (b) from an o-covering (O_i).

The formula yields nine valid collection order relations (Table 3), with overlappedness specified on neither side (r1), on either side (r2–r5), or on both sides (r6–r9), as indicated by the specific types.

COR	Specific type	Denotation	Terminology
r1	T	$<_S^{\mathrm{T}}$	*general total refinement*
r2	T-N	$<_S^{\mathrm{T\text{-}N}}$	*total refinement of a partition*
r3	T-O	$<_S^{\mathrm{T\text{-}O}}$	*total refinement of an o-covering*
r4	N-T	$<_S^{\mathrm{N\text{-}T}}$	*non-overlapped total refinement*
r5	O-T	$<_S^{\mathrm{O\text{-}T}}$	*overlapped total refinement*
r6	N-T-N	$<_S^{\mathrm{N\text{-}T\text{-}N}}$	*non-overlapped total refinement of a partition*
r7	O-T-N	$<_S^{\mathrm{O\text{-}T\text{-}N}}$	*overlapped total refinement of a partition*
r8	N-T-O	$<_S^{\mathrm{N\text{-}T\text{-}O}}$	*non-overlapped total refinement of an o-covering*
r9	O-T-O	$<_S^{\mathrm{O\text{-}T\text{-}O}}$	*overlapped total refinement of an o-covering*

Table 3: The collection order relations of total refinement.

Among these nine relations, T-N and T-O refinements (r2, r3) defined in equations 3 and 4 are two basic types revealing the sensitivity of the underlying IN_{Cp} (collection relation of IN) to Ro in total refinement.

Relations r6–r9 form a detailed JEPD set of total refinement. Figure 7 illustrates these four types with simplified geometric examples, where S and its granules are all unlabeled. The light-gray granules form a partition of each bounding box (S), while the translucent darker-gray ones turn four such partitions into *o*-coverings. The illustration has one example for each type, therefore the involved IN_{Cp} relations are not interpreted or distinguished as are in Figures 2–6.

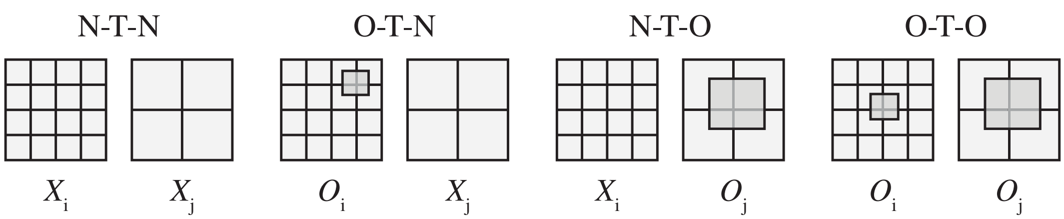

Figure 7: The four detailed JEPD types of total refinement (r6–r9).

4.2.3 Type C: Partial refinement

Type C refinement is *partial* as it combines Type A and Type B, duly splitting the two related coverings into corresponding non-empty proper subsets. The refined covering is divided into the *improperly refined subset* and the *totally refined subset*, and the refining covering is divided into the *improperly refining subset* and the *totally refining subset*. For covering V, refining or refined, the two subsets are referred to as *Subset A*, denoted by V^{A}, and *Subset B*, denoted by V^{B}, corresponding to the Type A and the Type B component, respectively. The coverage by V^{A} or V^{B} may be disconnected even if S is connected, and V^{A} and V^{B} make up a *set partition* of V as in the following formulation.

$$\mathrm{SetPartition}(\{V^{\mathrm{A}}, V^{\mathrm{B}}\}, V) := V^{\mathrm{A}} \subset V \wedge V^{\mathrm{B}} \subset V \wedge V^{\mathrm{A}} \cup V^{\mathrm{B}} = V \wedge V^{\mathrm{A}} \cap V^{\mathrm{B}} = \varnothing$$

The Type A and Type B components hold within two corresponding decomposable spaces, denoted by S^{A} and S^{B}, which are subsets of S obtained as relevant arbitrary unions, that is, $S^{\mathrm{A}} = \bigcup V^{\mathrm{A}}$ and $S^{\mathrm{B}} = \bigcup V^{\mathrm{B}}$, respectively. Despite the fact that V^{A} and V^{B} are disjoint sets of granules, S^{A} and S^{B} as decomposable spaces may overlap by having common interiors (Section 4.1) and thus make up a *set covering* of S, which is formulated as follows.

$$\mathrm{SetCovering}(\{S^{\mathrm{A}}, S^{\mathrm{B}}\}, S) := S^{\mathrm{A}} \subseteq S \wedge S^{\mathrm{B}} \subseteq S \wedge S^{\mathrm{A}} \cup S^{\mathrm{B}} = S$$

While the Type A component is improper (e1–e7), the Type B component as a total refinement is interpreted by r1–r9. Then a *P-N refinement* (*partial refinement of a partition*) from V_{i} to X_{j}, denoted by $V_{\mathrm{i}} <_S^{\text{P-N}} X_{\mathrm{j}}$, and a *P-O refinement* (*partial refinement of an o-covering*) from V_{i} to O_{j}, denoted by $V_{\mathrm{i}} <_S^{\text{P-O}} O_{\mathrm{j}}$, are each formulated as a combination of an improper and a total refinement (equations 7 and 8). Subsets $X_{\mathrm{j}}^{\mathrm{A}}$ and $X_{\mathrm{j}}^{\mathrm{B}}$ are both non-overlapped as proper subsets of a partition, thus forming partitions of S^{A} and S^{B} (equation 7) respectively. Subsets $O_{\mathrm{j}}^{\mathrm{A}}$ and $O_{\mathrm{j}}^{\mathrm{B}}$ as proper subsets of an *o*-covering may each be non-overlapped or overlapped, and therefore are treated as coverings of S^{A} and S^{B} in general, respectively (equation 8).

$$\begin{aligned} V_{\mathrm{i}} <_S^{\text{P-N}} X_{\mathrm{j}} := {} & \mathrm{Covering}(V_{\mathrm{i}}, S) \\ & \wedge \mathrm{SetPartition}(\{V_{\mathrm{i}}^{\mathrm{A}}, V_{\mathrm{i}}^{\mathrm{B}}\}, V_{\mathrm{i}}) \wedge \mathrm{SetPartition}(\{X_{\mathrm{j}}^{\mathrm{A}}, X_{\mathrm{j}}^{\mathrm{B}}\}, X_{\mathrm{j}}) \\ & \wedge V_{\mathrm{i}}^{\mathrm{A}} =_{S^{\mathrm{A}}}^{\text{-N}} X_{\mathrm{j}}^{\mathrm{A}} \wedge V_{\mathrm{i}}^{\mathrm{B}} <_{S^{\mathrm{B}}}^{\text{T-N}} X_{\mathrm{j}}^{\mathrm{B}} \\ & \wedge \mathrm{Covering}(X_{\mathrm{j}}, S) \wedge \text{Non-overlapped}(X_{\mathrm{j}}) \end{aligned} \tag{7}$$

$$\begin{aligned} V_{\mathrm{i}} <_S^{\text{P-O}} O_{\mathrm{j}} := {} & \mathrm{Covering}(V_{\mathrm{i}}, S) \\ & \wedge \mathrm{SetPartition}(\{V_{\mathrm{i}}^{\mathrm{A}}, V_{\mathrm{i}}^{\mathrm{B}}\}, V_{\mathrm{i}}) \wedge \mathrm{SetPartition}(\{O_{\mathrm{j}}^{\mathrm{A}}, O_{\mathrm{j}}^{\mathrm{B}}\}, O_{\mathrm{j}}) \\ & \wedge V_{\mathrm{i}}^{\mathrm{A}} =_{S^{\mathrm{A}}} O_{\mathrm{j}}^{\mathrm{A}} \wedge V_{\mathrm{i}}^{\mathrm{B}} <_{S^{\mathrm{B}}}^{\mathrm{T}} O_{\mathrm{j}}^{\mathrm{B}} \\ & \wedge \mathrm{Covering}(O_{\mathrm{j}}, S) \wedge \mathrm{Overlapped}(O_{\mathrm{j}}) \end{aligned} \tag{8}$$

A *P refinement* (*general partial refinement*), denoted by $V_{\mathrm{i}} <_S^{\mathrm{P}} V_{\mathrm{j}}$, is obtained through the disjunction of P-N and P-O refinements (equation 9).

$$\begin{aligned} V_{\mathrm{i}} <_S^{\mathrm{P}} V_{\mathrm{j}} := {} & \mathrm{Covering}(V_{\mathrm{i}}, S) \\ & \wedge \mathrm{SetPartition}(\{V_{\mathrm{i}}^{\mathrm{A}}, V_{\mathrm{i}}^{\mathrm{B}}\}, V_{\mathrm{i}}) \wedge \mathrm{SetPartition}(\{V_{\mathrm{j}}^{\mathrm{A}}, V_{\mathrm{j}}^{\mathrm{B}}\}, V_{\mathrm{j}}) \\ & \wedge V_{\mathrm{i}}^{\mathrm{A}} =_{S^{\mathrm{A}}} V_{\mathrm{j}}^{\mathrm{A}} \wedge V_{\mathrm{i}}^{\mathrm{B}} <_{S^{\mathrm{B}}}^{\mathrm{T}} V_{\mathrm{j}}^{\mathrm{B}} \\ & \wedge \mathrm{Covering}(V_{\mathrm{j}}, S) \end{aligned} \tag{9}$$

Then replacing the central component of the second tripartite formula (see equation 6) with P refinement gives rise to the third tripartite formula, from which any collection order relation of partial refinement is obtained as $V_{\mathrm{i}} <_S^{\text{[L-]P[-R]}} V_{\mathrm{j}}$ (equation 10).

$$V_{\mathrm{i}} <_S^{\text{[L-]P[-R]}} V_{\mathrm{j}} := [Lo\wedge]\ V_{\mathrm{i}} <_S^{\mathrm{P}} V_{\mathrm{j}}\ [\wedge Ro] \tag{10}$$

The formula yields nine types of partial refinement as collection order relations r10–r18 (Table 4).

COR	Specific type	Denotation	Terminology
r10	P	$<_S^{\mathrm{P}}$	*general partial refinement*
r11	P-N	$<_S^{\mathrm{P\text{-}N}}$	*partial refinement of a partition*
r12	P-O	$<_S^{\mathrm{P\text{-}O}}$	*partial refinement of an o-covering*
r13	N-P	$<_S^{\mathrm{N\text{-}P}}$	*non-overlapped partial refinement*
r14	O-P	$<_S^{\mathrm{O\text{-}P}}$	*overlapped partial refinement*
r15	N-P-N	$<_S^{\mathrm{N\text{-}P\text{-}N}}$	*non-overlapped partial refinement of a partition*
r16	O-P-N	$<_S^{\mathrm{O\text{-}P\text{-}N}}$	*overlapped partial refinement of a partition*
r17	N-P-O	$<_S^{\mathrm{N\text{-}P\text{-}O}}$	*non-overlapped partial refinement of an o-covering*
r18	O-P-O	$<_S^{\mathrm{O\text{-}P\text{-}O}}$	*overlapped partial refinement of an o-covering*

Table 4: The collection order relations of partial refinement.

Likewise, P-N (r11) and P-O (r12) refinements defined in equations 7 and 8 are two basic types from which the other relations in Table 4 are derived. While r15–r18 form a detailed JEPD set of partial refinement, the pattern of the O-P-O type varies with whether S^{A} and S^{B} overlap (by interiors), denoted by $ovS^{\mathrm{A}}S^{\mathrm{B}}$, which is defined with the DE5 coarse relations as follows.

$$ovS^{\mathrm{A}}S^{\mathrm{B}} := \begin{cases} \textit{true}, & \text{if } \mathrm{EQ}(S^{\mathrm{A}}, S^{\mathrm{B}}) \vee \mathrm{IN}(S^{\mathrm{A}}, S^{\mathrm{B}}) \vee \mathrm{IN}^{-1}(S^{\mathrm{A}}, S^{\mathrm{B}}) \vee \mathrm{OV}(S^{\mathrm{A}}, S^{\mathrm{B}}) \\ \textit{false}, & \text{if } \mathrm{OUT}(S^{\mathrm{A}}, S^{\mathrm{B}}) \end{cases}$$

The pattern of O-P-O refinement also varies with the specified overlappedness of the four subsets. Being equal, the two Subsets A have the same overlappedness, denoted by oA. Subset B of the refining covering and Subset B of the refined covering have their overlappedness denoted by $o\mathrm{I}^B$ and $o\mathrm{J}^B$ respectively. Similar to L and R, oA, $o\mathrm{I}^B$, and $o\mathrm{J}^B$ are each valued "N" or "O" respectively. Then, 16 (2^4) more-detailed combinations for r15–r18 are yielded by the 4-tuple as follows.

$$\langle ovS^{\mathrm{A}}S^{\mathrm{B}}, o\mathrm{A}, o\mathrm{I}^B, o\mathrm{J}^B\rangle$$

Figure 8 illustrates these 16 patterns for r15–r18, 13 of which are subtypes of O-P-O refinement. The granules in the Subsets A are drawn as the translucent white boxes. The granules in the Subsets B are represented by the light-gray boxes, which form partitions of S^{B}, and the translucent darker-gray ones each turning one of these light-gray partitions into an *o*-covering of S^{B}. The first two patterns (N-P-N and O-P-N) are with the same S^{A} and S^{B}. In the other 14 patterns, an enlarged S^{B} remains identical, while S^{A} is drawn with another three different extents, one for N-P-O and O-P-O (a) – O-P-O (e), and one for each of the two lower rows where S^{A} and S^{B} overlap in the bottom-right quadrant of each bounding box (S).

4.2.4 Proper (strict) and non-strict refinements

Type B and Type C refinements are proper by inducing strict collection orders. A *general proper refinement* from V_{i} to V_{j}, denoted by $V_{\mathrm{i}} <_S V_{\mathrm{j}}$, is therefore derived as the disjunction of a T and a P refinement (equation 11).

$$V_{\mathrm{i}} <_S V_{\mathrm{j}} := V_{\mathrm{i}} <_S^{\mathrm{T}} V_{\mathrm{j}} \vee V_{\mathrm{i}} <_S^{\mathrm{P}} V_{\mathrm{j}} \tag{11}$$

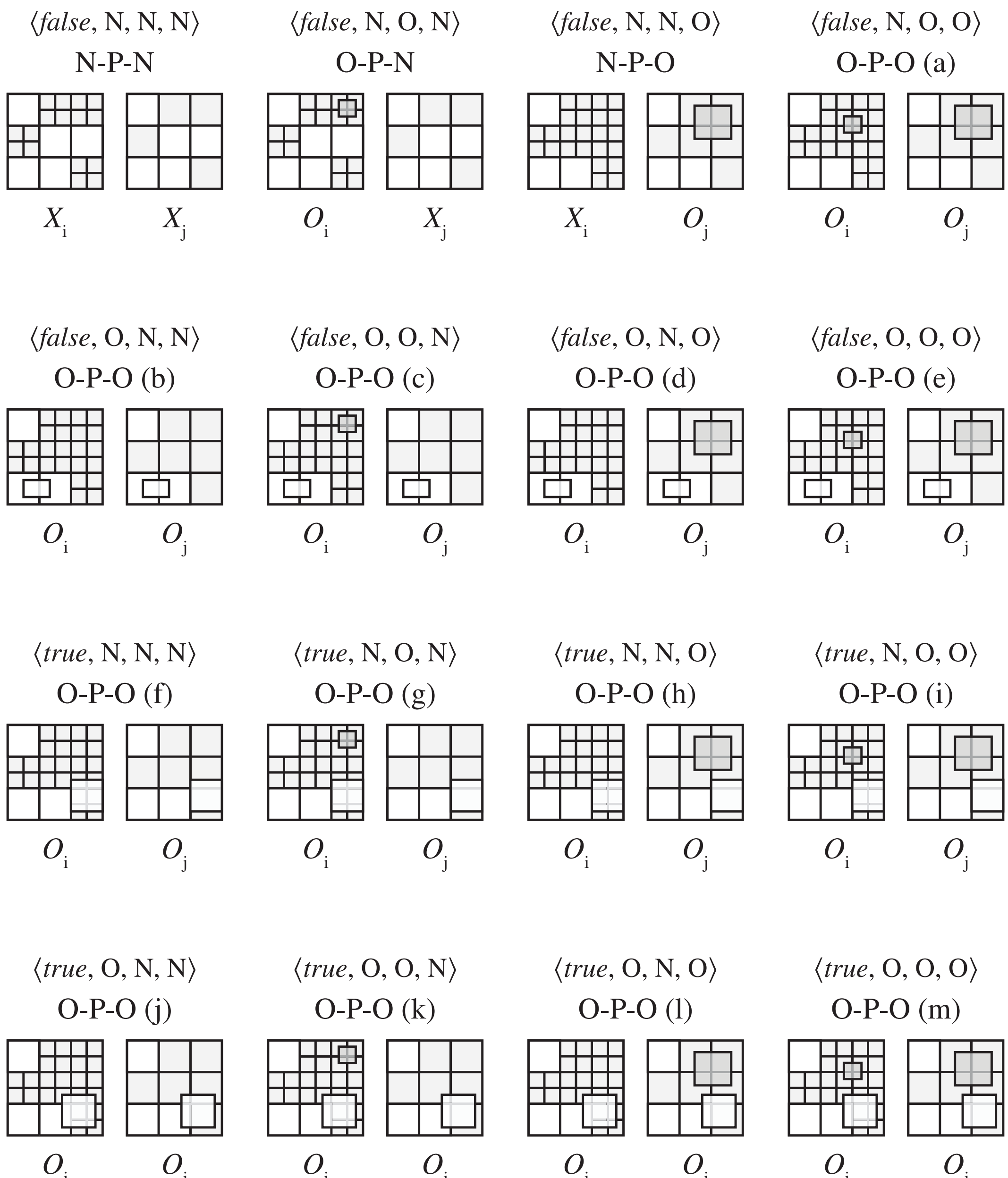

Figure 8: The 16 combinations for the four detailed JEPD types of partial refinement (r15–r18), of which 13 are subtypes of O-P-O refinement (r18).

Then the fourth tripartite formula is obtained for proper refinement by replacing the central component with $V_{\text{i}} <_S^{[*]} V_{\text{j}}$, where * is an optional truncation symbol for "T" or "P" indicating whether a proper refinement is specified as total or partial, referred to as *totalness*, or with totalness unspecified (equation 12).

$$V_{\text{i}} <_S^{\text{[L-][*][-R]}} V_{\text{j}} := [Lo\wedge]\ V_{\text{i}} <_S^{[*]} V_{\text{j}}\ [\wedge Ro] \tag{12}$$

The formula yields 27 (3^3) types of proper refinement including r1–r18 and nine types (r19–r27) with totalness unspecified (Table 5), with the latter being derived as relevant disjunctions. For example, *-N proper refinement* is the disjunction of T-N and P-N, *O- proper* combines O-T and O-P, and *N--O proper* is the generalization of N-T-O and N-P-O refinements.

COR	Specific type	Denotation	Terminology
r19	general proper	$<_S$	*general proper refinement*
r20	-N proper	$<_S^{\text{-N}}$	*proper refinement of a partition*
r21	-O proper	$<_S^{\text{-O}}$	*proper refinement of an o-covering*
r22	N- proper	$<_S^{\text{N-}}$	*non-overlapped proper refinement*
r23	O- proper	$<_S^{\text{O-}}$	*overlapped proper refinement*
r24	N--N proper	$<_S^{\text{N--N}}$	*non-overlapped proper refinement of a partition*
r25	O--N proper	$<_S^{\text{O--N}}$	*overlapped proper refinement of a partition*
r26	N--O proper	$<_S^{\text{N--O}}$	*non-overlapped proper refinement of an o-covering*
r27	O--O proper	$<_S^{\text{O--O}}$	*overlapped proper refinement of an o-covering*

Table 5: The collection order relations of proper refinement with totalness unspecified.

While total and partial refinements are proper (strict), refinements so called as *non-strict total* or *non-strict partial* are not valid as they would be generalized to *non-strict proper*, that is, *non-strict strict*, which is self-contradictory. As a result, non-strict refinements are only derived from the proper refinements that have totalness unspecified (r19–r27).

The most-general type of non-strict refinement, a *general refinement* from V_{i} to V_{j}, denoted by $V_{\text{i}} \leq_S V_{\text{j}}$, is obtained as the disjunction of r19 (general proper refinement) and e1 (general improper refinement) (equation 13).

$$V_{\text{i}} \leq_S V_{\text{j}} := V_{\text{i}} <_S V_{\text{j}} \vee V_{\text{i}} =_S V_{\text{j}} \tag{13}$$

Then any type of non-strict refinement is obtained as $V_{\text{i}} \leq_S^{\text{[L-][-R]}} V_{\text{j}}$ from the fifth tripartite formula, where the central component is $V_{\text{i}} \leq_S V_{\text{j}}$ (equation 14).

$$V_{\text{i}} \leq_S^{\text{[L-][-R]}} V_{\text{j}} := [Lo\wedge]\ V_{\text{i}} \leq_S V_{\text{j}}\ [\wedge Ro] \tag{14}$$

The formula yields each non-strict type as the disjunction of a pair of corresponding proper (r19–r27) and improper (e1–e7) types, except for O--N proper (r25) and N--O proper (r26), which do not have improper correspondents by precluding equality with opposite L and R. The formula then results in seven collection order relations (r28–r34) of non-strict refinement (Table 6).

In consequence, 34 collection order relations (r1–r34) of refinement—27 proper with additional 13 subtypes and seven non-strict—are derived from four basic proper types (r2, r3, r11, r12) and seven improper types (e1–e7), with eight types (r6–r9, r15–r18) forming a detailed JEPD set of proper refinement.

COR	Specific type	Denotation	Terminology
r28	general	$\leq_S$	*general refinement*
r29	-N	$\leq_S^{\text{-N}}$	*refinement of a partition*
r30	-O	$\leq_S^{\text{-O}}$	*refinement of an o-covering*
r31	N-	$\leq_S^{\text{N-}}$	*non-overlapped refinement*
r32	O-	$\leq_S^{\text{O-}}$	*overlapped refinement*
r33	N--N	$\leq_S^{\text{N--N}}$	*non-overlapped refinement of a partition*
r34	O--O	$\leq_S^{\text{O--O}}$	*overlapped refinement of an o-covering*

Table 6: The collection order relations of non-strict refinement.

4.3 Coarsening

Being the converse of refinement, coarsening falls into three corresponding broad, mutually exclusive types. As for each converse pair of specific coarsening and refinement types, their underlying collection relations and settings of overlappedness conversely correspond to each other.

The converseness of class relations [26, 28] applies to collection relations via model extension. Thus, if an individual relation R has a converse, a collection relation of it has a converse as well, which is of the converse individual relation R^{-1} with swapped left and right uniqueness and totality, that is, Rd for Ld, Rt for Lt, and vice versa. For example, $(\text{IN}_{Rd\ Lt\ Rt}(V_\text{i}, X_\text{j}))^{-1} = \text{IN}^{-1}{}_{Ld\ Lt\ Rt}(X_\text{j}, V_\text{i})$. Accordingly, proper (Type B and Type C) coarsening is based on each relevant $\text{IN}^{-1}{}_{Cp}$ as the converse of the corresponding IN_{Cp} in proper refinement.

Moreover, the two related coverings in each specific type of refinement have their positions swapped in the converse coarsening relation. Thus, the settings of Lo and Ro in the refinement are accordingly swapped in the coarsening, where Lo and Ro refer to *the overlappedness of the coarsening* (*covering*) and *the overlappedness of the coarsened covering* respectively. For example, the converse of an N- improper refinement is a -N improper coarsening, that is, $(X_\text{i} =_S^{\text{N-}} V_\text{j})^{-1} = V_\text{j} =_S^{\text{-N}} X_\text{i}$, with the two types belonging to the same set of relations (e1–e7) that interpret both refinement and coarsening of Type A.

Type B (total) coarsening implies left and right totality in the underlying $\text{IN}^{-1}{}_{Cp}$, which excludes $\text{IN}^{-1}{}_{Lt\ Rt-all}$. With its candidate IN_{Cp} depending on Ro, total refinement is *right sensitive*. Conversely, total coarsening is *left sensitive*, that is, the underlying $\text{IN}^{-1}{}_{Cp}$ is affected by Lo—the overlappedness of the coarsening instead of the coarsened—due to the swapped settings between Lo and Ro as described above.

Specifically, the converse of a T-N refinement, which implies right uniqueness (Rd), is an N-T coarsening implying left uniqueness (Ld). Then an *N-T* (*non-overlapped total*) *coarsening* from X_i to V_j, denoted by $X_\text{i} >_S^{\text{N-T}} V_\text{j}$, is underlain by $\text{IN}^{-1}{}_{Ld\ Lt\ Rt}(X_\text{i}, V_\text{j})$ (equation 15).

$$X_\text{i} >_S^{\text{N-T}} V_\text{j} := \text{Non-overlapped}(X_\text{i}) \wedge \text{Covering}(X_\text{i}, S) \wedge \text{IN}^{-1}{}_{Ld\ Lt\ Rt}(X_\text{i}, V_\text{j}) \wedge \text{Covering}(V_\text{j}, S) \tag{15}$$

Likewise, the converse of a T-O refinement is an O-T coarsening. Since the collection relations of the same individual relation are pairwise disjoint, for any pair of converse individual relations, at most one corresponding pair of converse collection relations holds between two particular coverings. Thus, the converse of the disjunction of the four candidate IN_{Cp} relations for T-O refinement is derived as the disjunction of the four converse

$\mathrm{IN}^{-1}{}_{Cp}$ relations as follows.

$$(\mathrm{IN}_{Ld\,Rd\,Lt\,Rt}(V_{\mathrm{i}}, O_{\mathrm{j}}) \vee \mathrm{IN}_{Ld\,Lt\,Rt}(V_{\mathrm{i}}, O_{\mathrm{j}}) \vee \mathrm{IN}_{Rd\,Lt\,Rt}(V_{\mathrm{i}}, O_{\mathrm{j}}) \vee \mathrm{IN}_{Lt\,Rt}(V_{\mathrm{i}}, O_{\mathrm{j}}))^{-1}$$
$$= \mathrm{IN}^{-1}{}_{Ld\,Rd\,Lt\,Rt}(O_{\mathrm{j}}, V_{\mathrm{i}}) \vee \mathrm{IN}^{-1}{}_{Rd\,Lt\,Rt}(O_{\mathrm{j}}, V_{\mathrm{i}}) \vee \mathrm{IN}^{-1}{}_{Ld\,Lt\,Rt}(O_{\mathrm{j}}, V_{\mathrm{i}}) \vee \mathrm{IN}^{-1}{}_{Lt\,Rt}(O_{\mathrm{j}}, V_{\mathrm{i}})$$

Then an *O-T* (*overlapped total*) *coarsening* from O_{i} to V_{j}, denoted by $O_{\mathrm{i}} >_S^{\text{O-T}} V_{\mathrm{j}}$, is obtained accordingly (equation 16).

$$\begin{aligned} O_{\mathrm{i}} >_S^{\text{O-T}} V_{\mathrm{j}} := {} & \mathrm{Overlapped}(O_{\mathrm{i}}) \wedge \mathrm{Covering}(O_{\mathrm{i}}, S) \wedge (\mathrm{IN}^{-1}{}_{Ld\,Rd\,Lt\,Rt}(O_{\mathrm{i}}, V_{\mathrm{j}}) \\ & \vee \mathrm{IN}^{-1}{}_{Ld\,Lt\,Rt}(O_{\mathrm{i}}, V_{\mathrm{j}}) \vee \mathrm{IN}^{-1}{}_{Rd\,Lt\,Rt}(O_{\mathrm{i}}, V_{\mathrm{j}}) \vee \mathrm{IN}^{-1}{}_{Lt\,Rt}(O_{\mathrm{i}}, V_{\mathrm{j}})) \\ & \wedge \mathrm{Covering}(V_{\mathrm{j}}, S) \end{aligned} \quad (16)$$

Similarly, equations 15 and 16 define two basic types of total coarsening. A further examination of coarsening, which is omitted here, leads to 34 collection order relations (c1–c34) as the converses of the corresponding relations of refinement.

In all, inclusion comparability transforms into refinement comparability through 75 collection order relations (seven improper, 34 refinement, and 34 coarsening), such that two coverings of S are comparable if any converse pair out of the 75 relations holds between them, otherwise they are incomparable.

5 Partial collection orders

Collection order relations of refinement and coarsening have their major potential applications in such geocomputations where spatial hierarchies of partially ordered levels, each formed by a covering, play a significant role. The usefulness of these relations, therefore, is closely related to their applicability as partial orders. A partial order is either strict or non-strict. A strict partial order is irreflexive, asymmetric, and transitive, while a non-strict partial order is reflexive, antisymmetric, and transitive. Although spatial refinement between partitions is conceptualized as partial orders in both strict and non-strict ways [14, 16], the question is whether such formulated collection order relations between spatial coverings based on collection relations, overlappedness, and totalness form partial orders. While such logical properties as reflexivity, irreflexivity, asymmetry, antisymmetry, and transitivity have not yet been fully studied for class relations [28], they are here analyzed with respect to the collection order relations, leading to the particular forms of partial orders between collections, referred to as *partial collection orders*.

5.1 Strict partial collection orders

Regarding proper (strict) refinement, the following analyses are on irreflexivity, asymmetry, and transitivity.

Interpreted by a strict collection order relation, any proper refinement, with totalness specified or unspecified, is irreflexive, that is, $\neg(V_{\mathrm{i}} <_S^{prop} V_{\mathrm{i}})$, where $prop$ denotes a combination of [L-][*][-R] in r1–r27.

For the same reason, proper refinement is asymmetric, that is, $V_{\mathrm{i}} <_S^{prop} V_{\mathrm{j}} \rightarrow \neg(V_{\mathrm{j}} <_S^{prop} V_{\mathrm{i}})$.

Transitivity as a more complex property is here examined separately regarding the three different settings of totalness. It is found that if an individual relation R is transitive, in all the four cases of totality, $R_{\text{all-1}}$, $R_{\text{all-2}}$, $R_{\text{all-12}}$, and $R_{\text{all-all}}$, which are equivalent to LT, RT, $LT \wedge RT$, and $R_{LT\ RT-all}$ [27, 28], respectively, the class relations are transitive [8]. Such transfer of transitivity conveniently applies to the collection relation $\text{IN}_{Lt\ Rt}$, with relation IN being transitive, and Lt and Rt being the set versions of $R_{\text{all-1}}$ (LT) and $R_{\text{all-2}}$ (RT), respectively. It is also provable that a transitive individual relation constrained by either or both cases of uniqueness (Ld and Rd) is transitive. Therefore, the other three candidate IN_{Cp} relations for total refinement, $\text{IN}_{Ld\ Lt\ Rt}$, $\text{IN}_{Rd\ Lt\ Rt}$, and $\text{IN}_{Ld\ Rd\ Lt\ Rt}$, are transitive as well.

Given tB as a combination of [L-]T[-R] for total refinement (r1–r9), tB refinement is transitive if $V_\text{i} <_S^{tB} V_\text{j} \wedge V_\text{j} <_S^{tB} V_\text{k} \rightarrow V_\text{i} <_S^{tB} V_\text{k}$. Since exactly one candidate IN_{Cp} holds for any total refinement between two particular coverings, such a V_i–V_j–V_k statement is true if only L and R in tB are unopposite (Section 4.2.1), such that the first two relations ($V_\text{i} <_S^{tB} V_\text{j}$ and $V_\text{j} <_S^{tB} V_\text{k}$) do not negate each other by opposite specified overlappedness of V_j. Consequently, seven (r1–r6, r9) of the nine total refinement types are transitive except O-T-N (r7) and N-T-O (r8) due to the reason stated above.

As to partial refinement, let $V_\text{i} <_S^{tC} V_\text{j}$ and $V_\text{j} <_S^{tC} V_\text{k}$, where tC denotes an applicable combination of [L-]P[-R] in r10–r18. The major issue on whether tC refinement is transitive, or its *transitiveness*, is the totalness of the third relation (from V_i to V_k). In fact, V_k^A (Subset A of V_k) in $V_\text{j} <_S^{tC} V_\text{k}$ may by chance be totally refined by a subset of V_i, turning V_i into a total refinement of V_k; otherwise, V_i partially refines V_k. Partial refinement, therefore, is not transitive due to the third relation's uncertain totalness.

Proper refinement with totalness unspecified (r19–r27) involves both total and partial refinements in its transitivity, which has five cases regarding the actual totalness of the three relations in the V_i–V_j–V_k statement as follows.

- $\text{T} \wedge \text{T} \rightarrow \text{T}$: a conjunction of two total refinements implying a third total refinement.
- $\text{T} \wedge \text{P} \rightarrow \text{T}$ or $\text{P} \wedge \text{T} \rightarrow \text{T}$: a conjunction of a total and a partial refinement, in either order, implying a total refinement.
- $\text{P} \wedge \text{P} \rightarrow \text{T}$: a conjunction of two partial refinements resulting in a total refinement.
- $\text{P} \wedge \text{P} \rightarrow \text{P}$: a conjunction of two partial refinements resulting in a third partial refinement.

Similarly, seven (r19–r24, r27) of the nine proper refinement types with totalness unspecified are transitive by each having unopposite L and R, with O--N proper (r25) and N--O proper (r26) being the two exceptions.

In summary, 14 types of proper refinement—seven total (r1–r6, r9) and seven with totalness unspecified (r19–r24, r27)—form *strict partial collection orders* by having irreflexivity, asymmetry, and transitivity.

5.2 Non-strict partial collection orders

For non-strict refinement, the following analyses are with respect to reflexivity, antisymmetry, and transitivity.

Let NS be a combination of [L-][-R] for non-strict refinement (r28–r34), and also for improper refinement (e1–e7) correspondingly. NS refinement involves a pair of proper and improper cases that have the same combined setting of Lo and Ro. NS refinement is reflexive if the improper case is reflexive on the proper case's domain (i.e., the set of refining coverings), such that whenever the proper case holds, the improper case holds from the refining covering to itself. This requires that the setting of R in NS is not more restrictive than that of L, as a more restrictive R results in a smaller domain of the improper case. For example, -N refinement ($\leq_S^{\text{-N}}$) is not reflexive, as the refining covering in its proper case ($<_S^{\text{-N}}$) may be a partition or an *o*-covering, while its improper case ($=_S^{\text{-N}}$) is restricted to hold between partitions only. Consequently, five (r28, r31–r34) of the seven non-strict types are reflexive, with -N (r29) and -O (r30) being the two exceptions.

NS refinement is antisymmetric if $V_\text{i} \leq_S^{NS} V_\text{j} \wedge V_\text{j} \leq_S^{NS} V_\text{i} \rightarrow V_\text{i} =_S^{NS} V_\text{j}$. By each having unopposite L and R, such that the first two relations do not negate each other by opposite specified overlappedness of V_j or V_i, and with proper cases holding between different coverings only, all the seven non-strict types are antisymmetric.

With improper refinement being transitive by conveying equality, the transitiveness of NS refinement comes down to the transitiveness of its proper case. Consequently, all the seven non-strict types (r28–r34) are transitive, with their respective proper cases being the seven transitive proper types with totalness unspecified (r19–r24, r27).

As such, five non-strict refinement types (r28, r31–r34) form *non-strict partial collection orders* by having reflexivity, antisymmetry, and transitivity.

According to the above analyses, 19 types of refinement—14 proper and five non-strict—turn out to be partial collection orders.

Similar results are with the corresponding types of coarsening as the converse of refinement, and therefore the examination is omitted here. Together, 38 types of strict or non-strict refinement and coarsening—19 of each—form a pool of partial collection orders for establishing spatial hierarchies with covering-based levels.

6 Conclusions and future work

This paper examined spatial refinement (and conversely, coarsening) between coverings, which differentiate by overlappedness into partitions (non-overlapped) and *o*-coverings (overlapped), and thereby offer enhanced ways of dealing with multiple spatial datasets and modeling real-world geographic features.

Refinement between spatial coverings varies with the overlappedness of the related coverings as well as their granule participation and correspondence in the inclusion relations that embody refinement. Granule correspondence is captured by the relations' cardinality properties involving uniqueness and totality, which result in collection relations of equality and proper inclusion when applied to relevant coarse topological relations. With granule participation, spatial refinement intuitively falls into three broad, mutually exclusive types: (A) improper (equality), (B) total, with each granule in one covering involved in proper inclusion with some granule in the other covering and vice versa, and (C) partial, a combination of A and B.

The paper then presented five tripartite formulae, from which each collection order relation (or specific type) of refinement is obtained as a conjunction of a central component, which conveys granule participation and correspondence, with optionally the overlappedness of the refining and the refined covering. The first tripartite formula yields seven relations of improper refinement, which are also the types of its equal converse, improper coarsening. The second and the third formula yield nine total and nine partial refinement relations respectively, with the fourth resulting in 27 proper (strict) types, which include the 18 total and partial relations as well as nine relations with totalness (total or partial) unspecified. The fifth tripartite formula further yields seven non-strict types by combining relevant proper and improper relations.

Later examinations show that 14 types of proper (strict) refinement (seven total and seven with totalness unspecified) form strict partial collection orders, and five non-strict types form non-strict partial collection orders. With coarsening, the converse of refinement, leading to corresponding results, out of the 75 collection order relations (seven improper and 34 pairs of strict or non-strict refinement and coarsening), 38 (19 pairs) provide a pool of partial collection orders.

Future work will explore various partial collection orders of spatial coverings on refinement, which result in *spatial refinement hierarchies*, and their relations with the corresponding partial orders of granules on inclusion, which form spatial inclusion hierarchies. Reflecting the refinement–inclusion association, a refinement hierarchy is simply the level structure of its induced inclusion hierarchy. Since hierarchies are established on partial orders, a simple refinement hierarchy is a chain of coverings, which is a totally ordered set (a special type of partially order sets), where the coverings are pairwise comparable with respect to refinement. In much simpler cases, the decomposable space S is one single granule, and a chain of coverings is on total refinement only, with its top element being the most coarse-grained partition of S, that is, the singleton $\{S\}$. Restricted as such, a *partition chain* on N-T-N refinement unfolds into a tree of granules ordered on inclusion, which, as the simplest structure of an inclusion hierarchy, is prevalent in current GIS and geographic information studies.

As to a *covering chain* with the same restrictions, the induced inclusion structure is not a tree if any granule is contained by at least two parent granules, which thereby overlap. Since a covering may have more than enough granules for a complete coverage, such a level in an inclusion hierarchy is often nested or unnecessarily complex for analysis or reasoning. Pairing with partitions, *non-redundant overlapped coverings*, or *nr-o-coverings*, are the overlapped options for single complete coverages where each granule is indispensible. An *nr-o*-covering can be used to generate an overlapped representation with the least number of granules or the smallest overlapped area. As such, an *nr-o*-covering is the best alternative when a partition is not obtainable or guaranteed, for example, in generating an optimal zoning with features from different datasets to alleviate a MAUP (Case 1, Section 1.2). An *nr-o*-covering is also suitable for representing geographic features as overlapping in transition zones (Case 4, Section 1.2). An inclusion hierarchy is free of nested levels or redundant granules if each of its levels is populated by either a partition or an *nr-o*-covering.

Complex hierarchies may arise from merging simple ones, and have structures based on *level trees* or *multiple level trees*. Hierarchical geocomputations switching between covering-based levels will benefit from operations that delaminate a nested level into partitions, *nr-o*-coverings, or other non-nested types of coverings, and as well from operations that combine them into a single covering. These operations are particularly useful in simplifying the level structure of a complex hierarchy, or splitting it into multiple simple structures. On the other hand, levels of inclusion may be reorganized to reflect other properties of the granules (e.g., spatial coarseness or geographic category). An investigation of these structures will lead to a deeper understanding of spatial hierarchies and granulated representations.

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