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# The Distribution of the Irreducibles in an Algebraic Number Field

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# THE DISTRIBUTION OF THE IRREDUCIBLES IN AN ALGEBRAIC NUMBER FIELD

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The objective of this thesis is to study the distribution of the number of principal ideals generated by an irreducible element in an algebraic number field, namely in the non-unique factorization ring of integers of such a field. In particular we are investigating the size of $M(x)$, defined as

$M(x) = \sum_{\substack{(\alpha) \\ \alpha \ irred. \\ |N(\alpha)| \leq x}} 1$, where $x$ is any positive real number and $N(\alpha)$ is the norm of $\alpha$. We finally obtain asymptotic results for $M(x)$.

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# 1 INTRODUCTION

In an abstract algebra course, the student learns that the concept of a prime and an irreducible element does not coincide in an integral domain without unique factorization.

This idea prompted us to ask for a characterization of irreducibles in familiar integral domains where unique factorization need not hold—namely, the ring of integers of an algebraic number field.

Moreover, having studied the distribution of prime numbers in an analysis course where we obtained the asymptotic result

$$\pi(x) = \sum_{\substack{p \le x \\ p \ prime}} 1 \sim \frac{x}{\log x}; \ x \to \infty \quad ,$$

we were prompted to study the distribution of irreducibles in an algebraic number field. Of course, after doing a literature search, we found an abundance of work already done in this area, mostly in the last thirty years or so.

The purpose of this thesis, then, is to further investigate the distribution of irreducibles and hence expand our knowledge on this particular subject. We shall obtain, as a result of our analysis, an asymptotic formula for the distribution which gives the already known main term and the second largest term which appears to be new.

## 2 THE DIRICHLET SERIES $\mu$(s)

Let $K$ be an algebraic number field, i.e. a finite degree extension of the rational number field, $\mathbf{Q}$, and let $\mathcal{O}_K$ denote its ring of integers. We denote by $N(x)$ the norm of an element $x$ from $K$ to $\mathbf{Q}$. Also, we denote by $Na$ the norm of an ideal $a$ of $\mathcal{O}_K$. Furthermore, let $\mathrm{Cl} = \mathrm{Cl}(K)$ denote the class group of $K$ and $h = h(K)$ the class number, i.e. the order of $\mathrm{Cl}(K)$.

In studying the distribution of the irreducibles, we introduce the following function.

**Definition 1**

$$\mu(s) = \sum_{\substack{(\alpha) \\ \alpha \text{ irred.}}} |N(\alpha)|^{-s},$$

*where $s$ is a complex number with real part $\sigma > 1$.*

Here the sum is over the principal ideals generated by irreducible elements of $\mathcal{O}_K$. This sum converges for $\sigma > 1$. We obviously do not wish to count all associates of an irreducible, since there are infinitely many when the unit group is infinite, i.e. anytime $K$ is not $\mathbf{Q}$ or an imaginary quadratic number field.

Ultimately, we shall be interested in the "summatory" function given by

**Definition 2**

$$M(x) = \sum_{\substack{(\alpha) \\ \alpha \text{ irred.} \\ |N(\alpha)| \leq x}} 1,$$

*where $x$ is any positive real number.*

We shall determine properties of $\mu(s)$ first and then use a Tauberian theorem to obtain information about the distribution of $M(x)$.

To this end, consider the following. Write $\mathrm{Cl} = \{c_1 = 1, c_2, \cdots, c_h\}$.

**Definition 3** *For each positive integer $m$, let*

$$\mathcal{D}_m = \{\underline{k} = (k_1, \cdots, k_h) \in \mathbf{N}_0^h \;:\; \prod_{j=1}^{h} c_j^{k_j} \overset{min}{=} 1,\ k_1 + \cdots + k_h = m\},$$

*where $\prod c_i^{k_i} \overset{min}{=} 1$ means that $\prod c_i^{k_i} = 1$ and if $\prod c_i^{\ell_i} = 1$ for some $\ell_i$ such that $0 \le \ell_i \le k_i$ for $i = 1, \cdots, h$, then $\ell_i = 0$ for all $i$ or $\ell_i = k_i$ for all $i$. We define $\mathbf{N}_0$ to mean $\mathbf{Z}^{\geq 0}$.*

Notice that $\overset{min}{=}$ is equivalent to guaranteeing that a product of elements is 1 but no nontrivial subproduct is 1. Hence the product gives a "minimal" representation of 1.

**Definition 4** *The Davenport constant of* Cl, *denoted by $D$ or $D(\mathrm{Cl})$, is the largest positive integer $m$ such that $\mathcal{D}_m$ is nonempty.*

The Davenport constant is defined as above for any finite abelian group. It is not known in general what the relation is between the Davenport constant and the structure of the group.

One fact we can see easily is the following lemma.

**Lemma 1** *The Davenport constant is not larger than the order of the group, i.e.*

$$D \le h.$$

*Proof.* Suppose $D > h$ and suppose for some integer $m > h$,

$$\prod_{j=1}^{m} a_j \overset{min}{=} 1$$

for $a_j$ elements of the group. Consider $A = \{a_1, a_1a_2, \cdots, a_1 \cdots a_m\}$. Since $A$ is a subset of the group

$$|\{a_1, a_1a_2, \cdots, a_1 \cdots a_m\}| \leq h$$

and hence the elements of $A$ are not distinct so we have $\prod_{j=1}^{k} a_j = \prod_{j=1}^{\ell} a_j$, for some $k < \ell$. Hence $\prod_{j=k+1}^{\ell} a_j = 1$, contradicting the minimal representation above. ◻

To help clarify the previous definitions, consider the following two examples.

Suppose that the class number $h$ of $K$ is 2. Then let $\mathrm{Cl} = \{1 = c_1, a = c_2\}$ where $a^2 = 1$. We start by determining $\mathcal{D}_m$. For $m = 1$, the only minimal representation of 1 is $1 \overset{min}{=} 1$; hence $\mathcal{D}_1 = \{(1,0)\}$. For $m = 2$, the only minimal representation is $aa \overset{min}{=} 1$; hence $\mathcal{D}_2 = \{(0,2)\}$. Notice that the Davenport constant, $D = 2$.

Now suppose that the class number of $h$ of $K$ is 3. Then let $\mathrm{Cl} = \{c_1, a = c_2, b = c_3\}$ where $a^2 = b$. We start by determining $\mathcal{D}_m$. For $m = 1$, the only minimal representation of 1 is $1 \overset{min}{=} 1$; hence $\mathcal{D}_1 = \{(1,0,0)\}$. For $m = 2$, the only minimal representation is $ab \overset{min}{=} 1$; hence $\mathcal{D}_2 = \{(0,1,1)\}$. Finally for $m = 3$, we have two minimal representations of the identity: $aaa \overset{min}{=} 1$ and $bbb \overset{min}{=} 1$, and thus $\mathcal{D}_3 = \{(0,3,0),(0,0,3)\}$. Notice that the Davenport constant, $D = 3$.

We now consider the following proposition which gives a connection between irreducibles and prime ideals. First, let us denote the set of nonzero prime ideals of $\mathcal{O}_K$ by $\mathcal{P}$.

**Proposition 1**

$$\mu(s) = \sum_{m=1}^{D} \sum_{\underline{k}\in\mathcal{D}_m} \prod_{i=1}^{h} \sum_{\substack{a_i \\ \exists\, p_{i1},\cdots,p_{ik_i}\in\mathcal{P}\cap c_i \\ a_i=p_{i1}\cdots p_{ik_i}}} N(a_i)^{-s},$$

*for any complex $s$ with $\sigma > 1$ and $\sum_{a_i}$ is defined to be 1 when $k_i = 0$.*

*Proof.* For $\underline{k} \in \mathcal{D}_m$, define

$$\mathcal{A}_{\underline{k}} = \{a \;:\; a = a_1 \cdots a_h, a_i = p_{i1} \cdots p_{ik_i}, \quad \text{some} \quad p_{ij} \in \mathcal{P} \cap c_i\}$$

where $a_i = 1$ if $p_{i1} \cdots p_{ik_i}$ is an empty product. Now let $\mathcal{A} = \cup \mathcal{A}_{\underline{k}}$ where the union is over all $\underline{k}$ in $\cup_m \mathcal{D}_m$. By the uniqueness of the factorization of ideals into prime ideals, we see that this union is disjoint. Moreover, by the multiplicativity of the norms, we have

$$\sum_{m=1}^{D} \sum_{\underline{k}\in\mathcal{D}_m} \prod_{i=1}^{h} \sum_{a_i} N a_i^{-s} = \sum_{a\in\mathcal{A}} N a^{-s},$$

where $a_i$ are as appears in the definition of $\mathcal{A}_{\underline{k}}$. Now notice that if $a \in \mathcal{A}$, then $a \in \mathcal{A}_{\underline{k}}$ for some $\underline{k} \in \mathcal{D}_m$. Thus the ideal class $[a]$ containing $a$ satisfies

$$[a] = [a_1][a_2] \cdots [a_h] = \prod_{i=1}^{h} [p_{i1} \cdots p_{ik_i}] = \prod_{i=1}^{h} c_i^{k_i} \overset{min}{=} 1,$$

by definition of $\mathcal{D}_m$. Hence $a = (\alpha)$ for some non-zero, non-unit integer $\alpha$ in $\mathcal{O}_K$. But notice that $\alpha$ must be irreducible for otherwise $[a] = \prod_{i=1}^{h} c_i^{k_i} = 1$ would not be a minimal representation of 1.

Conversely, if $\alpha$ is irreducible, then $(\alpha) \in \mathcal{A}_{\underline{k}}$ for some $\underline{k}$ by the uniqueness of the factorization of ideals into prime ideals; namely,

$$(\alpha) = \prod_{i=1}^{h} \prod_{j=1}^{k_i} p_{ij},$$

for some $k_i \in \mathbf{N}_0$ and $p_{ij} \in \mathcal{P} \cap c_i$. $\square$

Next, we examine the righthand sum in the proposition above.

**Proposition 2** *Let $k$ be a nonnegative integer and $c$ any class in* Cl. *Then*

$$\sum_{\substack{a \\ \exists p_1,\cdots,p_k \in \mathcal{P}\cap c \\ a=p_1\cdots p_k}} N(a)^{-s}$$

$$= \sum_{\substack{(i_1,\cdots,i_k)\in \mathbf{N}_0^k \\ i_1+2i_2+\cdots+ki_k=k}} \frac{1}{i_1!\cdots i_k!} \sum_{\substack{(p_{11},\cdots,p_{1i_1}, \\ \vdots \\ p_{k1},\cdots,p_{ki_k})\in(\mathcal{P}\cap c)^{i_1+\cdots+i_k} \\ distinct}} \prod_{j=1}^{k} N(p_{j1}\cdots p_{ji_j})^{-js},$$

*where the last sum is taken over tuples for which the components are distinct prime ideals of the class $c$ and if $k = 0$ then the empty sum is defined to be* 1.

*Proof.* Suppose the ideal $a = p_1 \cdots p_k$ for some $p_i \in \mathcal{P} \cap c$. For $j = 1, \cdots, k$, let $i_j$ be the number of distinct prime ideals among the $p_1, \cdots, p_k$ which occur exactly $j$ times in this product. Notice that $i_1 + 2i_2 + \cdots + ki_k = k$. Denote the $i_j$ prime ideals by $p_{j1}, \cdots, p_{ji_j}$. Hence

$$a = \prod_{j=1}^{k} (p_{j1} \cdots p_{ji_j})^j.$$

But this means that if for each $j$ we sum over $(p_{j1}, \cdots, p_{ji_j}) \in (\mathcal{P} \cap c)^{i_j}$, then $a$ will be counted $i_1! \cdots i_k!$ times. Thus we must divide by this number in each summand. $\blacksquare$

To help clarify the formulas, let us consider the following examples. Suppose that the class number $h$ of $K$ is 2. Then by what we did above, we know that the Davenport constant, $D = 2$. By Proposition 1 we have,

$$\mu(s) = \sum_{\substack{(\alpha) \\ \alpha \ irred.}} |N(\alpha)|^{-s}$$

$$= \sum_{p \in \mathcal{P} \cap c_1} N(p)^{-s} + \sum_{\substack{a_2 \\ \exists\, p_1, p_2 \in \mathcal{P} \cap c_2 \\ a_2 = p_1 p_2}} N a_2^{-s}$$

where the terms on the righthand side are determined by $(1,0),(0,2) \in \cup_m \mathcal{D}_m$, respectively. Now, by Proposition 2, for $i = 2$,

$$\sum_{\substack{a_i \\ \exists\, p_1, p_2 \in \mathcal{P} \cap c_i \\ a_i = p_1 p_2}} N a_i^{-s} = \frac{1}{2} \sum_{\substack{(p_1, p_2) \in (\mathcal{P} \cap c_i)^2 \\ distinct}} N(p_1 p_2)^{-s}$$

$$+ \sum_{p \in \mathcal{P} \cap c_i} N(p)^{-2s},$$

where the terms on the righthand side of this last equality are determined by the ordered pairs: $(2,0,),(0,1)$, respectively.

Now suppose that the class number $h$ of $K$ is 3. Then by what we did above, we know that the Davenport constant, $D = 3$. By Proposition 1 we have,

$$\mu(s) = \sum_{\substack{(\alpha) \\ \alpha \ irred.}} |N(\alpha)|^{-s}$$

$$= \sum_{p \in \mathcal{P} \cap c_1} N(p)^{-s} + \sum_{p_2 \in \mathcal{P} \cap c_2} N(p_2)^{-s} \sum_{p_3 \in \mathcal{P} \cap c_3} N(p_3)^{-s} +$$

$$\sum_{\substack{a_2 \\ \exists\, p_1, p_2, p_3 \in \mathcal{P} \cap c_2 \\ a_2 = p_1 p_2 p_3}} N a_2^{-s} + \sum_{\substack{a_3 \\ \exists\, p_1, p_2, p_3 \in \mathcal{P} \cap c_3 \\ a_3 = p_1 p_2 p_3}} N a_3^{-s},$$

where the terms on the righthand side are determined by

$(1,0,0),(0,1,1),(0,3,0),(0,0,3) \in \cup_m \mathcal{D}_m$, respectively. Now, by Proposition 2, for $i = 2,3$,

$$\sum_{\substack{a_i \\ \exists\, p_1,p_2,p_3 \in \mathcal{P}\cap c_i \\ a_i = p_1p_2p_3}} Na_i^{-s} = \frac{1}{6} \sum_{\substack{(p_1,p_2,p_3)\in(\mathcal{P}\cap c_i)^3 \\ distinct}} N(p_1p_2p_3)^{-s}$$

$$+ \sum_{\substack{(p_1,p_2)\in(\mathcal{P}\cap c_i)^2 \\ p_1 \neq p_2}} N(p_1)^{-s}N(p_2)^{-2s} + \sum_{p\in\mathcal{P}\cap c_i} N(p)^{-3s},$$

where the terms on the righthand side of this last equality are determined by the ordered triplets: $(3,0,0),(1,1,0),(0,0,1)$, respectively.

Now we come up with a convenient description of $\mu(s)$. To this end we define the following family of polynomials.

**Definition 5** *Let $k$ be a positive integer and $z_1,\cdots,z_k$ independent variables. Then*

$$P_k(\underline{z}) = P_k(z_1,\cdots,z_k) = \sum_{\substack{(\nu_1,\cdots,\nu_k)\in\mathbf{N}_0^k \\ \sum j\nu_j = k}} \frac{1}{\nu_1!\cdots\nu_k!\,1^{\nu_1}\cdots k^{\nu_k}} z_1^{\nu_1}\cdots z_k^{\nu_k}.$$

*Moreover, let*

$$P_0(\underline{z}) = 1.$$

We then have the following proposition.

**Proposition 3** *Let $k$ be a nonnegative integer and $c$ any class in* Cl. *Then*

$$\sum_{\substack{a \\ \exists\, p_1,\cdots,p_k\in\mathcal{P}\cap c \\ a = p_1\cdots p_k}} N(a)^{-s} = P_k(\underline{z}),$$

*where*

$$z_j = \sum_{p \in \mathcal{P} \cap c} Np^{-js}.$$

*Proof.* Let $S_k$ be the symmetric group on $\{1, \cdots, k\}$; let

$\underline{p} = (p_1, \cdots, p_k) \in (\mathcal{P} \cap c)^k$; and for any $\sigma \in S_k$ define $\sigma\underline{p} = (p_{\sigma(1)}, \cdots, p_{\sigma(k)})$. Let $C(\sigma)$ be the conjugacy class of $\sigma$ in $S_k$, i.e. $C(\sigma) = \{\gamma\sigma\gamma^{-1} \ : \ \gamma \in S_k\}$. Since every permutation is a product of disjoint cycles, let

$$\sigma = \prod_{j=1}^{k} \eta_{j1} \cdots \eta_{j\nu_j}$$

be a factorization of $\sigma$ into disjoint cycles, where $\nu_j \in \mathbf{N}_0$ and for each $j$ and $i = 1, \cdots, \nu_j$, $\eta_{ji}$ are the distinct $j$-cycles, say $\eta_{ji} = (a_{ji1} \cdots a_{jij})$ with $a_{ji\ell} \in \{1, \cdots, k\}$, and with the convention that 1-cycles are included so that $\cup_{j,i}\{a_{ji1}, \cdots, a_{jij}\} = \{1, \cdots, k\}$. Recall that $\tau \in C(\sigma)$ if and only if $\tau$ has the same type of cycle decomposition, i.e. if

$$\tau = \prod_{j=1}^{k} \eta'_{j1} \cdots \eta'_{j\nu'_j}$$

into disjoint cycles with the same conventions as above, then $\nu'_j = \nu_j$ for $j = 1, \cdots, k$ (see, for example [1]). Notice then that a conjugacy class in $S_k$ is determined uniquely by a $k$-tuple, $(\nu_1, \cdots, \nu_k) \in \mathbf{N}_0^k$ with $\sum_{j=1}^{k} j\nu_j = k$. Any permutation in the conjugacy class has a cycle decomposition determined by the $\nu_j$'s as above. Moreover, recall that

$$\#C(\sigma) = \frac{k!}{\nu_1! \cdots \nu_k! 1^{\nu_1} \cdots k^{\nu_k}},$$

since we can permute disjoint cycles and have cyclic permutations of elements in a cycle. Again see [1].

Next let $\delta \in S_k$, $\sum_{\underline{p}}$ be the sum over all $k$-tuples in $(\mathcal{P} \cup c)^k$, and $N\underline{p} = Np_1 \cdots p_k$. Now notice

$$\sum_{\substack{\underline{p} \\ \sigma\underline{p}=\underline{p}}} N\underline{p}^{-s} = \sum_{\substack{\underline{p} \\ \sigma\underline{p}=\underline{p}}} N\delta\underline{p}^{-s}$$

since $\delta\underline{p}$ is just a permutation of the $p_i$'s. Then notice that

$$\sum_{\substack{\underline{p} \\ \sigma\underline{p}=\underline{p}}} N\underline{p}^{-s} = \sum_{\substack{\underline{p} \\ \gamma\sigma\gamma^{-1}\underline{p}=\underline{p}}} N\underline{p}^{-s},$$

for any $\gamma \in S_k$, for

$$\sum_{\substack{\underline{p} \\ \gamma\sigma\gamma^{-1}\underline{p}=\underline{p}}} N\underline{p}^{-s} = \sum_{\substack{\underline{p} \\ \sigma\gamma^{-1}\underline{p}=\gamma^{-1}\underline{p}}} N\underline{p}^{-s} = \sum_{\substack{\underline{p} \\ \sigma\underline{p}=\underline{p}}} N\gamma\underline{p}^{-s} = \sum_{\substack{\underline{p} \\ \sigma\underline{p}=\underline{p}}} N\underline{p}^{-s},$$

by changing the variable of summation. But by what we did above we see that if $\tau \in C(\sigma)$, then

$$\sum_{\substack{\underline{p} \\ \sigma\underline{p}=\underline{p}}} N\underline{p}^{-s} = \sum_{\substack{\underline{p} \\ \tau\underline{p}=\underline{p}}} N\underline{p}^{-s}.$$

Also, notice that if $\sigma = \prod_{j=1}^{k} \eta_{j1} \cdots \eta_{j\nu_j}$ is a cycle decomposition as described above, then

$$\sum_{\substack{\underline{p} \\ \sigma\underline{p}=\underline{p}}} N\underline{p}^{-s} = \prod_{j=1}^{k} \left( \sum_{p \in \mathcal{P} \cap c} Np^{-js} \right)^{\nu_j}$$

since $\sigma\underline{p} = \underline{p} \iff \underline{p} = \lambda(p_{11}, \cdots, p_{1\nu_1}, p_{21}, p_{21}, \cdots, p_{2\nu_2}, p_{2\nu_2}, \cdots, \underbrace{p_{k\nu_k}, \cdots, p_{k\nu_k}}_{k\ times})$

where $p_{ij}$ are all distinct and $\lambda \in S_k$. Moreover since the order of $S_k$ is equal to the number of conjugacy classes times the number of elements in each class and since each $C(\sigma)$ in $S_k$ is determined uniquely by a $k$-tuple, $(\nu_1, \cdots, \nu_k)$ with $\sum\limits_{j=1}^{k} j\nu_j = k$,

$$\frac{1}{k!} \sum_{\sigma \in S_k} \sum_{\substack{\underline{p} \\ \sigma\underline{p}=\underline{p}}} N\underline{p}^{-s} = \frac{1}{k!} \sum_{C(\sigma)} \#C(\sigma) \sum_{\substack{\underline{p} \\ \sigma\underline{p}=\underline{p}}} N\underline{p}^{-s}$$

$$= \sum_{\substack{(\nu_1,\cdots,\nu_k)\in\mathbf{N}_0{}^k \\ \sum j\nu_j=k}} \frac{1}{\nu_1!\cdots\nu_k!1^{\nu_1}\cdots k^{\nu_k}} \prod_{j=1}^{k} \left(\sum_p Np^{-js}\right)^{\nu_j} = P_k(z_1,\cdots,z_k),$$

where $z_j = \sum_p Np^{-js}$ and where $\sum_{C(\sigma)}$ is the sum over all conjugacy classes of $S_k$ and $\sigma$ is an element of the class $C(\sigma)$.

Now we have

$$\frac{1}{k!}\sum_{\sigma\in S_k}\sum_{\substack{\underline{p} \\ \sigma\underline{p}=\underline{p}}} N\underline{p}^{-s} = \sum_{\underline{p}} \frac{1}{k!}\sum_{\substack{\sigma\in S_k \\ \sigma\underline{p}=\underline{p}}} N\underline{p}^{-s} = \sum_{\underline{p}} \frac{1}{k!}\ \#S_k(\underline{p})\ N\underline{p}^{-s},$$

where $S_k(\underline{p}) = \{\sigma \in S_k \ : \ \sigma\underline{p} = \underline{p}\}$, the stabilizer subgroup of $\underline{p}$. Given $\underline{p} \in (\mathcal{P}\cap c)^k$, define

$$\nu(\underline{p}) = \underline{\nu} = (\nu_1,\cdots,\nu_k),$$

where $\nu_j = \#\{p_{j1},\cdots,p_{j\nu_j}\}$ with $p_{ji}$ those components of $\underline{p}$ occurring exactly $j$ times. Hence $\sum j\nu_j = k$. Then notice that

$$\#S_k(\underline{p}) = (1!)^{\nu_1}\cdots(k!)^{\nu_k}.$$

Then we may write

$$\sum_{\underline{p}} \frac{1}{k!}\ \#S_k(\underline{p})N\underline{p}^{-s} = \sum_{\substack{\underline{\nu} \\ \sum j\nu_j=k}} \frac{1}{k!} \prod_{j=1}^{k}(j!)^{\nu_j} \sum_{\substack{\underline{p} \\ \underline{\nu}(\underline{p})=\underline{\nu}}} N\underline{p}^{-s}.$$

Now, let $\underline{p} = (jp_{ji})_{\substack{j=1,\cdots,k \\ i=1,\cdots,\nu_j}}$ be a element of $(\mathcal{P}\cap c)^k$ with $p_{ji}$ all distinct and where $jp_{ji}$ means $p_{ji}$ occurs $j$ times in $\underline{p}$. Notice that there are $k!/((1!)^{\nu_1}\cdots(k!)^{\nu_k})$ different permutations of $\underline{p}$. Hence if we use a bijection of $\mathcal{P}\cap c$ with $\mathbf{N}$ to put a well ordering $<$ on $\mathcal{P}\cap c$, then we have

$$\sum_{\substack{\underline{\nu} \\ \sum j\nu_j=k}} \frac{1}{k!} \prod_{j=1}^{k}(j!)^{\nu_j} \sum_{\nu(\underline{p})=\underline{\nu}} N\underline{p}^{-s} =$$

$$\sum_{\substack{\underline{\nu} \\ \sum j\nu_j = k}} \quad \sum_{\substack{(p_{11},\cdots,p_{1\nu_1}, \\ \vdots \\ p_{k1},\cdots,p_{k\nu_k}) \in (\mathcal{P}\cap c)^{\nu_1+\cdots+\nu_k} \\ \text{distinct} \\ p_{j1}<\cdots<p_{j\nu_j}}} \quad \prod_{j=1}^{k} N(p_{j1}\cdots p_{j\nu_j})^{-js}$$

since the $k!/((1!)^{\nu_1}\cdots(k!)^{\nu_k})$ different permutations of $\underline{p}$ cancelled with $\frac{1}{k!}\prod_{j=1}^{k}(j!)^{\nu_j}$.

Now if we ignore the ordering in the last sum then we would be counting each term $\nu_1!\cdots\nu_k!$ times more than we should. Therefore,

$$\sum_{\substack{\underline{\nu} \\ \sum j\nu_j = k}} \frac{1}{k!}\prod_{j=1}^{k}(j!)^{\nu_j} \sum_{\nu(\underline{p})=\underline{\nu}} N\underline{p}^{-s} =$$

$$\sum_{\substack{\underline{\nu} \\ \sum j\nu_j = k}} \frac{1}{\nu_1!\cdots\nu_k!} \quad \sum_{\substack{(p_{11},\cdots,p_{1\nu_1}, \\ \vdots \\ p_{k1},\cdots,p_{k\nu_k}) \in (\mathcal{P}\cap c)^{\nu_1+\cdots+\nu_k} \\ \text{distinct}}} \quad \prod_{j=1}^{k} N(p_{j1}\cdots p_{j\nu_j})^{-js} =$$

$$\sum_{\substack{a \\ \exists\, p_1,\cdots,p_k\in\mathcal{P}\cap c \\ a=p_1\cdots p_k}} N(a)^{-s},$$

by Proposition 2. This establishes the proposition. ◻

We now have the following useful corollary to Proposition 3.

**Corollary 1**

$$\mu(s) = \sum_{\substack{(\alpha) \\ \alpha \text{ irred.}}} |N(\alpha)|^{-s} = \sum_{m=1}^{D} \sum_{\underline{k}\in\mathcal{D}_m} \prod_{i=1}^{h} P_{k_i}(z_{i1},\cdots,z_{ik_i}),$$

*where*

$$z_{ij} = \sum_{p_i\in\mathcal{P}\cap c_i} Np_i^{-js}.$$

*Proof.* By Proposition 1,

$$\mu(s) = \sum_{m=1}^{D} \sum_{\underline{k} \in \mathcal{D}_m} \prod_{\substack{i=1 \\ k_i \neq 0}}^{h} \sum_{\substack{a_i \\ \exists\, p_{i1}, \cdots, p_{ik_i} \in \mathcal{P} \cap c_i \\ a_i = p_{i1} \cdots p_{ik_i}}} N(a_i)^{-s}.$$

By Proposition 3,

$$\sum_{\substack{a \\ \exists\, p_1, \cdots, p_k \in \mathcal{P} \cap c \\ a = p_1 \cdots p_k}} N(a)^{-s} = P_k(\underline{z}),$$

where $P_k(\underline{z}) = P_k(z_1, \cdots, z_k)$ and $z_j = \sum_{p \in \mathcal{P} \cap c} Np^{-js}$. Hence

$$\mu(s) = \sum_{m=1}^{D} \sum_{\underline{k} \in \mathcal{D}_m} \prod_{i=1}^{h} P_{k_i}(\underline{z}),$$

where $P_{k_i}(\underline{z}) = P_{k_i}(z_{i1}, \cdots, z_{ik_i})$ and where $z_{ij} = \sum_{p_i \in \mathcal{P} \cap c_i} N(p_i)^{-js}$. ◻

For the next proposition, write

$$z_{i1} = \sum_{p_i \in \mathcal{P} \cap c_i} Np_i^{-s} = \ell + g_i,$$

where

$$\ell = \frac{1}{h} \log(\frac{1}{s-1}),$$

and

$$g_i = g_i(s).$$

It is well known that $g_i(s)$ is regular at $s = 1$ (a stronger version of this result will by proven in the following section, see Proposition 5). We then have

**Proposition 4**

$$\mu(s) = \sum_{\mu=0}^{D} c_\mu\ \ell^\mu,$$

*where*

$$c_\mu = c_\mu(s) = \sum_{m=\max(1,\mu)}^{D} \sum_{\underline{k} \in \mathcal{D}_m} a_{\underline{k},\mu},$$

*where if* $\underline{k} = (k_1, \cdots, k_h)$*, then*

$$a_{\underline{k},\mu} = \underline{\underline{\sum_{\mu_1=0}^{k_1} \cdots \sum_{\mu_h=0}^{k_h}}} \prod_{i=1}^{h} b_{k_i,\mu_i},$$

*where the double bar indicates that the product of the sums is taken over* $\mu_1 + \cdots + \mu_h = \mu$ *and*

$$b_{k_i,\mu_i} = \sum_{\nu_{i1}=\mu_i}^{k_i} \frac{g_i^{\nu_{i1}-\mu_i}}{\mu_i!\,(\nu_{i1}-\mu_i)!}\,\rho_{k_i,\nu_{i1}},$$

*where*

$$\rho_{k_i,\nu_{i1}} = \sum_{\substack{(\nu_{i2},\cdots,\nu_{ik_i})\in \mathbf{N}_0^{k_i-1} \\ \sum j\nu_{ij}=k_i-\nu_{i1}}} \frac{1}{\nu_{i2}!\cdots\nu_{ik_i}!\;2^{\nu_{i2}}\cdots k_i^{\nu_{ik_i}}} z_{i2}^{\nu_{i2}}\cdots z_{ik_i}^{\nu_{ik_i}},$$

*if* $k_i > 1$*, and we define* $\rho_{1,1} = 1$*,* $\rho_{1,0} = 0$*, and* $\rho_{0,0} = 1$*.*

*Proof.* First use the definition of the polynomials $P_k(\underline{z})$ to expand $\mu(s)$ in Proposition 3, where the indices of summation are $\nu_{ij}$ for $i = 1,\cdots,h$ and $j = 1,\cdots,k_i$. Hence

$$\mu(s) =$$

$$\sum_{m=1}^{D} \sum_{(k_1,\cdots,k_h)\in\mathcal{D}_m} \prod_{i=1}^{h} \sum_{\substack{(\nu_{i1},\cdots,\nu_{ik_i}) \\ \sum j\nu_{ij}=k_i}} \frac{1}{\nu_{i1}!\cdots\nu_{ik_i}!1^{\nu_{i1}}\cdots k_i^{\nu_{ik_i}}}(\ell+g_i)^{\nu_{i1}} z_{i2}{}^{\nu_{i2}}\cdots z_{ik_i}{}^{\nu_{ik_i}}$$

where if $k_i = 0$ then $\sum_{(\nu_{i1},\cdots,\nu_{ik_i})} = 1$. Now in the righthand most sum above, sum over the $\nu_{i1}$ first in which case we get

$$\sum_{\substack{(\nu_{i1},\cdots,\nu_{ik_i}) \\ \sum j\nu_{ij}=k_i}} \frac{1}{\nu_{i1}!\cdots\nu_{ik_i}!1^{\nu_{i1}}\cdots k_i^{\nu_{ik_i}}}(\ell+g_i)^{\nu_{i1}} z_{i2}{}^{\nu_{i2}}\cdots z_{ik_i}{}^{\nu_{ik_i}} =$$

$$\sum_{\nu_{i1}=0}^{k_i} \frac{(\ell+g_i)^{\nu_{i1}}}{\nu_{i1}!}\rho_{k_i,\nu_{i1}},$$

with $\rho$ as defined in the statement of the proposition. Next expand

$z_{i1}^{\nu_{i1}} = (\ell + g_i)^{\nu_{i1}}$ using the binomial theorem as

$$\sum_{\mu_i=0}^{\nu_{i1}} \binom{\nu_{i1}}{\mu_i} \ell^{\mu_i} g_i^{\nu_{i1}-\mu_i}.$$

Then

$$\sum_{\nu_{i1}=0}^{k_i} \frac{(\ell + g_i)^{\nu_{i1}}}{\nu_{i1}!} \rho_{k_i,\nu_{i1}} = \sum_{\nu_{i1}=0}^{k_i} \frac{1}{\nu_{i1}!} \sum_{\mu_i=0}^{\nu_{i1}} \binom{\nu_{i1}}{\mu_i} g_i^{\nu_{i1}-\mu_i} \rho_{k_i,\nu_{i1}} \ell^{\mu_i} =$$
$$\sum_{\mu_i=0}^{k_i} b_{k_i,\mu_i} \ell^{\mu_i},$$

by switching the order of summations and where the $b's$ are defined as above.

But then

$$\prod_{i=1}^{h} \sum_{\mu_i=0}^{k_i} b_{k_i,\mu_i} \ell^{\mu_i} =$$
$$\sum_{\mu_1=0}^{k_1} \cdots \sum_{\mu_h=0}^{k_h} \prod_{i=1}^{h} b_{k_i,\mu_i} \ell^{\mu_1+\cdots+\mu_h} = \sum_{\mu=0}^{m} a_{\underline{k},\mu} \ell^{\mu},$$

with the given limits of summation since the largest value $\mu$ can take is

$m = k_i + \cdots + k_h$ and the smallest value it can take is 0 and with $a$ as defined

above.

But now

$$\sum_{\underline{k}\in\mathcal{D}_m} \sum_{\mu=0}^{m} a_{\underline{k},\mu} \ell^{\mu} = \sum_{\mu=0}^{m} \sum_{\underline{k}\in\mathcal{D}_m} a_{\underline{k},\mu} \ell^{\mu}.$$

Hence

$$\mu(s) = \sum_{m=1}^{D} \sum_{\mu=0}^{m} \sum_{\underline{k}\in\mathcal{D}_m} a_{\underline{k},\mu} \ell^{\mu} = \sum_{\mu=0}^{D} \left( \sum_{m=\max(1,\mu)}^{D} \sum_{\underline{k}\in\mathcal{D}_m} a_{\underline{k},\mu} \right) \ell^{\mu},$$

by switching the order of summation where we have $m = \max(1, \mu)$ since $m$

must be $\geq 1$ and $\geq \mu$. Thus

$$\mu(s) = \sum_{\mu=0}^{\mathcal{D}} c_{\mu} \ell^{\mu}$$

as desired. ◻

Now we rewrite the $a_{\underline{k},\mu}$ in Proposition 4 in a form more convenient for winning an explicit formula for $c_\mu$ for "large" $\mu$.

**Corollary 2**

$$\mu(s) = \sum_{\mu=0}^{D} c_\mu \; \ell^\mu,$$

*where*

$$c_\mu = c_\mu(s) = \sum_{\nu=\max(1,\mu)-\mu}^{D-\mu} \; \sum_{\underline{k}\in\mathcal{D}_{\mu+\nu}} a_{\underline{k},\mu},$$

*with*

$$a_{\underline{k},\mu} = \underline{\underline{\sum_{\nu_1=0}^{k_1}\cdots\sum_{\nu_h=0}^{k_h}}} \prod_{i=1}^{h} \frac{1}{k_i!} \sum_{\lambda_i=0}^{\nu_i} \frac{k_i!}{(\nu_i-\lambda_i)!(k_i-\nu_i)!} g_i^{\nu_i-\lambda_i} \rho_{k_i,k_i-\lambda_i},$$

*where the double bar indicates that the product of the sums is taken over* $\nu_1+\cdots+\nu_h=\nu$ *and (as above)*

$$\rho_{k_i,k_i-\lambda_i} = \sum_{\substack{(\nu_{i2},\ldots,\nu_{ik_i})\in\mathbb{N}_0^{k_i-1} \\ \sum j\nu_{ij}=\lambda_i}} \frac{1}{\nu_{i2}!\cdots\nu_{ik_i}!\; 2^{\nu_{i2}}\cdots k_i^{\nu_{ik_i}}} z_{i2}^{\nu_{i2}}\cdots z_{ik_i}^{\nu_{ik_i}}.$$

*Proof.* In Proposition 4 we have $\mu(s) = \sum_{\mu=0}^{D} c_\mu \; \ell^\mu$ and letting $\nu = m-\mu$ gives

$$c_\mu = \sum_{\nu=\max(1,\mu)-\mu}^{D-\mu} \; \sum_{\underline{k}\in\mathcal{D}_{\mu+\nu}} a_{\underline{k},\mu}.$$

Now letting $\lambda_i = k_i - \nu_{i1}$, we have

$$a_{\underline{k},\mu} = \underline{\underline{\sum_{\mu_1=0}^{k_1}\cdots\sum_{\mu_h=0}^{k_h}}} \prod_{i=1}^{h} \frac{1}{k_i!} \sum_{\lambda_i=0}^{k_i-\mu_i} \frac{k_i!}{(\nu_{i1}-\mu_i)!(\mu_i)!} g_i^{\nu_{i1}-\mu_i} \rho_{k_i,\nu_{i1}}$$

where the double bar is as defined in Proposition 4. Making a final change of variables where $\nu_i = k_i - \mu_i$, we have

$$a_{\underline{k},\mu} = \underline{\underline{\sum_{\nu_1=0}^{k_1}\cdots\sum_{\nu_h=0}^{k_h}}} \prod_{i=1}^{h} \frac{1}{k_i!} \sum_{\lambda_i=0}^{\nu_i} \frac{k_i!}{(\nu_i-\lambda_i)!(k_i-\nu_i)!} g_i^{\nu_i-\lambda_i} \rho_{k_i,k_i-\lambda_i},$$

where the double bar indicates that the product of the sums is taken over $\nu_1 + \cdots + \nu_h = \nu$ and $\rho_{k_i,k_i-\lambda_i}$ is as defined above. □

From this corollary we extract the following corollary.

**Corollary 3** *Let*

$$\mu(s) = \sum_{\mu=0}^{D} c_\mu \, \ell^\mu .$$

*Then*

*i)*

$$c_D = \sum_{\underline{k} \in \mathcal{D}_D} \prod_{i=1}^{h} \frac{1}{k_i!}.$$

*ii)*

$$c_{D-1} = \sum_{\underline{k} \in \mathcal{D}_{D-1}} \prod_{i=1}^{h} \frac{1}{k_i!} \; + \; \sum_{\underline{k} \in \mathcal{D}_D} \prod_{i=1}^{h} \frac{1}{k_i!} \sum_{j=1}^{h} k_j g_j .$$

*iii) If $D \geq 2$, then*

$$c_{D-2} = \sum_{\underline{k} \in \mathcal{D}_{D-2}} \prod_{i=1}^{h} \frac{1}{k_i!} \; + \; \sum_{\underline{k} \in \mathcal{D}_{D-1}} \prod_{i=1}^{h} \frac{1}{k_i!} \sum_{j=1}^{h} k_j g_j$$

$$+ \sum_{\underline{k} \in \mathcal{D}_D} \prod_{i=1}^{h} \frac{1}{k_i!} \left( \sum_{1 \leq j_1 < j_2 \leq h} k_{j_1} k_{j_2} g_{j_1} g_{j_2} \; + \sum_{j=1}^{h} k_j (k_j - 1) \left( \frac{1}{2} g_j^2 \; + \; \frac{1}{2} z_{j2} \right) \right) .$$

*Proof.* Since parts *i)* and *ii)* can be done similarly, we only present the proof for part *iii)*.

By Corollary 2 for $D \geq 2$,

$$c_{D-2} = \sum_{\underline{k} \in \mathcal{D}_{D-2}} a_{\underline{k},D-2} + \sum_{\underline{k} \in \mathcal{D}_{D-1}} a_{\underline{k},D-2} + \sum_{\underline{k} \in \mathcal{D}_D} a_{\underline{k},D-2} .$$

Now in the first sum (where $\nu = 0$),

$$a_{\underline{k},D-2} = \underbrace{\sum_{\nu_1=0}^{k_1} \cdots \sum_{\nu_h=0}^{k_h}} \prod_{i=1}^{h} \frac{1}{k_i!} \sum_{\lambda_i=0}^{\nu_i} \frac{k_i!}{(\nu_i - \lambda_i)!(k_i - \nu_i)!} g_i^{\nu_i - \lambda_i} \rho_{k_i, k_i - \lambda_i},$$

where the double bar indicates that the product of the sums is taken over $\nu_1 + \cdots + \nu_h = 0$. Thus for $\nu = 0$,

$$a_{\underline{k},D-2} = \prod_{i=1}^{h} \frac{1}{k_i!}$$

since $\rho_{k_i,k_i} = 1$.

For the second sum (where $\nu = 1$),

$$a_{\underline{k},D-2} = \underbrace{\sum_{\nu_1=0}^{k_1} \cdots \sum_{\nu_h=0}^{k_h}} \prod_{i=1}^{h} \frac{1}{k_i!} \sum_{\lambda_i=0}^{\nu_i} \frac{k_i!}{(\nu_i - \lambda_i)!(k_i - \nu_i)!} g_i^{\nu_i - \lambda_i} \rho_{k_i, k_i - \lambda_i},$$

where the double bar indicates that the product of the sums is taken over $\nu_1 + \cdots + \nu_h = 1$. So for exactly one $i$, $\nu_i = 1$ and the remaining $(h-1)$ $\nu's$ are 0. Thus for the first case,

$$\sum_{\lambda_i=0}^{1} \frac{k_i!}{(1 - \lambda_i)!(k_i - 1)!} g_i^{1-\lambda_i} \rho_{k_i, k_i - \lambda_i} = k_i g_i,$$

since $\rho_{k_i,k_i} = 1$ and $\rho_{k_i,k_i-1} = 0$ and for the latter case, the previous sum (with upper limit 0 now) is 1. Hence for $\nu = 1$,

$$a_{\underline{k},D-2} = \prod_{i=1}^{h} \frac{1}{k_i!} \sum_{j=1}^{h} k_j g_j$$

For the last sum (where $\nu = 2$),

$$a_{\underline{k},D-2} = \underbrace{\sum_{\nu_1=0}^{k_1} \cdots \sum_{\nu_h=0}^{k_h}} \prod_{i=1}^{h} \frac{1}{k_i!} \sum_{\lambda_i=0}^{\nu_i} \frac{k_i!}{(\nu_i - \lambda_i)!(k_i - \nu_i)!} g_i^{\nu_i - \lambda_i} \rho_{k_i, k_i - \lambda_i},$$

where the double bar indicates that the product of the sums is taken over $\nu_1 + \cdots + \nu_h = 2$. Hence we must consider two cases. For case 1, $\nu_i = 2$ for exactly one $i$ and the remaining $(h-1)$ $\nu's$ are 0. Whence

$$\sum_{\lambda_i=0}^{2} \frac{k_i!}{(2-\lambda_i)!(k_i-2)!} g_i^{2-\lambda_i} \rho_{k_i,k_i-\lambda_i} = k_i k_{i-1} \left[\frac{1}{2} g_i^2 + \frac{1}{2} z_{i2}\right],$$

since $\rho_{k_i,k_i-2} = \frac{1}{2} z_{i2}$.

For case 2, $\nu_i = 1$ for exactly two distinct $i's$ and the remaining $(h-2)$ $\nu's$ are 0. Whence

$$\left[\sum_{\lambda_{i_1}=0}^{1}\right] \left[\sum_{\lambda_{i_2}=0}^{1}\right] = k_{i_1} k_{i_2} g_{i_1} g_{i_2},$$

where each sum is over the same expression as in Corollary 2 with $\nu_i = 1$. Therefore for $\nu = 2$,

$$a_{\underline{k},D-2} = \prod_{i=1}^{h} \frac{1}{k_i!} \left( \sum_{1 \le j_1 \le j_2 \le h} k_{j_1} k_{j_2} g_{j_1} g_{j_2} + \sum_{j=1}^{h} k_j (k_j - 1) \left[\frac{1}{2} g_j^2 + \frac{1}{2} z_{j2}\right]\right)$$

Thus we have the desired form for $c_{D-2}$. ◻

We further obtain the following expressions for $\mu(s)$ for some fields with small class number.

**Corollary 4** *i) Suppose* $D = 1$ *whence* $h = 1$. *Then*

$$\mu(s) = \ell + g_1.$$

*ii) If* $D = 2$ *so* $h = 2$, *say* $\mathrm{Cl} = \{1 = c_1, c_2\}$, *then*

$$\mu(s) = \frac{1}{2}\ell^2 + (1 + g_2)\ell + \left(g_1 + \frac{1}{2} g_2^2 + \frac{1}{2} z_{22}\right).$$

*Proof.* In light of the formulas for the $c_\mu$ above, it suffices to compute $\mathcal{D}_m$ for each of the groups listed. As we have already computed these for the class group of order 2 we consider the other case.

Let Cl $= \{1 = c_1\}$. Then we have only one minimal representations of 1, namely $1 \overset{min}{=} 1$, implying that $\mathcal{D}_1 = \{1\}$. Using this with the previous corollary yields *i)*. ◻

# 3 THE SUMMATORY FUNCTION M(x)

Having established formal properties of the Dirichlet series $\mu(s)$, we now use well-known results relating a Dirichlet series to its associated summatory function as in [2].

Let

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series where $s = \sigma + it$ with $a_n$, $\sigma$, $t$ real numbers and $a_n \geq 0$.

As in [2], we have the following definition.

**Definition 6** *We let $\mathcal{A}$ be the set of those Dirichlet series $f(s)$ as above but satisfying the following three additional properties:*

*(i) for all $x, y \in \mathbf{R}$ such that $1 \leq x < y$,*

$$\sum_{x \leq n \leq y} a_n \leq (y - x) \log^{c_1} y + O(y^{\theta}),$$

*for some $c_1 > 0$, $\theta < 1$ where the constants depend on $f(s)$ only.*

*(ii) There exists a nonnegative integer $k$ and functions $g_j(s)$ for $j = 0, \cdots, k$, such that*

$$f(s) = \sum_{j=0}^{k} g_j(s) \log^j \left( \frac{1}{s-1} \right),$$

*for $\sigma > 1$ and such that $g_k(1) \neq 0$ and $g_j(s)$ is regular for $\sigma > 1$ and can be analytically continued to a regular function in the region $\mathcal{R}$ given by*

$$\mathcal{R} = \left\{ s = \sigma + it \ : \ \sigma > 1 - \frac{c_2}{\log(|t| + 2)} \right\}$$

*for some $c_2 > 0$.*

*(iii) In the region $\mathcal{R}$*

$$|g_j(s)| \ll \log^{c_3}(|t| + 3),$$

*for some $c_3 > 0$.*

**Lemma 2** *If $f(s)$ satisfies (i), then $f(s)$ converges for all complex $s$ with $\sigma > 1$.*

*Proof.* Let $x = 1$ in property *(i)* above. We then have,

$$\sum_{n \leq y} a_n \leq (y-1)\log^{c_1} y + O(y^\theta),$$

where $c_1$ and $\theta$ are as defined previously. So

$$\sum_{n \leq y} a_n \leq (y-1)\log^{c_1} y + O(y^\theta) = O(y^{1+\epsilon})$$

for any $\epsilon > 0$. Now applying Theorem 2, page 156 of [5], we have $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converges for $\sigma > 1$. ◻

We now present the following weaker form of Kaczorowski's "Main Lemma" given in [2], which will be sufficiently strong for our purposes.

**Theorem 1 (Corollary to Kaczorowski's Main Lemma)** *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ be a Dirichlet series in $\mathcal{A}$ as defined above. Let $S(x) = \sum_{n \leq x} a_n$, the summatory function associated with $f(s)$. Then there exists a constant $c_4 > 0$ such that for all $x \geq e^e$,*

$$S(x) = kg_k(1)\frac{x}{\log x}\left(\sum_{j=0}^{k-1} d_j(\log\log x)^j\right) + O\left(\frac{x}{\log^2 x}(\log\log x)^{c_4}\right),$$

*as $x \to \infty$, where the $d_j$ are complex numbers given by*

$$d_j = \frac{1}{kg_k(1)}\sum_{\nu=j}^{k} g_\nu(1)B_{\nu j},$$

*with*

$$B_{\nu j} = (-1)^{\nu-j}\binom{\nu}{j}\frac{1}{2\pi i}\int_C e^z(\log z)^{\nu-j}\ dz,$$

*where $C$ is the path of integration consisting of the segment $(-\infty,-1]$ of the lower side of the real axis (so that the argument of $\log z$ is $-\pi$), the circumference of the unit circle taken counter-clockwise, and the segment $[-1,-\infty)$ of the upper side of the real axis.*

The proof may be found in [2] where we take Case I and $q = 0$ in the Main Lemma.

**Lemma 3** *Let $m$ be an integer. Then*

*a)* $B_{mm} = 0$ *for all* $m \geq 0$,

*b)* $B_{m,m-1} = m$ *for all* $m \geq 1$,

*c)* $B_{m,m-2} = \binom{m}{2}\kappa_2$ *for all* $m \geq 2$ *and* $\kappa_2 = \frac{1}{2\pi i}\int_C e^z \log^2 z \approx 1.15$.

*Proof.* Since *a)* and *b)* can be done similarly, we only present the proof for *c)*. Hence from Theorem 1, we have

$$B_{m,m-2} = \binom{m}{m-2}\frac{1}{2\pi i}\int_C e^z(\log z)^2\ dz.$$

Now

$$\int_C e^z(\log z)^2\ dz = I_1 + I_2 + I_3$$

where

$$I_1 = \int_\infty^1 e^{ue^{-i\pi}}\log^2(ue^{-i\pi})e^{-i\pi}\ du,$$

$$I_2 = \int_{-\pi}^{\pi} e^{e^{i\theta}}\log^2(e^{i\theta})ie^{i\theta}\ d\theta,$$

$$I_3 = \int_1^\infty e^{ue^{i\pi}} \log^2(ue^{i\pi})e^{i\pi}\ du.$$

Since

$$I_1 = \int_\infty^1 e^{-u}[\log u - i\pi]^2(-1)\ du = -1\int_\infty^1 e^{-u}\left[\log^2 u - 2i\pi\log u + (i\pi)^2\right]\ du$$
$$= \int_1^\infty e^{-u}\left[\log^2 u - 2i\pi\log u - \pi^2\right]\ du$$

and

$$I_3 = \int_1^\infty e^{-u}[\log u + i\pi]^2(-1)\ du = -\int_1^\infty e^{-u}\left[\log^2 u + 2i\pi\log u + (i\pi)^2\right]\ du$$
$$= \int_1^\infty -e^{-u}\left[\log^2 u + 2i\pi\log u - \pi^2\right]\ du,$$
$$I_1 + I_3 = -4i\pi\int_1^\infty e^{-u}\log u\ du.$$

Now simplifying $I_2$ we have,

$$I_2 = \int_{-\pi}^{\pi} e^{\cos\theta + i\sin\theta}(i\theta)^2 ie^{i\theta}\ d\theta = -i\int_{-\pi}^{\pi}\theta^2 e^{\cos\theta}e^{i\sin\theta}(\cos\theta + i\sin\theta)\ d\theta$$
$$= -i\int_{-\pi}^{\pi}\theta^2 e^{\cos\theta}\left[\cos(\sin\theta) + i\sin(\sin\theta)\right]\left[\cos\theta + i\sin\theta\right]\ d\theta$$
$$= -i\int_{-\pi}^{\pi}\theta^2 e^{\cos\theta}p(\theta)\ d\theta, \qquad (*)$$

where $p(\theta) = \cos(\sin\theta)\cos\theta - \sin(\sin\theta)\sin\theta + i\left(\cos(\sin\theta)\sin\theta + \sin(\sin\theta)\cos\theta\right).$

Letting $\theta = -\theta$ we get,

$$I_2 = -i\int_{-\pi}^{\pi}\theta^2 e^{\cos\theta}p(-\theta)\ d\theta.$$

By adding $(*)$ to the last equation, we have

$$2I_2 = -2i\int_{-\pi}^{\pi}\theta^2 e^{\cos\theta}\left[\cos(\sin\theta)\cos\theta - \sin(\sin\theta)\sin\theta\right]\ d\theta.$$

Therefore,

$$I_2 = -i \int_{-\pi}^{\pi} \theta^2 e^{\cos\theta} \left[\cos(\sin\theta)\cos\theta - \sin(\sin\theta)\sin\theta\right]\, d\theta.$$

Hence

$$\kappa_2 = \frac{1}{2\pi i}\int_C e^z \log^2 z\, dz = -\frac{1}{2\pi}\int_{-\pi}^{\pi} h(\theta)\, d\theta - 2\int_1^{\infty} e^{-u}\log u\, du \approx 1.15,$$

where

$$h(\theta) = \theta^2 e^{\cos\theta}\left(\cos(\sin\theta)\cos\theta - \sin(\sin\theta)\sin\theta\right). \quad \blacksquare$$

**Corollary 5** *Let $d_j$ be defined as in the theorem above. Then*

*i) if $k \geq 1$,*

$$d_{k-1} = 1,$$

*ii) if $k \geq 2$,*

$$d_{k-2} = \frac{(k-1)}{k}\frac{g_{k-1}(1)}{g_k(1)} + \frac{(k-1)}{2}\kappa_2,$$

*where $\kappa_2$ is as defined in Lemma 3.*

*Proof.* Consider *i)*. From Theorem 1,

$$d_{k-1} = \frac{1}{kg_k(1)}\sum_{\nu=k-1}^{k} g_\nu(1)B_{\nu j} = \frac{1}{kg_k(1)}\left[g_{k-1}(1)B_{k-1,k-1} + g_k(1)B_{k,k-1}\right].$$

By Lemma 3, $B_{k-1,k-1} = 0$ and $B_{k,k-1} = k$. Hence $d_{k-1} = 1$.

Consider *ii)*. From Theorem 1,

$$d_{k-2} = \frac{1}{kg_k(1)}\sum_{\nu=k-2}^{k} g_\nu(1)B_{\nu j}$$

$$= \frac{1}{kg_k(1)}\left[g_{k-2}(1)B_{k-2,k-2} + g_{k-1}(1)B_{k-1,k-2} + g_k(1)B_{k,k-2}\right].$$

By Lemma 3, $B_{k-2,k-2} = 0$, $B_{k-1,k-2} = k-1$, and $B_{k,k-2} = \binom{k}{2}\kappa_2$. Hence

$$d_{k-2} = \frac{(k-1)}{k}\frac{g_{k-1}(1)}{g_k(1)} + \frac{(k-1)}{2}\kappa_2. \ \square$$

We now apply these results to $\mu(s)$ to obtain information about $M(x)$. Our goal at this point is to show that $\mu(s)$ belongs to the class $\mathcal{A}$. To this end, we need to review some facts from algebraic number theory.

Let $K$ be an algebraic number field of degree $n$ over $\mathbf{Q}$, where the degree of the extension is denoted as $n = [K : \mathbf{Q}]$, with class group $\mathrm{Cl}(K) = \mathrm{Cl}$ of order $h$. Let $\widehat{\mathrm{Cl}}$ denote the character group of Cl, i.e. the group of homomorphisms from Cl into the multiplicative group $\mathbf{C}^*$. As usual, we denote the principal character, i.e. the constant character 1, by either $\chi_0$ or simply by 1.

Let $\chi$ be an arbitrary character on Cl, then we define the $L$-series

$$L(s,\chi) = \sum_{a} \frac{\chi(a)}{Na^s} \quad (\sigma > 1),$$

where the sum is over all (nonzero) integral ideals of $K$.

If $\chi = 1$, the principal character, then

$$L(s,\chi_0) = \zeta_K(s),$$

the Dedekind zeta function of $K$.

As is well known, $L(s,\chi)$ converges absolutely and uniformly on compact subsets in the half plane $\sigma > 1$. Moreover, since the norm map $N$ is multiplicative on the set of ideals of $K$, we have

$$L(s,\chi) = \prod_{p} \left(1 - \frac{\chi(p)}{Np^s}\right)^{-1},$$

for all $\sigma > 1$ and where the product is taken over all (nonzero) prime ideals of $K$. It is also known that in the half plane $\sigma > 1 - 1/n$, where $n = [K : \mathbf{Q}]$, the series for $L(s, \chi)$ converges, if $\chi \neq \chi_0$, and $L(s, \chi)$ is regular there, see Theorem 7, page 163 in [5]. On the other hand, $\zeta_K(s)$ has a continuation into the same half plane but with a simple pole at $s = 1$ with (nonzero) residue $a_K$.

We now state two further properties of $L(s, \chi)$. See [2].

I) In the region $\mathcal{R}_K$ given by

$$\sigma > 1 - \frac{c_K}{\log(|t| + 2)}$$

$L(s, \chi)$ does not vanish, where $c_K$ depends on $K$ but not on $\chi$.

II) In the region $\mathcal{R}_K$ for $|t| \geq 1$, we have

$$|\log L(s, \chi)| \ll_K \log\log(|t| + e^e),$$

where the implied constant depends only on $K$.

Now, since $L(s, \chi)$ is nonzero in the region above, we see that $\log L(s, \chi)$ is defined and regular in this region.

**Proposition 5** *Let $c$ be an ideal class of* Cl. *Then*

$$\sum_{p \in c} \frac{1}{Np^s} = \frac{1}{h} \log \zeta_K(s) + \frac{1}{h} \sum_{\substack{\chi \\ \chi \neq 1}} \overline{\chi}(c) \log L(s, \chi) - \sum_{m=2}^{\infty} \sum_{\substack{p \\ p^m \in c}} \frac{1}{mNp^{ms}},$$

*for $\sigma > 1$.*

*Proof.* We first write

$$\log L(s, \chi) = \sum_{p} \sum_{m \geq 1} \frac{\chi(p^m)}{mNp^{ms}}.$$

This follows since

$$\log L(s,\chi) = \sum_p \log\left(1 - \frac{\chi(p)}{Np^s}\right)^{-1} = -\sum_p \log\left(1 - \frac{\chi(p)}{Np^s}\right)$$

$$= \sum_p \left[\frac{\chi(p)}{Np^s} + \frac{1}{2}\left(\frac{\chi(p)}{Np^s}\right)^2 + \frac{1}{3}\left(\frac{\chi(p)}{Np^s}\right)^3 + \cdots\right] \quad \left(\text{for } \left|\frac{\chi(p)}{Np^s}\right| < 1, \quad \sigma > 1\right)$$

$$= \sum_p \left[\frac{\chi(p)}{Np^s} + \frac{\chi(p^2)}{2Np^{2s}} + \frac{\chi(p^3)}{3Np^{3s}} + \cdots\right] = \sum_p \sum_{m\geq 1} \frac{\chi(p^m)}{mNp^{ms}}.$$

Using the orthogonality relations of characters, we have

$$\sum_\chi \overline{\chi}(c) \log L(s,\chi) = \sum_p \sum_\chi \overline{\chi}(c)\chi(p)\frac{1}{Np^s}$$

$$+\sum_p \sum_{m\geq 2} \sum_\chi \overline{\chi}(c)\chi(p^m)\frac{1}{mNp^{ms}}$$

$$= h\sum_{p\in c}\frac{1}{Np^s} + h\sum_{m\geq 2}\sum_{p^m\in c}\frac{1}{mNp^{ms}}.$$

But then on the other hand, notice that

$$\sum_\chi \overline{\chi}(c)\log L(s,\chi) = \log \zeta_K(s) + \sum_{\substack{\chi \\ \chi\neq 1}} \overline{\chi}(c)\log L(s,\chi).$$

Hence

$$\sum_{p\in c}\frac{1}{Np^s} = \frac{1}{h}\log\zeta_K(s) + \frac{1}{h}\sum_{\substack{\chi \\ \chi\neq 1}}\overline{\chi}(c)\log L(s,\chi) - \sum_{m=2}^{\infty}\sum_{\substack{p \\ p^m\in c}}\frac{1}{mNp^{ms}},$$

for $\sigma > 1$. □

Notice that this proposition allows us to analytically continue $\sum_{p\in c} Np^{-s}$ in the region $\mathcal{R}_K$.

**Corollary 6** *Let*

$$g_c(s) = \sum_{p\in c}\frac{1}{Np^s} - \frac{1}{h}\log\left(\frac{1}{s-1}\right).$$

*Then*

$$g_c(s) = \frac{1}{h}\log((s-1)\zeta_K(s)) + \frac{1}{h}\sum_{\substack{\chi\\ \chi\neq 1}} \overline{\chi}(c)\log L(s,\chi) - \sum_{m=2}^{\infty}\sum_{\substack{p\\ p^m\in c}} \frac{1}{mNp^{ms}},$$

*hence regular in $\mathcal{R}_K$. In particular,*

$$g_c(1) = \frac{1}{h}\log a_K + \frac{1}{h}\sum_{\substack{\chi\\ \chi\neq 1}} \overline{\chi}(c)\log L(1,\chi) - \sum_{m=2}^{\infty}\sum_{\substack{p\\ p^m\in c}} \frac{1}{mNp^{m}},$$

*where $a_K$ is the residue of $\zeta_K(s)$ at $s=1$.*

*Proof.* Write $\zeta_K(s)$ as $\frac{1}{s-1}(s-1)\zeta_K(s)$. Then

$$g_c(s) = \sum_{p\in c}\frac{1}{Np^s} - \frac{1}{h}\log\left(\frac{1}{s-1}\right)$$

$$= \frac{1}{h}\log\left(\frac{1}{s-1}(s-1)\zeta_K(s)\right) + \frac{1}{h}\sum_{\substack{\chi\\ \chi\neq 1}} \overline{\chi}(c)\log L(s,\chi) - \sum_{m=2}^{\infty}\sum_{\substack{p\\ p^m\in c}} \frac{1}{mNp^{ms}} - \frac{1}{h}\log\left(\frac{1}{s-1}\right)$$

$$= \frac{1}{h}\log((s-1)\zeta_K(s)) + \frac{1}{h}\sum_{\substack{\chi\\ \chi\neq 1}} \overline{\chi}(c)\log L(s,\chi) - \sum_{m=2}^{\infty}\sum_{\substack{p\\ p^m\in c}} \frac{1}{mNp^{ms}}.$$

Hence $g_c(s)$ is regular in $\mathcal{R}_K$. ◻

**Theorem 2** *The Dirichlet series $\mu(s)$ belongs to $\mathcal{A}$*

*Proof.* (See [2]) Write $\mu(s) = \sum_{n\geq 1} a_n n^{-s}$, where $a_n$ denotes the number of principal ideals $(\alpha)$, with $\alpha$ irreducible and $|N(\alpha)| = n$. Obviously, $a_n \geq 0$.

Now we show $\mu(s)$ satisfies conditions (i)-(iii) of class $\mathcal{A}$. Consider (i). By Landau, [4], we know

$$\sum_{Na\leq x} 1 = a_K\, x + O\left(x^{(n-1)/(n+1)}\right), \qquad x\geq 1,$$

where $a_K$ is the residue of $\zeta_K(s)$ at $s=1$ and $n = [K:\mathbf{Q}]$. But then for $1\leq x<y$,

$$\sum_{x\leq m\leq y} a_m \leq \sum_{x\leq Na\leq y} 1 = \sum_{Na\leq y} 1 - \sum_{Na\leq x} 1 = a_K(y-x) + O\left(y^{(n-1)/(n+1)}\right),$$

where the left hand side involves only principal ideals whose norm is within the given limits whereas the right hand side involves any ideal with specified norm. Thus $\mu(s)$ satisfies (i).

Now we consider (ii) and (iii). Let $\mathrm{Cl} = \{c_1, \cdots, c_h\}$. Then by Corollary 1,

$$\mu(s) = \sum_{\underline{\nu}} g_{\underline{\nu}}(s) z_{11}^{\nu_1} \cdots z_{h1}^{\nu_h},$$

where the sum is over $\underline{\nu} = (\nu_1, \cdots, \nu_h)$ with $0 \le \nu_i \le D$, $z_{i1} = \sum_{p \in c_i} Np^{-s}$, and $g_{\underline{\nu}}(s)$ are functions regular in the half plane $\sigma > 1/2$, since the functions are combinations of powers of $z_{ij}$ for $j \ge 2$. Hence by Proposition 5, $\mu(s)$ is a finite sum of terms of the form

$$G(s) := g(s) \log^k \zeta_K(s) \prod_{\chi \ne 1} \log^{k_\chi} L(s, \chi),$$

for which $g(s)$ is regular in the half plane $\sigma > 1/2$, $k$ and $k_\chi$ are nonnegative integers, and the product is over some subset of nonprincipal characters on Cl. Now write

$$\log^k \zeta_K(s) = \left( \log(\frac{1}{s-1}) + \log((s-1)\zeta_K(s)) \right)^k$$
$$= \sum_{\nu=0}^{k} \binom{k}{\nu} \log^\nu \left( \frac{1}{s-1} \right) \log^{k-\nu} \left( (s-1)\zeta_K(s) \right).$$

Therefore

$$g(s) \log^k \zeta_K(s) \prod_{\chi \ne 1} \log^{k_\chi} L(s, \chi) = \sum_{\nu=0}^{k} h_\nu(s) \log^\nu \left( \frac{1}{s-1} \right),$$

where

$$h_\nu(s) = \binom{k}{\nu} g(s) \log^{k-\nu}((s-1)\zeta_K(s)) \prod_{\chi \ne 1} \log^{k_\chi} L(s, \chi).$$

But then by property I) above, $h_\nu(s)$ also satisfies I) which implies $G(s)$ satisfies (ii). Moreover, by II) we have

$$|h_\nu(s)| \ll \log^{k-\nu+1}(|t|+3),$$

showing that $G(s)$ satisfies (iii). As any finite sum of these $G(s)$ also satisfy (ii) and (iii), we see that $\mu(s)$ does, too. Hence $\mu(s)$ is in $\mathcal{A}$, as desired. □

We now apply this result to $M(x)$.

**Theorem 3** *Let $K$ be an algebraic number field with class number $h$ and associated Davenport number $D$. Then*

$$M(x) = Dc_D h^{-D}\frac{x}{\log x}(\log\log x)^{D-1} + Dc_D h^{-D}\frac{x}{\log x}\sum_{j=0}^{D-2} d_j (\log\log x)^j$$
$$+O\left(\frac{x}{(\log x)^{3/2}}\right),$$

*where the $d_j$ are given in Theorem 1 with $g_j = h^{-j}c_j(1)$.*

*Proof.* By Proposition 4,

$$\mu(s) = \sum_{\mu=0}^{D} c_\mu(s)\left(\frac{1}{h}\log(\frac{1}{s-1})\right)^\mu.$$

Now apply Theorem 1 with $g_j = h^{-j}c_j(1)$. Hence

$$M(x) = Dc_D h^{-D}\frac{x}{\log x}(\log\log x)^{D-1} + Dc_D h^{-D}\frac{x}{\log x}\sum_{j=0}^{D-2} d_j (\log\log x)^j$$
$$+O\left(\frac{x}{(\log x)^{3/2}}\right),$$

since for any $\epsilon > 0$

$$\frac{(\log\log x)^{c_4}}{\log^2 x} \ll \frac{1}{\log^{2-\epsilon} x} \ll \frac{1}{\log^{3/2} x}. \quad \square$$

As an immediate corollary we have,

**Corollary 7**

$$M(x) \sim Dc_D h^{-D} \frac{x}{\log x} (\log\log x)^{D-1}.$$

Compare this with Theorem 1 of [3].

But we also get the following corollary.

**Corollary 8** *For $D \geq 2$,*

$$M(x) = \frac{x}{\log x} \left( C(\log\log x)^{D-1} + B(\log\log x)^{D-2} \right)$$

$$+O\left(E(x)\right),$$

*where*

$$C = D\, c_D\, h^{-D}$$

$$B = (D-1)c_{D-1}(1)h^{1-D} + C\,\frac{D-1}{2}\,\kappa_2,$$

*and where*

$$E(x) = \frac{x}{\log x} (\log\log x)^{D-3}$$

*if $D \geq 3$ and*

$$\frac{x}{(\log x)^{3/2}}$$

*if not.*

*Proof.* From Corollary 5,

$$d_{D-2} = \frac{(D-1)}{D} \frac{h^{-(D-1)} c_{D-1}(1)}{h^{-D} c_D} + \frac{D-1}{2} \kappa_2.$$

Let $B = Dc_D h^{-D} \cdot d_{D-2}$. Applying Theorem 3 gives us the desired result. ∎

As a special case we have the following corollary.

**Corollary 9** *Let $K$ be a number field with class number 2. Denote by $c$ the nonprincipal ideal class of* Cl. *Finally, let $\chi$ be the nonprincipal character on* Cl, *hence $\chi(c) = -1$. Then*

$$M(x) = \frac{1}{4}\frac{x}{\log x}\log\log x + \frac{1}{8}(4(1+g_c(1)) + \kappa_2)\frac{x}{\log x} + O\left(\frac{x}{(\log x)^{3/2}}\right),$$

*where*

$$g_c(1) = \frac{1}{2}\log a_K - \frac{1}{2}\log L(1,\chi) - \sum_{\substack{m\geq 3\\ m\equiv 1(2)}}\sum_{p\in c}\frac{1}{mNp^m}.$$

*Proof.* From Corollary 8 and Corollary 3 *i)*,

$$C = Dc_D h^{-D} = \frac{1}{2}c_2 = \frac{1}{4}.$$

Similarly from Corollary 8 and Corollary 3 *ii)*,

$$B = (D-1)c_{D-1}(1)h^{1-D} + C\frac{(D-1)}{2}\kappa_2 = \frac{1}{2}(1+g_c(1)) + \frac{1}{8}\kappa_2,$$

where $g_c(1)$ is as defined in Corollary 6. ◻

Notice that of all fields with class number two, the main term in the asymptotic expression of $M(x)$ is independent of the field whereas the second term seems to depend more on the arithmetic of $K$.

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## BIOGRAPHY OF THE AUTHOR

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