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Identifying and Addressing Specific Student Difficulties in Advanced Thermal Physics

Trevor I. Smith

IDENTIFYING AND ADDRESSING SPECIFIC STUDENT DIFFICULTIES IN ADVANCED THERMAL PHYSICS

By

Trevor I. Smith

M.S.T. University of Maine, 2007

B.S. University of Maine, 2005

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Advisory Committee:

John R. Thompson, Associate Professor of Physics, Advisor

Donald B. Mountcastle, Associate Professor of Physics

Michael C. Wittmann, Associate Professor of Physics

Susan R. McKay, Professor of Physics

Natasha M. Speer, Assistant Professor of Mathematics

DISSERTATION ACCEPTANCE STATEMENT

On behalf of the Graduate Committee for Trevor I. Smith, I affirm that this manuscript is the final and accepted dissertation. Signatures of all committee members are on file with the Graduate School at the University of Maine, 42 Stodder Hall, Orono, Maine.

Submitted for graduation in May, 2011

John R. Thompson, Associate Professor of Physics (Date)

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By Trevor I. Smith

Dissertation Advisor: Dr. John R. Thompson

An Abstract of the Dissertation Presented
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As part of an ongoing multi-university research study on student understanding of concepts in thermal physics at the upper division, I identified several student difficulties with topics related to heat engines (especially the Carnot cycle), as well as difficulties related to the Boltzmann factor. In an effort to address these difficulties, I developed two guided-inquiry worksheet activities (a.k.a. tutorials) for use in advanced undergraduate thermal physics courses. Both tutorials seek to improve student understanding of the utility and physical background of a particular mathematical expression. One tutorial focuses on a derivation of Carnot's theorem regarding the limit on thermodynamic efficiency, starting from the Second Law of Thermodynamics. The other tutorial helps students gain an appreciation for the origin of the Boltzmann factor and when it is applicable; focusing on the physical justification of its mathematical derivation, with emphasis on the connections between probability, multiplicity, entropy, and energy.

Student understanding of the use and physical implications of Carnot's theorem and the Boltzmann factor was assessed using written surveys both before and after

tutorial instruction within the advanced thermal physics courses at the University of Maine and at other institutions. Classroom tutorial sessions at the University of Maine were videotaped to allow in-depth scrutiny of student successes and failures following tutorial prompts. I also interviewed students on various topics related to the Boltzmann factor to gain a more complete picture of their understanding and inform tutorial revisions.

Results from several implementations of my tutorials at the University of Maine indicate that students did not have a robust understanding of these physical principles after lectures alone, and that they gain a better understanding of relevant topics after tutorial instruction; Fisher's exact tests yield statistically significant improvement at the $\alpha = 0.05$ level. Results from other schools indicate that difficulties observed before tutorial instruction in our classes (for both tutorials) are not unique, and that the Boltzmann factor tutorial can be an effective replacement for lecture instruction. Additional research is suggested that would further examine these difficulties and inform instructional strategies to help students overcome them.

IDENTIFYING AND ADDRESSING SPECIFIC STUDENT DIFFICULTIES IN ADVANCED THERMAL PHYSICS

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Physics education research (PER) is the study of how people think about, learn, understand, and teach topics in and related to physics. One goal of PER is to identify student difficulties with a particular topic and to develop curricular materials to address these difficulties. Results in PER show that guided-inquiry worksheet activities (a.k.a. tutorials) can be effective supplements to traditional lecture instruction in introductory physics classes. Recent research suggests that tutorials can also be useful within upper-division courses.

I developed two tutorials for use within advanced undergraduate thermal physics courses. One tutorial improves students' understanding of the relationship between heat engines (especially the Carnot cycle), entropy, and the Second Law of Thermodynamics. Heat engines are an integral part of many thermodynamics courses, as they provide a practical scenario in which all three laws of thermodynamics must be considered. Carnot's theorem is, in essence, a statement of the Second Law in

the context of heat engines, but my results indicate that students do not make this connection. My tutorial helps students by guiding them through a derivation of Carnot's theorem starting from a standard statement of the Second Law.

My second tutorial helps students gain an appreciation for the physical and mathematical origin of the Boltzmann factor and when it is applicable. The Boltzmann factor is a mathematical expression for the probability that a thermodynamic system has a certain energy. The Boltzmann factor may be used to determine many properties of the system and is, therefore, a cornerstone of statistical thermal physics. My results indicate that students often do not recognize situations in which the Boltzmann factor is appropriate, nor do they understand where this particular mathematical expression comes from.

Results from implementing my tutorials within the advanced thermal physics courses at the University of Maine indicate that students gain a better understanding of relevant topics after tutorial instruction, compared to lectures alone. Results from other schools indicate that difficulties observed before tutorial instruction in our classes are not unique, and that the Boltzmann factor tutorial can be an effective replacement for lecture instruction.

DEDICATION

To my wife Ashley: for always supporting me.

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Throughout my six years in grad school, and a total of ten years at the University of Maine, I've met and been helped by many different people (both personally and professionally). I'd like to take time to thank some of them. I sincerely apologize if I have left anyone out.

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Next I thank Don Mountcastle for checking my own understanding of the topics I was studying and allowing me nearly free rein in his courses. Don has always been a reality check to make sure I wasn't forgetting some subtle point that the students

needed to know (but that I may not have really understood myself). I also thank Don for his frequent reminders of where I started, as a student in the very courses that I have come to study, without necessarily having the robust understanding that I expect of my research subjects.

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Natasha Speer was also instrumental in helping me refine my dissertation to make it more readable. Her requests for additional details (while sometimes difficult to include) helped me to create a document that is much more concise and readable than its previous version. (You should probably thank Natasha and Michael, too.)

I would like to thank Susan McKay for pulling some of the strings to keep me in Maine for so long (even if it wasn't my original intent). As a M.S.T. student, Susan placed me in an office with her Ph.D. students, hoping that I would be enticed into staying for a Ph.D. myself. As much as it irks me that her scheming worked, I am very glad that I have stayed in Maine as long as I have. Staying here has given me many opportunities that I would not otherwise have had.

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gans. I greatly enjoyed my time in Bennett 224 and will probably never feel at home in an office without a beaded curtain in the doorway. I also thank the newest addition to the office that was my home: Kevin Roberge has a wonderful and treacherous knack for finding topics of conversation that completely derail any productivity in my immediate vicinity. I look forward to more of his expert "geek sniping" when I have a bit more time to truly appreciate his insights.

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Chapter 1
INTRODUCTION

The field of physics education research (PER) was born from a desire of university physics faculty to develop instructional strategies specifically for physics classes, and to better understand how people learn and think about physics concepts. Similar to other discipline-based forms of educational research, PER is primarily conducted by those who are knowledgeable in the specific content area: physicists and physics teachers. This allows the study of content-specific topics that may not be well understood by those outside the discipline.

The majority of the work that has been conducted in PER has focused on introductory physics students' understanding of basic topics in mechanics and electricity & magnetism.[2] A small fraction of PER studies have examined how upper-division physics students (primarily physics majors) reason about and understand concepts in advanced physics courses.[3–16] An even smaller fraction has looked at student understanding of topics in upper-division thermal physics courses.[17–23] This dissertation builds on the work that has been done in advanced thermal physics courses by identifying specific student difficulties and reporting on the development of curriculum materials designed to address these difficulties and enhance instruction in these courses.

The student population under investigation is particularly interesting, as they have completed several university-level physics courses and are beyond the level of entering introductory students, who may be considered novices; they are not, however, as advanced as professional physicists, who may be considered experts. They are somewhere in the middle: journeymen, as described by Bing and Redish.[24] Research in educational psychology has explored the differences between experts and novices in terms of their behaviors and reasoning strategies in various contexts.[25–29] Advanced undergraduate physics students provide a glimpse of physicists on their way to expertise. The examination of this journeyman population is enhanced by studying student understanding of thermal physics topics, as most of the students have not studied the field before. As I discuss in later sections, my research has identified situations in which these advanced undergraduate students display novice-like behavior, and situations in which they engage in expert-like reasoning

Previous studies have been conducted on student understanding of thermal physics topics both at the introductory and the advanced levels.[18–20, 30–32] Studies within the realm of classical thermodynamics have typically focused either on students' understanding of the First Law of Thermodynamics, state variables (internal energy, volume, pressure, temperature, etc.), and process variables (heat and work);[30, 31, 33] or on students' understanding of the Second Law of Thermodynamics and entropy.[18, 34] Little research has been conducted on student understanding of physical scenarios in which both the First and Second Laws of Thermodynamics are necessary for a full understanding. One such scenario is that of heat engines: devices that convert thermal energy into usable mechanical energy. The amount of mechanical energy that can be obtained per unit of thermal energy is restricted by all three laws of thermodynamics. Because a robust understanding of heat engines and Carnot's efficiency involves synthesizing the First and Second Laws of Thermodynamics appropriately — and because research shows that understanding either law by itself is not trivial (cf. Refs. 18, 32, & 35) — this seemed like a useful place for curriculum development efforts.

Studies within the realm of statistical mechanics have focused on student understanding of probabilities of discrete outcomes: coin flips, energy eigenstates, etc.[19, 20, 22] Little research has been conducted on student understanding of probability as it relates to continuous quantities.[21] The Boltzmann factor is a mathematical expression for probability as a function of energy, which can be either a discrete or a continuous quantity. An investigation of student understanding of the the Boltzmann factor, therefore, provides a natural extension to the work that has been conducted so far in the realm of statistical mechanics.

Other notable studies have examined students' (often lacking) understanding of the connections between physical processes and mathematical formalisms.[1, 36] Results from these studies indicate that the transition from using mathematics in a mathematics class to using it in a physics class is not trivial. This provides a rich area for research, as an understanding of the physical principles that underly mathematical formalism becomes more and more important in upper-division courses.

The goals of this dissertation are to identify and address student difficulties in two areas of thermal physics. In the realm of classical thermodynamics I examine student understanding of heat engines as they relate to both the First and Second Laws of Thermodynamics. In particular I investigate how students relate the Carnot efficiency and Carnot's theorem to the Second Law of Thermodynamics and its restrictions on changes of entropy. In the realm of statistical mechanics I examine student understanding of the applicability and origin of the Boltzmann factor. In particular I investigate students' use (or lack of use) of the Boltzmann factor in physically applicable situations as well as students' understanding of and appreciation for the physical reasoning and implications behind the mathematical expression for probability that is the Boltzmann factor.

With this research I attempt to answer several questions:

1. What specific difficulties do students display when answering questions and/or engaging in activities related to heat engines and the Carnot cycle?

2. What specific difficulties do students display when answering questions and/or engaging in activities related to the Boltzmann factor and the canonical partition function?

3. To what extent does instruction using guided-inquiry worksheet activities (tutorials) address these difficulties?

4. What difficulties persist after tutorial instruction, and what additional instruction may be necessary to address these difficulties?

The unifying theme of my research (apart from focusing on student understanding of topics in thermal physics) is the investigation of how students use particular mathematical expressions within relevant physical scenarios, as well as how they utilize the physical ideas embedded within those expressions in situations that do not require explicit use of the mathematics. The tutorials I have developed guide students through the mathematical derivations of two such expressions: Carnot's limit on thermodynamic efficiency, and the Boltzmann factor as an expression for probability. Starting from basic laws of physics and various definitions, I emphasize the physical connections between, and justification for, subsequent steps within a derivation.

Chapter 2 presents the basic physics relevant to heat engines and that relevant to the Boltzmann factor; more details are presented in Chapters 5 and 6 when they become an integral part of the research presented. Chapter 3 contains a review of relevant research that forms the basis for and background of my study. As mentioned above, little research exists on students' understanding of topics that relate to both the First and Second Laws of Thermodynamics or on students' understanding of the statistical treatment of complex systems. I present the work that has been previously conducted to identify student difficulties as well as that related to efforts made to address these difficulties.

Chapter 4 details the courses at the University of Maine in which my research was conducted, as well as several tutorials, previously developed by other researchers, that are used within those courses. These tutorials include student activities designed to enhance understanding of the First and Second Laws of Thermodynamics as well as tasks related to basic ideas of probability. I also discuss two courses at other universities where data were collected for my study. Chapter 4 also presents an overview of the research methods I used to conduct my study. Data were collected using written questions, videotaped classroom observations, teaching experiments, and clinical interviews. Data were examined using the framework of identifying specific student difficulties in order to develop instructional materials to address these difficulties.[37]

Chapter 5 presents evidence for the identification of several specific student difficulties with heat engines, as well as the details of the development of instructional materials (the *Heat Engines* tutorial) designed to improve student understanding of these concepts. While participating in the *Heat Engines* tutorial, students are led to recognize the need for an upper limit for thermodynamic efficiency based on the Second Law of Thermodynamics, and they derive Carnot's limit starting from the limit on changes in entropy dictated by the Second Law. Data suggest that students do not, in fact, gain a complete understanding of the importance and uniqueness of the Carnot cycle after lecture instruction alone, but that this understanding is enhanced by participating in the *Heat Engines* tutorial.

Similarly, Chapter 6 presents evidence for the identification of several specific student difficulties related to the Boltzmann factor, as well as the details of the development of the *Boltzmann Factor* tutorial, designed to improve student understanding of this mathematical expression. Within the tutorial students are guided through one derivation of the Boltzmann factor as it relates to multiplicity and, therefore, probability. Emphasis is placed on the connections between each mathematical step of the derivation rather than the final result; in this manner students are encouraged to develop a deeper physical understanding of the exponential relationship between energy and probability that is the Boltzmann factor. Data suggest that many students either do not use the Boltzmann factor when presented with relevant physical situations or that they do not recognize the physical meaning of the mathematical expression after lecture instruction. Evidence suggests, however that students gain an appreciation for when to use the Boltzmann factor, why it's useful, and its physical significance after engaging with the *Boltzmann Factor* tutorial.

Chapter 7 concludes by summarizing the findings of both halves of my dissertation, and discusses the results common to both halves. One such result is the usefulness of video data from classroom observations. These data provide evidence for the identification of several specific difficulties that were not evident from either written responses or student interviews; moreover, they provide evidence for which parts of the tutorials are useful for addressing student difficulties as well as areas that would benefit from revisions. Another common result was the instructional benefit of assigning a homework assignment to be completed before each tutorial. As I discuss in Chapters 5 and 6, these homework assignments were extremely beneficial in terms of cueing appropriate background information that is foundational to the derivations used in the tutorials. Moreover, data suggest that the advanced undergraduate population studied in my research differs from either the novice introductory student population or the expert physicist population. Suggestions are also discussed for future research related to difficulties identified from classroom observations and interviews that are not addressed by either the *Heat Engines* tutorial or the *Boltzmann Factor* tutorial.

Chapter 2
RELEVANT PHYSICS

The unifying theme of my research (apart from focusing on student understanding of topics in thermal physics) is the investigation of how students use particular mathematical expressions within relevant physical scenarios as well as how they utilize the physical ideas embedded within those expressions in situations that do not require explicit use of the mathematics. The tutorials I developed guide students through the mathematical derivations of two such expressions, starting from basic laws of physics and various definitions. I emphasize the physical connections between, and justification for, subsequent steps within a derivation. The two mathematical expressions I focused on are the limit on thermodynamic efficiency of heat engines (as it relates to the Carnot efficiency),

$$\eta \leq \eta_{\mathrm{C}} = 1 - \frac{T_{\mathrm{L}}}{T_{\mathrm{H}}}, \tag{2.1}$$

and the Boltzmann factor as an expression of probability,

$$P(E) \propto e^{\frac{-E}{k_B T}}. \tag{2.2}$$

This chapter presents the basic physics underlying each of these expressions. A more complete presentation is given in Chapters 5 and 6, as it becomes necessary to understand the physics as it relates to my research. I begin this chapter with a discussion of the basic laws of thermodynamics and their relation to heat engines. I then present the underlying ideas of statistical mechanics and how they relate to probability and the Boltzmann factor. A basic understanding of these principles is necessary to realize the implications of my results and those previously reported by other researchers.

2.1 Classical Thermodynamics

Classical thermodynamics is the study of the bulk properties of a system (pressure, volume, temperature, etc.) and how they change under various conditions. These changes are governed by the three laws of thermodynamics. The First Law of Thermodynamics (1^{st} Law) is a statement of energy transfer between different systems and is often considered a thermodynamic statement of energy conservation (provided one defines a "system" that includes all relevant and interacting bodies). The mathematical statement of the 1^{st} Law is given by,

$$\Delta U = Q - W, \tag{2.3}$$

where U is the internal energy of the system, Q is the heat transfer to the system, and W is the work done by the system.[1] The differential form of the 1^{st} Law may be written in terms of inexact differentials of Q and W or in terms of exact differentials of various state functions:

$$\mathrm{d}U = đQ - đW, \tag{2.4}$$

$$\mathrm{d}U = T\mathrm{d}S - P\mathrm{d}V + \mu\mathrm{d}N + \dots \tag{2.5}$$

where the exact differentials (d__) indicate that internal energy (U), entropy (S), volume (V), and number of particles (N) are all state functions of the system under investigation,[2] i.e., they are bulk properties of a system in thermodynamic equilibrium. The inexact differentials (đ__) indicate that heat and work are not state functions but rather depend on the specific process that is undergone to take the system from one thermodynamic state to another. Heat and work are, in fact, different types of energy transfer: heat being due to temperature differences between systems and work being due to a change in the external parameters (volume, number

[1] The 1^{st} Law may alternately be written as $\Delta U = Q + W$, with W defined as the work done *on* the system. I will exclusively define W as the work done *by* the system.

[2] As are temperature (T), pressure (P), and chemical potential (μ).

of particles, magnetization, etc.) of a system. The 1st Law essentially states that the change in internal energy of any thermodynamic system must be accounted for in one of these two ways and must be coupled to an equivalent change in energy of some other thermodynamic system.

The Second Law of Thermodynamics (2nd Law), on the other hand, allows for the conservation and creation of entropy, but not its destruction. Entropy is a thermodynamic state function that quantifies, in a sense, a system's disorder. Rudolph Clausius coined the term from the latin *entrepein* meaning "turning" or "changing" and proposed the definition,

$$\Delta S \equiv \int_{reversible} \frac{đQ}{T}, [38] \tag{2.6}$$

where ΔS is the change in entropy of a system due to a particular process, $đQ$ is the differential of the heat transfer during that process, and T is the temperature of the system. One additional factor is that the heat transfer in Eq. 2.6 must occur due to a reversible process for the definition to hold; in any spontaneous, irreversible process, the change in entropy will be greater than that described. One of the most interesting aspects of entropy is that even though it is defined by heat transfer, it is a state function of the system, i.e., the change in entropy only depends on the initial and final states of the system, not the process itself. The 2nd Law is embodied in the principle of maximizing entropy:

> The entropy of an isolated system increases in any irreversible [naturally occurring] process and is unaltered in any reversible [ideal] process;[38, p. 96]

or mathematically from the entropy inequality,

$$\Delta S_{universe} \geq 0, \tag{2.7}$$

where $S_{universe}$ is the total entropy of the universe, and the equality only holds for ideal reversible processes.[3] Eq. 2.7 may be expanded to consider the entropy changes of a thermodynamic system and its surroundings:

$$\Delta S_{system} + \Delta S_{surroundings} = \Delta S_{universe} \geq 0. \tag{2.8}$$

A canonical application of the 1st and 2nd Laws is a device that converts one form of energy into another, such as a heat engine or a refrigerator. A heat engine is a device that converts thermal energy into usable work. To accomplish this, a heat engine requires three things: a high-temperature (T_{H}) thermal reservoir, a low-temperature (T_{L}) thermal reservoir, and a working substance (e.g., a gas in a cylinder with a piston). The reservoirs are needed to energy with the working substance without being affected themselves (ideally, infinite heat capacity, no temperature change). The working substance is the stuff that actually does the work. A heat engine operates in a cycle so that the working substance repeatedly returns to its original thermodynamic state. In the course of this cycle, an amount of energy (Q_{H}) is transferred from the T_{H}-reservoir to the working substance; the working substance transfers energy to its surroundings by doing work (W); and some energy (Q_{L}) is transferred from the working substance to the T_{L}-reservoir. After one cycle the heat engine is essentially right back where it started, ready to do more work. If the working substance is considered to be the "system," then for each complete cycle the 1st Law becomes,

$$\Delta U_{\mathrm{ws}} = Q_{net} - W_{net} = Q_{\mathrm{H}} + Q_{\mathrm{L}} - W = |Q_{\mathrm{H}}| - |Q_{\mathrm{L}}| - W = 0, \tag{2.9}$$

since the working substance returns to its original state (defined by equilibrium values of U, V, P, T, S, etc.). This implies that,

$$Q_{net} = W_{net}, \quad \text{and} \quad |Q_{\mathrm{H}}| - |Q_{\mathrm{L}}| = W. \tag{2.10}$$

[3]In principle any isolated system could be chosen, but the universe is a natural choice as it is by definition isolated.

For this generic heat engine the change in entropy of the "universe" after one complete cycle will be the combination of the changes in entropy of the two reservoirs (the entropy of the working substance does not change after one complete cycle since entropy is a state function). Since the temperature of a thermal reservoir remains constant, Eq. 2.6 becomes,

$$\Delta S_{reservoir} = \frac{Q_{to\ reservoir}}{T_{reservoir}}, \tag{2.11}$$

and the total entropy change of the universe will be,

$$\Delta S_{uni} = \Delta S_{\mathrm{L}} + \Delta S_{\mathrm{H}} = \frac{|Q_{\mathrm{L}}|}{T_{\mathrm{L}}} - \frac{|Q_{\mathrm{H}}|}{T_{\mathrm{H}}}. \tag{2.12}$$

Combining Eqs. 2.7 & 2.12 one gets an expression of the of 2$^{\mathrm{nd}}$ Law for heat engines,

$$\frac{|Q_{\mathrm{L}}|}{T_{\mathrm{L}}} - \frac{|Q_{\mathrm{H}}|}{T_{\mathrm{H}}} \geq 0, \tag{2.13}$$

or,

$$\frac{|Q_{\mathrm{L}}|}{T_{\mathrm{L}}} \geq \frac{|Q_{\mathrm{H}}|}{T_{\mathrm{H}}}. \tag{2.14}$$

A refrigerator, in its simplest form, is a heat engine that is run backwards. The work W is done *on* the working substance to transfer the energy Q_{L} away from the T_{L}-reservoir and deliver the energy Q_{H} to the T_{H}-reservoir.

For any mechanical device, one typically wants to evaluate its cost-effectiveness, generally in the form of "what you get" ÷ "what you pay for." For the case of heat engines, "what you get" is the work (W), and "what you pay for" is the energy, Q_{H}, from the T_{H}-reservoir. This yields the definition of thermodynamic efficiency,

$$\eta \equiv \left| \frac{W}{Q_{\mathrm{H}}} \right|. \tag{2.15}$$

According to the 1$^{\mathrm{st}}$ Law, $0 \leq \eta \leq 1$, since more energy cannot be extracted from the working substance than was put in. As will be discussed below, however, the

2$^{\text{nd}}$ Law imposes a stricter upper limit on thermodynamic efficiency. An analogous quantity for a refrigerator is called the coefficient of performance:

$$\zeta \equiv \left| \frac{Q_{\text{L}}}{W} \right| \tag{2.16}$$

where "what you get" is the energy (Q_{L}) removed from the T_{L}-reservoir, and "what you pay for" is the work (W, often supplied by an electric outlet). A heat pump is a device that is identical to a refrigerator with the exception that the desired quantity is the energy, Q_{H}, delivered to the T_{H}-reservoir. The "coefficient of performance" for a heat pump is then

$$\xi \equiv \left| \frac{Q_{\text{H}}}{W} \right| . \tag{2.17}$$

One of the more elusive aspects of the 2$^{\text{nd}}$ Law is that it was first formulated before Clausius proposed his definition of entropy. This formulation is presented in several independent statements about the effect of the 2$^{\text{nd}}$ Law on heat engines and refrigerators, including Clausius' original statement:

> It is impossible to construct a device that operates in a cycle and whose sole effect is to transfer heat from a cooler body to a hotter body;[38, p. 90]

and the Kelvin-Planck statement (credited independently to Lord Kelvin and Max Planck):

> It is impossible to construct a device that operates in a cycle and produces no other effect than the performance of work and the exchange of heat with a single reservoir;[38, p. 90]

and Carnot's theorem:

> No heat engine operating between two reservoirs can be more efficient than a Carnot engine [defined by alternating reversible isothermal and adiabatic processes] operating between those same two reservoirs.[38, p. 91]

One can work out the consequences of applying each of these three statements of the 2nd Law to Eq. 2.13. The device proposed in the Clausius statement would require $Q_{\text{H}} = -Q_{\text{L}} > 0$. Defining, $Q \equiv |Q_{\text{H}}| = |Q_{\text{L}}|$, Eq. 2.13 becomes,[4]

$$\frac{-Q}{T_{\text{L}}} + \frac{Q}{T_{\text{H}}} = Q\left(\frac{1}{T_{\text{H}}} - \frac{1}{T_{\text{L}}}\right) \geq 0, \tag{2.18}$$

which is not true since $T_{\text{H}} > T_{\text{L}}$, by definition. The device proposed in the Kelvin-Planck statement would require $|Q_{\text{H}}| > 0$ and $|Q_{\text{L}}| = 0$ in Eq. 2.14, which is plainly false.

To realize the connection between Carnot's theorem and the entropy inequality form of the 2nd Law one must also consider the limits on thermodynamic efficiency alluded to above. Referring to Eq. 2.9 one may rewrite Eq. 2.15 as

$$\eta = \frac{|Q_{\text{H}}| - |Q_{\text{L}}|}{|Q_{\text{H}}|} = 1 - \frac{|Q_{\text{L}}|}{|Q_{\text{H}}|}. \tag{2.19}$$

To maximize this efficiency one must look at the limits imposed by the 2nd Law via Eq. 2.14 by obtaining the relationship,

$$\frac{|Q_{\text{L}}|}{|Q_{\text{H}}|} \geq \frac{T_{\text{L}}}{T_{\text{H}}}, \tag{2.20}$$

which indicates that,

$$1 - \frac{|Q_{\text{L}}|}{|Q_{\text{H}}|} \leq 1 - \frac{T_{\text{L}}}{T_{\text{H}}}, \tag{2.21}$$

so that,

$$\eta \leq \eta_{\text{C}} = 1 - \frac{T_{\text{L}}}{T_{\text{H}}}. \tag{2.22}$$

The right-hand-side of Eq. 2.21 (and Eq. 2.22) is defined as Carnot's efficiency (η_{C}) due to the fact that Carnot proposed a heat engine that consists of taking a working substance through an alternating sequence of reversible isotherms (constant temperature processes) and reversible adiabats (processes with no heat transfer) that

[4]Note, the signs have switched because heat transfer is now toward the T_{L}=reservoir and away from the T_{L}-reservoir.

achieves precisely this efficiency. Carnot did not have the benefit of our modern definition of entropy, but his proposed theoretical cycle allows the entropy of the universe to remain unchanged by using only ideal reversible processes. More on Carnot's formulation of this cycle is presented in Chapter 5.

It is clear from Eq. 2.21 that the only way to change the maximum efficiency of a Carnot cycle is to either lower the temperature of the T_L-reservoir or raise the temperature of the T_H-reservoir. The Third Law of Thermodynamics may now be applied which states that the entropy of a system is zero only when its absolute temperature is zero (i.e., $S(0\,\mathrm{K}) = 0\,\mathrm{J/K}$), and that it is impossible to achieve this temperature. The 3$^{\mathrm{rd}}$ Law is not directly applicable to my research as I do not ask students to lower the temperature of the T_L-reservoir while answering questions regarding heat engines.

2.2 Statistical Mechanics

Statistical mechanics is the probabilistic study of thermodynamic systems in terms of their constituent parts. The main premise is that, when examining a collection of particles, one may use information about the microscopic interactions of the individual particles to make claims about the probability that the macroscopic system is in a particular thermodynamic state. To explore this claim, consider the iconic binary example of flipping coins. The probability of flipping one coin and getting heads (or tails) is 50%. The probability of flipping two coins and getting two heads is the joint probability, or the product of the individual probabilities: $P(HH) = P(H)P(H) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. This is of course the same probability as each of the other three possible combinations: H-T, T-H, and T-T. More interesting conclusions may be drawn if one groups several of these combinations together, e.g., the H-T and T-H combinations will both be considered "one head, one tail." Grouping these outcomes together, one defines the "macrostate" by the number of heads (or tails) obtained from a series of coin flips. Each individual combination of flips is called a "microstate," and there are often many microstates for a given macrostate. The number of microstates that make up a particular macrostate is known as the "multiplicity," denoted by ω. Looking at the two-coin example, there are three different macrostates (zero, one, or two heads), but four distinct microstates (H-H, H-T, T-H, T-T), and the multiplicity depends on the macrostate: $\omega(0) = \omega(2) = 1$, $\omega(1) = 2$. The probability for any given macrostate is simply the multiplicity of that macrostate divided by the total number of microstates,

$$P(i) = \frac{\omega(i)}{\sum_j \omega(j)}. \tag{2.23}$$

The results get more interesting as the sample size increases. For example, in the case in which five fair coins are flipped, one possible macrostate is the outcome of getting three heads and two tails, one microstate that would yield this macrostate is the sequence H-H-T-H-T; nine other microstates would give the same macrostate. The fundamental assumption of statistical mechanics states that, "all accessible microstates are equally probable" for an isolated system in thermal equilibrium; this is equivalent to assuming all of the coin-tosses to be independent of one another (i.e., the exact sequence H-H-H-T-H is just as likely as the exact sequence T-H-H-T-T).[39] Consequently a macrostate that can be realized with relatively many different microstates (e.g., three heads: ten microstates) is more probable than a macrostate that can only be realized with relatively few microstates (e.g., five heads: one microstate), i.e., probability is proportional to multiplicity as indicated in Eq. 2.23. If the number of particles within the system is sufficiently large these probabilities become certainties, and one enters the realm of classical thermodynamics. In statistical mechanics a *macro*state is defined by the thermodynamic properties of the system, and a *micro*state is defined as one of the ways in which the constituent parts of the system can be arranged to achieve that macrostate.

One difficulty with the transition from statistical mechanics to classical thermodynamics is that, as the number of particles in a system increases, the multiplicity increases exponentially (e.g., flipping N coins yields 2^N distinct microstates). Moreover, since classical thermodynamics does not consider the constituent parts of a system, there is not an obvious connection between multiplicity and any of the thermodynamic properties discussed above. Thankfully, Ludwig Boltzmann provided us with the iconic connection between classical thermodynamics and statistical mechanics,

$$S = k_B \ln(\omega), \tag{2.24}$$

where S is of course entropy, ω is multiplicity, and k_B is a factor that provides the correct units and is known as Boltzmann's constant ($1.381\times10^{-23} J/K$). The logarithmic relationship depicted in Eq. 2.24 implies one of the most important aspects of statistical mechanics: since the total multiplicity of two independent systems is the product of the two individual multiplicities ($\omega_{total} = \omega_1\omega_2$), the total entropy is the sum of the two individual entropies ($S_{total} = S_1 + S_2$). In this case "independent" means that the microstate of one system is independent of the microstate of the other system, but the macrostate of one may still depend on the macrostate of the other. One example of this is two interacting many-particle thermodynamic systems: the total combined energy of the two systems may be fixed so that the energy of one system depends of the energy of the other ($E_1 = E_{total} - E_2$), but how that energy is distributed amongst the particles within each system is completely arbitrary. In this way the combined macrostate of the two systems will be defined by the same information that defines each of the macrostates of the individual systems (e.g., the values of E_1 and E_2 in the example). Eq. 2.24 also provides a more reasonable quantity by virtue of the logarithm; entropy is proportional to the number of particles, N, rather than an exponential of N, e.g., for the coin flips the total entropy is $S = k_B \ln\left(2^N\right) = k_B N \ln(2)$. This expression, however, assumes that the macrostate is only defined by the number of particles or events (N), *not* by the number that are in a particular state (e.g., "heads").

Two of the main questions in statistical mechanics are: How does one define "macrostate" and "microstate" within a particular physical context? and How does one calculate the multiplicity of each macrostate based on known quantities? The second of these questions is often more difficult to answer. In the context of coin flipping (or any other two-state system), the binomial distribution (discussed further in section 3.3.2) provides a mechanism for calculating the number of arrangements for a given number of heads. If instead one considers the physical context of a small thermodynamic system in equilibrium with a large energy reservoir, then the macrostate is defined by the energy of the system, and the microstate is defined by the energy distribution amongst all of the particles in both the system and the reservoir. The parameters of this scenario dictate that the small system of interest and the reservoir have the same constant temperature; that small energy transfers may occur across the boundary between the two; and that the number of particles in the system is fixed. Since the reservoir is so much larger than the system (by definition) the probabilistic effect of the multiplicity of the reservoir overshadows that of the system, and one need only be concerned with the microstates of the reservoir.[5]

The Boltzmann factor (right hand side of Eq. 2.2) is a mathematical expression of the multiplicity of the reservoir in terms of the energy of the system. Since

[5]In other words, since the system is so small, its multiplicity changes very little between different energy macrostates, but the multiplicity of the reservoir changes significantly because it is so much larger.

the probability of a macrostate is proportional to the corresponding number of microstates, the probability of the system being in a specific microstate, which has a particular energy, is proportional to the Boltzmann factor for that energy. More details on the derivation and origin of the Boltzmann factor are included in Chapter 6.

Chapter 3
RELEVANT RESULTS FROM PHYSICS EDUCATION RESEARCH

Physics education research (PER), broadly speaking, is the study of how people of all ages and all experience levels think about, understand, and learn topics in and related to physics. Much of the research in PER has consisted of examining physics students' responses to various written and/or verbal prompts, analyzing these responses for common themes, and attempting to improve the accuracy and consistency of students' statements.[2] As mentioned previously, my research fits well within this genre as I identify student difficulties within advanced thermal physics courses and develop materials to address those difficulties.

In this chapter I present my motivation for studying advanced thermal physics students: these students are neither novices who have never encountered a physics class before, nor experts who have mastered the field; they are somewhere in between. Furthermore, thermal physics is a novel subfield for many of students that provides an opportunity for both expert and novice behavior to be made manifest. I continue by giving an overview of student difficulties that have been previously identified by other researchers, some of the methods that they used to identify these difficulties, and the steps they have taken to address the various difficulties.

3.1 Advanced Physics Students: A Fascinating Population

The advanced undergraduate physics student population (typically within their third or fourth year of physics instruction) provides a fascinating opportunity for educational research. Considerable research over the past several decades has been dedicated to the study of expertise and the differences between experts and novices within particular domains.[25–29] In physics, introductory students may be considered novice physicists who have been previously exposed to very little of the field. On the other hand, professional physicists, physics professors, and even students nearly the end of their graduate studies may be considered expert physicists. But where do advanced undergraduate students fit in?

Expert physicists were all, at one time, advanced undergraduate students; and these advanced undergraduate students were all, at one time, novice introductory students. Due to the in-between status of these advanced undergraduate students, Bing and Redish have described them as "journeyman physicists": displaying some of the same behaviors as novices in certain situations, and the same behaviors as experts in other situations.[24] Their work examines students' problem-solving abilities within various upper-division undergraduate courses. They find that advanced students are much more fluid in their choice of problem-solving strategy than introductory students.[24] While introductory students stubbornly stick to one solution strategy (even in the face of utter failure), upper-division students switch strategies and justifications when their original line of reasoning seems to breakdown or come to a dead end. Most students in their study, however, failed to come to a satisfactory final solution, indicating that they have not yet mastered the material and cannot be considered expert physicists.

Within the context of thermal physics, Meltzer has identified some similarities and differences between upper-division and introductory students.[40] He provides evidence that upper-division students have the capacity for greater learning gains than their counterparts in introductory courses but admits that it is difficult to determine whether this potential comes from a transfer of knowledge or learning skills from previous courses, or the application of skills students have always possessed. Meltzer also warns that upper-division students have as much difficulty with unfamiliar topics as their introductory fellows.[40]

My research examines the behavior of advanced undergraduate, journeyman physicists in thermal physics courses. In particular, I identify specific difficulties that these students have with thermal physics topics. The following sections describe work that has previously been conducted by other researchers to identify student difficulties with thermal physics and related topics (at both the novice and journeyman levels) and efforts that have been made to address these difficulties.

3.2 Identifying Specific Student Difficulties

When discussing the ways in which students learn and the ways they think about their world, two well-known schools of thought repeatedly show up: concepts (conceptions,[41] conceptual frameworks[42]) and primitives (phenomenological primitives – p-prims,[43] facets,[44] resources,[45, 46] knowledge in pieces). Both of these frameworks for analyzing student performance in various situations involve attempting to describe mental structures that students apply in certain situations. The stability, applicability, and relevance of these structures varies greatly depending on the particular framework used.

Heron, however, has proposed a third framework for analyzing student performance: that of specific difficulties.[37] Heron describes specific difficulties as "incorrect or inappropriate ideas expressed by students, or flawed patterns of reasoning."[37] The identification of specific difficulties is driven by the goal of curriculum development: one lesson can be developed to address one difficulty. Difficulties may describe *what* a student does in a particular situation, *why* the student does it, or both, with the emphasis on describing a phenomenon before interpreting it. Furthermore, Heron emphasizes that her descriptions of students' "beliefs" are an assumption on the part of the researcher.[37]

Several researchers in PER have (either implicitly or explicitly) used the specific difficulties framework to analyze student performance.[30–32, 47] Identification of specific student difficulties has been especially useful for developing curricular materials that improve student understanding of a particular topic. As one of my main goals is the development of curricular materials for use in upper-division thermal physics courses, I analyze written and video data with an eye toward identifying specific difficulties that advanced undergraduate students have with thermal physics topics. Consistent with this perspective is the description of student responses with the "reasoning" used to back up a particular response.[37] There is some debate in the literature as to the cognitive implications of ascribing reasoning to students, rather than post-hoc justifications of the intuitive response.[48] However, this literature is referring primarily to introductory-level students with limited prior content knowledge, rather than the journeymen studied here. It is reasonable that advanced undergraduates will have specific ways of thinking about the concepts based on their prior learning in physics (and math, and other sciences). Furthermore, in this study I have been able to probe student understanding of topics with multiple instruments, using multiple methods, across multiple weeks for the same students. I present evidence across multiple questions suggesting that students at this level have consistent reasoning about a particular concept, supporting the notion that students are reasoning about the concepts in order to answer the question rather than justifying an intuitive response post-hoc.

A more detailed description of my analysis methods is included in Chapter 4. The following sections discuss some specific student difficulties that have already been identified in introductory and/or upper-division thermal physics contexts. These sections are meant to provide a foundation for my research by presenting the difficulties that have already been identified and suggesting topics that have not yet been fully explored.

3.2.1 Identification of Student Difficulties in Thermodynamics

Relatively few studies in PER have focused on students' understanding of topics related to thermal physics, but they shed light on many student difficulties at the pre-college, introductory undergraduate and advanced undergraduate levels. In this section I examines the work that has been conducted in the context of topics related to energy and the First Law of Thermodynamics, and topics related to entropy and the Second Law of Thermodynamics. I begin, however, by presenting a summary of the literature on pre-college students' understanding of thermal physics concepts.

Many studies report students' confusion between the basic concepts of heat and temperature. From a physicist's perspective, temperature is a measure of the average energy per particle of an object or substance, and heat is a measure of energy transfer due to a temperature difference between two objects or substances. Research shows, however, that students often do not make this distinction. As a summary report, Arons discusses the "well known" phenomenon that students do not discriminate between the concepts of heat and temperature and attributes this to their use in everyday language as "primitives" whose meaning is obvious.[49, p. 118] Harrison, Grayson, and Treagust, on the other hand, have directly identified several difficulties in high school students' ideas about heat and temperature including: 1) that heat and temperature are the same thing, and 2) that two objects in thermal equilibrium with each other could have different temperatures.[50]

Jasien & Oberem studied college students' and in-service teachers' ideas about heat and temperature.[51] They found that in-service middle and high school science teachers typically displayed higher levels of understanding in elementary concepts (compared to college students who had taken between one and four semesters of physical science courses), but even they were no more than 70% correct on the survey questions. Jasien & Oberem reported what they considered to be the most notable confusions found in their work: "1) the meaning of thermal equilibrium, 2) the physical basis for heat transfer and temperature change, and 3) the relationship between specific heat, heat capacity and temperature."[51] Yeo & Zadnik used the results from these and other studies to create a conceptual survey (similar to the *Force Concept Inventory* or *Force and Motion Conceptual Evaluation*) to assess students' understanding of basic concepts in heat and temperature.[52–54] They list possible student difficulties in four categories: 1) Students' conceptions about heat, 2) Students' conceptions about temperature, 3) Students' conceptions about heat transfer and temperature change, and 4) Students' conceptions about "thermal properties" of materials.[52] They also report the ways in which their survey assesses students' ideas and diagnoses their difficulties based on the various survey items. These studies represent a small fraction of the total work that has been done to investigate students' ideas about heat and temperature, but they provide a general basis for work in more advanced areas of thermal physics that is discussed below.

3.2.1.1 Difficulties with Energy and the First Law of Thermodynamics

Loverude, Kautz, and Heron report on physics students' ideas related to, and difficulties applying, the 1st Law. In particular they are interested in students' abilities to reason about temperature differences due to work in adiabatic processes (i.e., no heat transfer, $\Delta S = 0$).[32] Loverude *et al.* found that only between 10% (introductory) and 50% (upper-division) of students correctly used the 1st Law ($\Delta U = Q - W$) and the definition of thermodynamic work ($W = \int P \mathrm{d}V$) to predict the temperature change of an ideal gas due to an adiabatic process. Many students instead used the ideal gas law ($PV = nRT$) to reason about temperature changes even though insufficient information was provided to determine relative changes in pressure and volume.[32] Loverude *et al.* also report on students' tendency to discuss thermodynamic properties and changes in those properties interchangeably (e.g., P and

ΔP). This is often accompanied by students making statements about the "change in heat" or the "change in work" even though these are nonsensical terms, as heat and work themselves merely represent different types of energy transfers.[32] Additionally, up to 45% of students incorrectly state that the work done by a system depends only on the beginning and ending states, rather the specific process the system goes through.

As a follow-up to students' inappropriate use of the ideal gas law discussed above, Kautz, Heron, Loverude, and McDermott report on students' difficulties interpreting and applying the ideal gas law from both a macroscopic and a microscopic perspective.[30, 31] Kautz *et al.* identify several trends within student responses including the assumption that $P \propto 1/V$ or the assumption that $P \propto T$ for any process. They also identified several specific difficulties that students exhibited when discussing the concepts of pressure, temperature, and volume.[30] Kautz, Heron, Shaffer, and McDermott report on students' difficulties applying a microscopic perspective to ideal gas law scenarios as well. They found that as few as 10% of students gave correct answers using correct reasoning on questions requiring a microscopic model of an ideal gas.[31] Some common student errors identified by Kautz *et al.* include "assuming that lower (greater) particle density implies lower (greater) temperature," "assuming that molecular collisions generate kinetic energy," and "not recognizing the substance independence of the ideal gas law." Based on these findings Kautz *et al.* created a tutorial to help students better understand the implications of the ideal gas law (discussed further in section 3.3.1).[30, 31]

Monteyne, Gonzalez, and Loverude discuss difficulties that some chemistry students display while attempting to connect microscopic and macroscopic models of ideal gases. In particular they notice an asymmetry of sorts in that "students are more successful in linking the [microscopic] to macroscopic realms than the macroscopic to [microscopic] realms."[55]

Meltzer reports on students' understanding of heat, work, and the 1st Law when answering questions using P-V diagrams.[33] Graphs that show pressure on the ordinate and volume on the abscissa (P-V diagrams) are particularly prevalent and useful in classical thermodynamics. One benefit of this representation is that integrating pressure with respect to volume ($\int P\mathrm{d}V$) gives the thermodynamic work done during a particular process. Meltzer asked students to compare the work done, heat transfer, and change in internal energy for two different processes on a P-V diagram that had the same initial states and the same final states. The correct answer is that the change in internal energy must be the same for both since U is a state function, but the work done and heat transfer depend on the particular process, i.e., the different paths taken through the diagram represent different functions, $P(V)$. In particular, the process represented by a higher path on the P-V diagram generates more work since the area under the curve (representing the integral of the function) is greater. The comparisons of heat transfer may be determined by considering the work done in each process and the fact that the change in internal energy must be the same by applying the 1st Law.[33]

One prevalent difficulty that Meltzer reports is students' apparent belief that work and heat are state functions.[33] Up to 22% of students explicitly mention the path independence of work, and up to 44% mention the same for heat. Moreover, approximately a third of students who recognized that work is not a state function explicitly stated that heat transfer is path independent. Meltzer also found that up to 56% of students indicated that the net work done and the net heat transfer over a cyclic process would be zero. Furthermore, only 11% of students gave written explanations with their answers that indicated a correct use of the 1st Law; and only 22% of interview participants correctly generated a P-V diagram for a given scenario.[33]

3.2.1.2 Difficulties with Entropy and the Second Law of Thermodynamics

Studies to investigate children's understanding of concepts related to entropy have been conducted with students as young as kindergarten. Shultz and Coddington report that children as young as six years old have begun to develop intuitive ideas related to entropy when presented with a physical system that starts very far from equilibrium and asked to predict the state of the system after a given amount of time.[56] By age 15 most students correctly predict the physical system's tendency toward the equilibrium state; moreover, 15-year-olds display significantly more surprise when presented with a "trick" experimental outcome, in which equilibrium is not obtained, than their younger counterparts. Shultz and Coddington also report that students of all ages are better able to understand and articulate entropic concepts when relating to an experiment using discrete quantities (marbles of different colors being shaken together in a box) than when relating to those using continuous quantities (the water level in two different beakers connected by a thin tube).[56]

Kesidou and Duit also report that 15-year-olds have an intuitive sense of the Second Law of Thermodynamics (2^{nd} Law) and entropy as it relates to equilibration and irreversible processes.[57] They elaborate, however, that the framework on which these students base their claims is "much vaguer" than a physicist's and that "most students did not learn physics conceptions that would facilitate a deeper understanding of their conviction."[57, p. 97] In particular, students' frameworks often centered around a cause-and-effect scheme in which real processes are irreversible because causes do not exist to reverse them (e.g. a stone that falls to the ground does not spontaneously return to its initial height because there is no upward force present).[57] It is clear from these studies that students entering university physics courses without having been previously instructed regarding entropy and the 2^{nd} Law have some intuitive ideas about equilibration and irreversibility; these ideas are not, however, based on a robust understanding of the concept of entropy as a physicist would define it.

Several studies in recent years have focused on student understanding of entropy and the 2nd Law in both introductory and upper-division undergraduate physics courses.[17, 18, 35] Christensen, Meltzer, and Ogilvie investigated introductory undergraduate physics students' ideas regarding entropy in the context of real, spontaneous processes.[18] One focus of their study was students' abilities to arbitrarily define a thermodynamic "system" and its "surroundings" within a given problem. In thermodynamics the term "system" is merely shorthand for "the system of interest at the moment." The definition of what is considered the "system" may be different for each problem encountered and may change in the middle of a solution depending on the calculations being performed. One difficulty they found was an over-generalization of the 2nd Law statement, "During a spontaneous process, the entropy of an isolated system must always increase," in which the incredibly important designator "isolated" is ignored by some students, leading to the conclusion that the entropy of a system always increases. Christensen *et al.* report that more than a quarter of students in their introductory physics course stated that the entropy of some thermodynamic system would increase when insufficient information was given to determine the direction of heat transfer between the system and its surroundings (compared to 9% who said the entropy would decrease).[18] Another major finding of Christensen *et al.* was students' tendency to treat entropy as a conserved quantity. That is, many students prior to instruction believed that the total entropy of a system and its surroundings could not be changed during a real process, when in actuality the combined entropy of a system and its surroundings must *always* increase during a real process according to the 2nd Law.

Cochran & Heron report on student difficulties in introductory physics in the context of heat engines.[35] They found that many students did not apply the 2nd

Law correctly to determine whether or not a proposed device (e.g., heat engine or refrigerator) was physically possible. As discussed in section 2.1, for a heat engine (or a related device) to function, it must obey the three laws of thermodynamics, and the working substance must operate in a complete cycle (to repeatedly return to its initial state).

Cochran & Heron report that 25% of introductory students only used the 1st Law to determine the feasibility of various devices after lecture instruction. Another 15% of students claimed that a heat engine would function if the efficiency was less than 100%, and other students used arbitrary limits that were not explicitly related to the Carnot efficiency. They also report that some students seemed not to realize that Eq. 2.10 is derived from the 1st Law for a cyclic process.[35] The results reported by Cochran & Heron are directly applicable to my work, as I identify upper-division students' difficulties with heat engines.

Bucy, Thompson, and Mountcastle discuss students' understanding of entropy changes in upper-division thermal physics courses.[17] In particular they find that students have great difficulty applying the state function property of entropy to physical systems. They asked students to compare the change in entropy of two samples of an ideal gas due to two different processes: an isothermal expansion and a free expansion. The students are told that the two samples start in the same initial thermodynamic state and are given enough information to determine that they must end in the same final state as well, yielding the same entropy change for each gas. One incorrect idea expressed by students is that the change in entropy due to a free expansion is zero because no heat transfer occurs. Another incorrect idea was that the change in entropy of the gas due to the isothermal expansion was zero since it is a reversible process, and that due to the free expansion was positive because it is an irreversible process. These findings agree with Christensen's data indicating introductory students' over-generalization of the entropy inequality statement of the

2^{nd} Law.[17, 18] Bucy *et al.* also report that many students relate changes in entropy (correctly or incorrectly) to other more familiar thermodynamic quantities, such as temperature, work, and/or heat transfer.[34]

3.2.2 Identification of Student Difficulties in Statistical Mechanics

Woefully little research has been conducted on student understanding of topics in statistical mechanics. This lack is mostly due to the fact that statistical mechanics is necessarily an upper-division course and the lack of research on upper-division students' understanding of physics in general. Mountcastle, Bucy, and Thompson investigated student understanding of probability and uncertainty in upper-division statistical mechanics and laboratory courses.[21] Using several written questions, students were asked about the most likely outcome of an experiment as well as the uncertainty related to the number of experimental trials. In the context of coin flips, fewer than half of the students correctly predicted that increasing the number of trials (n) would reduce the relative uncertainty ($\frac{\Delta a}{a}$) of the most probable outcome (reported as $a \pm \Delta a$, where $a = \frac{n}{2}$). Up to 33% of students indicated that the relative uncertainty would not change as n increased, and 62% of them stated that it would cover all possible outcomes ($\Delta a = \frac{n}{2}$). In contrast, Mountcastle *et al.* found that all students correctly indicated that more experimental measurements reduced experimental uncertainty in the context of measuring rainfall during the same pretest.[21] This indicates that students are familiar with the concept of minimizing experimental uncertainty but do not necessarily apply this to the case of flipping coins. Finally, Mountcastle *et al.* report that students answer a corresponding coin toss uncertainty question correctly after instruction on probability within a statistical mechanics context.[21]

Loverude has conducted several studies on students' understanding of statistical topics within a one-semester thermal physics course that combines classical thermodynamics and statistical mechanics.[19, 20] In the first of these studies, Loverude investigated students' understanding of probability distributions within a binary system (flipping coins, gender of children, spin-$\frac{1}{2}$ Ising model, etc.) as well as their abilities to distinguish a microstate from a macrostate in a given context. He found that up to 45% of students were unable to correctly determine the probability of obtaining all heads in a series of coin flips. Loverude also found that up to 37% of students gave answers to free-response questions that indicated a confusion between microstates and macrostates (e.g., stating that all *macro*states are equally probable).[19]

In a subsequent study Loverude looked at students' abilities to reason about two interacting thermodynamic systems. He used the contexts of balls placed in boxes as well as interacting Einstein solids.[20] The Einstein solid is a model of a solid substance in which each atom in a simple cubic lattice is connected to its six nearest neighbors by identical springs. In the scenario in which two Einstein solids are in thermal contact, a macrostate is defined by the amount of energy within each solid (which is quantized in units based on the frequency of oscillation of the springs), and the microstate is defined by the way in which the energy is distributed among the individual oscillators in each solid. This physical system is more complicated than the binary system discussed above, as each atom may have no energy, one unit of energy, or many units of energy. The amount of energy any single oscillator can have is only limited by the total amount of energy in the two solids combined.

Loverude found that many students have difficulty determining the combined multiplicity of the two solids together based on each of their individual multiplicities.[20] Up to 60% of students added the multiplicities of constituent parts rather than multiplying them. Loverude also found that students often state that the most probable macrostate is the one in which each solid has the same amount of energy. This is true of two solids that are the same size but incorrect when discussing two interacting solids of different sizes. What these students didn't attend to is the fact that the total energy must be distributed evenly among all oscillators within the solids, not between the solids themselves.[20] Loverude's results provide the foundation for studies into students' understanding of the statistical treatment of systems whose states are defined by continuous (rather than discrete) quantities. Another such study that builds on this foundation is my investigation of students' understanding of the Boltzmann factor, presented in Chapter 6.

3.2.3 Identification of Student Difficulties in Mathematics

Along with identifying specific student difficulties with thermal physics topics, researchers have investigated student understanding of several related topics. As seen in Chapter 2, a full description of many interesting physics principles is impossible without mathematics accompaniment. In the following sections, I present some of the results from studies into student understanding of mathematics that are relevant to my research. First, I describe results from investigations into students' understanding of the mathematics underlying much of the physics in the upper-division courses. Second, I present some results from studies of pre-college students' reasoning about rational numbers. These results are relevant for the analysis of one of the questions that I have asked students during my research.

3.2.3.1 Difficulties with the Math-Physics Connection

Several studies document observations regarding student difficulty connecting physical intuition and interpretation in thermodynamics to appropriate mathematical formalism.[1, 58, 59] Thompson, Bucy, and Mountcastle discuss student successes and difficulties with partial differentiation in the context of the Maxwell relations. They report that students often succeed on problems in which only an algorithmic understanding is required but tend to fail when more conceptual understanding is needed.[59] Many students can produce various Maxwell relations but do not indicate that they understand the physical significance behind them. Furthermore, students are more likely to be able to give a verbal interpretation when presented with a particular mathematical expression of a partial derivative than to generate an appropriate derivative given a verbal description of a physical scenario.[59]

As part of a follow up study, Bucy, Thompson, and Mountcastle discuss further student difficulties with partial differentiation in the context of isothermal compressibility (κ) and thermal expansivity (β); in particular, students displayed great difficulty employing mixed second-order partial derivatives.[58] Since β and κ are defined using partial derivatives of volume (with respect to temperature and pressure, respectively) their derivatives are by definition second-order partial derivatives of volume. A common student difficulty in this context was stating that the mixed second-order partial derivatives (i.e., $\left(\frac{\partial \beta}{\partial P}\right)_T$ and $\left(\frac{\partial \kappa}{\partial T}\right)_P$) are identically zero, "since P has already been held constant for β and T has already been held constant for κ."[58] This difficulty was not observed in the context of Maxwell relations, however, a fact that Bucy *et al.* attribute to Maxwell relations (e.g., $\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial P}{\partial S}\right)_V$) being seen as first-order partial derivatives of thermodynamic variables (V, S, P, and T) rather than mixed second-order partial derivatives of the thermodynamics potentials (U, H, F, and G).[58]

Pollock, Thompson, and Mountcastle reexamined Meltzer's investigations of student understanding of the 1$^{\text{st}}$ Law and P-V diagrams by removing the physical context (examples shown in Figure 3.1).[1] The purpose of their study was to determine whether students' difficulties reasoning about the 1$^{\text{st}}$ Law using P-V diagrams were a result of their lack of understanding of the physical implications or a failure to correctly interpret the graph. Pollock *et al.* report several notable results. First, significantly more students answered the math-only question correctly when the two paths of the graph were labeled as two different functions (e.g., $f(y)$ and

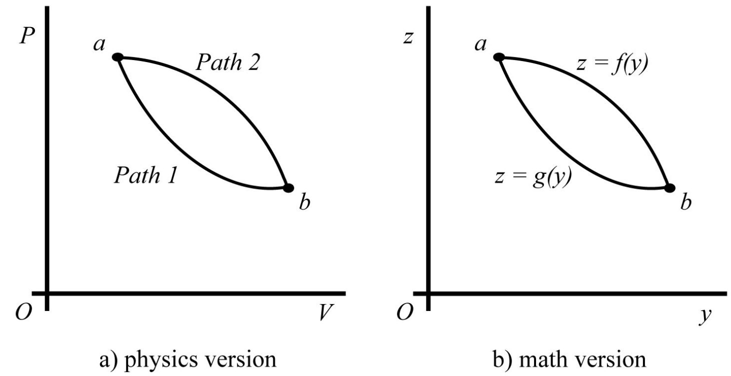


Figure 3.1. Physics and Math versions of a P-V diagram.[1]

$g(y)$). Second, correct mathematical understanding of the relationship between a graphed function and its integral seems to be a prerequisite to understanding the physics. And third, some students answer the math-only question correctly but inappropriately apply state function reasoning to the physics questions dealing with work.[1] More recent investigations by Thompson and Christensen have provided further evidence of the need for a robust understanding of the mathematics to fully appreciate the physical significance of P-V diagrams.[23, 36]

3.2.3.2 Difficulties with Rational Numbers

As mentioned above, students' understanding of mathematics concepts may have a profound impact on their abilities to understand related physics.[36] As I discuss in later sections, students' understanding of concepts related to rational numbers are particularly important to my research on their understanding of advanced thermal physics topics.

Many studies have been conducted on middle and high school students' understanding of rational numbers (cf. Refs. 60, 61, &62). Of particular interest to my research are those pertaining to students' understanding of order and equivalence of rational numbers. Smith reports that students who are competent in their abilities to reason about, use, and compare fractions use a variety of strategies to do so.[63] The use of these strategies is highly dependent on the salient features of the fractions in question, and students rarely use the same strategy in all contexts. The Numerator Principle is appropriately applied when two fractions have the same denominator and states that the fraction with the larger numerator is bigger (e.g., $3/14 > 2/14$). The Denominator Principle, on the other hand, is appropriately applied when two fractions have the same numerator and states that the fraction with the larger denominator is smaller (e.g., $7/11 > 7/13$). Smith reports that some competent students correctly apply the Denominator Principle in situations in which two fractions have numerators that are approximately equal (e.g., $8/11 > 7/15$).[1]

Smith also reports that students use the Compare Numerator-Denominator Differences strategy (in which the within-fraction difference between the denominator and the numerator is used as a comparative measure) when they feel appropriate.[63] One student applied this method to originally determine that $3/5 = 5/7$ since the difference between the denominator and the numerator for each fraction is 2. He later corrected this error by considering the fact that $2/3 > 1/2$ even though the difference between the denominator and the numerator in each of these fractions is 1. Another student stated that $14/24 > 7/12$ since $10 > 5$ but quickly realized that they were equivalent by multiplying $7/12$ by $2/2$. In each of these cases the student was led astray by comparing Numerator-Denominator Differences but was able to answer correctly by considering additional pieces of information available to them.[63, p. 30] It is unclear, however, how successful these students would have been if this additional information had not be available to them (e.g., in a situation in which the fractions were comprised of variables rather than numbers). In later

[1]Admittedly, however, the majority of students interviewed used $1/2$ as a reference point to compare to each fraction for this problem, i.e. $7/15 < 1/2$, $8/11 > 1/2$.[63, p. 30]

sections, I show that advanced undergraduate physics students compare fractions and ratios in a manner consistent with the strategies described by Smith.

3.3 Addressing Specific Student Difficulties

Once a specific difficulty has been identified, a natural subsequent goal is the creation of curricular materials to help eliminate this difficulty. Results from studies in educational psychology give several general suggestions for effective instructional strategies. Piaget's theory of assimilation and accomodation describes the need to account for students' prior knowledge base and recognize the difficulty of altering an already held idea; Vygotsky's theory of social learning suggests that learning occurs best when students work together with each other and that teachers must provide learning opportunities that are neither too mundane nor too challenging.[64] Consistent with these theories of learning, many researchers have developed curricular materials that are based on, and validated by, the results of physics education research.[65–73] These curricula are often characterized by students working together in an inquiry-based learning environment to answer questions and complete tasks.[2]

The Physics Education Group at the University of Washington has been particularly prolific in creating curricular materials based on the results of PER. They have published two sets of curricular materials for use in introductory physics courses: *Physics by Inquiry* and *Tutorials in Introductory Physics*.[66, 67] *Physics by Inquiry* (PBI) is a curriculum that suggests eliminating lectures and demonstrated problem-solving and replacing them with guided explorations and laboratory work. Students work in small groups (three or four students) to complete a series of experiments designed to help students guide their own learning of physics topics. Instructors act as facilitators, asking questions of students to probe their understanding and reveal specific difficulties with the material. PBI is meant to be used as a professional development tool for preservice and/or inservice teachers and focuses on the nature of science as an exploratory and explanatory process.

Instruction using PBI is very student-centered and progresses only as quickly as the students can properly work through the activities and explorations. As such, PBI is ill-suited as a replacement for many university courses, which demand fairly strict topical guidelines for each semester. Thus, the *Tutorials in Introductory Physics* were developed for just this purpose; as supplements for university physics courses. Tutorials are worksheets that implement guided-inquiry methods to encourage students to discover various physical principles for themselves. The tutorials act as a substitute for traditional recitations; their intent is to accomplish many of the same cognitive goals as PBI, without requiring additional time or course restructuring. The *Tutorials in Introductory Physics* were specifically designed for use in a calculus-based introductory physics course for scientists and engineers. Tutorials may also occasionally be used in place of lecture instruction if no formal recitation session exists. The goal is for students to complete each tutorial within a typical 50-minute class period.

Research at the University of Washington has verified the effectiveness of their curricular materials. Published results indicate that student success on conceptual survey questions typically improves by $\sim 50\%$ after interactive instruction (i.e., using tutorials or PBI) compared to lecture instruction alone.[30–32, 35, 74–76] Secondary implementations of the *Tutorials in Introductory Physics* and PBI have been successfully implemented at many institutions. In particular, the University of Colorado at Boulder has documented conceptual gains comparable to those reported at the University of Washington when using tutorials to supplement their introductory calculus-based physics courses.[77] Implementation at other institutions within both calculus-based and algebra-based introductory physics courses, however, has not been as successful.[78, 79] This suggests that tutorials may be an effective supplement to traditional lecture and laboratory instruction, but that student population is not a trivial concern when developing specific curricular materials. Additionally, the proper preparation of graduate teaching assistants (who are often the primary facilitators of tutorial instruction) is essential to the success of tutorial sessions and must not be overlooked.

Researchers at the University of Maryland, College Park have also developed many tutorials for introductory physics.[69, 72, 73] These tutorials, however, employ slightly different instructional methods than those utilized in the University of Washington varieties. The *Activity-Based Tutorials* use hands-on experimentation and computer-based data taking techniques to allow the students to demonstrate the principles of physics for themselves while engaging in guided-inquiry activities.[72, 73] Many of these activities are based on the *Tools for Scientific Thinking* materials created by Thornton and Sokoloff.[70] Students working on *Activity-Based Tutorials* often use force and motion sensors to take data while encountering various physical phenomena. The *Activity-Based Tutorials* also emphasize mathematics and calculation more than the *Tutorials in Introductory Physics.*

The *Maryland Open-source Physics Tutorials* employ yet another instructional strategy.[69] Based on the work of Hammer and Elby,[48] these tutorials use an epistemological basis of confronting students intuitions and beliefs about their surroundings. The *Maryland Open-source Physics Tutorials* treat students' intuitions about their world as valuable observations that may have been misinterpreted by the students themselves. An example from the tutorial for Newton's third law begins by asking students about a situation in which a heavy truck collides with a small car. The majority of students immediately respond that the truck exerts more force because the car "reacts" more. The tutorial guides the students through the process of recognizing that their intuitions and practical observations apply to the difference in acceleration of each of the vehicles, not necessarily the force. The students use Newton's second law ($F_{net} = ma$) to determine that the mass difference offsets the difference in acceleration and that the two objects exert the same amount of force on each other. The *Maryland Open-source Physics Tutorials* are the least well known of those mentioned, but they are growing in popularity. Both the *Activity-Based Tutorials* and the *Maryland Open-source Tutorials* have been shown to be effective supplements to traditional lecture-based instruction for introductory physics.[69, 80–86] Moreover, research suggests that the *Maryland Open-source Physics Tutorials* may be more effective than either the *Activity-Based Tutorials* or the *Tutorials in Introductory Physics* for helping students understand some topics in introductory physics.[87]

In addition to being useful in introductory courses, tutorials have been shown to be effective supplements to traditional lecture in upper-division undergraduate physics courses. Researchers have developed tutorials for use in classical mechanics,[3–7, 9–13, 16] quantum mechanics,[15] electricity & magnetism,[8] and thermal physics.[17, 19, 20, 22, 34] The following sections describe several tutorials that have been created to address many of the specific difficulties in thermal physics described in section 3.2. Some of these tutorials are intended to be used in introductory courses, and some are meant for advanced undergraduate courses.

3.3.1 Addressing Student Difficulties in Classical Thermodynamics

Kautz *et al.* describe a tutorial designed to address the difficulties they found with introductory students' understanding of the ideal gas law from both a macroscopic and a microscopic perspective.[30, 31] The first three parts of the tutorial help students to realize the connection between pressure and other aspects of an ideal gas (e.g., temperature, volume, the piston that contains it). After tutorial instruction, 50%–70% of students correctly answered questions requiring an understanding of the macroscopic aspects of the ideal gas law.[30] The fourth part of the tutorial uses the microscopic perspective to highlight the fact that the ideal gas law is not substance-specific (provided the substance is an ideal gas), i.e., the ideal gas law works for hydrogen gas as well as nitrogen gas.[31] Kautz *et al.* also report on the creation of a tutorial activity designed to help students consider the idea that intermolecular collisions within gases have no effect on the average kinetic energy of the particles and, therefore, have no effect on the temperature of the gas. After tutorial instruction, 70%–90% of students correctly answered questions requiring the application of a microscopic perspective of the ideal gas law.

To address many of the difficulties students have with entropy and the 2^{nd} Law discussed in section 3.2.1.2, Christensen *et al.* developed the *Entropy Spontaneous Process* (2-Blocks) tutorial to improve students' understanding of entropy and the 2^{nd} Law.[18] In particular, the purpose of this tutorial is to address students' difficulty arbitrarily defining a thermodynamic "system" and the documented overgeneralization that entropy always increases. The 2-Blocks tutorial asks students to consider heat transfer between two massive metal blocks ($V \sim 1\,\text{m}^3$, no appreciable temperature change) at different temperatures. Given the thermodynamic definition of entropy (Eq. 2.6), students are asked to calculate the total change in entropy of the two blocks for spontaneous heat transfer from the high-temperature block to the low-temperature block. They are also asked to calculate the total change in entropy in the limit that the blocks are the same temperature and for a hypothetical situation in which heat transfer spontaneously occurs from the low-temperature block to the high-temperature block. Combining the results from the three thought experiments, the students come to the realization that the entropy of the universe would decrease during an impossible process (heat transfer from the low- to the high-temperature block) and thus derive the 2^{nd} Law: $\Delta S_{uni} \geq 0$. Students also find that for any spontaneously occurring real process the entropy of the universe increases.

Another focus of the 2-Blocks tutorial is the arbitrary nature of the definitions of "system" and "surroundings" often used in thermodynamics texts.[18] Throughout a series of questions, students are first asked to define the "system" to be the low-temperature block, then the high-temperature block, and finally both blocks together. They are also asked to identify what the "surroundings" would be in each case. The goal is to help the students realize that the definition of a "system" depends on the question being asked and the calculations one wishes to perform.

The *Entropy State Function* (2-Processes) tutorial was also developed by Christensen *et al.* in an effort to address some of the difficulties with entropy identified by Bucy *et al.*citeChristensen2009Student,Bucy2006 The 2-Processes tutorial focuses on helping students gain a deeper appreciation for the state function property of entropy by comparing the changes in pressure, volume, temperature, and entropy of an ideal gas due to an isothermal expansion to the changes due to a free expansion. By realizing that both the initial and the final values of pressure, volume, and temperature of the gas are the same for each process, the students are guided to recognize that the change in entropy must be the same as well. This reinforces the idea that the change in entropy of a substance undergoing a thermodynamic process only depends on the initial and final states of that process.

Cochran & Heron developed two versions of the *Heat engines and the second law of thermodynamics* tutorial based on the difficulties discussed in section 3.2.1.2.[35] Both versions begin by motivating the need to consider more than the 1st Law (since, e.g., the 1st Law doesn't prohibit spontaneous heat transfer from a low-temperature object to a high-temperature object). Both versions also use heat engines as the context for investigating the 2nd Law. In the "Carnot version," students are given Carnot's theorem (that no heat engine can operate at an efficiency greater than that of a Carnot engine) and asked to apply it in various contexts. From an application of Carnot's theorem students are presented with the Kelvin-Planck statement of the 2$^{\text{nd}}$ Law: that it is impossible for a device to exist that operates in a cycle and whose sole effect is the conversion of heat from a single reservoir into usable work ($\eta = 100\%$). The "entropy version" of the tutorial takes a different approach by asserting the entropy inequality form of the 2$^{\text{nd}}$ Law shown in Eq. 2.7 and asking students to calculate changes in entropy of various parts of a heat engine (i.e., the working substance, and both reservoirs). The students are told that the Kelvin-Planck statement as well as Carnot's theorem may be derived from the entropy inequality but not shown those derivations. After tutorial instruction with either version, 70%–75% of students in a sophomore-level thermal physics course answered questions about heat engine functionality correctly using correct reasoning (compared to $\sim 30\%$ after lectures alone). Furthermore, some evidence suggests that the method students chose for applying the 2$^{\text{nd}}$ Law on post-tutorial assessments (comparing η to Carnot's efficiency or calculating ΔS_{uni}) was affected by the version of the tutorial they had experienced. It is unclear, however, the extent to which students appreciate the connections between the Kelvin-Planck and entropy inequality statements of the 2$^{\text{nd}}$ Law and Carnot's theorem that were presented in the two tutorial versions.

Given the success of tutorials for students studying both introductory and advanced thermodynamics, I decided to use a tutorial as an instructional strategy to improve student understanding of heat engines and the Carnot cycle. As I discuss in later sections, my tutorial differs in several meaningful ways from those created by Cochran & Heron for use in introductory physics courses.

3.3.2 Addressing Student Difficulties in Statistical Mechanics

Loverude has created several tutorials for use in upper-division statistical physics courses that address many of the difficulties discussed in section 3.2.2; he has reported on results from the *Counting States* tutorial and the *States in the Einstein Solid* tutorial.[19, 20] The *Counting States* tutorial asks students to consider the definitions of macrostate and microstate within the context of flipping coins. Students must also wrestle with concepts related to distinguishability (can we tell the coins apart?) and how to count microstates.[19] While going through the tutorial, students derive the binomial distribution, $\binom{n}{m} = \frac{n!}{m!(n-m)!}$, as the expression for the number of microstates in which m out of n total elements in a binary system have a desired characteristic (e.g., m heads out of n coin flips; the m^{th} macrostate). Preliminary results show that almost all of the students who participated in the *Counting States* tutorial answered a qualitative exam question requiring a distinction between micro- and macrostates correctly.[19] Only 50% of students, however, correctly applied these definitions in a quantitative case in which they were asked to determine the probability of a particular sequence of coin flips (a microstate).

The *States in the Einstein Solid* tutorial is in many ways a continuation of the *Counting States* tutorial in a different context, that of the Einstein solid.[20] The goals of the tutorial include being able to clearly define what constitutes a macrostate vs. a microstate in this context and to calculate the number of microstates within each macrostate. The *States in the Einstein Solid* tutorial uses a particular symbolic representation of each of the oscillators and units of energy (which is used in the course text, Ref.39) to draw an analogy between the oscillator-energy relation in the Einstein solid and the head-tail relation in coin flipping. In this manner students are guided to derive a method for computing the number of microstates for a given macrostate within the Einstein solid based on the concepts from the *Counting States* tutorial.[20]

An additional focus of the *States in the Einstein Solid* tutorial is the procedure for combining multiplicities of interacting systems. One of the final portions of the tutorial asks students to consider how the distribution of energy among the oscillators in one solid would affect the distribution in a connected solid (it wouldn't) to realize that the multiplicities must be multiplied to obtain the total. Preliminary results from tutorial implementation show mixed results. Students performed much better on tasks involving the calculation of joint multiplicities after tutorial instruction (90% correct), but many students still had difficulty on questions regarding the most probable distribution of energy between two interacting solids of unequal size (40% expressed the notion that the energy would split evenly between the two solids, implying an uneven distribution of average energy per particle).[20] Loverude suggests that additional attention may be needed to help students connect the particulate model of an Einstein solid with the concept of macroscopic thermodynamic equilibrium. These results indicate that the learning and teaching of statistical mechanics is a content area ripe for more research. The success of tutorials within advanced statistical mechanics courses has influenced my decision to use a tutorial as an instructional technique to improve student understanding of the Boltzmann factor.

3.4 Summary of Previous Results

Advanced undergraduate physics students provide an interesting population for investigation as they are neither novice nor expert physicists; they may be considered journeyman physicists, on a trajectory toward expertise. Several researchers have previously identified specific student difficulties that may be observed in both introductory and advanced thermal physics courses. Of particular interest is the result from Cochran & Heron that students do not appropriately apply the 2$^{\text{nd}}$ Law when answering questions about heat engines and related devices.[35] In Chapter 5, I present the results of my research into upper-division students' understanding of and difficulties with heat engines and the 2$^{\text{nd}}$ Law. Another noteworthy result is Smith's description of students' strategies for comparing rational numbers.[63] In Chapter 6, I present results from my research indicating that advanced thermal physics students compare ratios in a manner consistent with Smith's described strategies.

Tutorials have been shown to be effective supplements to traditional lecture instruction in both introductory and advanced courses. They have also been shown to successfully improve student understanding of topics in thermal physics as well as other areas. Of particular interest is the success of tutorials within advanced thermodynamics and statistical mechanics courses.[19, 20, 34] Given this success, I developed two tutorials to address several specific student difficulties within advanced thermal physics courses. Chapters 5 and 6 describe the difficulties I have identified and discuss the success of my tutorials with regard to addressing these difficulties.

Chapter 4
RESEARCH SETTING AND METHODS

Each of the tutorials I developed are intended for use in an upper-division thermal physics course. I assume that students in these courses will have gained a sufficient understanding of several underlying ideas before they attempt to engage in the tutorials. In the first part of this chapter I discuss this assumed background understanding in the context of describing the measures that are taken in the thermal physics courses at the University of Maine to ensure that students do, in fact, understand these prerequisite ideas.

In the second part of this chapter I describe the data gathering and analysis techniques that I used during this study. Three primary types of data were gathered: students' responses to written questions, videotaped classroom observations of students engaging with the tutorials, and student interviews on topics related to the Boltzmann factor. Each of these sources of data provided invaluable information regarding students' understanding of, and difficulties with, both heat engines and the Boltzmann factor, as well as insight into the effectiveness of the tutorials (both while students are engaging with the tutorial and after the fact).

4.1 Research Setting

The purpose of this section is to present my expectations about what students understand about thermal physics before encountering my tutorials, and to provide a brief description of the use of guided-inquiry tutorials at the University of Maine (UMaine) to ensure that students have this understanding. I begin, however, by presenting a few details about the thermal physics courses at UMaine and the student population who participated in this study. I also briefly describe the courses and student populations at two other universities where data were gathered.

The Department of Physics & Astronomy at UMaine offers two semester-long upper-division thermal physics courses. No thermal physics topics are included within the curriculum of the introductory sequence for scientists and engineers. The *Physical Thermodynamics* course (*Thermo*) is devoted entirely to classical, macroscopic thermodynamics and is offered every fall semester. I developed the *Heat Engines* tutorial for use in *Thermo*. The *Statistical Mechanics* course (*Stat Mech*) examines many of the same topics as *Thermo* from the microscopic (particulate) perspective and is offered every spring semester.[1] I developed the *Boltzmann Factor* tutorial for use in *Stat Mech*. Most students take *Thermo* during the fall semester of either their junior or senior year and take *Stat Mech* the following semester; however, some students (∼5–10%) take *Stat Mech* the spring before taking *Thermo*. The populations of *Thermo* and *Stat Mech* are mostly undergraduate physics majors in their junior or senior year. Other undergraduate students include engineering, mathematics, chemistry, and computer science majors. Occasionally physics graduate students will take *Thermo* and/or *Stat Mech* as a review of or supplement to their undergraduate education. Both courses meet for three 50-minute class periods each week. Most instruction uses lectures, but tutorials are used in place of lecture for between five and seven class periods each semester. Most students in *Thermo* and/or *Stat Mech* have previously participated in tutorial instruction within their intermediate mechanics course.[3–7, 9–13, 16]

Data were also gathered at two other universities: Rensselaer Polytechnic Institute (RPI, a private engineering university), and California Polytechnic State Uni-

[1] Course information was obtained from classroom observations and communication with the instructor.

versity, San Luis Obispo (Cal Poly, a four-year, comprehensive public university).[2] A written survey regarding heat engines (the engine entropy question described below) was administered to RPI students after all (lecture) instruction on heat engines in the *Thermodynamics and Statistical Mechanics* course: a single-semester upper-division lecture course (meeting for two 110-minute session per week) that combines topics in both classical thermodynamics and statistical mechanics; no tutorials are used during the course ($N = 38$). The course textbook is Carter's *Classical and Statistical Thermodynamics*.[38] The engine entropy question was administered at RPI during Year 2 of my research, after it had been implemented five times at UMaine.

Students at Cal Poly participated in the *Boltzmann Factor* tutorial and were given a related written survey both before and after tutorial instruction (the probability ratios question described below, $N = 32$). At Cal Poly, the probability ratios question was given as a true pretest, before any instruction on the Boltzmann factor, and the *Boltzmann Factor* tutorial was used in place of lecture in the *Thermal Physics I* course: the first of two semester-long upper-division lecture courses (meeting for three 50-minute sessions per week) that combines topics in both classical thermodynamics and statistical mechanics; the *Boltzmann Factor* tutorial was the only tutorial used in the course. The course textbook is Schroeder's *Thermal Physics*.[39] This is different from the treatment at UMaine in which both the probability ratios question and the *Boltzmann Factor* tutorial were given after lecture instruction on the Boltzmann factor. Both the *Boltzmann Factor* tutorial and the probability ratios question were administered at Cal Poly in Year 2 of my research; the *Boltzmann Factor* tutorial had previously been used once at UMaine and the probability ratios question had previously been asked six times at UMaine. More

[2]Information on courses and tutorial implementation was obtained through personal communication with the instructors.

detailed descriptions of these schools and the data gathered at each are included in Chapters 5 and 6.

4.1.1 Student Preparation for the *Heat Engines* Tutorial at the University of Maine

Topics of study in the *Thermo* course at UMaine include the first, second, and third laws of thermodynamics; phases of matter; definitions of heat, work, internal energy, entropy, etc., as they pertain to macroscopic thermodynamics; reversibility and irreversibility; heat engines; and thermodynamic potentials and Maxwell relations. Particular emphasis is placed on distinguishing between thermodynamic quantities that are and are not state functions, as well as distinguishing between those that are extensive and those that are intensive; connections between mathematical processes and physical interpretations; and the arbitrary but crucial nature of the definition of a thermodynamic "system." The textbook for the course is Carter's *Classical and Statistical Thermodynamics*.[38] The prerequisite courses for *Thermo* include Calculus III (multivariable and vector calculus) and introductory mechanics.

Before engaging with the *Heat Engines* tutorial, I expect that students understand several ideas: the difference between state functions and process variables and the importance of this distinction; the 1$^{\text{st}}$ Law and definitions of heat and work; and the 2$^{\text{nd}}$ Law, the thermodynamic definition for changes in entropy (Eq. 2.6), and the distinction between reversible and irreversible processes. Due to the difficult nature of these topics, as described in section 3.2.1, several tutorials are used in *Thermo* to ensure that students develop a full understanding of these topics.

The 2-Blocks tutorial, described in section 3.3.1, gives students the opportunity to use both the 1$^{\text{st}}$ Law and the definition for changes in entropy to derive the limitation on entropy changes due to the 2$^{\text{nd}}$ Law (i.e., $\Delta S_{uni} \geq 0$).[18] The 2-Blocks tutorial also gives students experience calculating changes in entropy due to different processes and reinforces the arbitrary nature of the definitions of a thermodynamic "system" and its "surroundings."[18] The 2-Processes tutorial helps students develop a further understanding of entropy as a state function and the implications inherent therein.[18] Students spend time during the 2-Processes tutorial negotiating the difference between quantities that depend on a particular process (heat and work) and those that only depend on the state of the system (internal energy, entropy, volume, etc.).

In later years the 2-Processes tutorial was replaced by an activity developed at UMaine in which students calculate the changes in, and/or magnitudes of, various thermodynamic quantities of an ideal gas as it is taken from a single initial state (a) to a single final state (b) via three different thermodynamic processes shown on a P-V diagram.[3] By calculating the internal energy, enthalpy, and entropy of the gas at various states on the P-V diagram, and the heat transferred and work done over each path, students see that the former quantities are state functions (and that changes in these variables are path-independent), and that heat and work depend on the particular process, represented by the path taken on the P-V diagram. By participating in both the 2-Blocks tutorial and either the 2-Processes tutorial or the 3-Paths activity, students are given the opportunity to gain an understanding of entropy including its state-function property and its limitations based on the 2$^{\text{nd}}$ Law. As will be discussed in Chapter 5, an understanding of these entropy concepts is necessary for a complete understanding of heat engines.

[3]This activity was inspired by Figure 4-8 in *Thermodynamics, Kinetic Theory, and Statistical Thermodynamics* by Sears & Salinger.[88, p. 117]

4.1.2 Student Preparation for the *Boltzmann Factor* Tutorial at the University of Maine

Topics of study in *Stat Mech* include the connection between multiplicity and probability, and how they relate to particulate models of thermodynamic systems; the statistical definition of entropy; density of states and quantum degeneracy; the canonical probability distribution and the Boltzmann factor; blackbody radiation; and properties of fermions and bosons. The textbook for the course is Baierlein's *Thermal Physics*.[89] The prerequisite courses for *Stat Mech* include Differential Equations and Introductory (Sophomore) Quantum Physics. The *Thermo* course is not a prerequisite for *Stat Mech*, but most students take them in sequential order.

In order for students to successfully complete and engage with the *Boltzmann Factor* tutorial, I expect them to understand several key topics: the difference between a microstate and a macrostate and how to define each for a given system; the fundamental assumption of statistical mechanics, which states that all accessible microstates are equally probable; and the resulting connection between the multiplicity (number of microstates) of a particular macrostate and the probability that the system occupies that macrostate. Due to the difficult nature of these ideas discussed in section 3.2.2, several guided-inquiry tutorials are used in *Stat Mech* to ensure that students gain a good understanding of these ideas. Within each tutorial, particular emphasis is placed on the definitions of microstate and macrostate for a particular physical system, as well as the calculation of multiplicity based on other known properties of the system.

Loverude's *Counting States* tutorial (described in section 3.3.2) uses the context of flipping coins to allow the students to derive the binomial distribution.[19] One important focus of this tutorial is the distinction between a microstate and a macrostate and how each is defined for a given physical scenario. As statistical mechanics is the probabilistic study of thermodynamic systems, this distinction is extremely important: a macrostate that contains a large number of microstates will be more probable than a macrostate that consists of relatively few microstates.

Two additional tutorials, which were developed at UMaine, are used in *Stat Mech.* In the *Binomial Distribution* tutorial, students examine the effects of sample size on probability distributions.[22] In particular it helps students examine the claim that, as sample size increases, the most likely outcome becomes "overwhelmingly probable."[38, 89] In actuality, the probability for the single most likely binomial macrostate decreases with increased sample size (e.g., in the binomial distribution $P(N/2) = \frac{N!}{2^N (N/2)!(N/2)!}$; $2^N > \frac{N!}{(N/2)!(N/2)!}$, for $N > 0$; where N is the number of experimental trials). If, however, one defines the most probable "outcome" to include all values within 1% of the most probable macrostate, then the probability of that "outcome" does, in fact, tend toward unity as N increases.[89, p. 26] The point that is often glossed over is that by including values within 1%, the textbook authors are, in essence, redefining the term "macrostate" to be a range of possible values, not just one (e.g., around 500 heads out of 1,000 flips). The *Binomial Distribution* tutorial helps highlight this distinction of what is considered to be a macrostate and encourages students to gain a deeper appreciation of the need for explicit definitions of microstates and macrostates.

The *Density of States* tutorial was developed at UMaine by Bucy to help students gain an appreciation for how the density of states function relates to multiplicity and probability.[34] The density of states function is an expression of microstate density (i.e., the number of microstates per unit energy) as a function of the energy of some thermodynamic system. By integrating the density of states function over a small range of energies, one obtains the number of microstates within the (redefined) macrostate defined by that energy range. The probability of a macrostate is then found by dividing by the total number of microstates (integral of density of states over the entire energy range). The *Density of States* tutorial helps students navigate what is meant by the term "macrostate" in this sense and motivates why an integral over a small range of energy values is needed. The *Density of States* tutorial is particularly relevant to my study, as an understanding of both the density of states function and the Boltzmann factor is necessary to make sense of our everyday observations of real-world phenomena (see section 6.7 for further discussion).

All three of the tutorials used in *Stat Mech* help students deal with two of the most fundamental questions in statistical mechanics: how do you define a microstate and a macrostate for a given thermodynamic system? and how do you count the number of microstates to determine the probability of a given macrostate? As is discussed in Chapter 6, an understanding of these topics is essential for students' preparation for, and success with, the *Boltzmann Factor* tutorial, as students begin to answer these same questions for interacting thermodynamic systems.

4.2 Research Methods

Throughout my investigation into student difficulties in thermal physics topics I employed several methods, both qualitative and quantitative, for collecting data on students' ideas. Data were primarily gathered using written surveys, videotaped classroom observations, and individual or group interviews. My main goal for collecting data in this manner was research oriented: to learn more about what students understood about the specific content of either heat engines and the Carnot cycle or the Boltzmann factor. A secondary goal was curriculum-development-oriented: to monitor the ways in which students engaged with the tutorials I created. Using videotaped classroom observations and interviews, I was able to observe whether or not students struggled when I wanted them to struggle and/or succeeded easily when I wanted them to succeed. This section contains a description of the methods I used for both collecting and analyzing data.

Preliminary versions of both the *Heat Engines* tutorial and the *Boltzmann Factor* tutorial were critiqued by members of the Physics Education Research Laboratory (PERL) at UMaine before being administered in class. Revisions after each classroom implementation were also critiqued by PERL members to ensure that they were intelligible to people not directly involved in tutorial development.

4.2.1 Written Questions

One of the primary data sources for my research was written questions administered either as homework, in class as ungraded surveys, or on a course examination in both *Thermo* and *Stat Mech*. Three primary questions were used to gather data on students' understanding of heat engines and the Carnot cycle in *Thermo*: the finite reservoirs question (Figure 5.1) assesses students' understanding of heat engines that operate between reservoirs that have finite heat capacity (not constant temperature) and was given as part of a graded homework assignment in six years; the engine entropy question (Figure 5.2) assesses students understanding of the Carnot cycle as a limiting case related to the 2$^{\text{nd}}$ Law and was given as an ungraded in-class survey both before and after tutorial instruction in three years; the engine feasibility question (Figure 5.6) assesses students' understanding of how the 1$^{\text{st}}$ and 2$^{\text{nd}}$ Laws restrict the energy transfers that occur during the operation of a heat engine and was given as part of a course examination in three years.

Three primary questions were also used to study students' understanding of the Boltzmann factor and related topics. The probability ratios question (Figure 6.2) and its analog (Figure 6.5) assess students' ability to recognize a situation in which the Boltzmann factor is applicable; they were given both as ungraded in-class surveys before tutorial instruction and as part of course examinations after tutorial instruction in two years. The Taylor series pretest (Figure 6.8) assesses students' ability to interpret the Taylor series of a function, given a graph of the function, and was given as an ungraded in-class survey before tutorial instruction in two years. The density of states question (Figure 6.9) assesses students' understanding of the graphical forms of the Boltzmann factor, the density of states, their product, and the canonical partition function and was given on a course examination after tutorial instruction in one year. The purpose of asking these questions is to identify specific difficulties that students have when answering questions about either heat engines or the Boltzmann factor. By asking several questions about similar topics, I had the opportunity to examine students' difficulties from different perspectives and add depth to my results. The identification of these specific difficulties focused the development of tutorials to help improve students' understanding of these topics and address these difficulties. Several of these surveys were chosen for use in my study because their use had been established within the *Thermo* and *Stat Mech* courses in years before tutorial development; others were developed during tutorial development to gain a fuller perspective of student understanding. Full details of the use of these questions and the results from their implementations are presented in Chapters 5 and 6.

Data for my study were collected over several years both before and during tutorial development. (A tabular form of this timeline is included in Tables 5.5 & 6.5.) The first set of written data consisted of two written questions spanning both *Thermo* and *Stat Mech* (the finite reservoirs question and the probability ratios question) and was collected three years before tutorial development began. These questions were asked every year before tutorial development, as well as every year during tutorial development. The *Heat Engines* tutorial was implemented in *Thermo* in three years during which the engine entropy question and engine feasibility question were administered. The engine entropy question was also administered at RPI in one year without tutorial instruction. The *Boltzmann Factor* tutorial was implemented in two years in *Stat Mech*; the Taylor series pretest was administered both years, and the density of states question was only administered the second year. The probability ratios question was also given at Cal Poly before tutorial instruction, and the analog question was given after tutorial instruction. All written data were photocopied and/or scanned to an electronic document before grading, and analysis was conducted without instructors' comments or grades. Each time a tutorial was used at UMaine, either the tutorial itself or the assessment questions were modified to some degree. I will, therefore, discuss each year separately in Chapters 5 and 6.

All of the above questions had one of two basic forms: multiple-choice with required explanation, or free-response. The multiple-choice questions all have a similar form in that students are presented with a physical scenario and asked to determine whether a particular quantity is positive, negative, or zero.[4] Furthermore, students were often given the option to state that the answer couldn't be determined using the given information.[5] For all multiple-choice surveys, students were asked to explain how they determined their answer. These explanations became a valuable source of data as some students gave incorrect responses using physically correct reasons and some students gave correct responses using incorrect ideas (see discussions in Chapters 5 & 6).

The free-response questions presented the students with a physical scenario and asked them to find an expression for a particular quantity (in terms of given variables and constants) or answer a more general question requiring mathematical

[4]Some questions ask students to determine whether a physical quantity increased, decreased, or remained the same after a particular process; or to determine whether one physical quantity was greater than, less than, or equal to another. I consider these answer sets to be isomorphic as each spans the space of relevant responses.

[5]Some students were explicit about not being able to answer a question even when not specifically given a "not determinable" option.

calculations to some extent. These free-response questions gave students the opportunity to express ideas and lines of reasoning that may not have been evident within multiple-choice questions. More details of the goals and use of both multiple-choice and free-response questions are presented in Chapters 5 and 6; all written surveys are included in the appendices.

Students' responses to written surveys were categorized in two ways: first by their answer chosen from one of the provided options (on multiple-choice questions), and second by the explanations that students provided. Analyzing these explanations, I used a grounded theory approach in which the entire data corpus was examined for common trends, and all data were reexamined to group them into the defined categories.[90, 91] One goal of my analysis was to focus on describing rather than interpreting students' explanations while defining the categories. As an example from a question regarding heat engines: students who explicitly stated that a heat engine was "reversible," were categorized into the *Reversible* reasoning category; no students who did not use the word "reversible" were placed in this category. In this way my analysis stays as true to the data as possible by limiting researcher biases and interpretations. This is consistent with Heron's identification of specific difficulties discussed in section 3.2.[37] More details of this categorization process are contained in Chapters 5 and 6.

Once students' responses had been categorized and counted, different data sets (e.g., pre- vs. post-tutorial instruction) were compared using a Fisher's exact test, a statistical test used with categorical data. The Fisher's exact test is similar to a χ^2-test for independent samples in that the distribution of responses within one population is compared to the combined distribution from all populations.[92–94] If both individual populations differ from the combined distribution in terms of the percentage of students occupying each category, then the populations are considered statistically different. The Fisher's exact test is more appropriate than the χ^2-test when the sample size is small (i.e., any categories with fewer than five occupants), which is the case for all of my data. The output statistic for the Fisher's exact test is the p-value, which ranges from 0 (completely different) to 1 (identical). Using the Fisher's exact test allowed me to make claims about whether or not the reasoning students used to answer a specific question changed after participating in one of my tutorials; as well as whether or not students from different universities give similar justifications for their responses. When testing two populations for differences (e.g., comparing pre- and post-instruction assessments), the threshold for significance was set at $\alpha = 0.05$, i.e., populations are considered significantly different with a result of $p < 0.05$. When testing two populations for similarities (e.g., pre-instruction data from two different universities), the threshold for significance was set at $\alpha = 0.10$, i.e., populations are considered statistically similar with a result of $p > 0.10$. Values of $0.05 < p < 0.10$ are considered an indication of approaching significance, i.e., not statistically significant but worth mentioning.

4.2.2 Classroom Observations

As mentioned previously, the focus of my data gathering and analysis was to examine ideas that student had regarding the content of the tutorials developed and to monitor their ability to efficiently and productively complete each of the tutorials. With this in mind, data from classroom observations were gathered by videotaping classroom episodes (one or two each semester) of students working in groups to complete one of my tutorials. Segments from these classroom episodes were selected for transcription and further analysis based on the content of student discussions. Given my focus on investigating students' understanding of particular topics, my methods of gathering video data align with Erickson's description of *manifest content* approaches, in which particular classroom sessions are selected to be videotaped based on the content being discussed.[95] I chose to videotape classroom sessions in which students were engaging in one of the tutorials that I had developed because I was primarily interested in their ideas about content related to the tutorials. During each tutorial session I videotaped one or two groups, each with three or four students working together, for a total of nine groups containing 27 students (about nine hours of video from five different semesters). To analyze video data, I watched each video in its entirety and made note of conversations that seemed interesting; I later watched these seemingly interesting segments many times and recorded both what was discussed and why I thought it was interesting. The results from these records are presented in sections 5.4.4 and 6.5.3. Quotations included in these sections were often selected for their uniqueness. Several students made comments and statements that indicated difficulties that were not expected and have not been previously documented. Data do not exist to verify the pervasiveness of these difficulties, but I feel their existence is noteworthy. In cases where more than one student displayed a similar difficulty, I have included multiple quotes to allow the reader to evaluate the similarities and differences between the data.

During analysis of classroom observations, attention was paid more to the physics content expressed during students' discussions than the broader social interactions evident within the video. While the data obtained could certainly be analyzed using existing literature on gestures and interpersonal interactions (cf. Ref. 96 and references therein), the focus of this overarching project, and my own interest in the data, lies in students' ideas regarding the conceptual and mathematical content of my tutorials and students' ability to negotiate tutorial prompts in an efficient and productive manner. For my purposes a "productive" student interaction is one in which they discuss topics related to the tutorial in a way that helps them progress through the tutorial tasks while seeming to gain a better understanding of those topics (discussing relevant concepts, synthesizing information, engaging with the connection between the mathematics and the physics, etc.). An "efficient" interaction is once that allows the students to complete the tutorial within the intended 50-minute class period. In some respects the categorization of student interactions is done with an eye toward the end justifying the means: an interaction cannot necessarily be considered productive or efficient without knowing the conversations that take place after that interaction. Other researchers at UMaine have used some of these videos to investigate group dynamics within upper-division physics courses;[97] however, I have limited my analysis of video data to that necessary to meet my goals. More details about the analysis of these videos is presented in sections 5.4.4 and 6.5.3. As is discussed in these sections, video data were instrumental in my investigation of students' understanding of tutorial concepts and their ability to successfully complete each of the tutorials.

4.2.3 Interviews

In an effort to delve further into students' ideas regarding concepts related to the *Boltzmann Factor* tutorial I conducted interviews with students both as individuals and in pairs. I conducted two rounds of interviews, each with a different goal. In one round my goal was to test instructional strategies used within the *Boltzmann Factor* tutorial; I therefore conducted interviews with four students in the style of a teaching experiment.[98, 99] It should be noted that the goal of the teaching interviews was not to determine students' understanding of the Boltzmann factor, but rather to examine how well they could complete instructional tasks based on previous knowledge related to the Boltzmann factor. As is discussed in Chapter 6, these teaching interviews were used after the initial tutorial implementation to inform tutorial revisions and improve instruction in subsequent years. According to Steffe and Thompson, "a teaching experiment involves a sequence of teaching episodes... [including] a teaching agent, one or more students, a witness of the teaching episodes, and a method of recording what transpires during the episode."[99] For my purposes I alternated roles as both teaching agent and witness during each interview. In a sense, the tutorial activities used during the interview may also be seen as a teaching agent as they asked students to perform tasks, and students interacted with the document in an intellectual manner. One of the unique aspects of a teaching experiment as an approach to interview procedures is that "it is an acceptable outcome... for students to modify their thinking" during the course of the interview.[98] My goals during these interviews were two-fold: to see how successful students would be at working through tutorial tasks; and, when difficulties arose, to see what interventions were necessary to help students succeed. Interviews were conducted in a think-aloud style in which students were encouraged to verbalize their thought processes while completing interview tasks. Additionally, these interviews were a valuable source of data on students' understanding of content presented within the tutorial. Field notes were taken during the interviews, and students' written work was collected afterward and examined in a manner consistent with my treatment of students' responses to written questions. A more detailed description of the teaching interview tasks and their results is presented in Chapter 6.

In other interview tasks, I was interested in investigating students' ideas about Taylor series expansions and the density of states (two topics that are closely related to the Boltzmann factor) without influencing them. With that in mind I used a clinical interviewing technique similar to those described by Piaget and Inhelder to examine five students' ideas about these topics.[98, 100] My goal in these interviews was to examine students' understanding of topics related to my tutorials more deeply than I could using either written surveys or classroom observations. In the clinical interview setting the students were asked a series of specific questions related to the concepts under investigation. Based on their responses to the original prompts, additional questions were asked to further probe their thought processes. A main goal of the clinical interview (and the primary difference between clinical interviews and teaching experiments) is to ascertain information about the student's ideas without altering those ideas during the interview. As such the interviewer must be careful when responding to student remarks so as not to encourage or oppress student ideas. The students must be given the opportunity to take up or put aside ideas based on their own criteria, not the interviewer's.

As with written surveys, I used a grounded theory approach for analyzing all of my video data (teaching interviews, clinical interviews, and classrooms observations) in an attempt to find interesting and common trends.[90, 91] With a data set so small (about 5 videos for each interview or tutorial), however, trends were not often apparent. As such, many videos are treated as case studies, and emphasis is placed on describing the data before interpreting them. The results from these case studies are presented in Chapters 5 and 6. A more detailed description of the interview tasks and their results is included in sections 6.6 and 6.7. Both the interview protocol used during the clinical interviews and the tutorial activity used during the teaching experiments are shown in Appendix C.

Chapter 5
THE *HEAT ENGINES* TUTORIAL

In this chapter I describe efforts made toward identifying specific difficulties that students display when responding to questions regarding the topic of heat engines, as well as efforts made to develop a tutorial for use in an upper-division thermal physics course to help address these difficulties. As discussed in previous chapters, a full and complete understanding of heat engines requires students to synthesize ideas relating to the 1st and 2nd Laws, as well as the definitions of various thermodynamic quantities and the properties and importance of quantities that are and are not state functions. Some of the difficulties relating to these underlying topics and efforts made to address them are included in Chapters 3 and 4. With these previously reported difficulties in mind, much of my effort was put toward identifying students' specific difficulties related directly to heat engines.

I begin by describing the physics of heat engines, emphasizing the various concepts that must be synthesized to gain a robust understanding. I present data from written surveys that indicate that few students gain a full understanding of the physical principles behind heat engines from lecture instruction alone. This result is consistent across two different universities. Using these data, I present the rationale for developing the *Heat Engines* tutorial as well as the details of the tutorial itself. Data from written surveys given after tutorial instruction provide interesting and somewhat mixed results: students' answers to some questions do not change overall after tutorial instruction, but, as I discuss in section 5.4.2, the reasoning they use to support their answers becomes more selective and sophisticated. Finally, I present the results from analysis of videotaped classroom observations of students working through the *Heat Engines* tutorial and describe how these results shed light on students' difficulties and successes, as well as inform tutorial revisions to improve instruction. I conclude with some implications for future research and curriculum development.

5.1 The Physics of Heat Engines

In order to discuss and appreciate student understanding of and difficulties with heat engines, one must first understand the physics underlying heat engines. As discussed in Chapter 2, a heat engine is a device that converts thermal energy into usable work. To accomplish this, a heat engine requires three things: a high-temperature (T_{H}) thermal reservoir, a low-temperature (T_{L}) thermal reservoir, and a working substance (e.g., a gas in a cylinder with a piston). The reservoirs are needed to exchange energy with the working substance without being affected themselves (ideally, no temperature change). The working substance is the stuff that actually does the work. A heat engine operates in a cycle so that the working substance repeatedly returns to its original thermodynamic state. In other words, after one complete cycle, all of the (equilibrium) state properties of the working substance — volume, pressure, temperature, internal energy, entropy, etc. — will return to their original values. In the course of this cycle, an amount of energy (Q_{H}) is transferred from the T_{H}-reservoir to the working substance; the working substance does work to transfer energy (W) to its surroundings; and some energy (Q_{L}) is transferred from the working substance to the T_{L}-reservoir. The efficiency of a heat engine is defined as the ratio of the work energy out to the heat energy in ($\eta \equiv \frac{W}{Q_{\mathrm{H}}}$). After one cycle the heat engine is essentially right back where it started, ready to do more work.

Heat engines are an integral part of the *Physical Thermodynamics* (*Thermo*) course at UMaine and are arguably one of the more practical applications taught in classical thermodynamics. From a historical perspective, the generic heat engine was the basis for the steam engine, the internal combustion engine, and any electrical power plant that burns fossil fuels or uses nuclear energy; it is therefore largely responsible for the industrial revolution. From a teaching perspective, heat engines provide a practical example in which all three laws of thermodynamics must be used together to solve problems in real-world scenarios. Today's engineering challenges include designing heat engines that achieve greater efficiencies while reducing harmful effects to the environment. The theoretical principles behind maximizing thermodynamic efficiency, however, have not changed in almost 200 years.

5.1.1 A Historical Perspective

In 1824 N. L. Sadi Carnot published a manuscript entitled "Reflections on the Motive Power of Fire, and on Machines Fitted to Develop that Power," which, at the time, was quite obscure, but has become one of the most influential writings on thermodynamics.[101] Carnot described a particular mechanism for operating a heat engine to obtain the most usable work under the constraint of two fixed-temperature thermal reservoirs. This mechanism consisted of taking the working substance through a four-step sequence of processes: an isothermal expansion (at the temperature of the higher temperature reservoir), an adiabatic expansion (to lower the temperature), an isothermal compression (at the temperature of the lower temperature reservoir), and an adiabatic compression (to raise the temperature back to its original state).[101, p. 74–75] Carnot's argument for this four-step process is best articulated in his claim that, "any change in temperature that is not due to a change in volume of [the working substance] is necessarily one in which the equilibrium of [heat] is restored profitlessly. Hence the necessary condition for the achievement of maximum effect is *that the bodies used to produce [work] should undergo no change in temperature that is not due to a change in volume*."[101, p. 70, original emphasis] In other words, a change in temperature of the working substance must be the result of a change in volume rather than a temperature gradient (i.e., the process must be adiabatic). Carnot also posits that the material used as a working substance must be inconsequential for the creation of work, using a perpetual motion machine as a counter-example.[1] He also proposes that greater differences between the temperatures of the two reservoirs will yield a higher capacity to perform work (and that this temperature difference is the only determining factor in the amount of work that could possibly be performed)[101, p. 103] but does not mention the importance of the ratio of the two temperatures.

Carnot's treatise, however, is not without its problems. Lacking much of the theoretical background we have since come to accept (e.g., the constancy of the ratio $\frac{c_{\mathrm{P}}}{c_{\mathrm{V}}} = \gamma$ for a particular material,[2] and the definition of entropy), he uses the experimental results on heat capacities of his contemporaries (e.g., Gay-Lussac, Dalton, etc.) to derive specific examples of how much work could be performed for a given working substance operating between two reservoirs at specified temperatures to which a particular amount of heat had been supplied. He also abundantly uses the term "caloric" which can be interpreted alternately as "entropy" or "heat" depending on the context.[101, p. 121–122] Carnot does not mention thermodynamic efficiency (as the modern definition had not yet been proposed), but his "theoretical results" agree to within 60% of modern theoretical treatments of his proposed scenario.[3]

[1] Two identical heat engines with different working substances operating between the same two reservoirs (one as an engine, one as a refrigerator) must have the same efficiency (unit work energy out per unit heat energy in), otherwise an infinite supply of energy would be available while the reservoirs are unaffected.

[2] Carnot ascribed to contemporary assertions that the difference, $c_{\mathrm{P}} - c_{\mathrm{V}}$, is, in fact, constant.[101, p. 80]

[3] Carnot claims to yield 1.112 units of work for 1000 units of input heat ($\eta = 1.112 \times 10^{-3}$) for an engine operating between reservoirs at 100°C and 99°C ($\eta_{\mathrm{C}} = 2.681 \times 10^{-3}$).[101, p. 98–99]

5.1.2 A More Modern Treatment

A full presentation of the physics related to heat engines from a contemporary perspective is included in Chapter 2. I present a cursory review here for readers familiar with thermodymanics. Our modern definitions of internal energy, entropy, and other thermodynamic variables help to streamline Carnot's arguments. As mentioned above, heat engines (like all devices) are subject to the three laws of thermodynamics. The First Law of Thermodynamics (1^{st} Law) may be written in terms of the energy transfers to and from the working substance of a heat engine to get

$$\Delta U = |Q_{\text{H}}| - |Q_{\text{L}}| - W. \tag{5.1}$$

The entropy inequality statement of the Second Law of Thermodynamics (2^{nd} Law) is given by,

$$\Delta S_{uni} \geq 0, \tag{5.2}$$

and changes in the entropy of any system are given by,

$$\Delta S \geq \int_{initial}^{final} \frac{đQ}{T}. \tag{5.3}$$

The equalities in Eqs. 5.2 & 5.3 hold only for reversible (ideal) processes. For the case of heat engines, the "universe" is considered to be comprised of only the working substance and the two reservoirs, and the total change in entropy of the universe is given by,

$$\Delta S_{uni} = \frac{|Q_{\text{L}}|}{T_{\text{L}}} - \frac{|Q_{\text{H}}|}{T_{\text{H}}}. \tag{5.4}$$

Thermodynamic efficiency for a heat engine is defined as,

$$\eta \equiv \left| \frac{W}{Q_{\text{H}}} \right|. \tag{5.5}$$

Combining Eqs. 5.1, 5.2, 5.4, and 5.5 one obtains the relationship:

$$\eta = 1 - \frac{|Q_{\text{L}}|}{|Q_{\text{H}}|} \leq 1 - \frac{T_{\text{L}}}{T_{\text{H}}} = \eta_{\text{C}}, \tag{5.6}$$

where η_{C} is Carnot's efficiency, and the equality holds only in the reversible case where the working substance goes through Carnot's four-step cycle of alternating isotherms and adiabats, during which the entropy change of the T_{L}-reservoir perfectly negates the entropy change of the T_{H}-reservoir. Conversely, an efficiency greater than Carnot's effieciency ($\eta > \eta_{\text{C}}$) would imply that the entropy of the universe would have to decrease ($\Delta S_{uni} < 0$), violating the 2$^{\text{nd}}$ Law. Thus our modern definitions of entropy and thermodynamic efficiency have allowed a direct connection between what we now know to be the 2$^{\text{nd}}$ Law and a succinct formulation of Carnot's theorem: No heat engine operating between two reservoirs can be more efficient than a Carnot engine operating between those same two reservoirs.

5.2 Identifying Student Difficulties with the Physics of Heat Engines

Because a robust understanding of heat engines and Carnot's efficiency involves synthesizing the 1$^{\text{st}}$ & 2$^{\text{nd}}$ Laws appropriately — and because research shows that this is not a trivial task (cf. Refs. 18, 32, & 35) — this seemed like a useful place for curriculum development efforts. Data exist that indicate that students do not, in fact, gain a complete understanding of the importance and uniqueness of the Carnot cycle after lecture instruction. In this section I present the evidence from analyses of student responses on several written questions assigned as homework, in ungraded in-class assessments, and/or on course examinations that suggest several specific difficulties.

5.2.1 Probing Student Understanding of (Ir)Reversibility in the Context of Heat Engines: The Finite Reservoirs Question

The finite reservoirs question (FRQ, shown in Figure 5.1) was developed by Donald Mountcastle, based on problems in both Carter's and Baierlein's texts, and included in course homework for several years in both *Thermo* and *Stat Mech* af-

The context is that of heat engines; in each case we wish to model the system as the working substance which is repeatedly taken through a very specific cycle used to convert energy absorbed in the form of heat into energy spent in the form of work. Heat is exchanged between the system and two finite thermal reservoirs, while the work is delivered somewhere else in the surroundings. During any one engine cycle, assume any change in temperature of the finite reservoirs is negligible.

Suppose you have four similar *finite* reservoirs, each of mass m and specific heat capacity c_P. Two of the reservoirs are initially at temperature T_1, the other two are initially at T_2 , where $T_2 > T_1$.

a) Devise and describe two different heat engines [name them Ralph (R) and Irv (I)], each to operate between a T_1 and a T_2 reservoir, until all deliverable energy has been exhausted; i.e., all heat flow ceases when thermal equilibrium of the two reservoirs is attained. Design Ralph to be the world's most efficient heat engine, and Irv to be the world's least efficient engine.

b) *Prediction*: Describe the final equilibrium state of each engine-reservoir pair (include the working substance and its pair of reservoirs), after each engine ceases operation. Do you expect all of the reservoirs to arrive at the same final temperature? Write down your prediction for the reservoir final temperatures along with a brief explanation.

c) For each engine-reservoir pair, calculate (in terms of m, c_P, T_1 , and T_2)

   i) the final temperature (T_{final}),

   ii) the total work delivered, and

   iii) the ΔS_{uni}, the entropy change of the universe.

d) Are the final temperatures for both engine-reservoir pairs the same or different? Compare with your prediction in part (b) above, and briefly explain.

Figure 5.1. The Finite Reservoirs Question (FRQ). Designed by Donald Mountcastle based on Carter's problem 7.8 and Baierlein's problem 3.6.[38, 89]. Given as a homework assignment in *Thermo* and *Stat Mech*.

ter lecture instruction on the Carnot cycle.[38, 89] The question asks students to consider two different heat engines operating between pairs of identical thermal reservoirs (all with mass, m, and finite specific heat capacity, c_P). Student difficulties with the FRQ were the strongest motivating factor in my decision to further investigate their understanding of heat engines and, subsequently, design the *Heat Engines* tutorial to address their difficulties. In the remainder of this section, I describe the FRQ and present the results that provided this motivation.

The solution to part (a) of the FRQ involves students recognizing that for "Ralph" to be the most efficient heat engine, it will have to be the reversible Carnot engine; and for "Irv" to be the least efficient heat engine, it will do no work ($\eta = 0$). Part (b) only requires the students to make predictions which, while informative, are not directly related to this study.

The key to part (c) of the FRQ is students recognizing that Irv has constant total internal energy, and Ralph has constant total entropy. The solution is that, since no work is done, no energy leaves Irv's two-reservoir system, and thus the total internal energy of the two reservoirs remains constant,

$$\begin{aligned} \Delta U_1 + \Delta U_2 &= \int_{T_1}^{T_{I_f}} mc_{\mathrm{P}}\, \mathrm{d}T + \int_{T_2}^{T_{I_f}} mc_{\mathrm{P}}\, \mathrm{d}T = 0, \\ &\therefore \\ 2\,T_{I_f} &- T_1 - T_2 = 0. \end{aligned} \tag{5.7}$$

This gives a final temperature of $T_{I_f} = \frac{1}{2}(T_1 + T_2)$. The entropy change of the universe, ΔS_{uni}, for Irv can be determined using the fact that any energy gained by the T_{L}-reservoir must have been lost by the T_{H}-reservoir; $\Delta S_{uni} = mc_{\mathrm{P}} \ln\left(\frac{\frac{1}{4}(T_1+T_2)^2}{T_1 T_2}\right)$. Ralph, on the other hand is the reversible Carnot cycle and will, therefore, have $\Delta S_{uni} = 0$,

$$\begin{aligned} \Delta S_1 + \Delta S_2 &= \int_{T_1}^{T_{R_f}} \frac{mc_{\mathrm{P}}}{T}\, \mathrm{d}T + \int_{T_2}^{T_{R_f}} \frac{mc_{\mathrm{P}}}{T}\, \mathrm{d}T = 0, \\ &\therefore \\ \ln\left(\frac{T_{R_f}}{T_1}\right) &+ \ln\left(\frac{T_{R_f}}{T_2}\right) = 0. \end{aligned} \tag{5.8}$$

Solving for T_{R_f} one obtains $T_{R_f} = \sqrt{T_1 T_2}$, which is lower than T_{I_f}. The total work is then the difference between the initial and the final internal energy of the R reservoirs ($U = 2mc_{\mathrm{P}}T_{R_{avg}}$); and since the initial internal energy of the R reservoirs (and the I reservoirs) is the same as the final internal energy of the I reservoirs: $W = 2mc_{\mathrm{P}}(T_{R_{avg,i}} - T_{R_{avg,f}}) = 2mc_{\mathrm{P}}(T_{I_f} - T_{R_f}) = 2mc_{\mathrm{P}}(\frac{T_1+T_2}{2} - \sqrt{T_1T_2})$.

In total, about half of the students for which data exist (20 out of 38) successfully completed the FRQ or a related question after lecture instruction alone. During the first years of implementation in *Thermo*, a preliminary version of the FRQ was used that only looked at the least efficient (Irv) engine ($N = 13$; Carter's problem 7.8).[38, p. 124] Examining students' written responses to this version of the FRQ, eight students successfully determined the final temperature of the reservoirs as well as the total change in entropy; the other five students had varying degrees of difficulty determining the changes in entropy due to this process. These difficulties ranged from including terms to calculate the entropy change of the surroundings outside of the reservoirs, to writing nonsense on the page (seemingly in a vain effort to get partial credit).

In later implementations in *Thermo* and all implementations in *Stat Mech*, the full FRQ was used as described above ($N = 25$).[4] Examining students' written responses in these years, 12 students successfully answered all parts of the question. Three other students successfully determined the final temperatures of all reservoirs as well as the total change in entropy of the universe in each case but made errors while calculating the total work done by Ralph. The remaining ten students made major errors that prohibited their successful calculation of either the final temperatures of the reservoirs or the total changes in entropy. One of the most glaring of these errors is a failure to use the fact that the entropy change of the universe for Ralph will be zero within their calculations: four students made this error. Another four stated that $\Delta S_{uni} = 0$, but did not use it productively to determine T_{R_f}. These errors are particularly noteworthy, as the uniqueness of the Carnot cycle (and the

[4]Some implementations of *Stat Mech* used a modified version of Baierlein's problem 3.6 that included all parts of the FRQ shown in Figure 5.1.[89, p. 72]

basis of its importance in thermodynamics) is that it is the only reversible heat engine that operates between two thermal reservoirs; this means that by definition, the entropy of the universe cannot change due to the operation of a Carnot cycle between any two reservoirs.

Possibly the most significant result is that even students who realized that Ralph would be the Carnot cycle did not necessarily recognize that the change in entropy of the universe would have to be zero. One student based his response on the relationship, $\frac{T_{\text{H}}-T_{\text{L}}}{T_{\text{H}}} = \frac{Q_{\text{H}}-Q_{\text{L}}}{Q_{\text{H}}}$, along with the premise that $T_{\text{H}} = T_2 - \frac{Q_{\text{H}}}{mc_{\text{P}}}t$ and $T_{\text{L}} = T_1 + \frac{Q_{\text{L}}}{mc_{\text{P}}}t$, where "$t$" is the time since the process started. He then solved for the time at which $T_{\text{H}} = T_{\text{L}}$, assuming that the same amount of energy, Q_{H} and Q_{L}, was transferred during each cycle (a false assumption, since less heat transfer will occur during each cycle as the temperatures get closer together). His final result for $\Delta S_{uni,\,R}$ was actually the negative of his (correct) result for $\Delta S_{uni,\,I}$: $\Delta S_{uni,\,R} = mc_{\text{P}} \ln\left(\frac{4T_1T_2}{(T_1+T_2)^2}\right)$. In a later section in which numerical values were supplied for T_1, T_2, m, and c_{P}, this student seemed to realize that his result was impossible and wrote, "This can't be right since [ΔS_{uni}] is negative, but I don't know why." Even though this student assumed that Ralph is the Carnot engine (as indicated by equating efficiency with Carnot's efficiency), he did not connect this to the fact that the entropy of the universe could not change due to the reversibility of this engine. Students' failure to relate the Carnot engine with a constant entropy of the universe was a strong motivating factor in the development of the *Heat Engines* tutorial, which emphasizes the connections between Carnot's theorem and the entropy inequality statement of the 2$^{\text{nd}}$ Law. In light of students' failures to use this connection to answer the FRQ and their difficulty determining the final temperatures of the reservoirs, other assessment tasks were created to evaluate their understanding of heat engines in different ways.

5.2.2 Probing Student Understanding of Entropy as a State Function and the Reversibility of the Carnot Cylce: The Engine Entropy Question

The engine entropy question (EEQ, shown in Figure 5.2) was developed by Warren Christensen while a postdoctoral researcher at UMaine to assess students' understanding of the connection between Carnot's theorem and the 2^{nd} Law. The EEQ asks students to consider the change in entropy of various "systems" for two heat engines, first as the result of one complete cycle of a Carnot engine, and second as a result of one complete cycle of a heat engine that is hypothetically *more* efficient than the Carnot engine. The students are asked about the change in entropy of the universe (working substance and both reservoirs) and then about the change in entropy of the working substance alone.[5]

To fully comprehend the correct answer, the students must understand and apply two ideas: 1) entropy is a state function, and 2) the Carnot cycle is reversible. The fact that entropy is a state function along with the fact that the working substance ends the cycle at the same thermodynamic state as it began (by definition of a cycle) indicate that the entropy of the *working substance* must be unchanged after one complete cycle. This statement is true for any heat engine regardless of its efficiency. The fact that the Carnot cycle is reversible means that the equality must hold in Eq. 5.2, so the entropy of the *universe* must also remain the same after one complete cycle of a Carnot engine. The fact that the Carnot cycle is reversible also indicates that to obtain a heat engine with an efficiency *greater* than the Carnot efficiency, the 2^{nd} Law must be violated. One may conclude that the entropy of the universe must decrease for this better-than-Carnot engine. Thus the correct response pattern for the EEQ is: same-same-decrease-same. At this point I should explain

[5] In later versions students were asked about the working substance before being asked about the universe.

For the following questions consider one complete cycle of a heat engine operating between two thermal reservoirs. The heat engine operates using an appropriate working substance that expands and compresses during each cycle.

For questions a) and b) consider (i.e., imagine) this engine to be a Carnot engine.

a) As a result of one complete cycle of the Carnot engine, will the entropy of **the universe** *increase*, *decrease*, *remain the same*, or is this *not determinable with the given information*? **Explain your reasoning.**

b) As a result of one complete cycle of the Carnot engine, will the entropy of **the working substance** *increase*, *decrease*, *remain the same*, or is this *not determinable with the given information*? **Explain your reasoning.**

For questions c) and d) consider (i.e., imagine) a heat engine that is *more* efficient than a Carnot engine.

c) As a result of one complete cycle of this new heat engine, will the entropy of **the universe** *increase*, *decrease*, *remain the same*, or is this *not determinable with the given information*? **Explain your reasoning.**

d) As a result of one complete cycle of this new heat engine, will the entropy of **the working substance** *increase*, *decrease*, *remain the same*, or is this *not determinable with the given information*? **Explain your reasoning.**

Figure 5.2. The Engine Entropy Question (EEQ). Developed by Warren Christensen and administered after lecture instruction and again after tutorial instruction.

that when I use terms like "correct response" or "correct answer," I intend this to be a shorthand for, "the answer one would give to the posed question if the solution was carried out correctly" (e.g. "same" to part (a) of the EEQ). I do not intend to imply that this short (usually single-word) answer is sufficient for determining whether or not a student understands the material or has any difficulties; for this, I also consider the reasoning that students give for their responses. Considering both the response and its reasoning provides a more complete picture of students' understanding and difficulties. I make this distinction between response and reasoning throughout this dissertation.

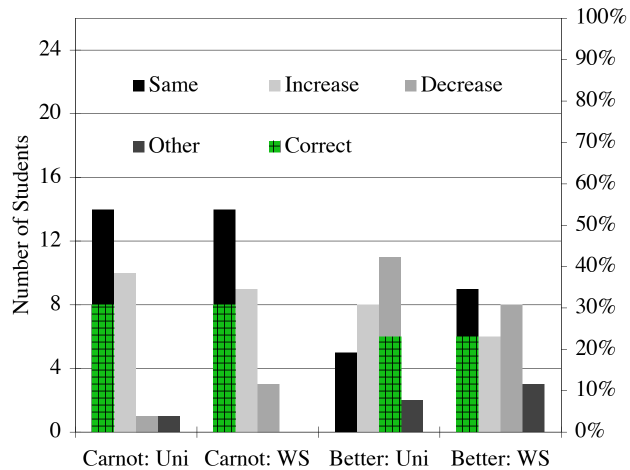


Figure 5.3. Response Frequencies: EEQ Pretest, UMaine. Data are the combination of those collected in *Stat Mech* in the spring before Year 1 and in *Thermo* in the fall of Years 1, 2 and 3. ($N = 26$)

5.2.2.1 Student Responses

The EEQ was first given in the spring of one year to students in the *Stat Mech* class, all of whom had previously completed *Thermo* ($N = 5$).[6] Several lectures had been spent on heat engines in *Thermo*, and emphasis was placed on the reversibility of the Carnot cycle. Student responses to the EEQ from this semester indicate that these students, who had completed an entire course on classical thermodynamics, did not have a good understanding of the connection between thermodynamic efficiency and changes in entropy: only two students correctly answered all four parts of the EEQ and provided appropriate reasoning for each.

To establish a baseline for tutorial instruction, the EEQ was administered in *Thermo* after all lecture instruction on heat engines for three consecutive years that the tutorial was administered ($N = 21$). After lecture instruction alone, none of the students used completely correct reasoning for their responses on all four parts

[6]Information about students and course content was provided by the course instructor.

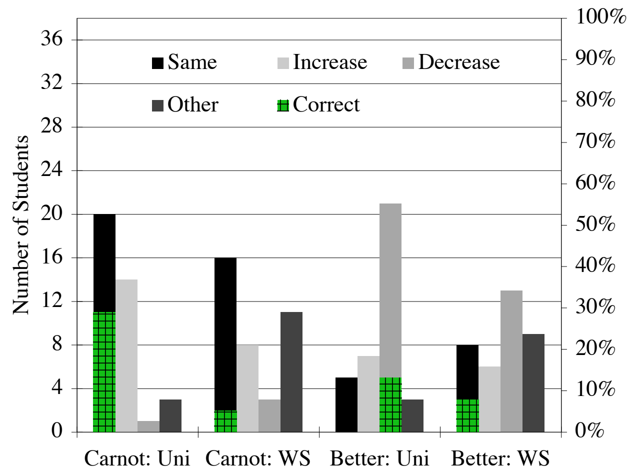


Figure 5.4. Response Frequencies: EEQ Pretest, RPI. Data collected in the spring of Year 2. ($N = 38$)

of the question. Figure 5.3 shows the response frequencies for the combined data corpus of all four semesters (spring before Year 1, and fall of Years 1, 2, and 3). The square-patterned bars (green) show the number of students who used correct reasoning for their response on each question.

The EEQ was also administered at Rensselaer Polytechnic Institute (RPI) in the Thermodynamics and Statistical Mechanics course in the spring of Year 2 ($N = 38$) after all (lecture-based) instruction on heat engines. Figure 5.4 shows the response frequencies from RPI. The data from RPI appear visually similar to that from

Table 5.1. Fisher's Exact Test: UMaine vs. RPI. Results are p-values from Fisher's Exact Test, $\alpha = 0.10$. Tests were done on the entire distribution of responses as well as on the distribution if all incorrect responses were combined.

	Carnot		Better	
Test	Uni	WS	Uni	WS
Response	0.95	0.001	0.78	0.34
Correct	1	0.01	0.34	0.15

UMaine, and a Fisher's exact test shows that the two populations are statistically similar in their response patterns for each sub-question ($p > 0.10$, see Table 5.1), with one exception. On part (b), which asks students about the change in entropy of the working substance of a Carnot cycle, the two populations are statistically significantly different ($p = 0.001$). The most salient difference between the two distributions is the large proportion of RPI students (11 compared to 0 at UMaine) who claimed that there is not enough information to answer the question ("other"). In fact, a *post-hoc* Fisher's exact test with the students who answered "other" removed yields a result that approaches significance ($p = 0.07$). This shows that the relative distribution of "increase," "decrease," and "stay the same" responses is approximately similar and that the difference between the two populations can almost entirely be attributed to some of the RPI students claiming that not enough information existed to answer the question. In the next section I discuss the reasoning that these students used for why they stated this.

The second row in Table 5.1 shows the results of a Fisher's exact test for which all incorrect answers have been combined (including those that gave the "correct" answer but did not use correct reasoning). The results of this test are the same in that the only significant differences are found when students are asked about the change in entropy of the working substance of the Carnot engine. Figure 5.5 shows the combined data from all semesters at UMaine and RPI ($N = 64$).

An interesting aspect of the RPI data is that while 16 students (42%) stated that the entropy of the working substance would remain the same for one complete cycle of a Carnot engine on part (b), only two of these students cited the state function property of entropy in their reason. Almost half of them (7 students) used the fact that the Carnot cycle is reversible to come to the correct conclusion while using inappropriate reasoning. This lack of using the fact that entropy is a state function to justify their response is evident in the dramatic drop (from 16

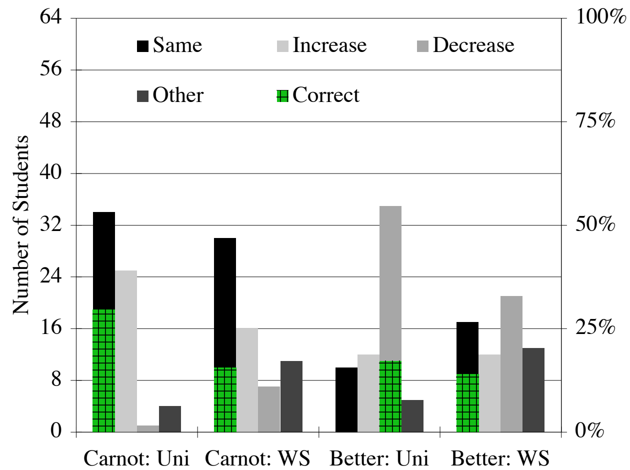


Figure 5.5. Response Frequencies: EEQ Pretest, All. Data are the combination of those collected at UMaine and RPI. ($N = 64$)

to 8) in "same" responses when asked about the entropy of the working substance of the better-than-Carnot engine on part (d). For this part only three students answered correctly using correct state function reasoning (two of whom are those that answered part (b) correctly using correct state function reasoning).

5.2.2.2 Student Reasoning

Shifting focus to examine more of the *reasoning* students used when answering the various parts of the EEQ pretest at UMaine and RPI, I have identified ten primary types of reasoning, described in Table 5.2. As described in Chapter 4, these categories were developed using a grounded theory approach in which I examined the data for common trends and then categorized the data based on these trends. These categories were not suggested by previous research into student understanding of heat engines but derived from the data themselves. My primary goal in developing these categories was to describe the data rather than to interpret students' thoughts. Examples of student responses that were categorized as each of the reasoning strategies are shown in Table 5.3. These reasoning schemes include considering the (ir)reversibility of a heat engine, the state function property of entropy, and tacitly or explicitly mentioning violations of the 1^{st} and/or 2^{nd} Laws. Some students used more than one of these reasoning types to answer various sub-questions of the EEQ; other categories were created for statistical analyses that indicate combinations of reasoning strategies. Along with those described, one type of response that is closely related to the *Statement* type of reasoning is the statement that the entropy of the universe always increases. This idea was expressed most often (5 out of 64 students) when answering part (a) of the EEQ, and all of these students used the same reasoning or simply stated their answer on part (c). The $\Delta S = \frac{Q}{T}$ reasoning was also accompanied by two related types of reasoning: one case where students

Table 5.2. Reasoning on the EEQ Pretest. Categories determined by an open analysis of students' written responses to the EEQ pretest at UMaine and RPI.

Label	Description
Reversible	Cite the reversibility of a heat engine
Irreversible	Cite the irreversibility of a heat engine
State Function	Entropy is a state function
Violate the 1^{st} *Law*	Energy is not conserved
Violate the 2^{nd} *Law*	Cite a violation of the 2^{nd} Law
Direction	The direction in which the device is operated (as a heat engine or a refrigerator) makes a difference
Balance	The change in entropy of a system must counteract that of its surroundings
$\Delta S = \frac{Q}{T}$	Cite that entropy is related to a ratio of heat transfer to temperature
Comparison	Compare to another heat engine (usually the Carnot engine)
Statement	No reasoning given; student merely stated an answer

related changes in entropy to heat only ($\Delta S \sim Q$), and one in which students relate changes in entropy to changes in temperature ($\Delta S \sim \Delta T$). These reasoning strategies are similar to those seen by Bucy, in which students reason about changes in entropy by discussing either changes in temperature or heat transfer.[34] These comparisons may or may not be valid methods for determining entropy change in a particular situation. The reasoning strategies that are considered correct for each sub-question are: a) *Reversible*, b) *State Function*, c) *Violate the* 2$^{\text{nd}}$ *Law*, and d) *State Function*. While it is true that the Carnot cycle is reversible and that entropy is a state function, only the former of these explains why the change in entropy of the universe is zero (part a), while the latter explains why the change in entropy of the working substance is zero (part b).

Table 5.4 shows the numbers of students at each institution who used each of these lines of reasoning and combinations of reasoning strategies on each sub-question. Many categories, however, are only occupied by a handful of students, as indicated by Table 5.4. Moreover, the distribution of the reasoning used differs between UMaine and RPI on some sub-questions. Using a Fisher's exact test to compare these distributions I found that students at both UMaine and RPI used similar reasoning on part (a) as well as on part (c). On part (a) this reasoning is most often the correct *Reversible* reasoning, but on part (c) students were most likely to simply state their answer without justifying it in any way (although mentioning violations of the 1$^{\text{st}}$ and 2$^{\text{nd}}$ Laws come in a close second, along with *Comparison* reasoning).

Results from a Fisher's exact test also show that students' responses to parts (b) and (d) were statistically different at UMaine and RPI ($p = 0.004$ for (b), and $p = 0.02$ for (d)). Examining Table 5.4 one may see that on part (b) students at UMaine most commonly used either the *State Function* (possibly combined with *Reversible*) or the $\Delta S = \frac{Q}{T}$ lines of reasoning, while students at RPI are most likely

Table 5.3. EEQ Pretest Reasoning: Student Examples. Sample responses that were categorized as each of the reasoning strategies.

Label	Example
Reversible	The entropy... will stay the same because the process is reversible.
Irreversible	As the Carnot cycle is a real process, the entropy of the universe will increase.
Rev. + Irr.	I need to know if the processes are reversible. If anything is irreversible then $\Delta S_{uni} > 0$.
State Function	Remain the same because S is a state variable and after one cycle the working substance is not changed.
Rev. + SF	Entropy will remain the same because it is a complete cycle of a reversible process.
Violate the 1st *Law*	You get more work out than input.
Violate the 2nd *Law*	This is contradictory to the 2nd Law.
Direction	The answer is not determinable because depending on the direction the... cycle takes the Δ entropy could be positive or negative.
$\Delta S = \frac{Q}{T}$	$\mathrm{d}S = \frac{đQ}{T}$
$\Delta S \sim Q$	Decrease, giving off heat.
$\Delta S \sim \Delta T$	The working substance is probably going from T_{H} to T_{L} so entropy will be decreasing.
Comparison	Because a less efficient engine increases entropy, it follows that a more efficient engine decreases entropy.
Statement	Decrease.

to use the $\Delta S = \frac{Q}{T}$, $\Delta S \sim Q$, or *Direction* reasoning. The *Direction* reasoning is particularly interesting as it is quite common at RPI (for both parts (b) and (d)), but it is not observed at all at UMaine. In fact the same seven students at RPI used this reasoning on both parts (b) and (d) to say that there was not enough information to determine the change in entropy of the working substance for either engine, indicating consistency across sub-questions, if not correctness. This use of

Table 5.4. Response Frequencies: EEQ Pretest Reasoning. The correct reasoning is shown in bold for each sub-question; the most common reasoning for each population is italicized.

Reasoning	Carnot				Better			
	a) Uni		b) WS		c) Uni		d) WS	
	UM	RPI	UM	RPI	UM	RPI	UM	RPI
Reversible	***10***	***12***	3	6	1	1	0	1
Irreversible	2	0	0	0	0	0	0	0
Rev. + Irr.	1	0	0	0	0	0	0	0
State Function	1	0	***6***	**2**	0	0	***7***	**3**
Rev. + SF	1	0	4	0	1	0	0	0
Violate the 1st *Law*	0	0	0	0	5	1	2	2
Violate the 2nd *Law*	0	1	0	0	**5**	**7**	0	2
Direction	0	0	0	*7*	0	0	0	*8*
$\Delta S = \frac{Q}{T}$	3	3	4	6	1	0	2	1
$\Delta S \sim Q$	1	3	1	*7*	1	0	1	3
$\Delta S \sim \Delta T$	0	1	1	1	0	1	1	0
Comparison	0	0	0	0	1	8	1	3
Statement	1	5	3	3	*6*	*10*	6	3

Direction reasoning is largely responsible for the comparatively high percentage of students at RPI claiming that there is not enough information to answer part (b) of the EEQ (mentioned above).

5.3 Tutorial Development and Implementation

To help students understand the connection between limits on thermodynamic efficiency, entropy, and the 2$^{\text{nd}}$ Law, I developed a tutorial to help address these issues. The primary goal of the *Heat Engines* tutorial is to help students develop an understanding of why the expression in equation 5.6 is the upper limit for thermodynamic efficiency and the conditions under which this efficiency is achieved. As mentioned in Chapter 4, the students are expected to have a sufficient (but not necessarily complete) understanding of several elements of thermodynamics before tutorial instruction, including:

- the 1st Law: $\Delta U = Q - W$, and definitions of U, Q, and W,
- the entropy inequality statement of the 2nd Law (Eq. 5.2),
- the arbitrary nature of the definitions of "system" and "surroundings,"
- definitions for calculating changes in entropy (Eq. 5.3),
- the state function property of energy and entropy,
- the fact that reversible heat transfer can only occur between two systems at the same temperature, and
- the definition of the "universe" being limited to a thermodynamic system and its surroundings of interest.

Students must apply these ideas in the context of heat engines to develop a better understanding of the Carnot cycle and Carnot's efficiency. To ensure that students have the appropriate prerequisite understanding, the 2-Blocks tutorial, the 2-Processes tutorial, and/or other activities described in Chapter 4 are used earlier in the course.

5.3.1 Content & Design

While first thinking of writing a tutorial on heat engines, I spent more time than I care to admit discussing exactly *why* the Carnot cycle is the most efficient heat engine with one of my collaborators. A simple statement of reversibility proved wholly unsatisfying. After all, any function on a P-V diagram that is integrated to find the work done and heat transfer during a specific process is necessarily reversible. Why must the process be an alternating series of isotherms and adiabats? The answer lies in Carnot's theorem itself. One of the key aspects of Carnot's theorem is that it applies to heat engines operating between *only two* specific reservoirs. Limiting the heat engine to having just two reservoirs makes all the difference, since reversible heat transfer only occurs between two objects/substances at the same temperature. An isochoric (constant volume) cooling process, for example, can be modeled as being reversible by imagining a series of reservoirs whose temperatures are infinitesimally close together. The process of the contents of a piston undergoing an isochoric cooling while in contact with a thermal reservoir of significantly different temperature, however, is inherently irreversible and will cause the entropy of the universe to increase. Restricting the thermal reservoirs to being two (significantly) different temperatures means that the only way to have reversible heat transfer to or from the working substance is by isothermal processes. This also means that the only way to reversibly change the temperature of the working substance is by adiabatic compression or expansion.

The tutorial begins by defining the quantities Q_{H}, Q_{L}, and W as stated in section 5.1. The students are asked to consider the changes in various properties of the working substance (all of which are state functions) as a result of one complete cycle of the heat engine. (In later implementations this task was included in a pre-tutorial homework assignment along with a discussion of the definition of thermodynamic efficiency. A discussion of the rationale for and content of this homework assignment can be found in section 5.4.4.)

The first part of the *Heat Engines* tutorial asks students to consider two extreme cases of heat engines. Cycle 1 is defined as doing no work ($W = 0$); Cycle 2 is defined as having no exhaust heat ($Q_{\text{L}} = 0$). The students use the 1$^{\text{st}}$ Law and the given information to calculate the thermodynamic efficiency (Eq. 5.5) of the two engines ($\eta = 0$ for Cycle 1, $\eta = 1$ for Cycle 2).

Students next use the expression for the change in entropy of a thermal reservoir given in Eq. 2.11 and the state function property of entropy to calculate ΔS_{uni} for Cycle 1 and Cycle 2 ($\Delta S_{uni} = Q(\frac{1}{T_{\mathrm{L}}} - \frac{1}{T_{\mathrm{H}}})$, and $-\frac{|Q_{\mathrm{H}}|}{T_{\mathrm{H}}}$, respectively, where $|Q_{\mathrm{H}}| = |Q_{\mathrm{L}}| = Q$ for Cycle 1). Given these values for ΔS_{uni}, students then invoke the entropy inequality form of the 2nd Law (Eq. 5.2) to determine whether or not each heat engine is physically possible. Since the combined entropy of the two reservoirs — and thus the universe — decreases for Cycle 2, students realize that it is impossible, effectively deriving the Kelvin-Planck statement of the 2nd Law (see section 2.1).

Part III of the tutorial asks the students to combine the entropy inequality with Eqs. 2.11 and 5.5 to derive the constraint on thermodynamic efficiency due to the 2nd Law found in Eq. 5.6. In this way my tutorial differs greatly from either of Cochran & Heron's versions.[35] Instead of presenting various forms of the 2nd Law and having students practice using them in the context of heat engines to show that they give the same results, I ask students to directly derive one statement of the 2nd Law from another. I feel that this provides a stronger connection than merely demonstrating that two principles yield the same results in a particular context.

Part IV has students examine the condition under which the equality in Eq. 5.6 holds, determining that a reversible cycle is needed that can only be created using an alternating sequence of isothermal and adiabatic processes. By completing the *Heat Engines* tutorial, students derive the maximum possible efficiency (equality in Eq. 5.6) as well as the processes necessary to create a heat engine to achieve this efficiency. It should be noted that the terms “Carnot engine,” “Carnot cycle,” and “Carnot efficiency” are not used in the tutorial worksheet until after the students have derived the maximum efficiency and the corresponding cycle. Some readers may also be interested in the fact that no pictorial or graphical representations of heat engines are used during the tutorial.

For homework, the students consider the P-V diagram for a Carnot cycle in which the working substance is an ideal gas and calculate the efficiency using appropriate expressions for W and Q_{H}. This problem is meant as an extension to the tutorial as it provides computational confirmation of the expression for efficiency in Eq. 5.6. The FRQ is also given as homework after completing the *Heat Engines* tutorial. In the second implementation, an additional homework problem was added that asks students to construct the T-S diagram for the Carnot cycle corresponding to the P-V diagram in the first problem. They are then asked to compare the physical interpretations of integrating $\oint PdV$ vs. $\oint TdS$ over the complete cycle — the former being the total work done, and the latter being the total heat transferred — and to discuss the generality of each of the diagrams (i.e., whether they are still useful if the working substance is some material other than an ideal gas).

5.3.2 Implementation

The *Heat Engines* tutorial was administered after lecture-based instruction on heat engines in *Thermo* in three consecutive years. The class in Year 1 consisted of ten students (primarily junior and senior physics majors and minors), six of whom completed the *Heat Engines* tutorial. The EEQ pretest was administered at the beginning of the tutorial session, and the homework consisted of the FRQ and direct calculations of Carnot's efficiency from the P-V diagram of a Carnot engine using an ideal gas. Students were given two 50-minute class periods to complete the tutorial. The EEQ was given again several weeks after tutorial instruction as part of an ungraded quiz. Furthermore, the engine feasibility question (described below) was included on a course examination. Table 5.5 summarizes the timeline of implementation as well as changes made in each year.

Table 5.5. *Heat Engines* Tutorial Implementation & Research Timeline. Entries show the changes that occurred each year.

Before Year 1	• FRQ given as homework assignment in *Thermo* and *Stat Mech* in four years after lecture instruction • EEQ given in *Stat Mech* in one year after lecture instruction on heat engines
Year 1	• EEQ given at the beginning of the tutorial period (after lecture instruction) • Tutorial: students given two 50-minute periods • FRQ given as homework after tutorial instruction • EEQ given as an ungraded quiz • EFQ given on course exam
Year 2	• EEQ given class before tutorial • Pre-tutorial HW included questions on state variables and definitions of efficiency • Tutorial: students given one 50-minute period plus 25 additional minutes • *P*-*V* diagram question moved to homework, added *T*-*S* diagram question • Additional engine added to EFQ • EEQ quiz given in following spring during *Stat Mech*
Year 3	• Pretest, pre-tutorial HW, and post-tutorial HW same as Year 2 • Tutorial: students given one 50-minute period • EFQ included refrigerator question • EEQ ungraded quiz given at the end of *Thermo*

In Year 2, ten students completed the *Heat Engines* tutorial. The EEQ pretest was administered one class period before the tutorial, and the students were given the pre-tutorial homework assignment to complete before the next class. Students were given one 50-minute class period and an additional 25 minutes in the next class to complete the tutorial. The *T*-*S* diagram homework question described above was added in Year 2, and the EEQ was given as an ungraded quiz in the first part of

Consider the following heat engine. The high temperature and low temperature reservoirs are at 600 K and 400 K, respectively. The heat transfer from the high temperature reservoir to the working substance during one complete cycle is 600 J. The heat transfer from the working substance to the low temperature reservoir during one complete cycle is 350 J. The work done by the working substance during one complete cycle is 250 J. A diagram of this heat engine is shown at the right.

Determine whether or not the engine could possibly function. Explain your reasoning.

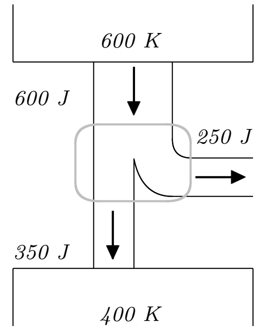


Figure 5.6. Engine Feasibility Question (EFQ). Adapted from Cochran & Heron;[35] used on course examinations after tutorial instruction.

Stat Mech the following spring.[7] The only change in Year 3 was that students were only given one 50-minute class period to complete the tutorial, and the EEQ was once again given at the end of *Thermo* as an ungraded quiz. Two groups of students (2–3 students per group) were videotaped each year as they worked through the *Heat Engines* tutorial.

5.3.3 Post-Tutorial Assessment Tools

Both the finite reservoirs question (FRQ) and the engine entropy question (EEQ) were administered after tutorial instruction to assess the effectiveness of the tutorial. As mentioned above, the FRQ was given as part of the tutorial homework, and the EEQ was given as an ungraded in-class quiz. In the interest of comparing my results with those of other researchers, the engine feasibility question (EFQ, Figure 5.6) was included as part of a course examination after each implementation of the *Heat Engines* tutorial as an additional measure of tutorial effectiveness. The EFQ, which asks students to determine whether a given heat engine is physically possible based

[7]Most students who had participated in the *Heat Engines* tutorial in *Thermo* were also in *Stat Mech*.

on the heat transfers and work done, was modeled after those developed by Cochran & Heron.[35] Using similar questions, Cochran & Heron found that only 30%–35% of introductory students were able to correctly determine whether or not a proposed heat engine would function after lecture instruction on heat engines and tutorial instruction on other aspects of thermodynamics.[8] They report that 25% used only the 1st Law to check heat engine feasibility, and 15% stated that an engine would function as long as the efficiency did not exceed 100%.[35] Other students seemed to use arbitrary thresholds for efficiency that were not explicitly connected with the Carnot efficiency. Cochran & Heron showed evidence of marked improvement in terms of student success on these types of questions after tutorial instruction on heat engines that focused on the equivalence of the various forms of the 2nd Law (Clausius Statement, Kelvin-Planck Statement, entropy inequality, etc.).

The correct answer to the EFQ in Figure 5.6 is that the heat engine will not function as described: the efficiency of the engine is greater than the Carnot efficiency for these reservoirs, and the entropy of the universe would decrease as a result of this heat engine. In Year 2, a second part was added to the EFQ that proposed a heat engine that *is* possible ($\eta < \eta_{\mathrm{C}}$, $\Delta S_{uni} > 0$), and asked students to determine its feasibility. In Year 3 an additional part was added to the EFQ that asked students if either of their answers to parts (a) or (b) would change if all of the energy transfers were reversed, i.e., if these devices were instead run as refrigerators with the same magnitudes of Q_{H}, Q_{L}, and W. The correct answer is that the device in part (a) that would not function as a heat engine *would* function as a refrigerator, and the device in part (b) that would function as a heat engine would *not* function as a refrigerator. In fact, the only device that will operate as

[8] I feel that comparisons between introductory students and our upper-division students are justified due to the similarities reported between various student populations in terms of their difficulties reasoning about questions about topics relating to entropy and the 2nd Law.[18, 40]

both a heat engine and a refrigerator (under a reversal of all energy transfers) is a Carnot engine (that's why it's "reversible"). The motivation for adding questions in Years 2 and 3 comes from the particular reasoning that students used in Year 1 to determine that the original engine would not function. These lines of reasoning and how they influenced the addition of questions is presented in section 5.4.3.

5.4 Results

We have many sources of data with which to assess the effectiveness of the *Heat Engines* tutorial at improving students' understanding of heat engines and (in particular) their connection to changes in entropy. These include the finite reservoirs question (FRQ): included on homework assignments after tutorial instruction; the engine entropy question (EEQ): given as an ungraded survey both before and after tutorial instruction; and the engine feasibility question (EFQ): administered as part of a course examination after each implementation of the *Heat Engines* tutorial. I begin by presenting the results that most closely address the stated instructional goals of the *Heat Engines* tutorial: students' understanding of the connections between heat engines, entropy, and the 2$^{\text{nd}}$ Law, as indicated by their responses to the FRQ homework question and the EEQ ungraded quiz. I then present the results from the engine feasibility exam question. I conclude this section by discussing in-class observations of tutorial sessions and their implications for tutorial revisions and success at generalizing and promoting productive student discussions about the physics. Data indicate that students gain a deeper understanding of the connection between entropy and heat engines after participating in the *Heat Engines* tutorial than after lecture instruction alone.

One of the strongest results from my study is that the number of students who answer the EEQ correctly does not significantly change after tutorial instruction;

however, the reasoning that students who participated in the tutorial use for each response is more selective and sophisticated than that given by students who had only had lecture instruction. This result is discussed more fully in section 5.4.2.

5.4.1 Finite Reservoirs Revisited: Tutorial Homework

After tutorial instruction, seven students completed the FRQ as part of the accompanying tutorial homework assignment. All seven students recognized that the total internal energy would not change ($\Delta U = 0$) for the "Irv" cycle (Cycle 1 from the *Heat Engines* tutorial) and that the total entropy would not change ($\Delta S = 0$) for the "Ralph" cycle (Carnot cycle); they also used this information to determine the final temperatures of all four reservoirs. Six of the students correctly calculated the final temperatures, and the remaining student made mathematical errors using $\mathrm{d}S$ to set up the integral in Eq. 5.8. It should be noted that all seven students correctly predicted that the Irv reservoirs would have a higher final temperature, citing the fact that all internal energy would be conserved. This success rate is much higher than the approximately 50% of students who correctly answered all parts of the FRQ after lecture instruction alone.

5.4.2 Engine Entropy Quiz Results

The response frequencies for the EEQ post-tutorial ungraded quiz are shown in Figure 5.7 ($N = 16$; once again the green bars indicate the students who gave the correct answer and used correct reasoning). Comparing to Figure 5.5, it seems clear that a greater fraction of students answer the various parts of the EEQ correctly with correct reasoning after tutorial instruction than after lecture instruction alone. A Fisher's exact test between the combined pre-tutorial data from UMaine and RPI and the post-tutorial data from UMaine, however, gives mixed results. As seen in Table 5.6, the only significant difference in the distribution of responses occurs in part

(d) when students are asked about the change in entropy of the working substance for the better-than-Carnot engine.[9] Another Fisher's exact test was performed by combining all of the incorrect responses into a single category; the results of this test indicate that students give the correct answer using correct reasoning significantly more often on parts (c) and (d), which ask about the better-than-Carnot engine (44% correct with correct reasoning after tutorial vs. 17% and 14%, respectively, after lecture alone). The decision to group the data in this manner for statistical analyses was motivated by the visual differences between Figures 5.7 and 5.5 and that fact that the differences between the populations were not statistically different. The "Correct" rows in Table 5.6 were created in an effort to say more about the differences between pre- and post-tutorial results than that the post-tutorial results look better.

[9]Statistical tests were not performed between the combined pre-tutorial data and the post-tutorial data for part (b) of the EEQ because the pre-tutorial UMaine and RPI response distributions were not similar.

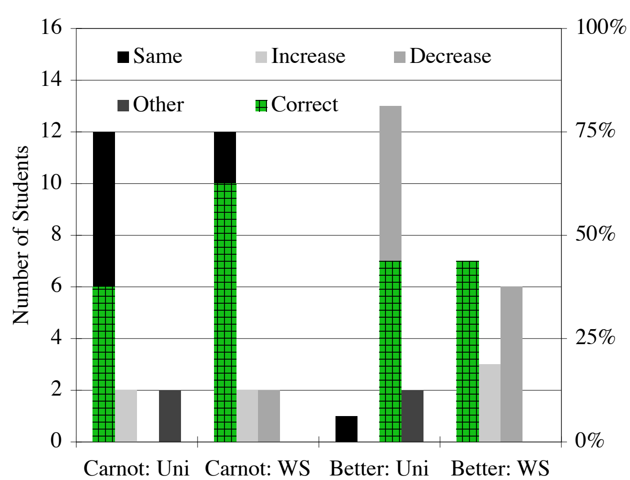


Figure 5.7. Response Frequencies: EEQ Post-test, UMaine. Data is the combination of those collected in Years 1, 2, and 3. ($N = 16$)

Table 5.6. Fisher's Exact Test: EEQ Post-test. Conducted using the full response pattern with the combined "pre-tutorial" data set from both RPI and UMaine as well as only using the data from UMaine, $\alpha = 0.05$. Significant differences are bolded. Tests were also conducted with all incorrect answers grouped together for both data sets. Tests were not conducted with the entire data set for part (b) (Carnot: WS) since the pre-tutorial populations are not statistically similar.

	Carnot		Better	
Test	Uni	WS	Uni	WS
Response	0.21	—	0.08	**0.03**
Correct	0.56	—	**0.04**	**0.02**
Response (UM)	0.29	0.20	0.08	0.34
Correct (UM)	0.74	0.06	0.19	0.19
Response (Matched)	0.23	**0.008**	0.11	0.43
Correct (Matched)	1	**0.009**	0.07	0.37

Similar tests were conducted between the pre-tutorial data at UMaine alone and the post-tutorial data. These results (also in Table 5.6) do not show any statistically significant differences (although $p = 0.06$ and $p = 0.08$ may be considered as approaching significance), but going from 31% to 68% correct with correct reasoning on part (b) is certainly noteworthy. I also conducted similar tests using only the matched data from students who participated in both the EEQ pretest and ungraded quiz at UMaine ($N = 12$, results in Table 5.6). These results are similarly discouraging in that only the post-tutorial data from part (b) is statistically different than the pre-tutorial data.

Looking at the reasoning that students used to answer the EEQ, one again finds mixed results. Figure 5.7 and Table 5.7 both seem to indicate that more students are using correct reasoning after instruction than before. A Fisher's exact test, however, does not show statistically significant differences between the reasoning used before and after the tutorial. Looking only at the students who participated in the *Heat*

Table 5.7. Response Frequencies: EEQ Post-test Reasoning. The correct reasoning is shown in bold for each sub-question; the most common reasoning is italicized.

Reasoning	Carnot		Better	
	a) Uni	b) WS	c) Uni	d) WS
Reversible	***6***	1	0	0
Irreversible	0	0	0	0
Rev. & Irr.	1	0	1	0
State Function	1	***9***	0	***7***
Rev. & SF	2	2	0	0
Violate the 1^{st} *Law*	0	0	2	1
Violate the 2^{nd} *Law*	0	0	**3**	0
Direction	0	0	0	0
$\Delta S = \frac{Q}{T}$	0	0	0	0
$\Delta S \sim Q$	0	1	1	1
$\Delta S \sim \Delta T$	0	0	0	0
Comparison	0	0	3	1
Statement	1	1	*5*	3

Engines tutorial in class and for whom there exist matched pre- and post-tutorial data ($N = 12$), there are still no statistically significant differences ($p > 0.05$, data shown in Table 5.8).

The interesting result is not, however, the difference between the reasoning used before and after instruction on a single sub-question; it is in fact the differentiation in the types of reasoning used by these students on each part of the EEQ. To fully examine this result, the remainder of this section considers only those students who participated in both the EEQ pretest before tutorial instruction and the ungraded quiz after tutorial instruction ($N = 12$). Consider an example of this differentiation of reasoning strategies: before tutorial instruction students were just as likely to use *Reversible* or $\Delta S = \frac{Q}{T}$ reasoning on part (a), and no clear dominant reasoning existed on part (b); after instruction, however, students were more likely to use

Table 5.8. Response Frequencies: EEQ Pre/Post Reasoning (Matched: $N = 12$). Data are that from students who participated in both the pretest and the ungraded quiz. The correct reasoning is shown in bold for each sub-question; the most common reasoning for each population is italicized.

Reasoning	Carnot				Better			
	Uni		WS		Uni		WS	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Reversible	***3***	***4***	1	0	1	0	0	0
Irreversible	0	0	0	0	0	0	0	0
Rev. & Irr.	1	1	0	0	0	1	0	0
State Function	1	1	***2***	***7***	0	0	**2**	***5***
Rev. & SF	1	2	*2*	2	1	0	0	0
Violate the 1st *Law*	0	0	0	0	2	1	1	1
Violate the 2nd *Law*	0	0	0	0	**1**	***3***	0	0
Direction	0	0	0	0	0	0	0	0
$\Delta S = \frac{Q}{T}$	*3*	0	*2*	0	1	0	2	0
$\Delta S \sim Q$	0	0	1	0	1	0	0	
$\Delta S \sim \Delta T$	0	0	1	0	0	0	1	0
Comparison	0	0	0	0	1	2	1	1
Statement	0	0	*2*	0	*3*	*3*	*4*	1

Reversible on part (a) and *State Function* on part (b), the correct reasoning for both. A Fisher's exact test showed that the distribution of reasoning strategies was statistically similar on parts (a) and (b) before tutorial instruction ($p = 0.56$) but statistically different after tutorial instruction ($p = 0.01$).

A similar trend existed between parts (a) and (c) ($p = 0.17$ before tutorial, and 0.003 after tutorial), where after tutorial instruction students were more likely to justify their responses by using *Reversible* on part (a) and *Violate the* 2nd *Law* on part (c).[10] Similarly, the distributions of reasoning strategies given were statistically

[10] Students still had a tendency to merely state their answers to part (c) without justification after tutorial instruction.

different between parts (c) and (d) after tutorial instruction ($p = 0.84$ before tutorial, and $p = 0.03$ after), with most students using *State Function* reasoning on part (d). The distributions of reasoning strategies for parts (b) and (d), however, were statistically similar both before ($p = 0.84$) and after ($p = 0.33$) tutorial instruction; but this result is not undesirable. In fact both parts (b) and (d) should be answered using *State Function* reasoning, the most common reasoning used after tutorial instruction.

These data indicate that after lecture instruction alone these students were not differentiating between the universe and the working substance or between a Carnot engine and a better-than-Carnot engine in terms of the reasoning they used to answer questions about changes in entropy. After tutorial instruction, however, they used different reasoning to answer questions about the Carnot cycle (universe vs. working substance), the universe (Carnot vs. better-than-Carnot), and a better-than-Carnot engine (universe vs. working substance). The only comparison that didn't show significant difference after tutorial instruction was looking at the reasoning used on entropy changes of the working substance (Carnot vs. better-than-Carnot). This is good, however, since students should answer these questions in the same way. In fact, after tutorial instruction, the most common reasoning on working substance questions was to cite the state function property of entropy. Additionally, a Fisher's exact test was performed by combining the distributions from parts (b) and (d) from the pre-tutorial data and comparing it to the combined distribution from the post-tutorial data. This test yielded a p-value of 0.02, indicating that the reasoning used after tutorial instruction is significantly different than after lectures alone. Before tutorial instruction, students were more likely to talk about the reversibility of the Carnot cycle or to give no explanation, while after tutorial instruction, they were more likely to justify their responses using the state function property of entropy.

5.4.3 Engine Feasibility Exam Results

As mentioned above, the engine feasibility question (EFQ) was included on course exams after tutorial instruction in an effort to facilitate comparisons between my results and those reported by Cochran & Heron, who found that many students do not invoke the 2$^{\text{nd}}$ Law (either by calculating changes in entropy or by comparing an engine's efficiency to the Carnot efficiency) when answering questions about heat engines and related devices.[35]

5.4.3.1 Impossible Engine

Looking first at students' successful completion of the task, 12 out of 20 students who participated in the *Heat Engines* tutorial (all three years) correctly determined that the heat engine in Figure 5.6 would not function using correct reasoning, and three other students used correct reasoning but came to incorrect conclusions. Examining student explanations on the EFQ yielded three main lines of reasoning used for determining if heat engines are feasible:

- *Compare Efficiencies* – Students calculate the efficiency of the engine and compare it to the efficiency of a Carnot cycle operating between the same two reservoirs.

- *Calculate* ΔS_{uni} – Students calculate the change in entropy for each of the reservoirs to determine if the inequality, $\Delta S_{uni} \geq 0$, is satisfied.

- *Compare Ratios* – Students calculate the ratios of heat transfers, $\frac{|Q_{\mathrm{L}}|}{|Q_{\mathrm{H}}|}$, and temperatures, $\frac{T_{\mathrm{L}}}{T_{\mathrm{H}}}$, and make a comparison.

In fact, all three of these reasoning strategies are appropriate ways to answer the question. One student correctly determined the efficiency of the engine in Figure 5.6 to be, "$\eta = \frac{W}{Q_{\mathrm{H}}} = \frac{250\,\mathrm{J}}{600\,\mathrm{J}} = \frac{5}{12}$," and the efficiency for a Carnot engine operating between the same reservoirs to be, "$\eta_{\mathrm{C}} = 1 - \frac{T_{\mathrm{L}}}{T_{\mathrm{H}}} = 1 - \frac{2}{3} = \frac{4}{12}$." He then used *Compare Efficiencies* to determine that the engine is "in violation of Carnot's theorem" and "cannot function as described." Another student correctly determined that the entropy change of the T_{H}-reservoir is, $\Delta S_{\mathrm{H}} = \frac{-|Q_{\mathrm{H}}|}{T_{\mathrm{H}}} = -1\,\mathrm{J/K}$, and that the entropy change of the T_{L}-reservoir is, $\Delta S_{\mathrm{L}} = \frac{-|Q_{\mathrm{L}}|}{T_{\mathrm{L}}} = 0.875\,\mathrm{J/K}$. He then used *Calculate* ΔS_{uni} to make the conclusion that, since $\Delta S_{uni} = -0.125 < 0$, the "2$^{\mathrm{nd}}$ Law [is] violated" and that the engine "cannot operate." In the case of *Compare Ratios*, the inequality, $\frac{|Q_{\mathrm{L}}|}{|Q_{\mathrm{H}}|} \geq \frac{T_{\mathrm{L}}}{T_{\mathrm{H}}}$, must be met for a heat engine to function. For the engine in Figure 5.6, $\frac{|Q_{\mathrm{L}}|}{|Q_{\mathrm{H}}|} = \frac{350\,\mathrm{J}}{600\,\mathrm{J}} \approx 0.58$, and $\frac{T_{\mathrm{L}}}{T_{\mathrm{H}}} = \frac{400\,\mathrm{K}}{600\,\mathrm{K}} \approx 0.67$; therefore, $\frac{|Q_{\mathrm{L}}|}{|Q_{\mathrm{H}}|} < \frac{T_{\mathrm{L}}}{T_{\mathrm{H}}}$, and the engine could not function. Of tutorial participants in all three years ($N = 20$), 11 students used *Compare Efficiencies* (seven correctly), nine students used *Calculate* ΔS_{uni} (six correctly), and only one student used *Compare Ratios* (incorrectly). The frequency of each line of reasoning is displayed in Figure 5.8 for all three parts of the EFQ.[11]

As suggested above, each of these strategies, while appropriate, can also be used incorrectly. Three students (15%) who used *Compare Efficiencies* stated that each engine would function if its efficiency was less than 100%, and one other student used an incorrect definition for efficiency ($\frac{W+Q_{\mathrm{L}}}{Q_{\mathrm{H}}}$; the ratio of all energy out to all energy in). Two students (10%) attempted to calculate the change in entropy of the universe, but used incorrect signs for the heat transfer *to* each of the reservoirs; one of these students also did not recognize that the change in entropy of the working substance would be zero (he also answered parts (b) and (d) of the EEQ incorrectly on the ungraded quiz). Despite various computational errors and difficulties with the definition of efficiency (one student, Arthur mentioned below, used $\eta = \frac{W+Q_{\mathrm{L}}}{Q_{\mathrm{H}}}$), 17 out of 20 students (85%) recognized that the 2$^{\mathrm{nd}}$ Law must be invoked in some

[11] One student gave no explanation as he could not answer the question, and two students are counted twice because they each used both *Compare Efficiencies* and *Calculate* ΔS_{uni}.

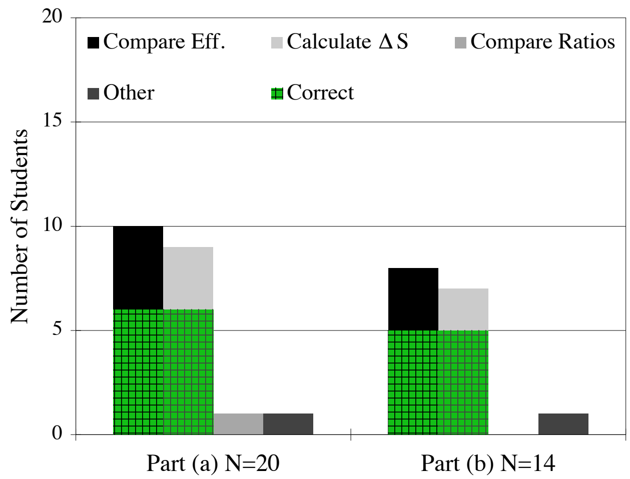


Figure 5.8. EFQ Reasoning Frequencies. Distribution of reasoning used by tutorial participants on each part of the EFQ. The green (cross-hatched) bars show the number of students who used each reasoning correctly to get the correct answer.

manner (either citing Carnot's efficiency or calculating ΔS_{uni}). No one used only the 1st Law and energy considerations to answer the EFQ. These results are comparable to those from Cochran & Heron's study after students participated in one of their tutorials and superior to their results based on lecture instruction alone.

All three students who used *Compare Ratios* in Year 1 (only one of whom had completed the tutorial) did so incorrectly by claiming that the heat engine would function only if the two ratios were equal (which is the case only for a Carnot engine, though none of them explicitly mentioned Carnot). The student who had participated in the tutorial wrote, "$\frac{|Q_\mathrm{L}|}{|Q_\mathrm{H}|} = \frac{T_\mathrm{L}}{T_\mathrm{H}} \Rightarrow \frac{350}{600} = \frac{400}{600}$, not OK," implying that the engine would *only* function if this equality was satisfied. This reasoning yielded the correct answer (that the engine would not function) but for incorrect (or at least incomplete) reasons. This phenomenon of getting the correct answer for incorrect reasons prompted the addition of a second engine feasibility exam question in Year 2.

5.4.3.2 Possible Engine

A second question was added in Year 2 in which students were meant to determine that a proposed heat engine *would* function as described. The values of T_{H}, T_{L}, and Q_{H} were the same as in Figure 5.6, but new values were assigned for W and Q_{L}: $W = 175\,\mathrm{J}$, $Q_{\mathrm{L}} = 425\,\mathrm{J}$. For this new engine: $\eta \approx 29\% < \eta_{\mathrm{C}} \approx 33\%$, $\Delta S_{uni} = 0.1875\,\mathrm{J/K} > 0$, and $\frac{|Q_{\mathrm{L}}|}{|Q_{\mathrm{H}}|} \approx 0.79 > \frac{T_{\mathrm{L}}}{T_{\mathrm{H}}} \approx 0.67$; therefore, each line of reasoning (if properly applied) yields a result indicating that the engine could function . The goals of this addition were to determine if students who used *Compare Ratios* for the first engine would do so appropriately for the engine that *would* function and to determine if any students would use different reasoning for each engine. Unfortunately no students used *Compare Ratios* while answering either part of the EFQ in Years 2 and 3, so the first of these questions is yet unanswered. Data indicate, however, that nearly all students (13 out of 14) in Years 2 and 3 used the same line of reasoning on both parts of the EFQ. The other student used *Calculate* ΔS_{uni} for both parts but also used *Compare Efficiencies* on part (b) as a check of his result that the engine would operate as described. Nine out of 14 students (in Years 2 and 3) answered this second question correctly using correct reasoning (all of whom had done so on the first question as well), and two students still used correct reasoning to come to incorrect conclusions.

5.4.3.3 Refrigerators

A third part of the EFQ was added in Year 3 to investigate students' understanding of refrigerators as well as heat engines. Students were asked to determine whether or not each of the proposed heat engines could operate as a refrigerator, i.e. under a reversal of all energy transfers. If students used *Calculate* ΔS_{uni} correctly to determine if each of the heat engines was feasible, then they should have little trouble reversing the energy transfers to see that, $\Delta S_{uni,\,HE} = -\Delta S_{uni,\,Ref}$ and realize that any device that functions as a heat engine (other than the Carnot engine) cannot function as a refrigerator, and vice versa. No students in Year 3, however, successfully used *Calculate* ΔS_{uni} on either of the first two parts of the EFQ; all used *Compare Efficiencies* (with varying success). In fact, only one student (out of four) correctly determined that the impossible heat engine *could* operate as a refrigerator by reversing all energy transfers and that the feasible heat engine could *not* operate as a refrigerator.

Two students (who, on the first two parts, determined that both heat engines were feasible since $\eta < 1$) failed to correctly determine that only the first device would operate as a refrigerator: one said that both would work, and the other calculated the coefficients of performance for each without commenting on feasibility. The other two students (who correctly determined that the first heat engine would not function and that the second would) both gave correct responses to this third question that were not as complete as those for which I had hoped. One student correctly calculated the coefficient of performance for each proposed refrigerator and compared it to that for a Carnot refrigerator between the same reservoirs; he correctly determined that the first device would function as a refrigerator and that the second would not. The other student stated that, "Only a reversible cycle can be reversed and run as a refrigerator with a simple reversal of the cycle."[12] Between the two of them, these students have given a wonderfully complete answer to this question. The first gave a completely correct answer, but did not comment on the generality of his findings (which was not required but would have been appreciated). The second provided the general restriction on devices that could be used as both a refrigerator and a heat engine (that they must be the reversible Carnot cycle), but failed to realize that, since one of the proposed devices could *not* have operated as

[12]His response to parts (a) and (b) of the EFQ made it clear that he equates reversibility with the Carnot cycle by indicating that "if cycle were rev. then $\eta = \frac{T_H - T_L}{T_H}$," the Carnot efficiency.

a heat engine, it *could* have operated as a refrigerator. It is possible that the first student also had this understanding of reversibility, but evidence only exists for his correct (yet not generalized) response.

5.4.4 Classroom Observations and Tutorial Revisions

In an effort to gain more information on student difficulties with heat engines, and the success of the tutorial at addressing these difficulties, I videotaped all tutorial sessions. The goals for videotaping the tutorial sessions were: a) to identify previously undocumented difficulties, b) to determine student success negotiating tutorial prompts, and c) to determine which aspects of the tutorial should be modified to better address students' difficulties with heat engines. In particular I wanted to document whether or not students had productive conversations while engaging with the tutorial.[13] Two groups of students were videotaped each year, for a total of 17 students.

This section is organized by difficulties that are defined by observations of the data. Each subsection discusses a particular difficulty — supported by evidence from student statements or discussions — and efforts that have been made to address it. One example of how these difficulties were addressed is the creation of a homework assignment to provide students with the opportunity to justify the definition of thermodynamic efficiency described in section 5.4.4.1.

The quotes presented in this section were chosen primarily for their uniqueness and the evidence they provide for previously undocumented difficulties. Many of these difficulties were only documented with one student, but with such a small sample size, data do not exist to comment on the prevalence of these difficulties. If, however, more than one student expressed the same (or a similar) difficulty, I present

[13] See section 4.2.2 for a description of what counts as either a "productive" or an "efficient" interaction.

all relevant quotes to allow the reader to evaluate the similarities and differences between the students in terms of the difficulties they express.

5.4.4.1 Thermodynamic Efficiency

During the first implementation of the *Heat Engines* tutorial, several students engaged in conversations that indicate difficulties with topics within the tutorial. One of these difficulties was with the definition of thermodynamic efficiency. The second day of tutorial implementation began with the students attempting to determine an expression for thermodynamic efficiency using only the quantities Q_{H} and Q_{L} (see the left-hand-side of Eq. 5.6; part III.A.1 of the tutorial). One student (Arthur)[14] proposed the difference of the two heat quantities as a plausible expression, but changed his mind when another student pointed out that this expression could easily be greater than 1 (100%). After several minutes of conversation in which the students did not come to a conclusion, the instructor intervened to ask them about their decision on part III.A.1. Arthur gave a response that seems to indicate a confusion between heat, internal energy, and temperature:[15]

Arthur – The most energy you could get into the working substance would be whatever the difference is between the two heat reservoirs. Like, . . . if there's a high temperature and a low temperature the most you could warm up the working substance would be; the biggest change you could get in the working substance would be the difference between the high one and the low one.

I – The biggest change in temperature?

Arthur – Yeah, in temperature or in internal energy.

[14] All names are pseudonyms.
[15] For all transcripts, "I" stands for the Instructor.

I – Well, which?

Arthur – In internal energy.

It seems clear from this excerpt that Arthur is not distinguishing quantities of temperature from internal energy. It is also noteworthy that Arthur would give this response, which never mentions heat transfer, after indicating that thermodynamic efficiency would be determined by a difference of heats ($Q_{\mathrm{H}} - Q_{\mathrm{L}}$).

The instructor proceeded to ask the students what the definition of thermodynamic efficiency is, expecting the expression from Eq. 5.5. Another student (Gary) spoke up (correctly) saying, "What you get divided by what you pay," and the third student in the group (Craig) added that this would be the "work over the heat" and clarified that the heat in question would be Q_{H}. At this point Arthur suggested that efficiency is the work divided by the net heat transfer ($\eta = \frac{W}{|Q_{\mathrm{H}}|-|Q_{\mathrm{L}}|}$), and Gary agreed. Craig was also easily swayed to this position. The instructor then engaged the students in a conversation about the energy transfers over an entire cycle of the working substance with the goal of having the students realize that their proposed definition of efficiency would be identically unity for *all* engines provided that energy is conserved. The fourth student of the group (Jake, who arrived in class at the end of the efficiency conversation) offered that "d-bar Q equals d-bar W ($đQ = đW$)," since "over a cycle dU would be zero." Arthur then proposed the relationship, $W = Q_{\mathrm{H}} - Q_{\mathrm{L}}$, and Gary and Craig realized that their expression for efficiency would be 100%, but they then attribute that to having the most efficient heat engine with $Q_{\mathrm{L}} = 0$. Only after the instructor's explanation that their expression would be unity regardless of the heat engine did Craig propose the definition, $\eta = \frac{|Q_{\mathrm{H}}|-|Q_{\mathrm{L}}|}{|Q_{\mathrm{H}}|}$, which was the desired response.[16]

[16]Unfortunately this discussion did not benefit all students in the long term as evidenced by Arthur's use of $\eta = \frac{W+Q_{\mathrm{L}}}{Q_{\mathrm{H}}}$ on the exam.

Since the students had used the definition of thermodynamic efficiency to answer part I of the tutorial, I expected part III.A.1 (which asks students to write an expression for η in terms of $|Q_{\mathrm{H}}|$ and $|Q_{\mathrm{L}}|$ alone) to be relatively easy and take the students approximately one minute to complete. Instead, these students needed 25 minutes to answer the question. I do not, however, consider this wasted time. These students obviously needed to consider other expressions for efficiency to be able to understand why the convention of $\eta = \frac{W}{Q_{\mathrm{H}}}$ exists. In light of this evidence, I created a pre-tutorial homework assignment that asks students to think about an alternative expression for efficiency ($\eta = \frac{W}{|Q_{\mathrm{H}}|-|Q_{\mathrm{L}}|}$) and determine why it would not be appropriate in the context of a cyclic process. The first part of the *Heat Engines* tutorial from Year 1 was also moved into this pre-tutorial homework. This section defines the quantities of heat, work, and temperature that are considered during the tutorial and asks the students to determine the changes in various state functions of the working substance as a result of one complete cycle. During tutorial implementation in Year 1, students were able to answer these questions with little difficulty. Due to the fact that they only completed through part II (of IV) on the first day of the tutorial, I decided that moving the first section into the pre-tutorial homework would help prepare the students to progress through the tutorial more quickly. Video data from Years 2 and 3 indicate that the completion of the pre-tutorial homework assignment helped to streamline tutorial implementation and allow the students to complete more of the tutorial successfully within one 50-minute period.

5.4.4.2 Impossible Cycles

Another unexpected difficulty occurred on the first day of tutorial implementation in Year 1. Jake (who was working with Gary and Moe) had great difficulty answering questions about Cycle II (in which $Q_{\mathrm{L}} = 0$). This difficulty was particularly prevalent when attempting to answer part II.C, which asks about changes in entropy due to this cycle. Moe proposed the desired response, that the change in entropy of the working substance would be zero, and that the change in entropy of the reservoirs (and the universe) would be $-\frac{|Q_{\mathrm{H}}|}{T_{\mathrm{H}}}$, which violates the 2$^{\mathrm{nd}}$ Law. Jake, however, did not agree that the change in entropy of the working substance would be zero because he claimed that, in order to to convert all of the heat from a single reservoir into work, one cannot use a cyclic process. In this Jake is absolutely correct, which Moe acknowledged by stating (after a very heated discussion in which Moe repeatedly tried to explain his point of view), "I'm thinking, *if* it's a cycle then it can't change all the energy to work. You're thinking, if it's changing all of the heat to work, then it can't be a cycle. We're thinking the same thing for different reasons." To which Jake replied, "Yeah, alright. I don't know, whatever. Not possible. I don't get this." A similar opinion is observed in Jake's response to parts (c) and (d) of the EEQ pretest which ask about changes in entropy for a better-than-Carnot heat engine: "I don't know. I thought the Carnot cycle is the most efficiency." After participating in the *Heat Engines* tutorial, however, this same student correctly used the reversibility of the Carnot cycle to state that the entropy of the universe would remain the same for the Carnot engine and decrease for a better-than-Carnot engine. He also used the state function property of entropy to claim that the entropy of the working substance does not change for either engine after one complete cycle. In this way, participating in the *Heat Engines* tutorial helped this student engage in the expert-like behavior of considering impossible scenarios.

Jake's intuition about situations that can and cannot exist appears to be very strong. In fact, his conviction that Cycle II in the *Heat Engines* tutorial could not exist is exactly the Kelvin-Planck statement of the 2$^{\mathrm{nd}}$ Law. Unfortunately this inability to consider hypothetical and impossible situations may hinder his reasoning abilities in situations in which his intuition is not as well developed. One of the tools that a physicist uses to support a proposition is to show that a counter-example violates known laws of physics. Bing and Redish suggest that using imagined (and impossible) situations to gain information about our physical world is a trait of expertise, but that many advanced undergraduate students may not have developed the capacity for this kind of reasoning.[24] Having students consider the implications of a heat engine that violates the Kelvin-Planck statement of the 2$^{\text{nd}}$ Law encourages this behavior and reasoning skill that is vital for physicists. To assist students in this, I added the term "imagine" to the EEQ: "consider (i.e., imagine) a heat engine that is more efficient than a Carnot engine." My hope is that students like Jake may be able to suspend their disbelief long enough to be able to think about *why* a particular process is impossible.

5.4.4.3 Differential vs. Net Change

Further difficulties were observed during tutorial implementation in Year 2. In particular two students (Jonah and Bill) engaged in a particularly interesting conversation when answering part I.A, discussing Cycle I ($W = 0$), the 1$^{\text{st}}$ Law, and efficiency:[17]

Jonah – What must be true to satisfy the first law, then?... [Bill – uh...] well... uh

Bill – dQ has to be equal to dU.

Jonah – *Has* to be. dU... must...

Bill – so dQ has to be [Jonah – equal d-bar Q ($đQ$)] zero.

Bill – and dU is zero, so ...dQ has to be zero...... That's the only thing I can think of.

[17]All transcriptions of differential vs. total change notation (d___ vs. đ___ vs. Δ___) is directly from the video. For example a transcription of "$đQ$" resulted from a verbal utteranced of "dee bar queue" from a student.

Jonah – Yeah, I mean, cause dU in a closed cycle, if it's not zero, then you're not conserving energy, so... [Bill – right] that's a problem. [Bill – yeah]

Bill – It's not a cycle if dU is not zero.

Jonah – Yeah.

Bill – So, dQ has to be zero

Jonah – Yeah. But thennnn...

Bill – No. Maybe, maybe it's Q_{H}, or T_{H} is equal to T-low, T_{L}.

Jonah – Yeah

Bill – Because then there'd be no Q, no, be heat, no heat transfer.

Jonah – Yeah, but isn't, now isn't the efficiency the work over the heat transfer or something?

Bill – Yeah, so it'd be zero.

Jonah – Well actually it would kinda be zero over zero wouldn't it? ...Undefined?

Bill – Yeah, I guess.

In this discussion Jonah and Bill have correctly related the heat transfer to the working substance and the change in its internal energy ($đQ = \mathrm{d}U$, since $đW = 0$), but they have incorrectly determined that there would have to be no heat transfer, requiring the reservoirs to have the same temperature. This has also lead to the enigmatic formulation for efficiency: $\eta = \frac{W}{Q_{\mathrm{H}}} = \frac{0}{0} =??$. The main problem in their reasoning seems to be the lack of distinction between dU and ΔU and between $đQ$ and Q_{net}. This is indicated in Bill's statement, "It's not a cycle if dU is not zero." It is absolutely true that for a cycle $\Delta U = \oint \mathrm{d}U = 0$, but if $\mathrm{d}U = 0$, then Bill and Jonah's assertion would be correct in this case, and there would be no heat transfer. This assertion, however, goes against the stated situation that heat transfer Q_{H} occurs from the T_{H}-reservoir to the working substance, and that heat transfer Q_{L} occurs from the working substance to the T_{L}-reservoir. What Bill and Jonah apparently do not realize is that since $đQ = \mathrm{d}U$, then $\oint đQ = |Q_{\mathrm{H}}| - |Q_{\mathrm{L}}| = 0$, and therefore, $|Q_{\mathrm{H}}| = |Q_{\mathrm{L}}|$. After much discussion the instructor was able to get Bill, Jonah, and Paul (the third group member who was silent during the above exchange) to realize that the total change in internal energy was equal to zero, and that they had to consider the net heat transfer, not just the differential, $đQ$.

The use of imprecise language in terms of differentials and net quantities was not unique to Jonah and Bill. As mentioned above, Jake (Year 1) stated that, "over a cycle $\mathrm{d}U$ would be zero." Furthermore, other students in Year 2 use similar language. One student (Sam) stated that, "... $\mathrm{d}U$ is zero for the cycle, so d-bar Q equals d-bar W ($đQ = đW$), which I took it as the net heat is equal to the net work." Using this reasoning, Sam discussed the pre-tutorial homework question proposing the alternate definition of efficiency mentioned above to correctly argue that η would be unity for all engines in that case. When asked by the instructor to articulate his reasoning again Sam clarified that "from the 1st Law, we know there's no change in energy for the cycle, so $\mathrm{d}U$ is zero, so d-bar Q equals d-bar W ($đQ = đW$), for the whole cycle; so the net heat is equal to the net work." In this case Sam is incorrectly stating that $\mathrm{d}U$=0, but his meaning is clear to his groupmates (as was Jake's): that the total change in internal energy over a cycle is zero. The use of precise language clearly would have benefitted Jonah and Bill, but apparently it was not necessary for Jake or Sam.

Jake did, however, express an insufficient understanding of why Q is not written as ΔQ since it is a form of energy transfer, and in the 1st Law Q is opposite ΔU across an equals sign. The instructor explained that, notationally, integrating an inexact differential yields a process-dependent quantity (e.g., $\int đQ = Q$), while integrating an exact differential yields a change in a state function (e.g., $\int \mathrm{d}U =$

ΔU); furthermore, the reason that heat has no "Δ" symbol is that heat only exists as a process quantity, not as an equilibrium property of a thermodynamic system. This explanation seemed to satisfy Jake, but one may wonder how many other students are disturbed by (or even recognize) this apparent lack of symbolic symmetry and are either unwilling or unable to express their discomfort.

Another interesting point is that Sam's group (in Year 2) was able to successfully negotiate part II.C of the *Heat Engines* tutorial, which caused Jake so much trouble, by first considering the fact that the total change in internal energy of the working substance over one complete cycle is zero.

I – [to Dave] So you were trying to relate the change in internal energy to the change in entropy.

Dave – Right, which is not going to work.

I – But could you say anything…

Dave – [unintelligible]

Sam – Didn't we say the change in entropy for a cycle is zero because it's a state...function...? On that one over there? [Points to homework assignment]

Dave – Yeah, we did.

Sam – Yeah, so can't we say for the substance that it goes through a cycle so it has zero change in entropy?

Dave – Yeah, that applies for every cycle… Yeah.

Sam – Yeah.

Rick – For the state functions.

Sam – [to Rick] Are you buying that?

Rick – For state functions or [Sam – Yeah] for a complete cycle…?

Sam – Well entropy's a state function so we do a full cycle on the substance and we're back where we started.

Dave – I suppose, but before we've always done one leg of a cycle, but if we're doing it for the whole cycle it would be zero.

Sam – Yeah, for the substance, not for... [Dave – Right, yeah] I like that argument.

They go on to correctly determine the change in entropy of the reservoirs (and the universe) to be, $-\frac{|Q_{\mathrm{H}}|}{T_{\mathrm{H}}}$, and that this violates the 2$^{\mathrm{nd}}$ Law. In this excerpt the students use their completed pre-tutorial homework to progress through the tutorial, indicating that it is a worthwhile use of their time outside of class. The students are also able (with instructor support) to fairly quickly apply the state function property of entropy to determine that the change in entropy for the working substance in Cycle II (and all other cycles) is zero. This is not, however, easy for all students: Bonnie and Claude had great difficulty expressing this idea as they worked through the *Heat Engines* tutorial in Year 3. When asked about the change in entropy of the working substance, Claude indicated that $\Delta S_{ws} = \frac{|Q_{\mathrm{H}}|-|Q_{\mathrm{L}}|}{T}$, but did not have a quick response as to which temperature "T" represented. After some intervention by the instructor, they agreed that "T" was the temperature of the working substance and that it changed throughout the process (and therefore that their expression couldn't be correct). The instructor proceeded to ask them what it meant for the working substance to complete a cycle. Bonnie volunteered that it would return to its original state, and Claude determined that its total change in entropy would have to be zero, because the heat flow would be zero ("d-bar Q ($đQ$) would be zero"). Bonnie and Claude only acknowledged the importance of the state function property of entropy and its implications for the working substance after direct instructor intervention. So, even though they eventually ended up at the same point as Sam, Dave, and Rick, their path was much more arduous and obviously frustrating, indicated by low voices, sighs, and holding their heads in their hands. Based on Bonnie and Claude's video data and generally poor performance on parts (b) and (d) of the EEQ pretest (which ask about the change in entropy of the working substance for each engine), it is clear that students struggle with the state function property of entropy and how it relates to cyclic processes even after the direct active-engagement instruction on state functions described in section 4.1.1.

5.4.4.4 An Unproductive Aside

The other three students in Year 3 worked very productively in a group together. They managed to get through parts I and II of the *Heat Engines* tutorial in about 25 minutes (twice as fast as anyone in Year 1, and on par with Sam's group in Year 2). When they got to the mock student discussion at the end of part II, however, their progress came nearly to a halt. The mock student discussion brings up issues of conservative vs. dissipative work; it was intended to help students who were struggling with the premise that the work done by either of the heat engines (Cycle I or Cycle II) did not increase the entropy of the surroundings. In all three years of tutorial implementation, no student has spontaneously expressed concern that this might be the case. In fact this group in Year 3 was discussing all of the changes in entropy for each cycle very well without worrying about the work causing any change in entropy. Once they read the mock student discussion, they were mired in a conversation about dissipative vs. conservative work for the better part of ten minutes, the results of which did not appear to benefit their understanding of heat engines or entropy and severely hindered their progress through the tutorial. By the end of the period, these students had already derived Carnot's efficiency and were discussing the last questions in part III that ask them to consider the conditions under which a heat engine would achieve this efficiency (reversibility). It seems clear that the time spent discussing dissipative and conservative work would have better been spent considering the implications of reversibility and giving these students the chance to discuss the processes that must comprise a cycle that would operate at the upper limit of thermodynamic efficiency. Students in previous years did not get as bogged down with the mock student discussion as these students in Year 3, but I believe this portion of the tutorial is unnecessary as no students in any year have expressed concern or confusion at not considering entropy changes due to the work done by a heat engine. The mock student discussion regarding dissipative and conservative work will be omitted from all future implementations of the *Heat Engines* tutorial.

5.5 Discussion

The results from implementations of the *Heat Engines* tutorial show many good trends. On the engine feasibility exam question (EFQ), all students recognized the need to invoke the 2$^{\text{nd}}$ Law in some capacity rather than relying only on the restrictions of the 1$^{\text{st}}$ Law. On the finite reservoirs homework question (FRQ), even in years before tutorial instruction, most students were able to correctly determine the final temperature of the reservoirs for the least efficient heat engine (which does no work). In years including tutorial instruction, all students correctly determined the changes in temperature for the reservoirs of both the least efficient and the most efficient (Carnot) engines. They also correctly determined the total entropy change of the universe for each engine and calculated the work done by the Carnot engine. Along with these successes, however, student difficulties emerged, many of which were reduced after tutorial instruction.

Examining the data, I have identified several specific difficulties that students display (even after lecture instruction) when answering questions about or discussing topics related to heat engines. First, students don't recognize that, since entropy is a state function, the entropy change of the working substance after one complete cycle of *any* heat engine is unequivocally zero. This difficulty is most noticeably observed in students' responses to two parts of the engine entropy question (EEQ) which ask about the change in entropy of the working substance for two different heat engines: fewer than 20% of all students answered these questions correctly using correct reasoning after lecture instruction. This is especially noticeable in student responses that claim that the direction in which a cycle is operated (as a heat engine or a refrigerator) determines the entropy change of the working substance. This difficulty is also observed within videotaped classroom observations of tutorial sessions. Bonnie and Claude tried to calculate the entropy of the working substance using the net heat transfer and an ill-defined temperature. One student also tried to calculate the entropy change of the working substance for the first part of the engine feasibility exam question. Additionally, student reasoning for responses on the EEQ indicate that they do not discriminate between whether to consider the state function property of entropy or the reversibility of the Carnot engine when answering questions about the entropy change of either the working substance or the universe. This result indicates that students do not typically appreciate the implications of the state function property of entropy or the ramifications of the reversibility of the Carnot cycle. The *Heat Engines* tutorial helps address this issue as indicated by 40%-65% of students answering the working substance parts of the EEQ correctly using correct reasoning after tutorial instruction. The reasoning students used to answer the Carnot portion of the EEQ also indicates that they are using more sophisticated ways of discriminating between relevant and irrelevant information. After tutorial instruction students generally recognize that the state function property of entropy ensures that the entropy of the working substance doesn't change after one full cycle, and that the reversibility of the Carnot cycle ensures that the entropy of the universe stays the same.

A related difficulty (observed after lecture instruction) is that students do not universally equate the Carnot cycle with reversibility. Student responses to the finite reservoirs homework question in several years before tutorial development indicate that they do not recognize that the Carnot cycle is (by definition) a reversible set of processes, and that, therefore, the entropy of the universe will not change whether it is operated once or many times: only half of the students correctly applied this logic to come to a correct conclusion. This trend is seen again in the EEQ pre-tutorial survey in which some students cited the irreversibility of the Carnot engine (or that the Carnot engine is a real engine) to justify their claims about the entropy change. Some students also claimed that the entropy of the universe always increases, which completely misses the important distinction between processes that are reversible and those that are irreversible. Two of these students also claimed that the entropy of the universe for the hypothetically *more* efficient engine would increase as well, indicating a failure to recognize the importance of the Carnot cycle as a reversible engine *and* a limiting case. In general, these difficulties were reduced after tutorial instruction: no students claimed that the Carnot cycle was a real engine on the EEQ post-tutorial survey, and all students recognized that the entropy of the universe would not change due to the "Ralph" (Carnot) cycle on the FRQ tutorial homework. One possible exception to this success is two students who claimed that the impossible engine in the EFQ exam would, in fact, function because its efficiency was less than 100%. This does not necessarily indicate a failure to recognize that the Carnot engine is reversible, but it does indicate a failure to realize that the Carnot engine is a limiting case, even after tutorial instruction.

A third difficulty that was observed several times was students' use of an inappropriate definition of thermodynamic efficiency. This was first observed in videotaped classroom observations of students participating in the *Heat Engines* tutorial, in which some students proposed the expression, $\eta = \frac{W+Q_{\mathrm{L}}}{Q_{\mathrm{H}}}$, which is the ratio of the total energy out of the working substance to the total energy in. This difficulty was again observed in students' responses to the EFQ exam; one student claimed that the engine could not function (which is the correct answer) because it had an efficiency of 100%, which "is not possible unless the low-temperature reservoir is at 0 K." Later implementations of the *Heat Engines* tutorial included a pre-tutorial homework assignment which asked students to consider an alternative definition of thermodynamic efficiency and justify its rejection based on considerations of energy conservation and the 1^{st} Law. Videotaped observations from these years indicate that students successfully completed this homework assignment before class and were able to use it within the tutorial session to justify and use the conventional definition of efficiency. No students in later years used inappropriate definitions of efficiency on the EFQ exam. These data indicate that the pre-tutorial homework assignment is a productive use of students' time outside of the classroom environment.

A fourth difficulty was observed primarily within classroom observations and consisted of students being sloppy about their use of differentials (both exact and inexact) and net changes or quantities. Some students in all years used "$đQ$" to represent the total heat transfer. In most cases this did not hinder students' discussions, but (as discussed above) one group of students erroneously concluded that, if an engine does no work, then no heat transfer can occur, and the reservoirs must have the same temperature. In this case the students confused the infinitesimal heat transfer ($đQ$) with the total heat transfer ($Q = \int đQ$). Students in another year had a similar difficulty understanding why the expression "ΔQ" is inappropriate and, in fact, redundant. The subtle notational differences between $Q = \int đQ$ and $\Delta U = \int \mathrm{d}U$ were not clear to these students without explicit instruction. Additionally, students often casually use the term "dQ" rather than "đQ" within their conversations, indicating a lack of rigor when discussing state variables (which use exact differentials) and process variables (which use inexact differentials).

5.6 Summary and Implications for Future Research

Research indicates that students do not develop a robust understanding of the connection between the 2$^{\text{nd}}$ Law and heat engines after lecture instruction alone. This has been shown by Cochran & Heron in introductory physics courses, and I have reported the same result within upper-division thermal physics courses at two different institutions.[35] Student performance on ungraded surveys at both schools indicate that lecture instruction is insufficient for students to gain a complete understanding of the Carnot cycle and its implications. Results from student performance on written questions indicate that their understanding of the connection between the Carnot cycle and the 2$^{\text{nd}}$ Law improves if students participate in the *Heat Engines* tutorial.

Some students demonstrate an improvement in their understanding of entropy and heat engines on the engine entropy question after tutorial instruction. The most notable improvement is in students' use of different reasoning to answer questions about changes in entropy. After lecture instruction alone, students often do not discriminate between strategies that are useful when making conclusions about the change in entropy of the universe, and those that are useful for considering the working substance. After tutorial instruction, students are more likely to use the reversibility of the Carnot cycle to determine that the change in entropy of the universe does not change after one of its cycles, and to use the state function property of entropy to determine that the entropy of the working substance does not change after one complete cycle of any heat engine. Students also discriminate between appropriate lines of reasoning more after tutorial instruction when answering questions about the Carnot engine as compared to a better-than-Carnot engine. These increases in discrimination between reasoning strategies are statistically significant at the $\alpha = 0.05$ level.

Results from student performance on the finite reservoirs homework question support my claim that the *Heat Engines* tutorial benefits students' understanding of heat engines. All students who completed the homework following tutorial participation correctly used the fact that the entropy of the universe does not change due to the Carnot cycle to answer questions about a heat engine operating between real, finite-heat-capacity reservoirs. After lecture instruction alone only 50% of students correctly answered all parts of the question, and as many as a third of them did not realize that the entropy of the universe would remain the same or how this fact impacts that final temperature of the reservoirs. It seems clear that completion of the *Heat Engines* tutorial helps students gain a better understanding of the connection between Carnot's theorem and the entropy inequality of the 2$^{\text{nd}}$ Law than lecture instruction alone.

Student performance on the engine feasibility exam question also indicates that the majority (85%) of students recognize the need to consider the 2$^{\text{nd}}$ Law in some fashion to determine whether or not a device could function as described. Other students still considered the efficiency of the proposed heat engine, but stated that anything less than 100% was possible. In contrast to the students in Cochran & Heron's study who only received lecture instruction, no one who participated in the *Heat Engines* tutorial used only the 1$^{\text{st}}$ Law to determine the feasibility of a heat engine.[35] Student understanding of the connection between heat engines and refrigerators in terms of feasibility still remains unclear. Only one student correctly determined that the device that could not function as a heat engine could function as a refrigerator (and vice versa), but his response was largely algorithmic with no comment on this apparent flip in feasibility. It is, therefore, impossible to determine how deeply he understands this relationship. One other student commented that only the reversible Carnot cycle could be operated as both a heat engine and a refrigerator, but he failed to articulate that a device that could *not* be operated as a heat engine *could* be operated as a refrigerator without violating the constraint of reversibility.

Video data from in-class tutorial sessions provide evidence for student difficulties with the terminology used in thermodynamics. A lack of specificity between $\mathrm{d}U$, U, and ΔU, and between $đQ$ and Q was observed in all three years of tutorial implementation. The mixed usage of these terms may go unnoticed by some students, and may not cause problems (as with Sam, Dave, and Rick), but not being clear about their distinction could lead to misunderstandings and ludicrous conclusions (as with Jonah and Bill). Other in-class observations indicate that students do not necessarily understand the need for the "Δ" symbol when talking about changes in state functions, and its absence when discussing heat and work as types of energy transfer.

Video data also provide valuable information as to the logistical aspects of administering the tutorial. Student discussions during the tutorial in Year 1 indicate difficulties with understanding the definition of thermodynamic efficiency as well as its applicability. This difficulty manifested itself within the exam data when one student used an incorrect definition for thermodynamic efficiency ($\eta = \frac{W+Q_{\mathrm{L}}}{Q_{\mathrm{H}}}$) while solving the engine feasibility question. Giving students the chance to wrestle with this definition before coming to the tutorial session seems to have alleviated this problem as no students in Years 2 or 3 displayed difficulty using the standard definition of efficiency either during the tutorial session or on the exam. The pre-tutorial homework assignment also provides students with the opportunity to refresh their knowledge of relevant background topics (including state functions) before coming to class. Doing this ahead of time seems to provide students with extra classroom time needed to progress through the tutorial in 50 minutes.

One avenue for further research would be an investigation into students' understanding of the distinction between exact (d__) and inexact (đ__) differentials and their use. Much research has been conducted on student understanding of the path dependence of Q and W and the path independence of ΔU (cf. Ref.33), but little to no studies exist that investigate student understanding of their respective differentials and how and why they are used. This is obviously a difficulty for some students, who confused a differential change in a variable with the net change of that variable over a given process. It is also interesting to note that most students (during tutorial conversations) seem to be able to use differential and total change language interchangeably. Further studies may be necessary to determine whether these students merely used a wrong word or if they really have conceptual difficulties with differentials.

Another interesting investigation would be student understanding of reversibility in the context of heat engines and refrigerators. As mentioned above, no students in Year 3 completely recognized the importance of the Carnot cycle as the divider between devices that could operate as heat engines and devices that could operate as refrigerators. This could prove to be a valuable instructional opportunity to provide students with a concrete example of a reversible process being the only one of its kind that can literally be reversed and still obey all the laws of physics.

Chapter 6

THE *BOLTZMANN FACTOR* TUTORIAL

In this chapter I present evidence that suggests that students express several specific difficulties when answering questions and completing tasks related to the Boltzmann factor. Some of these difficulties are conceptual and/or related to mathematical concepts; others have to do with less tangible issues, but issues that are more important for developing expertise. Results from written survey questions indicate that students do not recognize situations in which the Boltzmann factor is appropriate to use, even after lecture instruction. Moreover, when students do not recognize that the Boltzmann factor is relevant, they often resort to novice-like behaviors to solve the problem in a manner consistent with children's treatment of mathematics.[63]

The motivating premise for a tutorial on the Boltzmann factor derivation is that guiding students through the derivation in peer groups will help them appreciate the physical significance of the Boltzmann factor more than relatively passive lecture instruction, and thus improve student recognition of appropriate situations for use of the Boltzmann factor. Videotaped classroom observations have been instrumental in my investigation of student difficulties as well as their successes with my tutorials. I present several examples from these observations and discuss how they informed curricular revisions.

In addition to my research into students' understanding of the Boltzmann factor and the effect of the *Boltzmann Factor* tutorial, I have conducted several related studies that serve as extensions to this work. First, I have used written surveys and clinical interviews to study students' understanding of the Taylor series. The Taylor series is a mathematical tool that is used in the *Boltzmann Factor* tutorial; my research suggests that many students are familiar with the Taylor series, but that they do not use it fluently in physical contexts. Results from teaching interviews and several years of tutorial implementation suggest that a pre-tutorial homework assignment provides students with the necessary opportunity to refresh their memory of what exactly a Taylor series is and how to use it in physical contexts.

Second, I have conducted a preliminary investigation of students' understanding of the density of states function and how it relates to the multiplicity of a thermodynamic system as well as the probability that the system occupies a particular macrostate. As mentioned in Chapter 4, an understanding of the density of states function, the Boltzmann factor, and how they relate to one another is critical for a robust understanding of the statistical underpinnings of real-world observations. Results from clinical interviews suggest that, by the end of the *Stat Mech* semester, many students have all of the information necessary to develop this understanding, but that they have not synthesized this information into a coherent framework of how the density of states and the Boltzmann factor complement each other to accurately predict known observations.

I conclude the chapter with a summary of the work that I have conducted regarding student understanding of the Boltzmann factor and related topics and suggest several avenues for future research in this area.

6.1 The Physics of the Boltzmann Factor

The Boltzmann factor is a mathematical expression for the probability that a system at a fixed temperature is in a particular energy state given the energy of that state,

$$P(\Psi_j) \propto e^{-E_j/kT}, \tag{6.1}$$

where Ψ_j denotes the energy eigenstate with eigenvalue E_j, k is Boltzmann's constant ($1.381 \times 10^{-23} J/K$), and T is the temperature of the system. The underlying assumption of the Boltzmann factor is that the "system" under investigation is very small with respect to a thermal energy reservoir with which it is in thermal contact (free to exchange energy but not particles). The canonical partition function (Z) is the result of the normalization constraint that the sum of $P(\Psi_j)$ over all j must be unity:

$$\begin{gathered} \sum_j P(\Psi_j) = \sum_j \frac{e^{-E_j/kT}}{Z} = 1 \\ \therefore \\ Z = \sum_i e^{-E_i/kT}, \end{gathered} \tag{6.2}$$

where Z is a constant with respect to energy. For systems for which the energy of the system is a continuous quantity over some range, $\{E\}$, the Boltzmann factor and the canonical partition function become,

$$\begin{gathered} \int\limits_{\{E\}} P(E)\,\mathrm{d}E = \int\limits_{\{E\}} \frac{D(E)\,e^{-E/kT}}{Z}\,\mathrm{d}E = 1 \\ \therefore \\ Z = \int\limits_{\{E\}} D(E)\,e^{-E/kT}\,\mathrm{d}E, \end{gathered} \tag{6.3}$$

where $D(E)$ is the density of states function, which accounts for all of the eigenstates that have the same particular energy.

The Helmholtz free energy of a system may be written as a function of Z,

$$F = kT\ln(Z), \tag{6.4}$$

and derivatives of F yield information about the system's entropy, pressure, magnetization, and many other thermodynamic variables. Moreover, the average energy

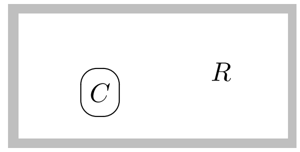


Figure 6.1. Sample system for the Boltzmann factor instructional sequence. An isolated container of an ideal gas is separated into a small system (C) and a large reservoir (R). The label "C" is used to avoid confusion with entropy.

of a system may be expressed as a derivative of the natural logarithm of the Boltzmann factor. In this way the canonical partition function and the Boltzmann factor are cornerstones of statistical mechanics, and a thorough understanding of when they are useful (i.e., when examining a small system in thermal contact with a large reservoir) is essential to the study of the field.

To understand the mathematical form of the Boltzmann factor, consider the interactions between the system under investigation (I call this C to avoid confusion with entropy, S) and the thermal reservoir (R; see Figure 6.1).[1] Recall from section 2.2 that the probability of finding the system in a particular state will depend on the total multiplicity of the system-reservoir combination ($P(E_C) \propto \omega_{tot}$), and that this is the product of the individual multiplicities of the system and the reservoir ($\omega_{tot} = \omega_C\,\omega_R$). In fact, if one considers a small enough system (perhaps a single particle) the energy of the system may only occupy a handful of discrete, non-degenerate energy levels ($E_C \in \{E_j\} = \{E_1,\, E_2,\, \ldots\}$). The system would therefore have a constant multiplicity, $\omega_C = 1$. The total multiplicity of the system-reservoir combination will then be exactly equal to the multiplicity of the reservoir:

$$\omega_{tot} = \omega_R \omega_C = \omega_R. \tag{6.5}$$

[1]The derivation presented here follows that in many thermal physics textbooks, cf. Refs. 89 &102.

The challenge now is to determine an expression for ω_R in terms of E_C (the defining parameter of the macrostate). To accomplish this one must first relate E_C to the properties of the reservoir.

It is reasonable to assume that the system-reservoir combination is isolated from the rest of the universe such that its total energy,

$$E_{tot} = E_C + E_R, \tag{6.6}$$

remains constant. The energy of the system, however, may fluctuate about some average value,

$$E_C = \langle E_C \rangle \pm \delta E. \tag{6.7}$$

The magnitude of these energy fluctuations (δE) may be relatively large compared to $\langle E_C \rangle$, but insignificant compared to $\langle E_R \rangle$, thus we are justified in considering R a reservoir as its energy does not change appreciably. Qualitatively, by conservation of energy, as the energy of the system decreases, the energy of the reservoir must increase, increasing ω_R and ω_{tot}, yielding a higher probability; therefore, lower energy states are more probable than higher energy states.

One must now be concerned with the precise mathematical form of multiplicity as it relates to energy, but while energy is an extensive variable, multiplicity is neither extensive nor intensive. This dilemma is solved by the immortal equation carved on Boltzmann's tombstone,

$$S = k \ln(\omega), \tag{6.8}$$

where S (entropy) is an extensive variable which may also be written as a function of other extensive variables (e.g., $S_R(E_R)$).[2] Since the system is so much smaller than the reservoir, it is clear that $E_R \approx E_{tot}$, and that a Taylor series expansion is

[2] As discussed in section 2.2 this logarithmic relationship allows us to consider both the additive nature of entropy (and energy) and the multiplicative nature of multiplicity (and probability).

appropriate to approximate $S_R(E_R)$ about the point $E_R = E_{tot}$:

$$S_R(E_R) = S_R(E_{tot}) - \left.\frac{\partial S_R}{\partial E_R}\right|_{E_{tot}} E_C + \ldots \approx S_R(E_{tot}) - \frac{E_C}{T}, \tag{6.9}$$

where $\left(\frac{\partial S}{\partial E}\right)_{V,N} = \frac{1}{T}$ from the differential form of the 1$^{\text{st}}$ Law (Eq. 2.5) and $E_R = E_{tot} - E_C$. Thus one obtains an expression for S_R as a function of E_C and constants. Revisiting Eq. 6.8 one obtains,

$$\omega_R \propto e^{\frac{-E_C}{kT}} \;\therefore\; P \propto e^{\frac{-E_C}{kT}}, \tag{6.10}$$

giving the desired result of $P(E_C)$, from Eq. 6.3.

It should be noted, however, that the above derivation is not the only method for obtaining the Boltzmann factor. Schroeder, for example, uses an approximation of what he calls the "thermodynamic identity" (1$^{\text{st}}$ Law, Eq. 2.5) rather than a Taylor series expansion to determine an expression for S_R in terms of E_C.[39] Carter, on the other hand, uses the method of Lagrange multipliers to maximize $\ln(\omega)$ with the constraints that the average energy and number of particles in the system are both fixed; this derivation does not require the assumption of a large thermal reservoir as the multiplicity of the reservoir is never used.[38] The derivation presented in this section was chosen for use within the *Boltzmann Factor* tutorialas it is presented in the course textbook (Ref. 89) as well as several other commonly used texts (cf. Ref. 102 & 103).

6.2 Student Recognition of When to Use the Boltzmann Factor

One desired result of teaching students about the Boltzmann factor is that they will recognize applicable situations and use it to make claims about probabilities.[3] The probability ratios question (PRQ, shown in Figure 6.2) probes their ability to

[3]See sections 2.2 & 6.1 for descriptions of situations in which the Boltzmann factor is applicable as well as an explanation of its connection to probability.

Consider a particle (Particle A) in a system with three evenly spaced energy levels, as seen in the figure at right. The probability that Particle A is in the n^{th} energy level is $P_A(n)$.

$n = 3$ ——— $0.10\,\text{eV}$
$n = 2$ ——— $0.05\,\text{eV}$
$n = 1$ ——— $0.00\,\text{eV}$

A. Is the <u>ratio</u> of the probabilities $\frac{P_A(3)}{P_A(2)}$ *greater than*, *less than*, or *equal to* the <u>ratio</u> of the probabilities $\frac{P_A(2)}{P_A(1)}$? Please explain your reasoning.

B. Consider a second single particle, Particle B, that can also only be in three states. The energies of the three states of each system are listed in the table at right. Both systems are in equilibrium with a reservoir at temperature T. Is the <u>ratio</u> of the probabilities $\frac{P_B(3)}{P_B(2)}$ for Particle B *greater than*, *less than*, or *equal to* the <u>ratio</u> of the probabilities $\frac{P_A(3)}{P_A(2)}$ for Particle A? Please explain your reasoning.

n	Particle A	Particle B
1	$0.00\,\text{eV}$	$-0.05\,\text{eV}$
2	$+0.05\,\text{eV}$	$0.00\,\text{eV}$
3	$+0.10\,\text{eV}$	$+0.05\,\text{eV}$

Figure 6.2. Probability Ratios Question (PRQ).

do this. The correct solution to the PRQ requires students to recognize three pieces of information:

- The probability of the particle being in each state is proportional to the Boltzmann factor for that state
- A ratio of exponential functions is the exponential of the difference of their exponents
- The energy difference between the states for each given ratio is the same ($\Delta E_{n,n-1} = 0.05\,\text{eV}$)

The first two items indicate that each ratio of probabilities will be an exponential function of the energy difference between the two states. The third item reveals that both pairs of ratios in the PRQ are equal.

6.2.1 Student Use of the Boltzmann Factor

The PRQ was given to students in the *Statistical Mechanics* course (*Stat Mech*) at UMaine after they had completed all lecture instruction on the Boltzmann factor and the canonical partition functionin five years.[4] The PRQ was also given once in the Thermal Physics I course at California Polytechnic State University, San Luis Obispo (Cal Poly) as a true pretest, before any instruction on the Boltzmann factor had occurred.[5] Student responses were coded in two ways: first by the response given (equal to, greater than, less than, or other), second by whether or not the Boltzmann factor was used. Figure 6.3 shows the response frequencies for the entire five-year data corpus from UMaine, and Figure 6.4 shows the response frequencies from Cal Poly. Green bars (dark cross-hatch) indicate students who gave the correct answer (equal to) and included correct reasoning; teal bars (light cross-hatch)

[4]See Section 4.1.2 for course details.

[5]Information on courses and tutorial implementation was obtained through personal communication with the instructors.

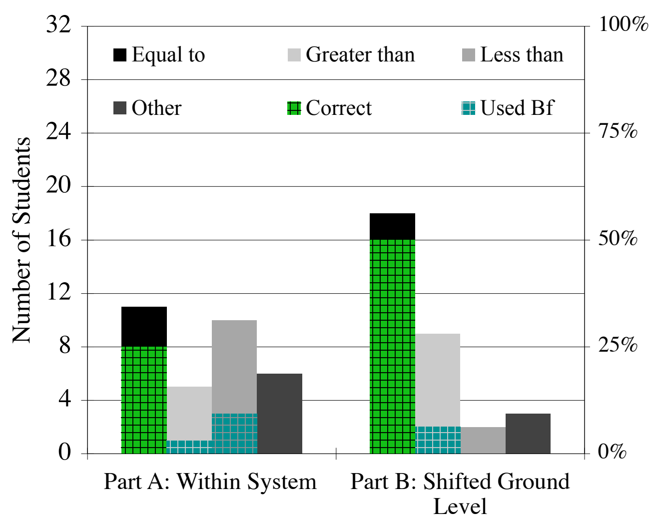


Figure 6.3. PRQ results: UMaine pre-tutorial. After lecture instruction on the Boltzmann factor in five years ($N = 32$).

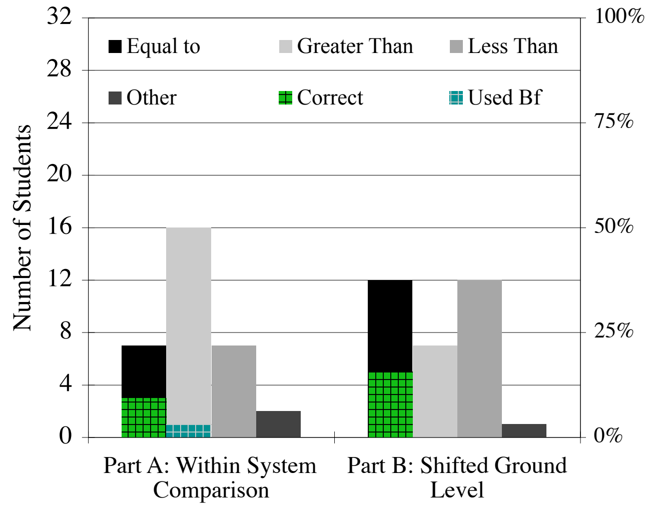


Figure 6.4. PRQ results: Cal Poly pre-tutorial. Before any instruction on the Boltzmann factor in one year ($N = 32$).

indicate students who gave an incorrect answer but recognized that the Boltzmann factor was needed to answer the question.[6]

The data represented in Figures 6.3 and 6.4 suggest several questions: 1) What percentage of students at each school use the Boltzmann factor, whether or not they get the right answer? and 2) What are students doing if they're *not* using the Boltzmann factor? To answer the first of these questions, I define four categories of responses:

- Correct response (equal to) with correct Boltzmann factor reasoning
- Correct response without correct Boltzmann factor reasoning
- Incorrect response with correct Boltzmann factor reasoning
- Incorrect response without correct Boltzmann factor reasoning

[6]This color scheme will be used for all presentations of response frequency data for probability ratio tasks.

By reducing to this coding scheme I am able to highlight the number of students who are and are not using the Boltzmann factor to answer the PRQ. A natural question associated with this coding scheme is, what does it look like for someone to use the Boltzmann factor but get an incorrect response? The answer to this question has several facets. On one hand, this person could have made a math error in computing ratios of exponentials. On the other hand, this person could have compared the wrong ratios, but done so correctly using the Boltzmann factor. Data exist that also indicate that some students impose degeneracy terms when using the Boltzmann factor to answer the PRQ. Basically, if a student wrote that probability is related to a decaying exponential of the energy, he was given credit for using the Boltzmann factor no matter what answer he got in the end.

Table 6.1 shows the percentages of students who occupy each of these categories at each school for both parts of the PRQ. From this presentation of the data it is clear that students at UMaine are using the Boltzmann factor more than students at Cal Poly on the PRQ pretest. In fact, a Fisher's exact test shows that this is a statistically significant difference (see Table 6.2), although not a surprising one: students at UMaine had had some instruction on the Boltzmann factor while students at Cal Poly hadn't had any. One the other hand, only 37% of students at

Table 6.1. PRQ results: Combined. Tabulated results from the PRQ pretest at both UMaine ($N = 32$) and Cal Poly ($N = 32$).

		Part A			Part B		
		Correct	Incorrect	Total	Correct	Incorrect	Total
UMaine	Used Bf	25%	12%	37%	50%	6%	56%
	No Bf	9%	54%	63%	6%	38%	44%
Cal Poly	Used Bf	9%	3%	12%	16%	0%	16%
	No Bf	13%	75%	88%	22%	63%	84%

Table 6.2. Pretest Comparison – UMaine vs. Cal Poly: Fisher's Exact Test. Numbers shown indicate the p-value for each comparison (significance set at $p < 0.05$).

	Total Distribution	BF Use	$>$ vs. $<$
Part (A)	0.02	0.06	0.05
Part (B)	0.001	0.001	0.03

UMaine are using the Boltzmann factor on part A and only 56% on part B. These are certainly not numbers to celebrate.

Going back to the question of what the students do when they don't use the Boltzmann factor, the reader may notice that the most common incorrect response at UMaine for both parts of the PRQ is the idea that, $\frac{P(0.10eV)}{P(0.05eV)} < \frac{P(0.05eV)}{P(0.00eV)}$ ("less than" for part A and "greater than" for part B). These answers are considered consistent since the second and third energy levels in particle B have the same numerical values as the first and second energy levels in particle A. A Fisher's exact test shows the distribution of "less than" and "greater than" responses to be significantly different on part A as compared to part B ($p = 0.02$).

The results from Cal Poly, shown in Figure 6.4, appear quite different from those at UMaine. In fact a Fisher's exact test for independence shows that the two

Table 6.3. Pretest Comparison: UMaine vs. Cal Poly. Numbers shown indicate the percentage of *incorrect* responses at each of the two schools.

	Part A		Part B	
	greater than	less than	greater than	less than
UMaine	21%	42%	56%	13%
Cal Poly	55%	24%	26%	44%

populations are significantly different for both parts of the PRQ ($p < 0.05$, see Table 6.2). The first noticeable difference is that there are fewer correct responses to either part at Cal Poly than at UMaine. This is not surprising, as the UMaine students had received some instruction on the Boltzmann factor while the Cal Poly students had not. The more interesting difference between the populations (summarized in Table 6.3) is the relative distribution of “greater than” and “less than” incorrect responses. While at UMaine 42% of incorrect responses to part A were “less than,” 55% of incorrect responses were “greater than” at Cal Poly. A similar trend exists in part B, where UMaine students were more likely to give the “greater than” response, and Cal Poly students were more likely to give the “less than” response. These results are reported as percentages of incorrect responses only. This is necessary since significantly more students answered the PRQ correctly at UMaine than at Cal Poly. Only by looking at the percentages of incorrect responses can meaningful comparisons be made.

Table 6.2 shows the results from six Fisher’s exact tests used to compare the response distributions from UMaine to those from Cal Poly on each part of the PRQ. These results indicate that the distribution of responses at Cal Poly is statistically significantly different than the distribution of responses at UMaine. Since the students at these two schools had received different levels of instruction before answering the PRQ, several different Fisher’s exact tests were completed to get a fuller picture of the similarities and differences between the populations. The “Total Distribution” column takes into account all eight categories (based on response and correctness of explanation). The “BF Use” column groups the data into two categories: one in which students used the Boltzmann factor correctly, and one with everyone else. The “$>$ vs. $<$” column only looks at the students who answered either “greater than” or “less than” and compares their relative distribution across schools.

Table 6.2 shows that the results of almost all of these comparisons are statistically significant ($p < 0.05$). The comparison of Boltzmann factor use between UMaine and Cal Poly for part A is not significant at the $\alpha = 0.05$ level, but at $p = 0.06$ I feel comfortable asserting that the students at UMaine used the Boltzmann factor more after lecture instruction than the Cal Poly students did before any instruction. Two Fisher's exact tests were also performed between the populations at UMaine and Cal Poly by correlating individual students' responses across parts A and B. The first of these tests compared the distribution of all nine different response pairs ($=$, $>$, or $<$ for each part) and yielded a nearly significant result ($p = 0.06$).[7] The second additional Fisher's exact test only looked at the response pairs in which both responses are either "greater than" or "less than" (pairs: $>$,$>$; $>$,$<$; $<$,$>$; and $<$,$<$). This second test also yielded a nearly significant result ($p = 0.06$), providing further evidence that the distribution of responses given by students at UMaine are different from that given by students at Cal Poly.

Figure 6.3 also shows that students at UMaine are more likely to answer part B correctly (which discusses an effective shift in the ground state energy of a system) than part A (comparing two different sets of probabilities for states within the same system). One student at UMaine justified his response for part B in stating that, "...it does not matter what the 'baseline' is, just the amount of energy added." This apparent increase in correctness could be a result of the coding scheme used in that explanations which involve comments about the arbitrariness of the ground state energy were considered correct for part B regardless of the student's response to part A. A Fisher's exact test looking at the distribution of correct and incorrect responses at UMaine was inconclusive ($p = 0.21$) as to whether a significant difference exists between the responses to parts A and B. I would be remiss, however, not to point

[7]The "other" responses were omitted during analyses using correlated responses, as one student's "other" response may have been completely different than another's.

out a difference of 25% correct with a correct reasoning on part A compared to 50% on part B. This phenomenon is not significantly observed at Cal Poly (see Figure 6.4, Fisher's exact test yields $p = 0.45$).

6.2.2 Incorrect Reasoning on the PRQ

The reasoning that students use when answering can be sorted into several categories; as described in Chapter 4, these categories were developed using a grounded theory approach in which I examined the data for common trends and then categorized responses based on those trends. At UMaine 12 students used the Boltzmann factor within their explanation of their answers to the PRQ. Only four students at Cal Poly used the Boltzmann factor. Of the remaining students 12 (out of 20) at UMaine and 13 (out of 28) at Cal Poly used a ranking of probabilities as their primary line of reasoning (e.g., $P_A(1) > P_A(2) > P_A(3)$). An additional five students at Cal Poly used a similar line of reasoning by stating that the lowest energy is most probable, but they did not make claims about the relative probabilities of energy states 2 and 3. This use of an explicit or implied probability ranking is the most common incorrect reasoning at both schools. Additionally, six of the students at UMaine who used a probability ranking, and seven students at Cal Poly (half of all students who used explicit rankings) made claims about the relative difference in probability between states 1 and 2 and between states 2 and 3. These claims were made in either sentence form, as one student from Cal Poly wrote, "... it is more likely that the system will have less energy so the difference between [state] 3 & [state] 2 is less than [between states] 2 and 1;"[8] or as a mathematical expression such as given by a student at UMaine, "$P_A(1) - P_A(2) > P_A(2) - P_A(3)$." Both of these statements lead to the idea that $P_A(1) \gg P_A(2) > P_A(3)$.

[8]Students' emphasis.

It should be noted that the probability rankings discussed above are completely accurate, and that the difference in probability between states 1 and 2 is in fact greater than that between states 2 and 3, but this information is not sufficient to determine the relationship between the probability ratios. The more interesting result is how students answer the question about probability ratios based on this reasoning. All seven students at Cal Poly who expressed the idea that $P_A(1) \gg P_A(2) > P_A(3)$, used it to claim that $\frac{P_A(3)}{P_A(2)} > \frac{P_A(2)}{P_A(1)}$, which is not necessarily true. This reasoning seems reminiscent of the Compare Numerator-Denominator Differences (NDD) strategy discussed in section 3.2.3.2.[63] Three students at UMaine also appear to use the NDD strategy although they come to three different conclusions:

$$P_A(1) \gg P_A(2) > P_A(3) \rightarrow \frac{P_A(3)}{P_A(2)} > \frac{P_A(2)}{P_A(1)}; \tag{6.11}$$

$$P_A(1) > P_A(2) \gg P_A(3) \rightarrow \frac{P_A(3)}{P_A(2)} < \frac{P_A(2)}{P_A(1)}; \tag{6.12}$$

$$P_A(1) \gg P_A(2) \gg P_A(3) \rightarrow \frac{P_A(3)}{P_A(2)} = \frac{P_A(2)}{P_A(1)}. \tag{6.13}$$

Interestingly, the third of these students uses this strategy to justify a correct response, possibly indicating a more sophisticated understanding of ratios and multiplicative vs. additive relationships than some of his classmates. Additionally, three students at UMaine compared the differences between the probabilities of adjacent states but came to conclusions that do not seem to follow from their reasoning via the NDD strategy for comparing ratios. For example, one student justified his claim that, "$\frac{P_A(3)}{P_A(2)} < \frac{P_A(2)}{P_A(1)}$" by stating that, "$P_A(1) - P_A(2) > P_A(2) - P_A(3)$, and $P_A(1) > P_A(2) > P_A(3)$." Using the NDD strategy with this ranking, however, would yield a "greater than" response rather than the "less than" that was given. Two other students used the ranking shown in Eq. 6.12 to justify a "greater than" response to part A of the PRQ. The exact thought process used by each of these students is unclear, as their conclusions do not logically follow from their claims.

It seems that without knowing an exact relationship between the numerators and denominators of the ratios in question (as the students did in Smith's study, Ref. 63 discussed in section 3.2.3.2), the students who used the NDD strategy to answer the PRQ were not able to recognize that their conclusions were not logically sound. It seems as though these advanced undergraduate students resorted to novice-like methods for comparing fractions when they did not have a well-defined expression for probability as a function of energy. An external evaluator used my categories to independently classify the data from Cal Poly. We initially agreed on 72% of students; after discussion and negotiation, we completely agreed on 91% of students and at least partially agreed on 97% of students (one person placed students simultaneously in two categories, the other person only agreed on one of the categories).

While only three students (two from UMaine, and one from Cal Poly) explicitly ranked the differences in probabilities of adjacent states, I will reasonably assume that the responses of all students who indicated that the probabilities were not evenly spaced may be classified as using the NDD strategy. The remaining students who justified their responses using the first-order probability ranking, $P_A(1) > P_A(2) > P_A(3)$, seemed to use a mixture of either the Numerator Principle, the Denominator Principle or Larger Components to come to their conclusions. No student, however, admitted to exclusively using either the numerator or the denominator of each ratio to compare the two; therefore, I cannot conclude that students used these strategies, only that the students' final responses are consistent with their use.

The key difficulty identified so far is that many students do not use the Boltzmann factor when it is appropriate to do so, even after lecture instruction. Instead, these students revert to novice-like behavior for comparing fractions. The following sections describe the *Boltzmann Factor* tutorial, which was designed to help students understand the criteria for determining when the Boltzmann factor is appropriate to use by recreating its derivation.

6.3 Tutorial Development and Design

Given students' apparent inabilities to properly use the Boltzmann factor, I created a tutorial to guide students through its derivation and encourage deep connections between the physical quantities involved. The derivation chosen for use in the *Boltzmann Factor* tutorial is found in many widely used textbooks, including the one used at UMaine.[89] The *Boltzmann Factor* tutorial begins by asking students to consider an isolated container of an ideal gas. They are guided to recognize that the container will have a fixed internal energy (E_{tot}) and that all microstates are equally probable.[9]

Once the properties of the isolated container have been established, the students are presented with a scenario in which the container of ideal gas is separated into relatively small and large sections (shown in Figure 6.1). The system of interest (C) is said to be in thermal equilibrium with the reservoir (R), and the students are asked to compare the values of various thermodynamic properties of C to those of R to highlight the fact that the intensive properties (e.g., temperature, pressure) will have the same value for both C and R, but the values of the extensive properties (e.g., volume, number of particles, internal energy) of C will be much smaller than those of R.

The third section of the tutorial uses the fact that the multiplicities of C and R are so different ($\omega_C \ll \omega_R$) to justify a single-particle toy model in which $\omega_C = 1$ ($\omega_{tot} = \omega_C \, \omega_R = \omega_R$), and the energy of C can only take on a handful of discrete values, $E_C \in \{E_j\} = \{E_1, E_2, \dots\}$ (see Table 6.4). The students are asked to determine which macrostate (denoted by j) is most probable and which is least probable. The desired solution is that the macrostate in which R has the largest number of microstates (multiplicity) would be the most probable (E_4 below) since

[9]The entire *Boltzmann Factor* tutorial is included in Appendix B.

all microstates are equally likely. Careful consideration of the relative probabilities of each macrostate leads to the proportionality between the probability of the j^{th} macrostate and the multiplicity of the reservoir ($P(\Psi_j) \propto \omega_R(\Psi_j)$).

Table 6.4. Sample energy & multiplicity values for toy model system. Presented within the tutorial (see Figure 6.1).

E_C	ω_C	E_R	ω_R
E_1	1	$E_{tot} - E_1$	3×10^{18}
E_2	1	$E_{tot} - E_2$	5×10^{19}
E_3	1	$E_{tot} - E_3$	4×10^{17}
E_4	1	$E_{tot} - E_4$	1×10^{20}
E_5	1	$E_{tot} - E_5$	7×10^{18}

The final section of the *Boltzmann Factor* tutorial is the derivation of the Boltzmann factor itself. The students are asked to perform a Taylor series expansion of $S_R(E_R)$ about the value $E_R = E_{tot}$ to obtain the expression for S_R as a linear function of E_C given in Eq. 6.9. They are explicitly asked to consider the physical significance of each term and to determine the final linear expression on their own. Then, using the relationship between entropy and multiplicity ($S = k\ln(\omega)$), they derive an expression for ω_R as being proportional to a decaying exponential of the energy of C (the Boltzmann factor):

$$\omega_R = e^{S_R/k} \approx e^{S_R(E_{tot})/k - E_C/kT} \tag{6.14}$$

and, since $S_R(E_{tot})$ is a constant,

$$\omega_R \propto e^{-E_C/kT}. \tag{6.15}$$

Students have now found that $P(\Psi_j) \propto \omega_R(\Psi_j)$ and that $\omega_R(\Psi_j) \propto e^{-E_j/kT}$, leading to the proportionality in Eq. 6.1. Finally, they are asked to normalize the probability, thus deriving Z (see Eq. 6.2).

The post-tutorial homework assignment is an application of the Boltzmann factor to a three-state system with unevenly spaced energy levels. Students are asked various questions about the ratios of probabilities of the system being in a particular state. These questions are similar to the PRQ but the students are given specific values for T and N and asked to determine numerical values for the probability ratio rather than compare two different ratios. They are also asked to determine an expression for the generic ratio between the probabilities of any two energy levels. This homework assignment was used as a continuation of the tutorial, not as an assessment or research tool. A full recreation of the tutorial and all pre- and post-tutorial assignments and assessments can be found in Appendix B.

6.4 Tutorial Implementation

The *Boltzmann Factor* tutorial was administered in *Stat Mech* after all lecture instruction on the Boltzmann factor in two consecutive years. Students were given one 50-minute class period to complete the *Boltzmann Factor* tutorial. The course instructor and one TA were available during the tutorial session as observers and facilitators. No course credit is offered for the tutorial itself, but the course grade includes in-class participation. In tutorial sessions, two groups were videotaped, and written data were collected in the form of pre-tutorial assessments and post-tutorial exam questions.

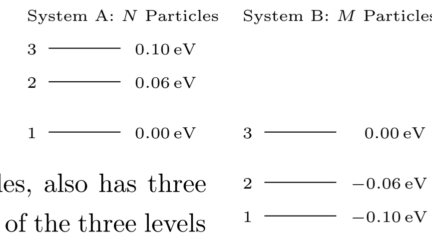
During the first year of tutorial implementation, students only completed about half of the tutorial; therefore, I conducted individual interviews with four students after classroom instruction to determine their familiarity with the Boltzmann factor, its applications, and its origin. The interviews were conducted in the style of teaching interviews (as discussed in Section 4.2.3) and consisted of asking students to complete the second half of the *Boltzmann Factor* tutorial starting with looking at

Table 6.5. *Boltzmann Factor* Tutorial Implementation Timeline

Before Year 1	• PRQ given after lecture instruction at UMaine in three years
Year 1	• PRQ given after lecture instruction at UMaine • Tutorial administered at UMaine – included pre- and post-tutorial homework assignments • Teaching interviews conducted at UMaine as follow-up to the tutorial • PRQ given on course exam after tutorial
Year 2	UMaine • PRQ given after lecture instruction • Tutorial administered after lecture • PRQ Analog given on course exam after tutorial • Clinical Interviews conducted on related topics Cal Poly • PRQ given before any instruction • Tutorial administered instead of lecture • PRQ Analog given on course exam after tutorial

how probability relates to multiplicity in the divided container (C-R) scenario (see Figure 6.1). The goal of the interviews was not to determine students' understanding of the Boltzmann factor but rather to examine how well they could complete instructional tasks based on previous knowledge related to the Boltzmann factor. Students worked individually, and I solicited explanations for their work and gave assistance when required. Two interview participants had participated in the first half of the *Boltzmann Factor* tutorial during class, and the other two had not seen the first half. Field notes were taken during the interviews, and students' written work was collected afterward.

Systems A and B are both at the same temperature T. System A has N identical particles, each of which must be one of the three energy levels shown. In thermal equilibrium, the numbers of particles in the three levels are n_1, n_2, and n_3. System B, with M identical particles, also has three energy levels, as shown. The numbers of particles in each of the three levels of system B are m_1, m_2, and m_3.



Which is the true statement?

I. The ratio n_3/n_2 in system A is *greater than* the ratio m_2/m_1 in system B.
II. The ratio n_3/n_2 in system A is *equal to* the ratio m_2/m_1 in system B.
III. The ratio n_3/n_2 in system A is *less than* the ratio m_2/m_1 in system B.
IV. There's *not enough information* to compare n_3/n_2 in system A to m_2/m_1 in system B.
Explain your reasoning. If you answer IV, also say what additional information you would need.

Figure 6.5. PRQ Analog. Administered on a course exam at UMaine in Year 2 and at Cal Poly. Question developed by instructor at Cal Poly.

In Year 2, the *Boltzmann Factor* tutorial was also administered in the Thermal Physics I course at Cal Poly in place of lecture instruction in one quarter. Students were given one 50-minute class period and an additional 20 minutes during the next period to complete the tutorial.[10] Written data were collected in the form of pre-tutorial assessments and post-tutorial exam questions. The PRQ was administered as a pretest at both institutions before tutorial instruction. Table 6.5 shows the timeline of tutorial implementation at both UMaine and Cal Poly.

The PRQ was given on a course examination after the first implementation of the *Boltzmann Factor* tutorial at UMaine. A similar question was developed by the instructor at Cal Poly and asked on a course exam in Year 2 at UMaine and at Cal Poly. This second question, referred to as the PRQ Analog (shown in Figure 6.5), requires students to apply the same knowledge as is used to correctly answer the

[10]Personal communication with course instructor.

Table 6.6. Comparison of pre- and post-tutorial performance at UMaine. Number correct on probability ratios assessments; percentages shown in parentheses.

	N	Pre-tutorial	Post-tutorial
Undergrad w/ tutorial	6	1 (17%)	5 (83%)
Undergrad w/ tutorial, no pretest	5	n/a	4 (80%)
Grad w/ tutorial	4	3 (75%)	4 (100%)
Undergrad no tutorial	5	2 (40%)	3 (60%)

PRQ, but it involves comparing ratios between systems that have unevenly spaced energy levels.

6.5 Results

I begin by presenting the results that most closely address the stated instructional goals of the *Boltzmann Factor* tutorial: students' use of the Boltzmann factor on applicable exam questions. I will then discuss in-class observations of tutorial sessions and their implications as well as the results from teaching interviews conducted to supplement tutorial instruction in Year 1.

6.5.1 Exam Results

As mentioned above, the PRQ (Figure 6.2) was administered on a course examination in Year 1 at UMaine, and the PRQ Analog (Figure 6.5) was administered on a course examination in Year 2 at UMaine and at Cal Poly. From the two implementations at UMaine there are 15 sets of matched (pre-/post-tutorial) data: six undergraduate physics majors and four graduate students in physics who participated in the *Boltzmann Factor* tutorial, and five undergraduates who did not participate in the *Boltzmann Factor* tutorial. Additionally, five undergraduate students who participated in the *Boltzmann Factor* tutorial and/or the teaching interviews did not complete the PRQ pretest but did answer the exam question. Table 6.6 shows how

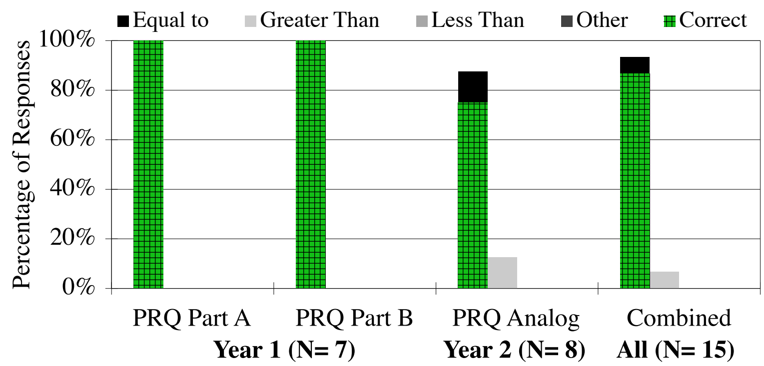


Figure 6.6. Ratio Questions Results: UMaine. Post-tutorial results from PRQ and PRQ Analog, administered during course examinations. Green bars indicate the fraction of "correct" responses that included correct explanations.

many students in each of these groups answered correctly with correct reasoning either before or after tutorial instruction. Figure 6.6 shows the exam data from all students who participated in the tutorial broken down by year. These data provide evidence that the *Boltzmann Factor* tutorial helps students recognize the utility of the Boltzmann factor and how to apply it properly in the context of these questions.

The most striking feature of Figure 6.6 is that all seven students in Year 1 gave the correct answer with appropriate reasoning on both parts of the PRQ. In Year 2, seven out of the eight tutorial participants gave the correct answer to the PRQ Analog, and six of them gave appropriate explanations. This is a marked improvement over lecture instruction alone. A Fisher's exact test shows that the exam results do not differ significantly between the two years ($p = 1$).[11] In order to perform statistical analyses to compare the exam results with the pretest results, data were grouped into the four categories discussed in section 6.2.1.

This reduced coding scheme is necessary because, since I essentially asked three questions at various times (PRQ parts A and B, and the PRQ Analog), the "greater

[11]Since the results from parts A and B in Year 1 are identical, these were combined into one result for statistical analyses. All UMaine exam data is combined for further statistical tests.

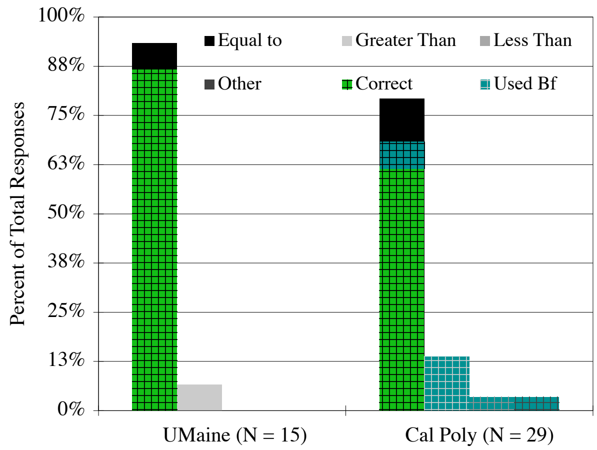


Figure 6.7. Ratio Questions Results: Cal Poly & UMaine. Post-tutorial results from PRQ Analog administered during a course examination. Data from UMaine are the combination of results from the PRQ and PRQ Analog used on course examinations after tutorial instruction. Green bars indicate the fraction of "correct" responses that included correct explanations.

than" responses to the various questions (for example) cannot necessarily be considered the same response. As such, the only categories available for grouping responses are either the correct response or one of the incorrect response; along with this we have the dimension of whether or not a student used the Boltzmann factorappropriately to justify their response, yielding the four categories above. These general categories do not allow claims to be made about how reasoning patterns differ within the incorrect responses, but they do allow comparisons of the frequency with which students use the correct Boltzmann factor reasoning and whether or not it yields a correct response. Using these categories, a Fisher's exact test showed that the results from the exams at UMaine are statistically significantly better than the results on part A of the PRQ pretest at UMaine ($p = 0.0003$) and nearing statistical significance on part B ($p = 0.06$).

Table 6.7. PRQ results: Combined. Tabulated results from the PRQ and PRQ Analog exam questions at both UMaine ($N = 15$) and Cal Poly ($N = 29$).

		Correct	Incorrect	Total
UMaine	Used Bf	87%	0%	87%
	No Bf	7%	7%	13%
Cal Poly	Used Bf	69%	21%	90%
	No Bf	10%	0%	10%

Results from tutorial implementation at Cal Poly are similarly promising. Figure 6.7 shows the response frequencies from the PRQ Analog at Cal Poly as well as the combined results from Years 1 and 2 at UMaine. Table 6.7 shows these data categorized by either correct or incorrect response and by whether or not the Boltzmann factor was used. A total of 90% of the students at Cal Poly and 87% of the students at UMaine recognized the need for the Boltzmann factor and used it appropriately on the exam question, though some of these students made mathematical or other procedural errors. In fact, a smaller percentage of students at Cal Poly answered completely correctly with correct reasoning than at UMaine, but a number of students used the Boltzmann factor appropriately to come to alternative conclusions. For example, two students used the Boltzmann factor correctly to compare the wrong pair of ratios. Another three students made math errors while using the Boltzmann factor. Comparing Figure 6.7 to Figures 6.3 & 6.4 and Table 6.7 to Table 6.1, the differences are substantial: from 32% consistently using the Boltzmann factor after lectures to nearly 90% after tutorial instruction. Furthermore, a Fisher's exact test (using the categories described above) shows that the exam results from Cal Poly are not significantly different from those at UMaine ($p = 0.37$) and that the exam results at Cal Poly are statistically significantly better than the results of both part A ($p = 9 \times 10^{-10}$) and part B ($p = 2 \times 10^{-8}$) of the PRQ pretest at

Cal Poly. These results suggest that the *Boltzmann Factor* tutorial helps improve student understanding of how and when to use the Boltzmann factor when it is used instead of lecture instruction (Cal Poly) as well as when it is used in addition to lecture instruction (UMaine).

6.5.2 Understanding the Origin of the Boltzmann Factor

Results from the teaching interviews conducted at UMaine in Year 1 provide further evidence of the need for the *Boltzmann Factor* tutorial, especially with regard to the origin of the Boltzmann factor itself. One student (Joel,[12] who had participated in the second and third sections of the *Boltzmann Factor* tutorial in class) was very familiar with the applications of the Boltzmann factor and seemed to be just as familiar with its origin.[13] When asked to determine the most probable macrostate in Table 6.4, Joel wanted to use the Boltzmann factor rather than thinking about multiplicities, even though no information had been given about the relative energy values.[14] The interviewer asked Joel to show where the Boltzmann factor came from before applying it to this situation, at which point Joel quoted the textbook derivation of the Boltzmann factor practically verbatim. The final portion of Baierlein's mathematical derivation is as follows, [89, p. 92][15]

$$P(\Psi_j) = \text{const} \times \begin{pmatrix} \text{multiplicity of reservoir} \\ \text{when it has energy } E_{tot} - E_j \end{pmatrix} \tag{6.16}$$

$$P(\Psi_j) = \text{const} \times \exp\left[\frac{1}{k} S_R(E_{tot} - E_j)\right] \tag{6.17}$$

$$P(\Psi_j) = \text{const} \times \exp\frac{1}{k}\left[S_R(E_{tot}) + \left.\frac{\partial S_R}{\partial E_R}\right|_{E_{tot}} \times (-E_j)\right] \tag{6.18}$$

$$P(\Psi_j) = (\text{new constant}) \times \exp\left(-E_j/kT\right). \tag{6.19}$$

[12] All names are pseudonyms.

[13] Teaching interviews were not audio or video recorded so as to provide a more informal atmosphere. Analysis is based on interviewer fields notes and students' written work.

[14] Joel had also provided this reasoning during the in-class tutorial session.

[15] Eq. 6.18 is not explicitly shown in Ref. 89, but Joel wrote it during his interview.

When asked how the multiplicity of the reservoir relates to the Boltzmann factor, however, Joel was at a loss. During his replication of the derivation of the Boltzmann factor he had implicitly written that it was proportional to ω_R (connecting Eqs. 6.16 and 6.19), but without explicit help from the interviewer, Joel could not recognize that the multiplicity of the reservoir when it has energy, $E_{tot} - E_j$ (RHS of Eq. 6.17), is proportional to the exponential function, $e^{(E_j/kT)}$ (RHS of Eq. 6.19). Furthermore, Joel had great difficulty relating the physical example used in the text (a "bit of cerium magnesium nitrate... in good thermal contact with a relatively large copper disc"[89, p. 91]) to the ideal gas example used during the interview. He was unable to recognize and articulate the important physical characteristics of each scenario that make the Boltzmann factor applicable. Joel's failure to make these connections suggests an incomplete understanding of the physical reasoning used to derive the Boltzmann factor, even after memorizing the textbook derivation.

Data from videotaped in-class observations of tutorial implementation at UMaine in Year 2 provide evidence that students gain an appreciation for the origin of the Boltzmann factor while participating in the *Boltzmann Factor* tutorial. Two students (Sam and Bill, who worked in a group on their own) participated in several conversations throughout the tutorial session that indicate their contemplation of relevant physical ideas. During the *Boltzmann Factor* tutorial they discussed which macrostate (from table 6.4) will be most probable:

Bill – Probably the one with more microstates

Sam – Yeah... the one with the highest multiplicity

...

Bill – "Give a general expression for the probability of the system"... so probably just use omega R (ω_R), so we'd say omega R j (ω_{R_j}) over the sum of all of them.

Sam – Yeah, that's what I said: omega R j over the sum of omega R j ($\frac{\omega_{R_j}}{\sum \omega_{R_j}}$).

Later in the tutorial, after completing the Taylor series expansion (with instructor intervention), interpreting the physical quantities involved, and relating their expression for multiplicity to the Taylor series of entropy, Sam and Bill had a realization:

Sam – That's cool. Look, see, you get the Boltzmann factor. You solve for omega (ω): e to the minus E over k T ($e^{-E/kT}$).

...

Bill – I guess that's where it comes from.

Sam – 'Cause I didn't know where it came from.

Bill – I had no idea.

Sam – I was just like, "OK."

These excerpts indicate that Sam and Bill are discussing relevant physical quantities and principles and gaining an appreciation for the origin of the Boltzmann factor as a result of the *Boltzmann Factor* tutorial. In particular they are correctly relating the Boltzmann factor of the system with the multiplicity of the reservoir as an indicator of probability. It should be noted that before tutorial instruction, Sam answered both parts of the PRQ correctly using correct reasoning, and Bill used the Boltzmann factor correctly but made errors in his calculations. These data indicate that students who are able to successfully use the Boltzmann factor after lecture instruction may not have a good understanding of the conceptual meaning behind the mathematics they are using; furthermore, these same students can gain an appreciation for the physical significance of the Boltzmann factor after participating in the *Boltzmann Factor* tutorial.

6.5.3 Tutorial Observations and Revisions

Data from videotaped classroom tutorial sessions and teaching interviews at UMaine provide valuable information on students' abilities to complete tutorial tasks. The tutorial sessions at Cal Poly were not videotaped, so detailed analyses of student conversations and time-on-task data are not possible. The instructor, however, provided detailed feedback on students' abilities to perform tutorial tasks as well as specific places where they had particular difficulty. All of these data were used to inform tutorial revisions and modifications.

It should be noted that data do not exist to determine the precise effect that each of these tutorial modifications has on student learning and understanding of the Boltzmann factor. Data are presented, however, indicating increased student efficiency in completing tutorial tasks during later implementations, allowing them to complete more of the tutorial in the time allotted. Increased efficiency benefits students by giving them the opportunity to get to the "punchline" of the *Boltzmann Factor* tutorial: the derivation of the Boltzmann factor itself.

During the in-class tutorial session in Year 1 at UMaine several unanticipated difficulties were observed. The first occurred while students completed the first page of the tutorial on which it asked them to "estimate (to order of magnitude) how many *microstates* (molecular configurations) exist such that the total energy of the gas [in the isolated container] is E_{tot}." This language cued the students to attempt to find a formula for calculating the multiplicity of the gas based on its energy.[16] The intent of the task, however, was for the students to recognize that there would be many many molecular configurations that would have a total energy of E_{tot} and to just write down any appropriately large number. Students spent four minutes on this task before asking the instructor for help. (This wasn't expected to take very long; a rigorous calculation was neither intended nor possible, and thus it should only have taken about a minute.) The wording of the question was therefore altered in subsequent implementations to ask the students, "How many *microstates* (molecular configurations) would you estimate exist such that the total energy of the gas is E_{tot}: 1, 1000, 10^N?" Data from the second tutorial implementation at UMaine indicate that students found this order-of-magnitude estimate much easier than the year before.

One observation noted during the teaching interviews was that some students focused strongly on a relationship between multiplicity and energy ($\omega \propto V^N E^{\frac{3}{2}N+1}$) that was given in an introductory paragraph of the interview (and the tutorial section). The intent of the statement was to connect the *Boltzmann Factor* tutorial to the density of states function, $D(E)$, which they had recently learned about and motivate the notion that $\omega_C \ll \omega_R$ (since $V_C \ll V_R$ and $E_C \ll E_R$). Students tried to use this expression, however, to relate the multiplicities given in Table 6.4 to the energies. One student (Jake, who had participated in the first three sections of the tutorial in class) even stated that since the $E_C = E_3$ macrostate has the lowest multiplicity ($\omega_R = 4 \times 10^{17}$, rightmost column in Table 6.4), E_3 lowest energy (of C) and, therefore, be the most probable. What he failed to consider is that the multiplicity of the *reservoir* is the lowest, making E_R the lowest, and E_3 the highest value (by conservation of energy). Jake's reasoning, in fact, reached the exact opposite conclusion of what was intended.

[16]Discussions centered around trying to remember the density of states function and the binomial distribution.

The intent of the tutorial section is to motivate the connection between multiplicity of the reservoir and probability of the system being in the corresponding macrostate. The students were meant the see that the $E_C = E_4$ macrostate is the most probable since it has the largest corresponding multiplicity for the reservoir and later conclude that E_4 must be the lowest energy of the system because E_R must be at its highest value. Two other interview participants displayed this tendency to latch onto the given expression relating mulitiplicity to energy, and it was observed during the in-class tutorial session to a lesser extent. The statement reminding students about the connection between multiplicity and energy mentioned above was removed from later implementations of the *Boltzmann Factor* tutorial along with most of the original introductory paragraph. Data from the implementation at UMaine in Year 2 indicate that the information originally included in this introductory paragraph is not needed for successful completion of the *Boltzmann Factor* tutorial.

Results from teaching interviews motivated several revisions to the tutorial document (including the removal of the explicit relationship: $\omega \propto V^N E^{\frac{3}{2}N+1}$). Data from the implementation in Year 2 show that these revisions (along with those motivated from in-class observations) helped students navigate the *Boltzmann Factor* tutorial more efficiently (as determined by time on task and correctness of responses) without any adverse effects evident in post-tutorial assessments.

Other in-class observations indicated that students did not always refer to their own work from previous sections of the tutorial when answering more difficult questions later. In particular, when answering questions about multiplicity concerning the divided container (see Figure 6.1), students did not necessarily refer to the conclusions they had made about the original undivided container. Specific references to previous tutorial sections were added to encourage students to make these connections and build on knowledge they had previously constructed.

The instructor at Cal Poly reported that students were quite confused by the double-subscript notation used in the tutorial (e.g., E_{C_1} to denote the value of E_C when C is in the $j = 1$ state). This notation was used in place of that found in Table 6.4 in an effort to be explicit about which energy (that of C or R) was being discussed. A short interview study may be necessary to determine the notation that would be most transparent and informative for students.

The most evident observation from all implementations of the *Boltzmann Factor* tutorial is that students could not complete the tutorial in the 50-minute session allotted. As a result of poorly worded questions and some distracting information, the students at UMaine in Year 1 were only able to complete the first three sections of the tutorial, ending in an expression indicating, $P(\Psi_j) \propto \omega_R(\Psi_j)$. They did not have the opportunity to even begin the Taylor series expansion that would lead to the derivation of the Boltzmann factor (the portion of the tutorial that I expected to be the most difficult). After revising the tutorial to address the specific difficulties discussed above, students at UMaine in Year 2 were able to successfully complete the first four sections of the tutorial, culminating with the derivation of the Boltzmann factor. They did not, however, have sufficient time to complete the normalization of probability to determine an expression for the canonical partition function. A similar result was reported at Cal Poly in that six out of seven groups of students (≈ 4 students per group) were able to derive the Boltzmann factor after the entire 70 minutes allotted,[17] but only 1-2 groups had enough time to derive Z as well.

Based on the overwhelming majority of students not finishing the entire tutorial during all three implementations, I removed the fifth section from the tutorial, in which students derive the canonical partition function, and added it as the first question in the post-tutorial homework assignment. Students at Cal Poly (and

[17]Students at Cal Poly were given an additional half of a class period to complete the tutorial.

during the teaching interviews at UMaine) who got to that portion of the tutorial had little trouble normalizing their expression for probability to get Z. It seems likely, therefore, that students will be able to perform this task on their own as part of the homework. The tutorial now ends with the derivation of the Boltzmann factor as well as a comment on the term "Boltzmann factor" and a reference to the homework assignment in which they will determine an exact expression for the probability rather than just a proportionality.

6.6 Related Difficulties — The Taylor Series

Before the first implementation of the *Boltzmann Factor* tutorial I expected students to have difficulty with some aspects of the derivation. In particular I expected that students might not be able to generate a Taylor series expansion of entropy as a function of energy. This concern led to the use of a pretest on the graphical interpretation of a Taylor series expansion and a pre-tutorial homework assignment for the students to complete at home on their own and bring to class on the day of the tutorial.

In the Taylor series pretest, developed by Warren Christensen while a postdoc at UMaine and based on a suggestion by Andrew Boudreaux (Western Washington

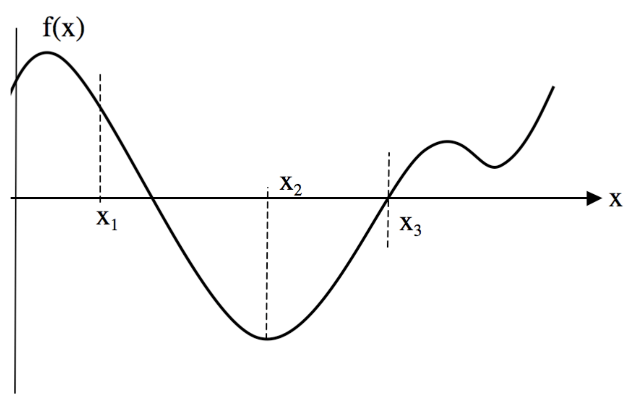


Figure 6.8. Graph used in the Taylor series pretest at UMaine.

University), students interpret the terms of a Taylor series expansion based on a given graph of a function, $f(x)$, shown in Figure 6.8.[104] The Taylor series expansion about the point $x = x_1$ is given to the students as,

$$f(x) = a_1 + b_1(x - x_1) + c_1(x - x_1)^2, \tag{6.20}$$

and they are asked to determine whether each of the quantities a_1, b_1, and c_1 are positive, negative, or zero and explain their reasoning based on the graph with x_1 clearly marked. The same question is asked of two other locations on the graph, x_2 and x_3. The correct solution requires students to recognize that a is the value of the function at the specified point, b is the slope of the function (corresponding to the first derivative), and c is proportional to the concavity (corresponding to the second derivative). In two years that this question was given at UMaine before tutorial instruction, 9 students out of 16 correctly determined the signs of the various quantities and gave appropriate reasons. This suggests that about half of the students in *Stat Mech* are familiar with the meaning of the various terms in the Taylor series.

The pre-tutorial homework assignment (shown in Appendix B) asks the students to write a Taylor series expansion of entropy as a function of energy (including no more than five terms) about the value $E = E_0$,

$$S(E) = S(E_0) + \left.\frac{\partial S}{\partial E}\right|_{E_0}(E - E_0) + \frac{1}{2!}\left.\frac{\partial^2 S}{\partial E^2}\right|_{E_0}(E - E_0)^2 + \frac{1}{3!}\left.\frac{\partial^3 S}{\partial E^3}\right|_{E_0}(E - E_0)^3 + \ldots \tag{6.21}$$

The homework assignment also asks the students to give an interpretation of how each of the terms in the Taylor series relates to a given graph of S vs. E. The goal of the pre-tutorial homework assignment was to give the students the opportunity to look up the generic form of the Taylor series and have it with them in class to facilitate the application of the Taylor series to the specific physical situation presented in the tutorial. The graphical interpretation question was included to encourage students to think about the meaning of the terms in the Taylor series rather than copying down abstract symbols devoid of meaning.

6.6.1 Student Use of Taylor Series

During the teaching interviews conducted at UMaine in Year 1 Joel (mentioned above) was unique in that he was the only student interviewed who successfully spontaneously generated a Taylor series expansion of entropy as a function of energy as it relates to the given physical scenario, the necessary step to go from Eq. 6.17 to Eq. 6.18. Unfortunately the interviewer did not probe Joel's understanding of the Taylor series further; it therefore remains unclear whether or not this was another case of Joel successfully memorizing the text without developing a complete understanding of its meaning or implications (as he did for the derivation of the Boltzmann factor). Jake and one other student were able to generate the appropriate expansion when given the generic mathematical expression for a Taylor series (Eq. 6.21), but the final student (of four) was unable to make any connections between the generic Taylor expansion and the physical scenario without explicit instruction from the interviewer. These results indicate that student understanding of the motivation for a Taylor series expansion (a crucial part of this derivation of the Boltzmann factor) cannot be taken for granted. When combined with the data from the Taylor series pretest discussed above, these results indicate that many students will be able to interpret and apply a Taylor series that is given to them, but they may not be able to generate an appropriate expansion in a novel context. This underscores the need for the pre-tutorial homework assignment in which students are asked to generate the Taylor expansion in Eq. 6.21. Along with several other researchers at UMaine, I have found this pre-tutorial homework strategy to be worthwhile, even necessary, for implementing tutorials in upper-division physics courses. This marks a distinct difference from typical tutorial implementation within introductory courses, as far more prerequisite knowledge is assumed at the upper division, including a robust understanding of concepts in both physics and mathematics.

One unexpected difficulty observed during the tutorial session in Year 2 at UMaine is that two students (Sam and Bill, discussed above) did not correctly construct the Taylor series expansion asked for in the pre-tutorial homework assignment. Instead of constructing the appropriate expansion as seen in Eq. 6.21, they used the terms "S_1" and "S_2" (constant entropy terms) in place of the "$(E - E_0)$" and "$(E - E_0)^2$" terms, respectively (i.e., $S = S_0 + S_1 S' + \frac{1}{2} S_2 S'' + \ldots$, where S_0, S_1, and S_2 were said to be constants), making it impossible to obtain entropy as a function of energy. These students did, however, recognize that their expression lacked an energy term, and once the instructor intervened to discuss the appropriate form of the Taylor series with them, they were able to use it correctly to complete the derivation of the Boltzmann factor. This is further evidence that the successful completion of the pre-tutorial homework assignment is crucial to student success with the tutorial, but that the assignment itself is not trivial, as some students may not be successful in completing the task.

As reported by the instructor at Cal Poly, many students in this course had great difficulty using the Taylor series expansion in the tutorial context even after having completed the pre-tutorial homework. The instructor assigned specific study of Taylor series between the two class periods, and reported that a short lecture on the use of Taylor series expansion was necessary at the beginning of the second tutorial period to allow students to successfully complete the tutorial.

6.6.2 Further Investigation into Student Understanding of the Taylor Series

After the tutorial implementation at UMaine in Year 2, I conducted clinical interviews with four students who had participated in the tutorial, one of whom was accompanied by a student who had not participated in the tutorial. One of the goals of the interview was to determine how familiar the students were with Taylor series expansions in terms of when they are applicable and how they are used. Some common uses of Taylor series include numerical computations, evaluations of definite integrals and/or indeterminate limits, and approximations.[105] Approximations are particularly useful in physics at times when a solution in its exact form is unnecessary or too difficult; in situations where information is known about various derivatives of a function at a specific point, but nothing more is known about the function itself; or in situations in which one is investigating small fluctuations about an average value.

All students interviewed had a reasonable understanding of situations in which the Taylor series is an appropriate tool. All students spontaneously used terms like "approximation" and "estimation" when describing how to use a Taylor series expansion, and all students were able to list one or more specific areas of physics in which Taylor series expansions are useful. One interesting aspect of the interviews is that all students at some point during the interview *spontaneously* referred to the kinematic equation ($x = x_0 + v_0 t + \frac{1}{2}at^2$) as a Taylor series. This had been mentioned during lecture as an example of a Taylor series expansion with which everyone would be familiar (even if they had never thought of it as a Taylor series). Their acknowledgment of the kinematic equation as a Taylor series seemed to influence their responses to various interview prompts.

One of the main questions the interviewees were asked about the Taylor series was how they knew when to truncate the series. A common response involved knowledge about the functional form of any higher order derivatives; i.e., if one of the derivatives is constant, then all higher derivatives will be identically zero. One student (Malcolm, a graduate student in physics) used this reasoning to justify why the kinematic equation has only three terms: "Usually acceleration's constant, so we don't have a jerk. If we had a jerk running around messing things up, we'd need more terms." When prompted about situations in which no information was known about the derivatives, however, Malcolm said that he would use different "rules of thumb" depending on the application. If only a "ballpark" estimate was needed, for example, only one or two terms would be necessary, but he indicated more terms would be needed as desired precision increased (e.g., to 16 decimal places). Malcolm also expressed the idea that looking very close to the value about which he was expanding would require fewer terms than if he were to try to examine a value far away from the expansion point. Finally, Malcolm stated that he would examine the deviation between the Taylor series expansion and any experimental data available and keep enough terms to have a reasonable fit (although he did not specify how close he would require the expansion to match experimental data). This relation to experimental data was echoed by Jayne (another graduate student in physics) who initially had trouble articulating a good rationale for truncating a series but eventually referred to different needs for different experimental tasks. Jayne also cited a threshold for truncation of three or four orders of magnitude, i.e., terms that are 3–4 orders of magnitude smaller than the linear term are not necessary.

Two undergraduate physics majors who were interviewed together (Paul, who participated in the *Boltzmann Factor* tutorial, and Jonah, who did not) also cited constant acceleration as the reason why the kinematic equation only has three terms and knowledge of constant derivatives as the primary reason to truncate a Taylor series. After several prods and questions about series truncation they started using "estimation" language to discuss the possibility of starting with a "ballpark" estimate and keeping terms until the results were close enough (using a guess-and-check type of method). Paul also argued that the purpose of a Taylor series is to estimate something that is more complex and that the first few terms must be the most significant while the higher order terms die out.

All students interviewed were able to list some areas of physics in which Taylor series might be useful other than the kinematic equation (examples in quantum mechanics, solid state physics, statistical mechanics, etc.), but no one elaborated on *how* exactly a Taylor series would be useful in these various situations. Malcolm came closest by citing the use of Taylor series to approximate a potential in quantum mechanics as a harmonic potential, a task he implied he had completed in the past. It is still unclear, however, what (if anything) would motivate these students to spontaneously use a Taylor series expansion in a given physical situation. I do not have evidence that they are able to generalize their knowledge to state the general conditions under which a Taylor series is appropriate, and when to terminate one. It seems as though their past experience has been based on various instructors and texts indicating when a Taylor series is appropriate and how many terms are necessary.

6.7 The Boltzmann Factor and its Relationship to The Density of States

An additional portion of the post-tutorial interview sessions in Year 2 asked the students to relate graphs of the density of states ($D(E)$) and the Boltzmann factor as functions of energy to the probability of various energy levels. The graphs (shown in Appendix C) were presented one at a time (with $D(E)$ given first), and students were asked to determine which values of energy, if any, were more probable and which were less probable. They were also asked to articulate how these graphs related to probability. I was looking for evidence of student understanding of three pieces of information that could be synthesized to gain a picture of total probability in this context: 1) that $D(E)$ is related to the multiplicity of the *system*, based on the energy of the system; 2) that the Boltzmann factor is proportional to the multiplicity of the *reservoir* for a particular energy of the system; and 3) that the total multiplicity

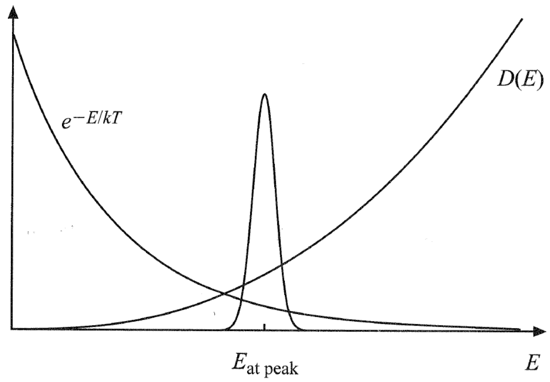


Figure 6.9. Graph of the Boltzmann factor and the density of states. Graph also shows the product of the two as a function of energy. Taken from Figure 5.4 in Ref. 89, p. 100. Figure used without "$e^{-E/kT}$" and "$D(E)$" labels on final exam in 2010 (see Appendix C).

needed to determine probabilities is the product of the multiplicities of the system and the reservoir, and therefore qualitatively shown by the product of $D(E)$ and the Boltzmann factor.[18]

Students in all four interviews were able to articulate how $D(E)$ determines the multiplicities and, therefore, the relative probabilities of various energy levels, but Paul and Jonah were not very confident in this relationship. Students in three out of the four interviews also indicated in some fashion that the Boltzmann factor is related to the multiplicity of the reservoir, but only one, Kyle, seemed to have a robust understanding of this relationship. All students had an understanding that combining multiplicities requires multiplication. Several students articulated this fact by stating that "multiplicity is multiplicative," a succinct and memorable expression. Moreover, all students recalled a figure from Baierlein which overlays the Boltzmann factor on a graph of $D(E)$ as well as graphing the product of the two (recreated in Figure 6.9).[89, p. 100] Several students referred to this figure even before the graph of the Boltzmann factor was introduced, often discussing the "bell-

[18]This is an extension of the situation used in the tutorial (described in section 6.3); now $\omega_C \neq 1$, and one must calculate it using $D(E)$ in order to find $\omega_{tot} = \omega_R \omega_C$.

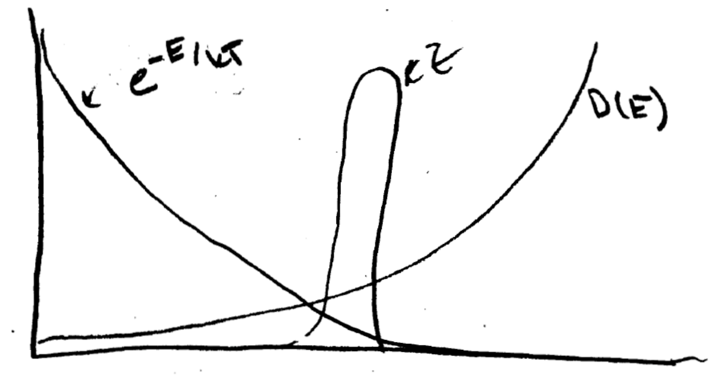


Figure 6.10. Student work – Boltzmann factor vs. $D(E)$: Bill. Incorrect interpretation of Z. Accompanied by the explanation, "The partition function, Z, . . . is the peak gaussian shaped curve."

curve" shape. Paul had a particular fixation on this graph. He could not remember the details of what the various curves represent or the paragraph explaining why it was important, but he mentioned the figure almost as soon as the graph of $D(E)$ was first presented, and he continued to refer to it throughout the interview (often lamenting his lack of memory).

Out of all of the interviewees, Kyle seemed to have the best overall understanding of the material based the time it took him to answer questions and the accuracy of his responses. He was confident in both how $D(E)$ may be used to determine the multiplicity of the system and that the Boltzmann factor is proportional to the multiplicity of the reservoir. He was articulate about the multiplicative nature of multiplicities; he had good intuitions regarding practical limits on energy values; and he spontaneously reflected on topics with which he was and was not comfortable and confident. With all of these desireable cognitive and metacognitive traits, however, Kyle still had not synthesized all of the information that was available to him to articulate that the bell-curve shape in Figure 6.9 results from considerations of the multiplicity of the entire system-reservoir combination and, therefore, requires the product of the two functions that give the multiplicities for each constituent part.

The weakest interviewees by far were Paul and Jonah. Between the two of them they expressed many good ideas about thermodynamics and statistical mechanics that were relevant to the interview scenario. Their confidence in any of these ideas, however, was extremely low. Every time a new piece of information seemed to contradict what they had previously said, they would dismiss one or the other as incorrect or try to reconcile them without concern for factors which they had previously expressed (e.g., appropriate units). In fact the only fact they were completely sure about was that the Boltzmann factor is proportional to probability. After much prodding Paul expressed the need to maximize the entropy of the system-reservoir combination as a whole to determine the most probable state, indicating that all of the necessary information to succeed was available to them, and that their lack of confidence in the physical meaning of the various terms and functions is what most hindered their understanding of the combined probability distribution.

These interview results indicate that with an understanding of how both the density of states function and the Boltzmann factor relate to the multiplicities of different systems and a remembered image of a graph of the product of the two, one may still not have synthesized this information to gain a robust understanding of the physical reasoning behind taking the product of the two functions to determine the total probability.

As a follow-up and broadening of the interview task, a question on the final exam in Year 2 at UMaine presented the graph from Figure 6.9 without any labels (see Appendix C). Students were also given expressions for the canonical partition function (e.g., $Z = \int e^{-E/kT} D(E)\,\mathrm{d}E$) and asked to indicate what aspects of the graph corresponded with various items in the equations. Ten out of 12 students correctly labeled the graphical representation of both the Boltzmann factor and the $D(E)$, and eight of these students related the gaussian-shaped curve to the product of the two. Only one student (Kyle) correctly interpreted Z as the area under

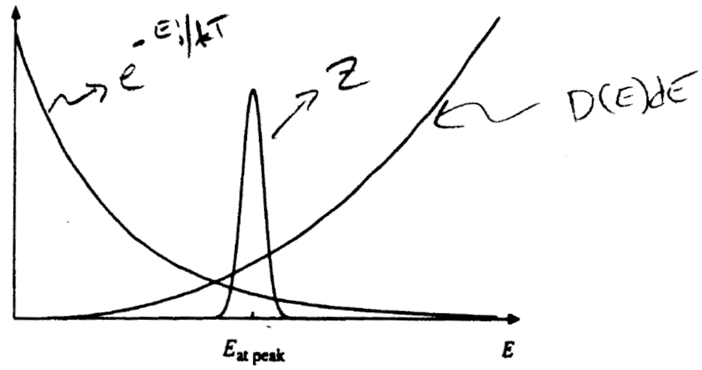


Figure 6.11. Student work – Boltzmann factor vs. $D(E)$: Jayne. Incorrect interpretation of Z. Accompanied by the explanation, "...the 'hump' is the product of the two [$D(E)$ and Boltzmann factor], otherwise known as Z, the partition function."

the curve of the graph of the product. Two students indicated that the gaussian-shaped graph would be Z (shown in Figures 6.10 & 6.11), while eight students made no mention of Z in their response. These exam results strengthen the claim that students recognize graphs of the Boltzmann factor, $D(E)$, and their product, but may not have a robust understanding of the physical implications (as evidenced by their failure to properly interpret the graph to represent Z). This failure to recognize an integral as represented by the area under a graph of a function has been documented by several researchers in thermal physics education.[1, 23, 33, 36]

6.8 Summary and Implications for Future Work

Preliminary research shows that students often do not use the Boltzmann factor when answering questions related to probability in applicable physical situations after lecture instruction on the Boltzmann factor and the canonical partition function. These results have been replicated over several years. Students instead tend to use statements about a ranking of the relative probabilities to make claims about probability ratios, consistent with literature in math education. This is a common error among students at UMaine who had received lecture instruction and among students at Cal Poly who had received no instruction on the Boltzmann factor. To address students' failure to appropriately apply the Boltzmann factor, I created the *Boltzmann Factor* tutorial to improve their understanding of situations in which the Boltzmann factor is appropriate by providing them with the opportunity to engage in the physical reasoning behind the derivation of the Boltzmann factor.

Results from tutorial implementation indicate that students are far more likely to use the Boltzmann factor properly after tutorial instruction than after lecture instruction alone. I've shown that the *Boltzmann Factor* tutorial could be an effective supplement to (as at UMaine) or replacement of (as at Cal Poly) lecture instruction. Further investigation into students' understanding of the Boltzmann factor has revealed that even a student who takes the time to memorize the derivation of the Boltzmann factor from a textbook may not gain a full appreciation of the physical implications of the mathematical formalism. I have also shown, however, that by participating in tutorial instruction, students can gain this appreciation and make connections between the physical situation and mathematical expressions that had previously eluded them.

Additional studies on student understanding of Taylor series expansion as it applies to physics have provided mixed results. Many students display the ability to interpret a Taylor series of a function given the graph of that function. Results from interviews and classroom observations, however, indicate that many students struggle with generating a Taylor series expansion using physical quantities (i.e., entropy and energy). Once provided with a generic Taylor series using physical quantities most students are able to apply it to a specific situation, but this does not appear to be a trivial task for them. Results from further interviews on student understanding of the applicability of Taylor series expansions show that many students recognize that Taylor expansion is a relevant mathematical tool in various areas of physics, but they often lacked a sense of when its use is appropriate. Students also did not have rigorous criteria for determining how many terms should be kept (except when one of the derivatives is a constant, resulting in all higher derivatives being identically zero).

A follow-up study on student understanding of the Boltzmann factor and how it relates to the density of states function has shown several things. Students often have a good understanding of how both the Boltzmann factor and the density of states function relate to probability; they may also be able to relate how the density of states function relates to the multiplicity of a thermodynamic system. Students often cannot, however, articulate how or why these two expressions for probability should be combined. Several students interviewed mentioned a graph from the textbook that showed the product of the two but had difficulty explaining why the product was necessary. Results from an exam question asking students to interpret an unlabeled graph of the Boltzmann factor, the density of states, and their product, indicate that many students recognize these graphs and label them appropriately, but most do not correctly determine the graphical representation of the canonical partition function. This result suggests that students do not understand the physical rationale for multiplying the Boltzmann factor and the density of states to determine probability. This result may also suggest difficulty with the idea of an integral being represented by the area between the graph of a function and the horizontal axis.

The results from my research suggest several avenues for future studies. The first of these pertains to student understanding of Taylor series expansion. While my research shows that many students are able to graphically interpret a Taylor series expansion and that many recognize that Taylor series is appropriate in specific physical contexts, it is still unclear what factors would motivate students to spontaneously use a Taylor expansion to solve a particular problem. In other words, under what conditions do students choose to use a Taylor series expansion without instructor intervention? And, what aspects of a physical scenario should be highlighted to encourage its use? The answers to these questions may benefit instructional sequences (such as the *Boltzmann Factor* tutorial) in which students' use of Taylor series expansion is desired.

Another continuing study that could benefit statistical mechanics instruction would be on student understanding of the physical connection between the Boltzmann factor and the density of states. In virtually all physically interesting systems one must consider the interactions between a system and its surroundings and how the entropy and multiplicity of each affects thermodynamic equilibrium. Though a necessary first step toward understanding more complex systems, the Boltzmann factor on its own is only applicable in a handful of cases; otherwise one needs knowledge of the multiplicity (or degeneracy) of the system that may be obtained from the density of states function. It is unclear at this point how well students understand the physical connection between these two mathematical expressions. It is also unclear how well they understand why the product of the two (rather than the sum or any other combination) yields an expression for the probability of a system having a particular energy. The bell-shaped curve that is the graph of this product, however, is virtually the definition of thermodynamic equilibrium, with the vast majority of particles in a system being within a small ranges of values of an average energy. A robust understanding of the product of the Boltzmann factor and the density of states and why they are physically relevant is, therefore, vital to the understanding of statistical mechanics.

Chapter 7
CONCLUSIONS

I have identified several specific difficulties that student express in the context of either heat engines and the Carnot cycle or the Boltzmann factor. Furthermore, I have developed two guided-inquiry tutorial activities that address these difficulties and are intended for use within advanced thermal physics courses. The decision to identify specific difficulties rather than classify student understanding in some other way is consistent with Heron's description of the utility of the specific difficulties framework for the development of instructional materials.[37] My use of tutorials as an instructional strategy is consistent with many other researchers who have studies student difficulties in thermal physics at both the introductory and advanced levels.[17–20, 22, 30–32, 34]

Advanced thermal physics students provide an interesting population of journeyman physicists who are no longer novice but have not reached full expertise.[24] My results indicate that, when faced with an unfamiliar situation, these upper-division students display novice-like behavior (see section 6.2.2). This result is consistent with Meltzer's comparison of upper-division and introductory students.[33] Furthermore, engaging in either the *Heat Engines* tutorial or the *Boltzmann Factor* tutorial promotes expert-like skills (e.g., appropriately using mathematics and understanding the physical implications of the results, and using impossible situations as a counterexample in a proof).

Data from multiple sources (e.g., the finite reservoirs question and the engine entropy question) indicate that similar difficulties may be observed across different contexts. This suggests that these difficulties may not be instinctual responses, but the result of semi-stable beliefs that some students have about thermal physics.

Discussions of the major results from both halves of my dissertation are included in sections 5.5 and 6.8. In this chapter I summarize those discussions and present common themes from both. In particular I highlight three common themes: the benefits of using classroom video data from a research perspective; the benefits of using pre-tutorial homework assignments from an instructional perspective; and the aspects of the advanced undergraduate population that separate them from both novice introductory students and expert physicists. I conclude with a summary of the implications of my research on future studies within advanced thermal physics courses.

7.1 Identifying and Addressing Specific Difficulties with Heat Engines

Data from written questions and videotaped classroom observation provide evidence for several specific student difficulties with heat engines. One of these (the one that the *Heat Engines* tutorial was designed to address) is students' failure to use the fact that the Carnot cycle is reversible (and that, therefore, the entropy of the universe does not change) to answer questions about heat engines. On the finite reservoirs question (given at UMaine after lecture instruction), a third of students (8 out of 25) did not use the reversibility of the Carnot cycle to correctly determine the final temperature of the reservoirs. Four of these students even wrote that $\Delta S_{uni} = 0$, but did not use this fact appropriately to determine the final temperature. On the engine entropy question (given at UMaine and RPI after lecture instruction), less than a third of students (18 out of 64) correctly used the reversibility of the Carnot cycle to determine that the entropy of the universe doesn't change after one complete cycle of a Carnot engine. The most common incorrect answer of this part was the claim that the entropy of the universe would increase; these answers were usually accompanied by statements about the Carnot engine being a real engine and/or that entropy always increases. Observing this difficulty within both data sets suggests that some students consistently have trouble with the idea that the Carnot engine is the only ideal, reversible heat engine. Furthermore, instructional activities that encourage students to derive Carnot's efficiency, and the steps necessary for the Carnot cycle, starting from the constraint of reversibility (like those found in the *Heat Engines* tutorial) could be of great benefit to these students.

Data from pre- and post-tutorial assessments indicate that the *Heat Engines* tutorial helps students gain an understanding of how Carnot's theorem relates to and can be derived from the entropy inequality statement of the 2^{nd} Law. Data also indicate that students become more selective with their reasoning on questions pertaining to heat engines and entropy after tutorial instruction. Before tutorial instruction, students were just as likely to use *Reversibility* reasoning as they were to use *State Function* reasoning when answering questions about the entropy change of the working substance during one complete cycle of the Carnot engine. After tutorial instruction, students are much more likely to use *Reversibility* reasoning when answering questions about the change in entropy of the universe and use *State Function* reasoning when answering questions about the change in entropy of the working substance. Students' failures to properly recognize the implications of Carnot's theorem and the state function property of entropy in the context of heat engines are evident across several years of data-taking and at two different schools.

On the engine feasibility question, all students invoked the 2^{nd} Law: either explicitly, by calculating ΔS_{uni} and checking if it is positive, or implicitly, by either calculating the efficiency of the engine and comparing it to the Carnot efficiency for the same two reservoirs, or by comparing the ratios of $\left|\frac{Q_{\mathrm{L}}}{Q_{\mathrm{H}}}\right|$ and $\frac{T_{\mathrm{L}}}{T_{\mathrm{H}}}$. No students used only the 1^{st} Law to determine whether or not a device could function. Students' abilities to determine whether or not a proposed device will function as described are comparable to introductory students' achievement after instruction using a different tutorial reported by Cochran & Heron.[35] Additionally, the *Heat Engines* tutorial gives students the opportunity to directly show that Carnot's theorem and the Kelvin-Planck statement may both be derived from the entropy inequality statement of the 2nd Law.

Data from videotaped classroom observations suggest several additional difficulties related to heat engines and the *Heat Engines* tutorial. One is students' use of improper definitions of thermodynamic efficiency. Students in Year 1 used the ratio of net output energy ÷ net input energy as a definition for efficiency ($\eta = \frac{W}{|Q_{\mathrm{H}}|-|Q_{\mathrm{L}}|}$) while working through the *Heat Engines* tutorial. After instructor intervention, these students realized that this ratio would be exactly unity for all heat engines satisfying the 1st Law and, therefore, that it is not useful. One of these students, however, used an inappropriate definition of efficiency of the engine feasibility question after tutorial instruction. This indicates that students' difficulties with the definition of thermodynamic efficiency are robust in that they persist over time and after direct instructional intervention. By asking students in Year 2 to consider why this is an inappropriate expression for efficiency *before* the tutorial session, I gave them the opportunity to wrestle with this dilemma and consequently work through the tutorial more efficiently. In fact, when one student expressed the desire to use this ratio in Year 2, one of his group-mates referred to the homework assignment to explain why it would be inappropriate.

Two additional difficulties were identified using data from classroom observations but are not explicitly addressed by the *Heat Engines* tutorial. The first of these is some students' difficulty imagining impossible situations. In Year 1, one student had great difficulty considering a heat engine that operated in a cycle to convert heat from only one reservoir into work (an impossible process that violates the 2nd Law). Even with explicit help from his group-mate, who seemed to understand his dilemma, this student had trouble considering this impossible process. Bing and Redish suggest that the ability to use impossible situations to gain information about our physical world is a trait of an expert physicist that upper-division students may not have yet acquired.[24] By asking students to engage in this type of reasoning, I hope to help develop their skills as physicists.

The second difficulty that has not yet been addressed is students' failure to articulate (and confusion surrounding) the difference between differential change and net change. For some students, this difficulty seems to be nothing more than sloppy language: they use terms like "$đQ$" to mean the "net heat," seemingly without difficulty. Other students, however, seem to become confused when this distinction is not articulated. Jonah claimed that, since the net heat transfer to the working substance over one cycle of a particular heat engine is 0, then there is not heat transfer; he then concluded that the reservoirs must be the same temperature if no heat transfer occurs. This difficulty is not addressed by the *Heat Engines* tutorial, and more investigation may be needed to inform appropriate instructional strategies.

7.2 Identifying and Addressing Specific Difficulties with the Boltzmann Factor

Data from written surveys, videotaped classroom observations, teaching interviews, and clinical interviews provide evidence for several specific student difficulties with topics related to the Boltzmann factor. The primary difficulty that the *Boltzmann Factor* tutorial was designed to address is students' failure to use the Boltzmann factor to determine the probability of a particular macrostate in a canonical ensemble, i.e., a system at constant temperature and occupying one of several possible energy states. Data from pre- and post-tutorial assessments indicate that students who participate in the *Boltzmann Factor* tutorial are significantly more likely to use the Boltzmann factor when answering probability ratio questions that require its use than students who receive lecture instruction alone. After lecture instruction, only 25% (8 out of 32) of students at UMaine used the Boltzmann factor to answer the probability ratios question. Others students used a ranking of the probabilities to come to a conclusion about the ratios using strategies consistent with those reported by Smith.[63] After tutorial instruction, nearly 90% of students used the Boltzmann factor to answer similar exam questions. These post-tutorial results have been replicated over three implementations at two different institutions. Results from Cal Poly suggest that the *Boltzmann Factor* tutorial may be used successfully as a replacement of (rather than a supplement to) lecture instruction on the Boltzmann factor.

Participation in the *Boltzmann Factor* tutorial also helps students gain an appreciation for the origin and derivation of the Boltzmann factor even if they were able to use it correctly after lectures alone. Results from teaching interviews highlight the importance of giving students the opportunity to engage in mathematical derivations of physical expressions within the classroom, as one student had memorized and could recite the textbook derivation of the Boltzmann factor but had very little understanding of the relationships between the various equations he wrote. Data from classroom observations suggest that students who use the Boltzmann factor appropriately to answer the probability ratios question after lecture instruction might not appreciate its conceptual meaning; these students may gain a better understanding of the physical significance of the Boltzmann factor by engaging with the derivation in the *Boltzmann Factor* tutorial.

Several difficulties have again been identified that are not explicitly addressed by the tutorial. The first of these is students' inability to use the Taylor series appropriately in physical contexts. Data from teaching interviews indicate that only one out of four participants could generate a Taylor series of entropy as a function of energy for a particular situation (and this one may have only memorized the Taylor series from the course textbook as he had the entire derivation); however, all students could use a generic Taylor series (provided by the interviewer) to create the situation-specific variety, with varying degrees of interviewer assistance. Data from classroom observations at both UMaine and Cal Poly, however, indicate that some students who complete the pre-tutorial homework and bring the generic Taylor series with them to class still have trouble using it in the tutorial. Additionally, data from clinical interviews suggest that many students know that Taylor series is a relevant mathematical tool in physics but have not developed sophisticated heuristics for *when* it should be used.

A second difficulty observed during clinical interviews is students' failure to synthesize the information they already knew about the Boltzmann factor and the density of states function into a complete model of how they compliment each other to predict real-world observations. The most noteworthy result of this aspect of the interviews is that all of the interview participants possessed the necessary pieces of knowledge to develop this understanding (i.e., the relationship between the Boltzmann factorand multiplicity of a reservoir, the relationship between the density of states and the multiplicity of the system, and the fact that the product of the multiplicities of constituent parts determines the total multiplicity of the whole), but they did not assemble them appropriately without interviewer prompts. This may be indicative of novice-like behavior as experts are more likely to perceive large meaningful patterns than novices.[27]

7.3 Research Benefits of Videotaped Classroom Observations

From a methodological point of view, data from videotaped in-class tutorial sessions were immensely beneficial to my study of the effectiveness of tutorials. Written surveys were useful to see a snapshot of student understanding as evidenced by their responses to various written prompts, but video data afforded me the opportunity to revisit how students engaged with the tutorials and identify areas that were difficult for them and areas that were unproductively easy in more detail than simply observing and taking field notes.

In addition to providing evidence for the difficulties described above, video data provided evidence that the mock student discussion portion of the *Heat Engines* tutorial is unnecessary since no students worry about the consequences of work on the entropy of the universe until asked to do so (a topic which is not needed for the successful completion of the tutorial). With respect to the *Boltzmann Factor* tutorial, video data provided evidence that students who recognized when the Boltzmann factor is appropriate and used it properly after lecture instruction alone had not, in fact, developed an understanding or an appreciation of what the mathematical expression of the Boltzmann factor represents physically; moreover, there is some evidence that these same students realized the physical significance of the Boltzmann factor after participating in the *Boltzmann Factor* tutorial. Without video data, these and other incredibly valuable observations would have eluded me and been lost as fleeting moments in time. Analyzing classroom video data has allowed me to uncover student difficulties that had been previously undocumented (based on written data); to confirm whether or not students engaged with tutorial prompts as they were intended; and to demonstrate the benefits of tutorial participation beyond the original intent.

7.4 Instructional Benefits of Pre-tutorial Homework Assignments

One of the most striking results from both halves of my research is the benefit of assigning a homework activity to be completed by the students and brought to class on the day of the tutorial. In Year 1, before the creation of the pre-tutorial homework assignment, students got stuck working through the *Heat Engines* tutorial when they did not understand the justification for defining thermodynamic efficiency in the conventional way ($\eta = \left|\frac{W}{Q_{\mathrm{H}}}\right|$). Students in later years had the chance to ponder this definition before the tutorial and were, therefore, not encumbered by long discussions about the definition of efficiency in class. In fact, one student used his pre-tutorial work to explain to another student why the conventional definition is the most appropriate. Students were also able to refer back to the pre-tutorial homework while working through the *Heat Engines* tutorial to help answer questions about the change in entropy of the working substance due to a complete cycle of a heat engine.

As mentioned above, results from teaching interviews indicate that most students could not spontaneously generate a Taylor series expansion of entropy as a function of energy when asked to do so in the *Boltzmann Factor* tutorial. Giving students the chance to refresh their memory of Taylor series before coming to class helped alleviate this problem, as most students were able to refer to their homework and generate a Taylor series for the tutorial situation. Some students, however, had trouble creating the necessary Taylor series even after completing the pre-tutorial homework; it is clear that more research is needed to understand students' thought processes regarding Taylor series and to design appropriate instructional materials that help them to recognize situations in which Taylor expansion is useful, and to apply the mathematics appropriately.

7.5 Advanced Undergraduates: Journeyman Physicists

As mentioned above, several pieces of evidence suggest that the advanced undergraduate physics students who were the research subjects of this study are not novices, but neither are they expert physicists. My research suggests that, if these students do not recognize the appropriate physical concepts that apply in a particular situation (e.g., not using the Boltzmann factor to determine probability for a canonical ensemble), they may resort to novice-like behavior (e.g., using general strategies for comparing fractions observed in middle and high school students).[63] Another example of novice-like behavior is students' failure to refer to their own work while engaging in later parts of the *Boltzmann Factor* tutorial.

Video data from classroom observations of students engaging in the *Heat Engines* tutorial suggest that some advanced students have not developed the expert-like skill of using impossible situations as counterexamples in order to gain more information about reality.[24] This claim is strengthened by the evidence that before tutorial instruction, one of these students (Jake) did not answer the parts of the engine entropy question that ask students to consider an engine that was more efficient than the Carnot engine. His only response was that this engine was impossible, and he did not make any claims about the change in entropy of either the universe or the working substance due to this engine. After engaging in this process during the tutorial, however, this same student correctly used the reversibility of the Carnot cycle to state that the entropy of the universe would remain the same for the Carnot engine and decrease for a better-than-Carnot engine. He also used the state function property of entropy to claim that the entropy of the working substance does not change for either engine after one complete cycle. In this way, participating in the *Heat Engines* tutorial helped this student engage in the expert-like behavior of considering impossible scenarios.

The display of non-expert-like behavior, however, does necessarily mean that advanced undergraduate students are novices. I treat upper-division students as more advanced than their introductory counterparts. I expect that they have the mathematical sophistication to be able to productively engage in the derivations that are presented in the tutorials. Teaching interviews and classroom observations suggest that they are able to engage with these derivations successfully if given the proper background cues. I also trust upper-division students to have the intellectual integrity and work ethic to complete homework assignments in time to use them during the tutorial sessions. These pre-tutorial homework assignments represent a deviation from the typical tutorial model used in introductory physics classes, but they are, I feel, a very beneficial addition to tutorials in upper-division courses. In introductory physics courses, tutorials typically ask students to rely only on their own everyday experiences or on observations made during the tutorial session. Upper-division students, on the other hand, are expected to have a wealth of knowledge and understanding gained from years of physics and/or mathematics instruction. They may not, however, have developed sophisticated heuristics for selecting the appropriate background knowledge needed within a specific physical scenario (as would be expected of expert physicists)[27]; therefore, it is the instructor's responsibility to help students sort through their vast understanding to find the needed nugget of knowledge. Within the tutorial setting, in which students progress through the class period with minimal instructor intervention, this cueing may be possible through a pre-tutorial homework assignment in which students are asked to apply certain pieces of background knowledge to answer relatively simple problems, i.e., problems that do not require much cognitive thinking, but that may require more time than would be available within the classroom. My use of pre-tutorial homework assignments has greatly benefitted student understanding of relevant topics as well as their abilities to progress through the tutorials in an efficient and effective manner. This is particularly important as the desired learning result is achieved by completing the last section of the tutorial.

7.6 Implications for Future Research

The results of my research suggest several avenues for continuing studies. Video data from student participation in the *Heat Engines* tutorial indicate that students often do not understand (or at least articulate) the distinction between exact (d__) and inexact (đ__) differentials and their use. Much research has been conducted on student understanding of the path dependence of Q and W and the path independence of ΔU, but few to no studies exist that investigate student understanding of their respective differentials and how and why they are used.

Another interesting investigation would be student understanding of reversibility in the context of heat engines and refrigerators. Exam data indicate that no students in Year 3 articulated the importance of the Carnot cycle as the divider between devices that could operate as heat engines and devices that could operate as refrigerators. This unique aspect of the Carnot engine may be overlooked by many students who do not recognize the literal meaning of the Carnot cycle being "reversible."

Student understanding of the connection between the density of states function and the Boltzmann factor is another area that could benefit greatly from further research. My preliminary interview results indicate that students often possess all of the necessary components to develop a complete understanding of this relationship, but they do not synthesize these pieces of information into a coherent whole without explicit intervention and/or guidance. Exam results show that many students have difficulty representing the partition function graphically when given a graph the product of the density of states and the Boltzmann factor. Additional research on students' understanding of how each of these expressions relates to the multiplicity of either a thermodynamic system or its accompanying reservoir, as well as how they in turn relate to the probability of of the system occupying a particular macrostate. A study of students' understanding of the Boltzmann factor and the density of states would also be quite interesting as students attempt to connect discrete quantities (e.g., quantum energy eigenstates used to calculate the Boltzmann factor) with continuous quantities (e.g., the continuous energy spectrum of the density of states). This connection between discrete and continuous quantities is vitally important in modern physics, as properties of subatomic particles that must be treated quantum mechanically are being used to predict phenomena that are observed at the macroscopic level and can be described classically.

Finally, an investigation into student understanding of Taylor series expansion could be incredibly beneficial. This has not been documented from a physics perspective, even though the Taylor series is a mathematical tool that is used extensively in many branches of physics (including statistical and classical mechanics). Of particular interest would be an examination of expert physicists' spontaneous use of the Taylor series. By learning when expert physicists choose to use Taylor series and how they make that decision, one could (in principle) design an instructional sequence that could enhance student understanding of physics and useful mathematical tools within many different courses typically taught in the undergraduate sequence. This would be an excellent stepping stone for undergraduate physics majors on their way to expertise.

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APPENDIX A

THE *HEAT ENGINES* TUTORIAL

Course | **Heat Engines – Survey** | Name____________________
Semester | (EEQ Pretest) | Date____________________

For the following questions consider one complete cycle of a heat engine operating between two thermal reservoirs. The heat engine operates using an appropriate working substance that expands and compresses during each cycle.

For questions a) and b) consider (i.e. imagine) this heat engine to be a Carnot engine:

a) As a result of one complete cycle of the Carnot engine, will the entropy of **the working substance** *increase*, *decrease*, *remain the same*, or is this *not determinable with the given information*? **Explain your reasoning.**

b) As a result of one complete cycle of the Carnot engine, will the entropy of **the universe** *increase*, *decrease*, *remain the same*, or is this *not determinable with the given information*? **Explain your reasoning.**

For questions c) and d) consider (i.e. imagine) a heat engine that operates between two thermal reservoirs and conserves energy but is *more* efficient than a Carnot engine:

c) As a result of one complete cycle of this new heat engine, would the entropy of **the working substance** *increase*, *decrease*, *remain the same*, or is this *not determinable with the given information*? **Explain your reasoning.**

d) As a result of one complete cycle of this new heat engine, would the entropy of **the universe** *increase*, *decrease*, *remain the same*, or is this *not determinable with the given information*? **Explain your reasoning.**

Course
Date

Introduction to Heat Engines
(Pre-tutorial Homework)

Name____________________

The concept of a heat engine is to use some working substance to transform heat energy from a thermal reservoir (at temperature T_H) into mechanical work to perform a task, such as lifting a weight. Some energy in the working substance will also be released into a different thermal reservoir (at temperature $T_L < T_H$) as exhaust heat energy. In order to complete this process without compromising the integrity of our heat engine the process must be cyclic, i.e. the working substance returns to its original state defined by pressure, volume, temperature, etc. For our purposes we can imagine the working substance as being an ideal gas contained within a piston. Heat energy may be transferred to or from the working substance by placing the piston in contact with the T_H or T_L reservoir, respectively. As a matter of sign convention, $|Q_H|$ is the magnitude of the heat transfer from the higher temperature reservoir to the working substance, $|Q_L|$ is the magnitude of the heat transfer from the working substance to the lower temperature reservoir, and W is the work done by the working substance.

A. Will the value of each of the following properties of the working substance *increase, decrease,* or *return to its original value* after the completion of one full cycle?
   1. Pressure

   2. Temperature

   3. Internal energy

   4. Entropy

The effectiveness of a heat engine is determined by how much of the energy extracted from the thermal reservoir can be used to do work. We must, however, be very clear what we mean by "effectiveness." How well a heat engine operates is quantified by the ***thermodynamic efficiency***, defined as the ratio of the work done by the working substance to the heat transfer from the high temperature reservoir ($\eta = W/|Q_H|$).

B. Some would suggest that efficiency would be better defined as the ratio of the work done by the working substance to the net heat transfer to the working substance ($\eta = W/(|Q_H|-|Q_L|)$). Do you agree with this suggestion? Why or why not?
   (*Hint*: consider the first law of thermodynamics.)

Course
Date

Heat Engines
(Tutorial)

Name____________________

I. The First Law and Thermodynamic Efficiency

Consider the following proposed heat engines. For our purposes any work done by the working substance will be conservative and stored outside the engine as potential energy (gravitational, spring, etc.) for later use.

A. The first heat engine (Cycle I) transfers heat energy, $|Q_H|$, from the T_H reservoir but does no work ($W = 0$).

1. What must be true for this heat engine to satisfy the First Law for a complete cycle?

2. Determine the efficiency of this engine.

B. The second heat engine (Cycle II) transfers heat energy from the T_H reservoir but does not transfer any heat energy to the T_L reservoir ($|Q_L| = 0$).

1. What must be true for this heat engine to satisfy the First Law for a complete cycle?

2. Determine the efficiency of this engine.

II. The Second Law and Entropy

In a previous tutorial we considered heat transfer between two massive blocks and found that processes that are <u>impossible</u> (e.g. spontaneous heat transfer from a lower temperature reservoir to a higher temperature reservoir) could still satisfy the First Law. We concluded that considerations of entropy and the Second Law of Thermodynamics could be used to validate the possibility of a proposed process. We will now use the Second Law to discuss Cycle I and Cycle II from above. Recall that, $\Delta S_{universe} \geq 0$ and $\Delta S = \int dQ_{rev}/T$.

A. In general, what must be true about a process for the entropy *of the universe* to stay the same ($\Delta S_{universe} = 0$)?

In general, what must be true about a process for the entropy *of the universe* to increase ($\Delta S_{universe} > 0$)?

B. Give an expression for the change in entropy during the completion of one cycle of Cycle I for each of the following parts of the heat engine. Explain your reasoning in each case.

1. The working substance

2. The thermal reservoirs

3. Determine the total change in entropy of the universe. According to the Second Law, is this cycle possible?

C. Give an expression for the change in entropy during the completion of one cycle of Cycle II for each of the following parts of the heat engine. Explain your reasoning in each case.

1. The working substance

2. The thermal reservoirs

3. Determine the total change in entropy of the universe. According to the Second Law, is this cycle possible?

D. Consider the following student discussion about the work done by the working substance:

Leslie: "What about the fact that the second engine does some work on the outside environment? Isn't that going to increase the entropy of the surroundings?"

Janice: "But the work isn't dissipative. It's just lifting a block or something. It's not going to make the block more disordered or make it hotter, it's just going to lift it higher."

Do you agree with either of these two students? Why or why not?

III. Limitations on Efficiency

In II.C.3 we concluded that, according to the Second Law, a heat engine cannot possibly operate at 100% efficiency. This result is generally credited (independently) to Lord Kelvin and Max Planck and is summarized by the Kelvin-Planck statement of the Second Law: "It is impossible to construct a device that operates in a cycle and produces no other effect than the performance of work and the exchange of heat [energy] with a single reservoir." It may now seem obvious that a heat engine of 100% efficiency is unattainable. So let's investigate the maximum efficiency we could obtain while satisfying both the First and Second Laws.

A. Determining the upper limit on efficiency

1. First, write an expression for efficiency solely in terms of the heat transfer between the working substance and the two reservoirs ($|Q_H|$, $|Q_L|$).

2. Based on your answers to parts II.B and II.C derive a relationship between the heat transfer during one cycle of an arbitrary heat engine and the temperature of the reservoirs (T_H, T_L) based on the Second Law ($\Delta S_{universe} \geq 0$).

3. Combine the results from III.A.1 and III.A.2 to determine an inequality for the efficiency of a heat engine operating between two thermal reservoirs with temperatures T_H and T_L.

   Under what condition(s) will a heat engine operate at the upper limit of efficiency?

❖ Check your results with an instructor before proceeding to the next section.

B. We now want to design a heat engine (Cycle III) that will operate *at* the upper limit of efficiency. We saw in a previous tutorial that any heat transfer between two objects at (discernibly) different temperatures will be spontaneous and inherently irreversible ($\Delta S_{universe} > 0$). We'll now consider how we could design a cyclic process that restricts heat transfer to occur reversibly between each of the two thermal reservoirs and the working substance.

1. First, what kind of process will allow reversible heat transfer from the T_H reservoir to the working substance?

   What kind of process will allow reversible heat transfer from the working substance to the T_L reservoir?

2. What needs to happen to the working substance to complete a thermodynamic cycle while including the two processes described in part 1 above?

   What condition(s) must be placed on the working substance during the remainder of the cycle for the engine to operate at the upper limit of efficiency?

   What kind of process(es) would accomplish this?

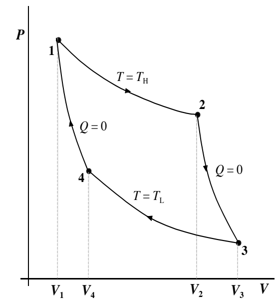
3. Using these ideas, determine how many processes we need to complete Cycle III. List each of them.

The cycle you've just developed is known as the Carnot cycle (or Carnot engine) after Nicolas Léonard Sadi Carnot who first derived the upper limit of efficiency that you found in part III.A.3. The formal statement of this result is known as Carnot's Theorem: "No engine operating between two reservoirs can be more efficient than a Carnot engine operating between those same two reservoirs."

Heat Engines - Homework

Name____________________

1) ***The Carnot Cycle for an Ideal Gas:*** The *P-V* diagram for a Carnot cycle in which the working substance is an ideal gas is shown at the right. Use the Ideal Gas Law and the First Law of Thermodynamics to derive an expression of the efficiency of a Carnot cycle.



A. First, derive an expression for the efficiency of a Carnot cycle operating between thermal reservoirs at temperatures T_H and T_L in terms of the temperatures of the reservoirs, the volumes at the beginning and end of each process, and constants.

B. Now, use the known result that PV^γ = constant for an ideal gas going through an adiabatic process to show that the ratios of the volumes at the beginning and end of the isothermal processes are equal, i.e. $\frac{V_2}{V_1} = \frac{V_3}{V_4}$.

C. Use the relationship between these ratios of volumes to simplify your expression for efficiency in part A so that it does not depend on volume.

D. Compare your calculated efficiency from part C to the upper limit you found in III.A.3 of the tutorial. Resolve any discrepancies.

2) A. See the P-V diagram for the Carnot heat engine on p. 1. Sketch the appropriate temperature-entropy (T-S) diagram for the Carnot engine here, labeling the isotherms as T_H and T_L, and the process intersection points ("*nodes*") as 1, 2, 3, 4 to correspond to the P-V diagram (p. 1).

B. Evaluate $\int TdS$ for the full cycle ($1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$) on your T-S diagram in A above.

What is the physical interpretation of the value of $\int TdS$ for each complete cycle of the working substance?

Show the graphical representation of that integral on the T-S diagram.

C. Consider the quantity $\int PdV$ for the full cycle ($1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$) as shown in the P-V diagram.

What is the physical interpretation of the value of $\int PdV$ for each complete cycle of the working substance?

Show the graphical representation of that integral on the P-V diagram.

How do the magnitudes, signs, and dimensions of $\int PdV$ and $\int TdS$ compare? Explain how you know.

D. See the P-V diagram for the Carnot cycle on p. 1. Does that diagram represent a Carnot cycle for any working substance in addition to an ideal gas? Explain.

E. See your T-S diagram in part A above. Does that diagram represent a Carnot cycle for any working substance in addition to an ideal gas? Explain.

3) ***Real Reservoirs (FRQ):*** Now we change the context; the least efficient heat engine (Cycle I from the tutorial) and the most efficient heat engine (Cycle III from the tutorial, the Carnot cycle) are each now set up to operate between identical pairs of *real* (*finite*) thermal reservoirs, rather than the ideal, constant-temperature reservoirs used in ideal engines. In each case the higher temperature reservoir is initially at temperature T_H, and the lower temperature reservoir is initially at temperature T_L. All four reservoirs have mass m and specific heat capacity c_P. (What is the heat capacity of an ideal thermal reservoir?) During any one cycle, we may assume that any change in temperature of the reservoirs is negligible. The cumulative change in temperature due to many cycles, however, will not be negligible. Consider the situation in which each heat engine operates for many cycles until all available energy has been exhausted, and each pair of reservoirs has come to thermal equilibrium. Use additional sheets of paper if necessary.

A. *Without doing any calculations!* Will the final temperature of the working substance and both of its reservoirs in Cycle I be *greater than, less than,* or *equal to* the final temperature of the working substance and both of its reservoirs in Cycle III? Explain your reasoning.

B. For the least efficient heat engine, calculate the following quantities in terms of m, c_P, T_H and T_L.
1. The total work done by the working substance
2. The final equilibrium temperature of the heat engine
3. The change in entropy of the working substance, reservoirs, and universe for the entire process

C. For the most efficient heat engine, calculate the following quantities in terms of m, c_P, T_H and T_L.
1. The change in entropy of the working substance, reservoirs, and universe for the entire process
2. The final equilibrium temperature of the heat engine
3. The total work done by the working substance

D. Compare your answers to parts B.2 and C.2 with your prediction from part A. If your prediction was incorrect, qualitatively describe what you didn't initially consider.

E. How do the *arithmetic mean temperature* T_a and the *geometric mean temperature* T_g of T_H and T_L relate to your answers for parts B.2 and C.2 above?

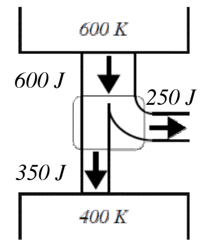
Course
Date

Engine Feasibility Question
(Year 3 version)

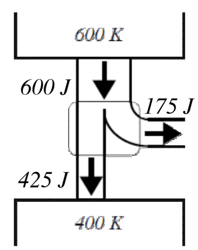
Name____________________

Consider two heat engines that operate between the same two high-temperature and low-temperature reservoirs, at 600 K and 400 K, respectively.

a) For the first engine: The heat transfer from the high temperature reservoir to the working substance during one complete cycle is 600 J. The heat transfer from the working substance to the low temperature reservoir during one complete cycle is 350 J. The work done by the working substance during one complete cycle is 250 J. A diagram of this heat engine is shown at the right.
Determine whether or not this engine could operate as described.
Explain your reasoning.



b) For the second engine: The heat transfer from the high temperature reservoir to the working substance during one complete cycle is 600 J. The heat transfer from the working substance to the low temperature reservoir during one complete cycle is 425 J. The work done by the working substance during one complete cycle is 175 J. A diagram of this heat engine is shown at the right.
Determine whether or not this engine could operate as described.
Explain your reasoning.



c) Determine whether each of the devices above could operate as a refrigerator (all energy transfers reversed).
Explain your reasoning.

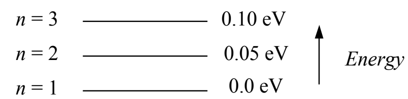
Adapted from Cochran, M. J. and Heron, P. R. L., *Am. J. Phys.,* (2006)

APPENDIX B

THE *BOLTZMANN FACTOR* TUTORIAL

Course	Probability Survey	Name____________________
Date	(PRQ Pretest)	

1. Consider a particle (Particle A) in a system with three evenly spaced energy levels, as seen in the figure at right. The probability that Particle A is in the n^{th} energy level is $P_A(n)$.



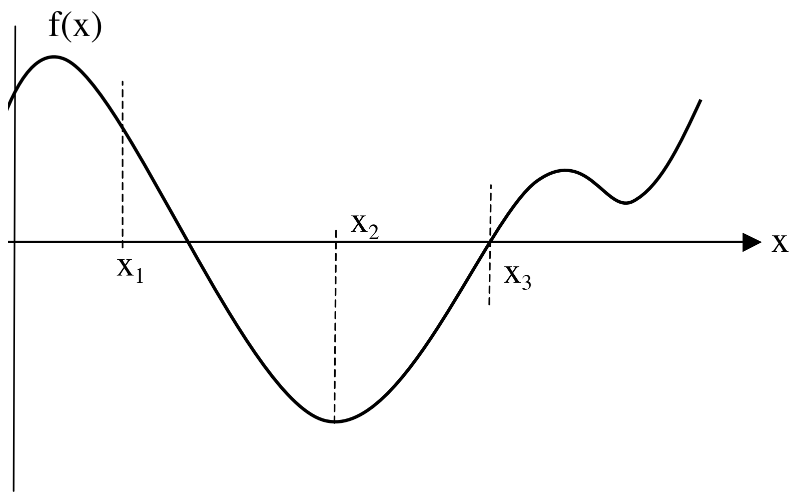
   A. Is the <u>ratio</u> of the probabilities $P_A(3)/P_A(2)$ *greater than, less than,* or *equal to* the <u>ratio</u> of the probabilities $P_A(2)/P_A(1)$? Please explain your reasoning.

   B. Consider a second single particle, Particle B, that can also only be in three states. The energies of the three states of each system are listed in the table at right. Both systems are in equilibrium with a reservoir at temperature T.

n	Particle A	Particle B
1	0.0 eV	–0.05 eV
2	+0.05 eV	0.0 eV
3	+0.10 eV	+0.05 eV

   Is the ratio of the probabilities $P_B(3)/P_B(2)$ for Particle B *greater than, less than,* or *equal to* the ratio of the probabilities $P_A(3)/P_A(2)$ for Particle A? Please explain your reasoning.

2. Below is a sketch of the function f(x).



The function f(x) is expanded in a Taylor series about $x = x_1$, $x=x_2$ and $x=x_3$. For each point, state whether the given coefficients are *positive*, *negative*, *zero*, or *not determinable with the given information*. Explain how you determined your answers using words and/or sketches on the graph above.

$x = x_1 \rightarrow f(x) = a_1 + b_1(x-x_1) + c_1(x-x_1)^2$

a_1:

b_1:

c_1:

$x = x_2 \rightarrow f(x) = a_2 + b_2(x-x_2) + c_2(x-x_2)^2$

a_2:

b_2:

c_2:

$x = x_3 \rightarrow f(x) = a_3 + b_3(x-x_3) + c_3(x-x_3)^2$

a_3:

b_3:

c_3:

Course Date

Application of Taylor Series Expansion

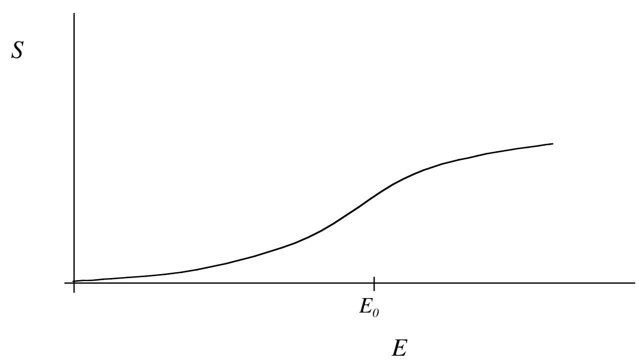
(Pre-tutorial Homework)

Name__________________

Complete and bring to next class period.

Taylor expansion of entropy: Use the fact that entropy is a function of energy to write a Taylor series expansion (include at least three terms) about the point $E = E_0$ i.e. $S(E) = S(E_0) + \ldots$

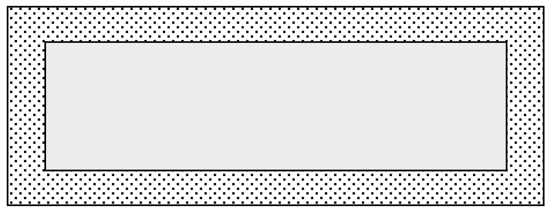
Give an interpretation for each of the terms in your Taylor series expansion as it relates to the graph of entropy vs. energy shown below. Give as much graphical detail as possible.



The Boltzmann Factor
(Tutorial)

I. An Isolated Container

Consider a container of gas molecules that is isolated from its surroundings and has a uniform spatial density. The total internal energy of the gas is initially E_{tot}.



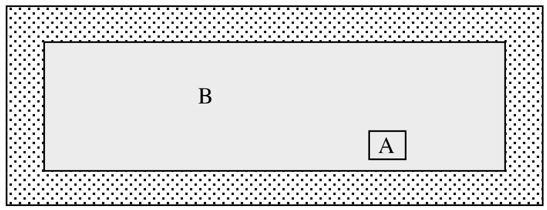
A. After a long time, what is the probability that the total internal energy of the gas will still be E_{tot}? Explain your reasoning.

B. How many *microstates* (molecular configurations) would you estimate exist such that the total energy of the gas is E_{tot}? 1, 1000, 10^{N_A}? In contrast, how many *macrostates* exist such that the total energy is E_{tot}?

C. What is the probability of finding the gas in a particular microstate? Is there any reason to expect that one microstate would be more probable than another?

II. A Divided Container

Looking again at the container, we realize that it is actually divided into a very small section (with variable energy E_A) and a relatively large section (with variable energy E_B). The two sections are divided by a partition that allows heat transfer but keeps the particles in each section separated. As such, the gas in both sections will be in thermal equilibrium with each other.



A. If the variable energy E_A is measured and found to have the value E_{A1}, what would a measurement of E_B yield?

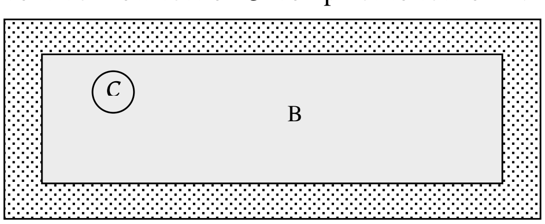
   How will this value of E_B compare to E_{A1}?

B. If the energy E_A is measured later and found to be a slightly different value (E_{A2}), what will a measurement of E_B yield now and how will it compare with E_{A2}?

C. For each of the following thermodynamic properties determine whether the value for section A is *greater than, less than,* or *equal to* the value of the same property for section B. If there is not enough information to answer, state so explicitly. Explain your reasoning for each.

   1. Volume
   2. Number of Particles
   3. Temperature
   4. Pressure

D. Consider a different small section that is in an arbitrary location within the container and has an arbitrary size and shape. Explain why your answers to part II.C. <u>will not</u> change if you consider how section C compares to section B.



III. Systems and Reservoirs

Since section B is so much bigger than section $\mathcal{C}$ we may consider B to be a thermal reservoir designated by $\mathcal{R}$, and $\mathcal{C}$ will be our system of interest. We may also conclude that the multiplicity of $\mathcal{R}$ will be very much larger than the multiplicity of $\mathcal{C}$ (i.e., $\omega_{\mathcal{R}} >> \omega_{\mathcal{C}}$). As such, we will make the approximation that $\omega_{\mathcal{R}}/\omega_{\mathcal{C}} \approx \omega_{\mathcal{R}}$ which leads to $\omega_{\mathcal{C}} \approx 1$. For the remainder of this section we will investigate a model in which $\omega_{\mathcal{C}} = 1$ and $E_{\mathcal{C}}$ can be a handful of discrete values.

The table at right shows a scenario in which there exist only 5 possible values for $E_{\mathcal{C}}$, each with corresponding values for $E_{\mathcal{R}}$, $\omega_{\mathcal{C}}$, and $\omega_{\mathcal{R}}$. Each value for $E_{\mathcal{C}}$ has a corresponding index j, where $1 \leq j \leq 5$.

$E_{\mathcal{C}}$	$\omega_{\mathcal{C}}$	$E_{\mathcal{R}}$	$\omega_{\mathcal{R}}$
$E_{\mathcal{C}1}$	1	$E_{tot} - E_{\mathcal{C}1}$	3×10^{18}
$E_{\mathcal{C}2}$	1	$E_{tot} - E_{\mathcal{C}2}$	5×10^{19}
$E_{\mathcal{C}3}$	1	$E_{tot} - E_{\mathcal{C}3}$	4×10^{17}
$E_{\mathcal{C}4}$	1	$E_{tot} - E_{\mathcal{C}4}$	1×10^{20}
$E_{\mathcal{C}5}$	1	$E_{tot} - E_{\mathcal{C}5}$	7×10^{18}

A. What is the total number of *microstates* for the entire container (system + reservoir) in our scenario? How do you know?

B. Are any of the *microstates* more probable than any other? Consider your answer to part I.C. on the first page.

C. Using your answer to part III.B. which of the above *macrostates* is *most* probable? Why?

Which *macrostate* is *least* probable?

D. Give a general expression for the probability of the system being in macrostate j designated by energy $E_{\mathcal{C}j}$.

IV. Energy, Entropy, and Probability

You should see from your answer to part III.D. that the probability of the system $\mathcal{C}$ having energy $E_{\mathcal{C}j}$, $P(E_{\mathcal{C}j})$, is *proportional* to the multiplicity of the reservoir for that state, labeled $\omega_{\mathcal{R}j}$. We now want to find an expression for $\omega_{\mathcal{R}j}$ in terms of the properties of $\mathcal{C}$.

A. First write an expression for the entropy of the reservoir ($S_{\mathcal{R}j}$) in terms of $\omega_{\mathcal{R}j}$.

B. Now use a Taylor series expansion and the fact that the entropy of the reservoir is a function of the energy of the reservoir ($S_{\mathcal{R}j} = S_{\mathcal{R}j}(E_{\mathcal{R}j})$) to write an approximation for $S_{\mathcal{R}j}$ as a linear function of $E_{\mathcal{C}j}$. (See your homework for reference.)
Consider: About what value of energy should we expand?

What is the *physical* interpretation of the first term in the Taylor expansion? Does this fit with what you know about Taylor series? Rename the first term to reflect this interpretation.

What is the *physical* interpretation of the partial derivative in the second term? Consider the Thermodynamic Identity (the differential form of the First Law of Thermodynamics).

D. Equate your two expressions for $S_{\mathcal{R}j}$ from parts IV.B. and IV.C. to get an expression for $\omega_{\mathcal{R}j}$ in terms of the other variables and constants.

Which of these quantities will change with different values of *j*?

❖ Check your results with an instructor before proceeding to the next section.

V. The Canonical Partition Function

A. Now that you have an expression for $\omega_{\mathcal{R}j}$ determine an expression for the probability of finding the system in state j, labeled $P(E_{Cj})$.

B. Consider the constraint on the sum of all probabilities. Does your expression fulfill this constraint?

The denominator in your expression is often called Z, for the German *Zustandsumme* meaning "sum over states."

C. Is your new expression for Z a constant? (i.e. does it depend on the state of the system?) How does your expression for Z compare with the normalizing factor for the binomial distribution (2^N)?

Course
Date

The Boltzmann Factor- Homework

Name___________________

E_1	5 eV
E_2	10 eV
E_3	20 eV

Comparing Probabilities – Consider a system in which the temperature and the number of particles are held fixed (T = 300K, N = 10^3), but the internal energy is allowed to fluctuate between three discrete values. The allowed energy levels are shown in the table at right.

a) Which energy state is the most probable? What is the probability of finding the system in this state?

b) How does the probability of finding the system in state 1 compare to the probability of finding the system in state 2? (i.e. calculate the ratio $P(E_1)/P(E_2)$)

c) How does this ratio compare with the ratio $P(E_2)/P(E_3)$?

d) Give a general expression for the ratio of probabilities between two energy states $P(E_i)/P(E_j)$.

Describe in words how the ratio $P(E_i)/P(E_j)$ depends on the energies E_i and E_j.

e) Under what conditions is $P(E_i)/P(E_j)$ greater than 1? Less than 1?

f) Under what condition(s), if any, will this ratio be the same for any two adjacent levels? Can you think of any physical situations that meet this (these) condition(s)?

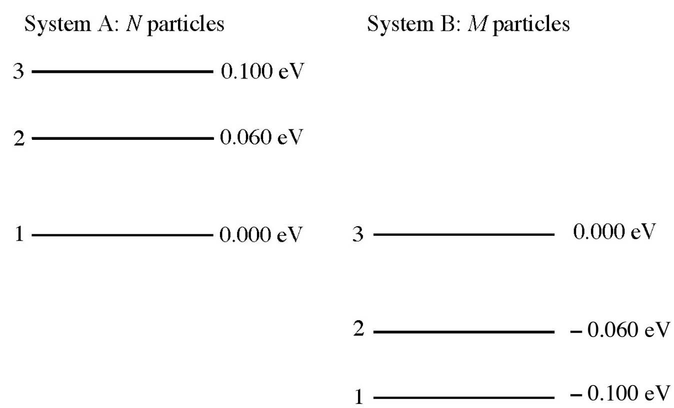
Course | Probability Exam Question | Name____________________

Date | (PRQ-Analog, UMaine)

Probability Exam Question

(PRQ-Analog, UMaine)

Systems A and B are both at the same temperature T. System A has N identical particles, each of which must occupy one of the three energy levels shown. In thermal equilibrium, the numbers of particles in the three levels are $\boldsymbol{n_1}$, $\boldsymbol{n_2}$, and $\boldsymbol{n_3}$. System B, with M identical particles, also has three energy levels, as shown. The numbers of particles occupying each of the three levels of system B are $\boldsymbol{m_1}$, $\boldsymbol{m_2}$, and $\boldsymbol{m_3}$.



Is the ratio $\boldsymbol{n_3/n_2}$ in system A *greater than*, *less than*, or *equal to* the ratio $\boldsymbol{m_2/m_1}$ in system B? If there is not enough information, what else would you need to know? Explain your reasoning.

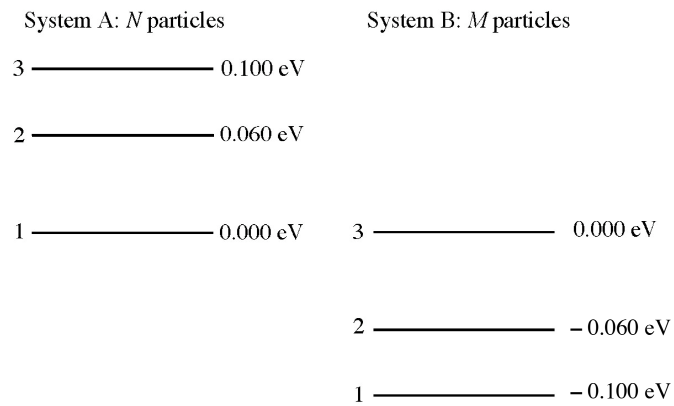
Course

Date

Probability Exam Question
(PRQ-Analog, Cal Poly)

Name____________________

Systems A and B are both at the same temperature T. System A has N identical particles, each of which must be one of the three energy levels shown. In thermal equilibrium, the densities of the particles in the three levels are n_1, n_2, and n_3. System B, with M identical particles, also has three energy levels, as shown. m_1, m_2, and m_3 are the densities of the particles in the three levels of system B.



Which is the true statement?

I. The density ratio n_3/n_2 in system A is greater than the density ratio m_2/m_1 in system B.

II. The density ratio n_3/n_2 in system A is equal to the density ratio m_2/m_1 in system B.

III. The density ratio n_3/n_2 in system A is less than the density ratio m_2/m_1 in system B.

IV. There's not enough information to compare n_3/n_2 in system A to m_2/m_1 in system B.

Explain your reasoning. If you answer IV, also say what additional information you would need.

APPENDIX C

INTERVIEW MATERIALS

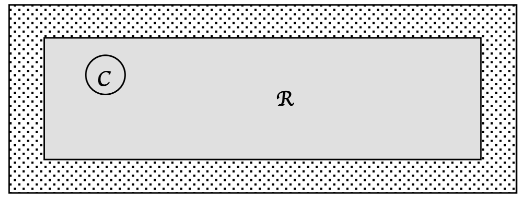
Course

Teaching Interview Activity

(Year 1)

Name____________________

Systems and Reservoirs



Consider a container of an ideal gas isolated from its surroundings (shown at right). The container is divided into two sections: a relatively small section ($\mathcal{C}$) that will be our system of interest and a relatively large section ($\mathcal{R}$). The two sections are in thermal equilibrium and have uniform spatial density, and the combined energy is equal to E_{tot} (i.e., $E_{\mathcal{C}} + E_{\mathcal{R}} = E_{\text{tot}}$). Since $\mathcal{R}$ is so much larger than $\mathcal{C}$ we will treat $\mathcal{R}$ as a thermal reservoir. We know from chapter 4 and the density of states tutorial that the energy of a system in thermal equilibrium may fluctuate around an average value ($E_{\mathcal{C}} = E_{\text{Ave}} \pm \delta E$). We also know that the multiplicity of an ideal gas is related to the volume of the gas, its internal energy, and the number of particles ($\omega \propto V^N E^{3N/2}$). Therefore we may conclude that the multiplicity of $\mathcal{R}$ will be very much larger than the multiplicity of $\mathcal{C}$ (i.e., $\omega_{\mathcal{R}} >> \omega_{\mathcal{C}}$). As such, we will make the approximation that $\omega_{\mathcal{R}}/\omega_{\mathcal{C}} \approx \omega_{\mathcal{R}}$ which leads to $\omega_{\mathcal{C}} \approx 1$. For the remainder of our discussion we will investigate a model in which $\omega_{\mathcal{C}} = 1$ and the fluctuations in $E_{\mathcal{C}}$ will yield a handful of discrete values ($E_{\text{Ave}} \pm \delta E = E_j = E_1, E_2, E_3, \ldots$). The table below shows a scenario in which there exist only 5 possible values for $E_{\mathcal{C}}$, each with corresponding values for $E_{\mathcal{R}}$, $\omega_{\mathcal{C}}$, and $\omega_{\mathcal{R}}$.

$E_{\mathcal{C}}$	$\omega_{\mathcal{C}}$	$E_{\mathcal{R}}$	$\omega_{\mathcal{R}}$
E_1	1	$E_{\text{tot}} - E_1$	3×10^{18}
E_2	1	$E_{\text{tot}} - E_2$	5×10^{19}
E_3	1	$E_{\text{tot}} - E_3$	4×10^{17}
E_4	1	$E_{\text{tot}} - E_4$	1×10^{20}
E_5	1	$E_{\text{tot}} - E_5$	7×10^{18}

A. What is the total number of *microstates* for the entire container (system + reservoir) in our scenario?

B. Are any of the *microstates* more probable than any other?

C. Using your answer to part B, which of the above *macrostates* is *most* probable? Why?

Which *macrostate* is *least* probable?

D. Give a general expression for the probability, $P(E_j)$, of $E_{\mathcal{C}} = E_j$.

Energy, Entropy, and Probability

You've now determined that the probability of the system C having energy E_j, $P(E_j)$, is *proportional* to the multiplicity of the reservoir for that state, labeled $\omega_{\mathcal{R}j}$. (Compare this to the probability you've found previously for getting M heads from flipping N coins.) But what if we don't explicitly know $\omega_{\mathcal{R}j}$, as will often be the case in real systems? In this case, we need an expression for $\omega_{\mathcal{R}j}$ that depends on properties of C (i.e., $\omega_{\mathcal{R}j} = \omega_{\mathcal{R}j}(E_j, T_C, V_C, \ldots)$).

A. Is state j a macrostate or a microstate? How do you know?

B. Write an expression for the entropy of the reservoir ($S_{\mathcal{R}j}$) in terms of $\omega_{\mathcal{R}j}$.

C. Now use Taylor series expansion and the fact that entropy is a function of energy ($S_{\mathcal{R}j} = S_{\mathcal{R}j}(E_{\mathcal{R}j})$) to write an approximation for $S_{\mathcal{R}j}$ as a linear function of E_j.

   What is the *physical* interpretation of the first term in the Taylor expansion? Does this fit with what you know about Taylor series? Rename the first term to reflect this interpretation.

   What is the *physical* interpretation of the partial derivative in the second term? Consider the differential form of the first law of thermodynamics.

D. Equate your two expressions for $S_{\mathcal{R}j}$ from parts B and C to get an expression for $\omega_{\mathcal{R}j}$ in terms of the other variables and constants.

   Which of these quantities will change with different values of j?

E. Since $P(E_j) \propto \omega_{\mathcal{R}j}$ we can group any constant coefficients together. Write an expression for $P(E_j)$ as a function of E_j eliminating any constant terms and dividing by a normalizing term Z. (Remember, a function of a constant is a constant.)

F. Determine an expression for Z and rewrite your expression for $P(E_j)$. Consider the constraint on the sum over all probabilities $P(E_j)$.

G. Is your new expression for Z a constant? (i.e., does it depend on the state of the system?) How does your expression for Z compare with the normalizing factor for the binomial distribution (2^N)?

The normalizing factor for the probability is known as the canonical partition function. The symbol Z comes from the German *Zustandsumme* meaning "sum over states."

Taylor Series

1. What do you know about the Taylor series? That is, when I say "Taylor series," what comes to mind?
2. Can you write out a Taylor series for position as a function of time? (provide one if they can't)
    a. What are each of your terms? Do they have physical significance?
    b. What are the units of each term? Each item in each term?
    c. Are there any terms or items that can be considered constants?
3. How do you know how many terms to write?
4. Why do we even care about the Taylor series? Is it applicable in physics?
    a. If so, when?
    b. How does it relate to perturbation theory?
    c. Can you think of an instance in which your approximation of x(t) above would be useful?

Density of States vs. Boltzmann Factor

1. These are graphs of probability distributions due to the density of states and the Boltzmann factor for a many-particle system of a monatomic ideal gas in thermal equilibrium with a large reservoir.
    a. According to the density of states graph, which energy value(s) are more probable?
    b. According to the Boltzmann factor graph, which energy value(s) are more probable?
2. So, the probability distribution due to the density of states seems to indicate that higher energy values are more probable. But the corresponding graph of the probability distribution due to the Boltzmann factor seems to indicate that lower energy values are more probable. [Is this a problem? Is it OK?] [What do you make of this?]
    a. Are they both applicable in the same situations?
        i. What factors would make one more applicable than the other?
    b. In order to know the probability that a system is in a particular state, do these individual probability distributions help? How?"
        i. (If they say multiply) Why do you multiply them together?
            1. What would the resulting distribution look like?
            2. Which energy/energies would you expect to have the highest probability? (proceed to c.)
        ii. (If they say add) Why do you add them together?
            1. What would the resulting distribution look like?

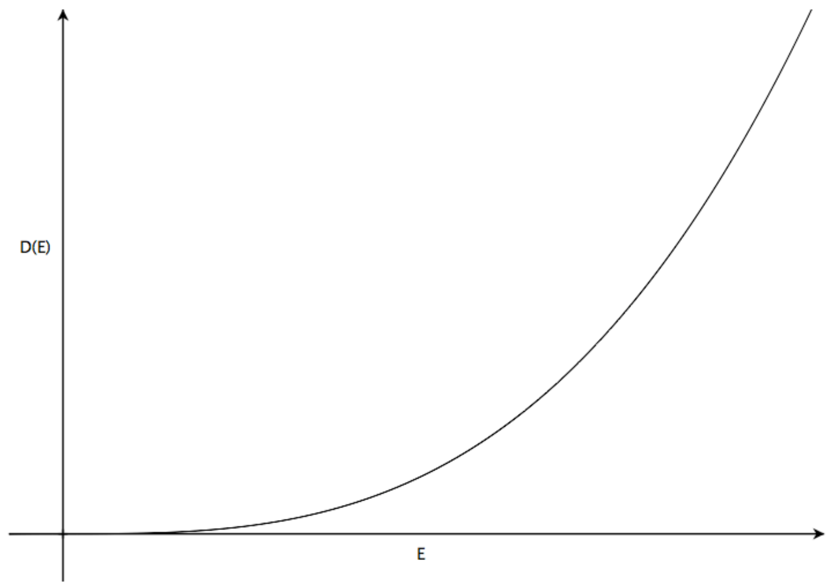
2. Which energy/energies would you expect to have the highest probability?
3. In this case, the very high and the very low energy states are equally probable; can you explain the physical meaning of that result [or “how can you make sense of that result?”]?
4. How else could we combine them? (go to c. if they don’t know)

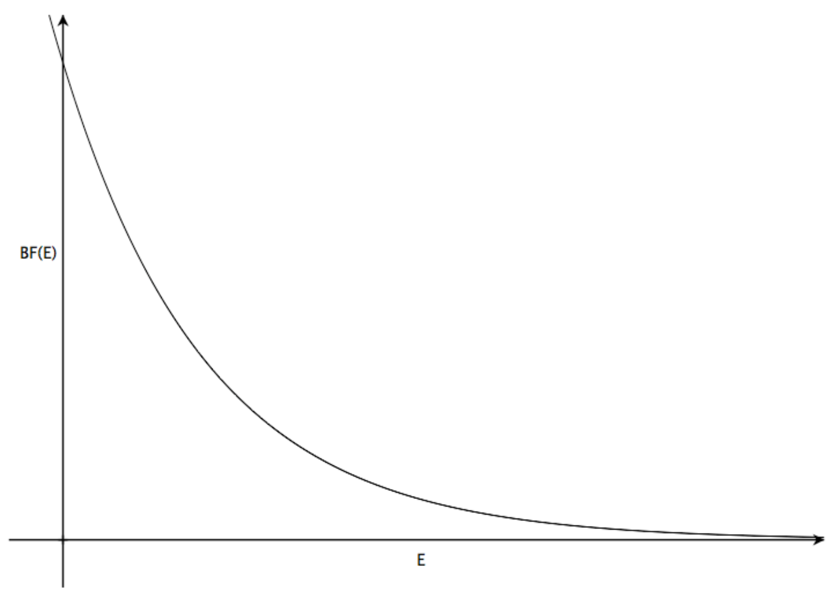
iii. (If they have no clue) proceed to c.

c. Where do the density of states and Boltzmann factor come from? Why are they related to probability?

i. How does the density of states D(E) relate to the probability of a thermodynamic system occupying a state with a particular value of energy?

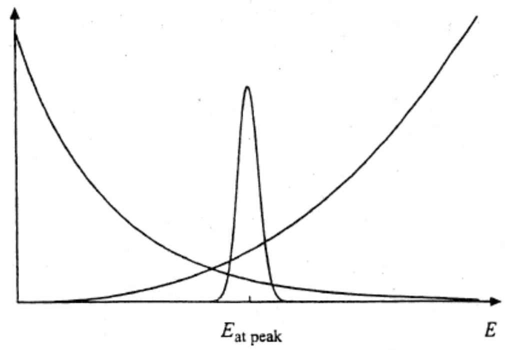
ii. How does the Boltzmann factor relate to the probability of a thermodynamic system occupying a state with a particular value of energy?


D(E)
E


BF(E)
E

2. Below are reproduced three different representations taken from Chapter 5 of Baierlein's textbook.

$$Z = \sum_j \exp(-E_j/kT)$$
$$= \int e^{-E/kT} D(E)\, dE.$$



$$\int e^{-E/kT} D(E)\, dE = [e^{-E/kT} D(E)]|_{E=\langle E\rangle} \delta E$$
$$= e^{-\langle E\rangle/kT} D(\langle E\rangle) \delta E.$$

Other than single letters (e.g., k, T), identify anything and everything that appears more than once in these three representations, and state (describe) what each of them refer to; e.g., are there one or more *functions* displayed graphically?

BIOGRAPHY OF THE AUTHOR

Trevor Ian Smith spent his childhood in Carlisle, PA after being born in Harrisburg, PA. In 2001, he graduated from Carlisle Area High School and came to the University of Maine as an undergraduate. He earned a Bachelor of Science in Physics in May, 2005 and a Master of Science in Teaching in May, 2007, both from the University of Maine. Trevor is a member of the American Physical Society, the American Association of Physics Teachers, and the Sigma Pi Sigma physics honors society. He has published several papers relating to his work as both a Masters and a Doctoral candidate.[87, 106–109] This dissertation marks the culmination of a decade spent in Maine.

While at UMaine, Trevor has been an enthusiastic participant in the music program, performing with the University Symphonic Band, Percussion Ensemble, the Screamin' Black Bears Pep Band, and the Pride of Maine Black Bear Marching Band, as well as performing in several pit ensembles (including that for the Rocky Horror Show in 2002). While an undergraduate, Trevor spent four summers touring with the Bluecoats Drum & Bugle Corps based in Canton, OH. Upon entering graduate school, Trevor became the percussion instructor for the Black Bear Marching Band, a position that he held for six years.

During his time at UMaine, Trevor has made numerous friends through his involvement in both music and physics; been a departmental representative to, and an officer for, the Graduate Student Government; met and married his wife, Ashley; and, most recently, became a father: Maxwell Eli Smith was born on April 5, 2011.

Trevor I. Smith is a candidate for the Doctor of Philosophy degree in Physics from The University of Maine in May, 2011.