2019

Some results on partial difference sets and partial geometries

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SOME RESULTS ON PARTIAL DIFFERENCE SETS AND PARTIAL GEOMETRIES

By

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A THESIS

Submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

In Mathematical Sciences

MICHIGAN TECHNOLOGICAL UNIVERSITY

2019

This thesis has been approved in partial fulfillment of the requirements for the Degree of MASTER OF SCIENCE in Mathematical Sciences.

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Author Contribution Statement

Non-existence of two types of partial difference sets [7] was published in *Discrete Mathematics*, volume 340, pages 2130-2133 in September, 2017 collaboratively with Dr. Stefaan De Winter and the advisor to this thesis, Dr. Zeying Wang. Elsevier's Author's Rights allows the material contained within to be republished as part of a thesis or dissertation given that the user is an author of the paper, so long as the thesis references the paper directly. This paper makes up the majority of the content within Chapter 3 of this thesis, and has been cited within as such.

Chapters 4 and 5 of this thesis are also original research developed in collaboration with Dr. Wang. We are currently preparing this content in order to submit it to a combinatorial journal.

Abstract

This thesis shows results on 3 different problems involving partial difference sets (PDS) in abelian groups, and uses PDS to study partial geometries with an abelian Singer group. First, the last two undetermined cases of PDS on abelian groups with $k \leq 100$, both of order 216, were shown not to exist. Second, new parameter bounds for k and Δ were found for PDS on abelian groups of order p^n, p an odd prime, n odd. A parameter search on p^5 in particular was conducted, and only 5 possible such cases remain for $p < 250$. Lastly, the existence of rigid type partial geometries with an abelian Singer group are examined; existence is left undetermined for 11 cases with $\alpha \leq 200$. This final study led to the determination of nonexistence for an infinite class of cases which impose a negative Latin type PDS.

Chapter 1

Introduction

This thesis studies partial difference sets in abelian groups and partial geometries with abelian Singer groups, and proves a number of nonexistence results on those two combinatorial structures. A partial difference set (in short, PDS) is a subset of a group which satisfies certain combinatorial conditions. In 1994, S. L. Ma wrote a survey paper on partial difference sets [15]. This paper includes many results on partial difference sets, including a list of all possible PDSs parameter sets with $k \leq 100$ which pass a set of necessary conditions (k is the size of the PDS). There were 32 cases for which existence was undetermined in that table, and in a 1997 paper [16], S.L. Ma was able to prove nonexistence for 13 of these cases, leaving 19 cases. One of these remaining 19 cases, the (512, 73, 12, 10)-PDS, was shown to exist in 1998 [10]. The other 18 cases remained open until 2016, when S. De Winter, Z. Wang, and E. Kamischke derived a local multiplier theorem [6], which they combined with variance techniques and some other ad-hoc methods to show nonexistence of every case remaining except for two: the (216, 40, 4, 8)-PDS and (216, 43, 10, 8)-PDS. Chapter 3 of this thesis shows the nonexistence of these two cases, a result which was published in 2017 [7] and completed the classification of PDS which have $k \leq 100$.

This result was proved using the local multiplier theorem, a similar variance method to the one used by De Winter, Wang, and Kamischke [6], combined with a weighted mapping of subsets of elements to the projective plane of order 3.

Motivated by these results on the two cases of order 216 (which is $8 * 3^3$), in 2018 S. De Winter and Z. Wang showed the nonexistence of nontrivial PDS in abelian groups of order $8p^3$, where p is any odd prime [9]. Only a few other general classification results are known on PDS in abelian groups, see for example [1], [8], or [17]. Additionally, very few PDSs have been shown to exist in abelian groups of order p^n, where p is prime and n is odd. Motivated by this observation, Chapter 4 of this thesis studies the question of existence of partial difference sets of order p^n, where p is an odd prime, in hope to discover more about the cases where n is also odd. Using some results of Ma, identities relating PDS parameters, and some ad-hoc methods, more strict bounds on k were established for PDSs on abelian groups of order p^n, p prime. This bound on k led to the development of some bounds on Δ using parameter integrality conditions, examination of discriminants of quadratics obtained by observing functions of parameters, and other ad-hoc methods (Δ is a very useful function of three of the parameters of a PDS). These parameter restrictions were used to perform a computer search for parameter sets which passed all necessary conditions in groups of size p^5, p prime. This search showed that the number of possible parameter sets of non-Paley type partial difference sets in an abelian group of order p^5 is sparse, with 5 potential parameter sets which pass the necessary conditions among the 54 primes $p < 250$.

Partial geometries were introduced in 1963 by R. Bose to study association schemes of partially balanced incomplete block designs using graph theoretical methods [2]. The point graph of a partial geometry, where the vertex set V is the set of points in the geometry, and two vertices are connected by an edge if and only if they lie on a common line, is a strongly regular graph. Thus, the study of partial geometries is connected to both design theory and graph theory and can proceed using techniques from both fields. In 2006, S. De Winter [4] divided partial geometries with an abelian Singer group into three classes based on the size of the stabilizer of each line from an abelian group of automorphisms acting regularly on the geometry: spread type, rigid type, and mixed type. In 2013, E. Kamischke [11] used a set of necessary conditions for rigid type partial geometries with an abelian Singer group and determined that there were only 11 possible parameter sets with $3 \leq \alpha \leq 8$ (if a point x is not on a line L, there are α lines through x which intersect L). This thesis shows the nonexistence of 10 of those 11 cases. It also presents an expanded table using a new computer search which includes a necessary condition from a 2008 paper by K. Leung, S.L. Ma and B. Schmidt [13], reaching $\alpha \leq 200$. An infinite class of partial geometries with an abelian Singer group which pass the necessary conditions and whose point graph induces a negative Latin square type PDS is presented; nonexistence is proved for this class. The computer search showed 20 potential parameter sets for rigid type partial geometries with an abelian Singer group which have $\alpha \leq 200$; 6 of these 20 cases are shown not to exist by the infinite class nonexistence result, one case is previously known to exist, and the nonexistence of two additional cases is shown through other methods, leaving 11 undetermined cases with $\alpha \leq 200$.

Chapter 2

Preliminaries

2.1 Relevant group theory

A large number of results throughout this thesis use the language of group theory, particularly actions on abelian groups. We begin by defining groups and stating a few specific results on groups and important classes of groups. Multiplicative notation will be used throughout this thesis and as such, groups are defined in that manner.

Definition 2.1.1 *A group $(G, *)$ is a set of elements G equipped with an operation $*$ that satisfy three properties:*

- *There exists an identity, often denoted 1 or e, such that for every $g \in G$, $g * 1 = g$.*
- *For each $g \in G$, there exists an inverse of g, denoted g^{-1}, such that $g^{-1} * g = 1$.*
- *For any $g, h, k \in G$, the associative property holds; that is $(g*h)*k = g*(h*k)$.*

Further the group is called abelian if for each $g, h \in G$, g and h commute, that is $g * h = h * g$.

Usually, since we use multiplicative notation, the multiplication is implied and the operation is assumed. We now define a group action:

Definition 2.1.2 *Let G be a group, and X a set. Then the mapping $G \times X \to X : g \cdot x \mapsto gx$ is called a (left) action of G on X if:*

- *The identity $e \in G$ maps such that $e \cdot x = x$ for all $x \in X$.*
- *For all $g, h \in G$, $x \in X$, we have $(gh) \cdot x = g \cdot (h \cdot x)$.*

One specific type of action is the *natural* action, which maps $G \times G \to G$, and the properties of an action hold by the associativity of the group. We need one more specific type of action in this thesis:

Definition 2.1.3 *Let X be a set and G be a group. Let $G \times X \to X$ be a group action.*

- *If for each pair of $x, y \in X$, there exists a $g \in G$ such that $g \cdot x = y$, then the action is called transitive.*
- *If the property that $g \cdot x = h \cdot x$ implies $g = h$ holds, then the group is called free.*
- *An action which is both transitive and free is called sharply transitive.*

We will deal with sharply transitive natural actions on abelian groups throughout this thesis. These generate automorphisms of the group involved, which when applied to geometries are called Singer groups after James Singer [20], as we do on partial geometries in this thesis.

2.2 Partial difference sets, strongly regular graphs, and partial geometries

We refer to Ma's 1994 survey paper [15] for our definitions (as well as a number of important results). These

Definition 2.2.1 *A partial difference set of order v, denoted $(v, k, \lambda, \mu) - PDS$, on a group G, with $|G| = v$, is a subset $\mathcal{D} \subset G$, with $|\mathcal{D}| = k$ such that the expressions $gh^{-1}; g \neq h; g, h \in \mathcal{D}$ represent every non-identity element in $\mathcal{D}$ λ times and every non-identity element not in $\mathcal{D}$ μ times.*

This definition generalizes difference sets, which are simply partial difference sets where $\lambda = \mu$.

Definition 2.2.2 *A strongly regular graph of order v, denoted $srg(v, k, \lambda, \mu)$ is a k-regular graph such that any two adjacent vertices share λ neighbors, and any two non-adjacent vertices share μ neighbors.*

If a partial difference set is not a difference set, we immediately can determine some structural information.

Theorem 2.2.3 *Let $\mathcal{D}$ be a $(v,k,\lambda,\mu)-PDS$ with $\lambda \neq \mu$. Define $\mathcal{D}^{(-1)} = \Big\{d^{-1} | d \in \mathcal{D}\Big\}$. Then, $\mathcal{D} = \mathcal{D}^{(-1)}$.*

Proof. Let $d \in \mathcal{D}$; we want to show $d^{-1} \in \mathcal{D}$. There are λ pairs of elements $g, h \in \mathcal{D}$ such that $gh^{-1} = d$. But, taking the inverse of both sides, $hg^{-1} = d^{-1}$, so there are λ pairs generating d^{-1}. Since $\lambda \neq \mu$, $d^{-1} \in \mathcal{D}$.

If the PDS satisfies that property, we notice that including or excluding the identity of the group only changes the parameters.

Theorem 2.2.4 *Let $\mathcal{D}$ be a $(v,k,\lambda,\mu) - PDS$ on an abelian group G where $\lambda \neq \mu$, and the identity of G, $e \in \mathcal{D}$. Then, $\mathcal{D} \setminus \{e\}$ is also a PDS. Further, if $k > 1$, then $\mathcal{D} \setminus \{e\}$ is a $(v, k-1, \lambda-2, \mu) - PDS$ (otherwise, the new PDS is a trivial PDS with $\mathcal{D} = \varnothing$ and paramater set $(v,0,0,0) - PDS$).*

Proof. We notice immediately that if $k = 1$ that e is the only element in $\mathcal{D}$, so removing it gives $\mathcal{D} \setminus e = \varnothing$. Assume otherwise that $k > 1$. Recall that $e^{-1} = e$; thus the differences excluded from $\mathcal{D} \setminus e$ are those of form $g \cdot e$ and $e \cdot g^{-1}$. Furthermore, if $g \in \mathcal{D}$, then since $\lambda \neq \mu$, we have $g^{-1} \in \mathcal{D}$ by 2.2.3, so this implies we are removing the products $g^{-1} \cdot e$ and $e \cdot g$ as well, for any $g \in \mathcal{D}$. Therefore, we have decreased the number of representations of nonidentity elements in $\mathcal{D}$ by 2, while not affecting how many times any element not contained in $\mathcal{D}$ appear.

These results motivate us to define a regular PDS. This definition will allow us to look at differences as the products $g \cdot h$ rather than $g \cdot h^{-1}$ (since $h^{-1} \in \mathcal{D}$ will imply $h \in \mathcal{D}$, while studying partial difference sets as an independent entity of difference sets. We will make the structural convention to exclude the identity from the PDS, which will prove to be useful in working with strongly regular graphs:

Definition 2.2.5 *A partial difference set $\mathcal{D}$ on group G is regular if the identity of G is not in $\mathcal{D}$ and $\mathcal{D}^{(-1)} = \mathcal{D}$.*

Next, we want to establish a connection between partial difference sets and strongly regular graphs, lending meaning to the shared parameters given in their definitions.

Definition 2.2.6 *Let G be a group and $\mathcal{D}$ be a subset of G such that the identity of G is not in $\mathcal{D}$. The Cayley graph $\Gamma(G, \mathcal{D})$ is the directed graph with an edge from g to h if and only if $gh^{-1} \in \mathcal{D}$. In other words, for each $g \in G$ and $h \in \mathcal{D}$, the edge $(g, g \cdot h) \in \Gamma(G, \mathcal{D})$.*

For $\mathcal{D}$ a regular partial difference set, it follows that the Cayley graph is undirected; if $g, h \in \mathcal{D}$, then $g^{-1}, h^{-1} \in \mathcal{D}$, so if $(g, h^{-1}) \in \mathcal{D}$, then $(h^{-1}, g) \in \mathcal{D}$. This gives new insight as to the reason for the shared parameters in definitions 2.2.1 and 2.2.2; the Cayley graph induced by a regular partial difference set $\mathcal{D}$ is a strongly regular graph with the same parameters. The converse construction does not always exist; not all strongly regular graphs imply the existence of a partial difference set (regular or otherwise).

Theorem 2.2.7 *Let $\mathcal{D}$ be a regular $(v, k, \lambda, \mu) - PDS$ in a group G. Then, the Cayley graph $\Gamma(G, \mathcal{D})$ is an $srg(v, k, \lambda, \mu)$.*

Proof. Since the vertex set of Γ is G, it follows immediately that Γ has v vertices. Similarly, we can see that each vertex lies on k edges, since it lies on the edge (g, gh) for each $h \in \mathcal{D}$.

Next, we want to show λ is the same for both structures. Let $g, h \in G$ be adjacent. Therefore, there is an $a^{-1} \in \mathcal{D}$ such that $ga^{-1} = h$, or $gh^{-1} = a$. Since $a \in \mathcal{D}$, there are λ ways to write $a = c^{-1}d$ with $c, d^{-1} \in \mathcal{D}$. If g and h share a neighbor z, that implies there is a way to write $z = gx = hy$, or alternatively, $z = x^{-1}y = gh^{-1}$, with $x, y \in \mathcal{D}$. But since $gh^{-1} = a$, and $a \in \mathcal{D}$, this difference occurs λ times.

Lastly, we need to show that μ is the same for both structures. Let $g, h \in G$ be non-adjacent, that is, there is no $a \in \mathcal{D}$ such that $ga = h$. We want to count the pairs such that there is a solution to $z = x^{-1}y = gh^{-1}$, with $x, y \in \mathcal{D}$. However, since there is no solution to $ga = h$, which implies $gh^{-1} = a^{-1}$, and since $a^{-1} \in \mathcal{D}$, this difference occurs μ times in $\mathcal{D}$. $\square$

Two general classes of partial difference sets and strongly regular graphs come from Latin squares, constructed by Dr. P.J. Cameron and Dr. J.H. van Lint [3]. We encounter the so-called negative Latin square type PDS class in our study of partial geometries.

Definition 2.2.8 *A partial difference set of the form* $(n^2, r(n+1), -n+r^2+3r, r^2+r) - PDS$*, where* n *and* r *are positive integers, is called a negative Latin square type partial difference set.*

A partial difference set of the form $(n^2, r(n-1), n+r^2-3r, r^2-r) - PDS$*, where* n *and* r *are positive integers, is called a Latin square type partial difference set.*

This local multiplier theorem has been very helpful in showing the nonexistence of many partial difference sets recently, and plays an important role in showing the nonexistence of the $(216, 40, 4, 8)$-PDS and the $(216, 43, 10, 8)$-PDS in abelian groups:

Proposition 2.2.9 [LMT [6]] Let D be a regular (v, k, λ, μ)-PDS in an abelian group G. Furthermore assume $\Delta = (\lambda - \mu)^2 + 4(k - \mu)$ is a perfect square. Then $g \in G$ belongs to D if and only if $g^s \in D$ for all s coprime with $o(g)$, the order of g.

For our development of the theory on partial difference sets of orders p^k where k is odd and p is prime, we need the following results:

Proposition 2.2.10 *No non-trivial PDS exists in*

- *an abelian group G with a cyclic Sylow-p-subgroup and $o(G) \neq p$;*
- *an abelian group G with a Sylow-p-subgroup isomorphic to $\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^t}$ where $s \neq t$.*

We need some results on adjacency matrices of strongly regular graphs in order to obtain our nonexistence results in Chapter 3.

Definition 2.2.11 *Let $G = (V, E)$ be a graph on v vertices, named $(k_1, k_2, ..., k_v)$. Then, the adjacency matrix of G, A, is the $v \times v$ matrix with $A_{ij} = 1$ if v_i and v_j are connected and $A_{ij} = 0$ otherwise.*

For strongly regular graphs, the spectrum of the adjacency matrix has been computed. These values are important in our setup for the variance method throughout Chapter 3.

Proposition 2.2.12 *Let A be the adjacency matrix of a strongly regular graph G, srg(v, k, λ, μ). Then, the eigenvalues of A are:*

$\nu_1 = k$, *with multiplicity* $m_1 = 1$,

$\nu_2 = \frac{1}{2}\big(\lambda - \mu + \sqrt{\Delta}\big)$, *with multiplicity* $m_2 = \frac{1}{2}\big(v - 1 - \frac{2k+(v-1)(\lambda-\mu)}{\sqrt{\Delta}}\big)$, *and*

$\nu_3 = \frac{1}{2}\big(\lambda - \mu - \sqrt{\Delta}\big)$, *with multiplicity* $m_3 = \frac{1}{2}\big(v - 1 + \frac{2k+(v-1)(\lambda-\mu)}{\sqrt{\Delta}}\big)$.

If G is not a conference graph (i.e. it is not an srg$(v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4})$), then these eigenvalues are integers.

The next two propositions were very important in showing the nonexistence of cases in [6]. Both give us information about how many elements of $\mathcal{D}$ come from different subgroups of G; we will use the second of the two multiple times in our study of the cases of order 216.

Proposition 2.2.13 *Let $H = \mathbb{Z}_p^r$ be a subgroup of G, where p is prime. Let $\mathcal{D}$ be a partial difference set on G, and call $|H \cap \mathcal{D}| = s$. Let U be a common eigenspace for the group of matrices $\{P_1, P_2, ..., P_v\}$ constructed by performing the action of each element of G on the adjacency matrix A of the Cayley graph of G. Let x be the number of vectors in U with eigenvalue 1. Additionally, let a_1 be the multiplicity of the eigenvalue $\nu_2 - \nu_3$ in the matrix $P(A - \nu_3 I)$, where P is n element of order p in $\mathcal{D}$. Lastly, let a_1' be the multiplicity of $\nu_2 - \nu_3$ in the matrix $P(A - \nu_3 I)$, where P is n element of order p which is not in $\mathcal{D}$. Then,*

$$m_2 + sa_1 + (p^r - 1 - s)a_1' = xp^r + (m_2 - x)p^{r-1}.$$

We can compute a_1 and a_1' as the solutions to the system:

$$a_1(\nu_2 - \nu_3) - a_p(\nu_2 - \nu_3) = -\nu_3(f - 1) + g - k$$

$$a_1 + (p - 1)a_p = m_2,$$

where f is the number of fixed points by some automorphism ϕ on the point graph of G, and g is the number of points mapped to their image. Taking ϕ to be a sharply transitive morphism, we can force $f = 0$ and $g = v$ for $g \in \mathcal{D}$ (the case when calculating a_1), and $g = 0$ for $g \notin \mathcal{D}$ (the case when calculating a_1').

This means we can readily compute $m_2, m_3, \nu_2, \nu_3, a_1$, and a_1' for any given parameter set. Since s and x must be nonnegative integers, this often gives us some restrictions on the size of particular subgroups which we can use to our advantage when we apply the LMT, and seems to be more useful for showing nonexistence when p is larger. More useful for small primes (particularly $p = 2$, which is vital in studying the cases of order 216) is the following result, again from S. Ma [15]:

Proposition 2.2.14 *Let D be a nontrivial regular (v, k, λ, μ)-PDS in an abelian group G. Suppose $\Delta = (\lambda - \mu)^2 + 4(k - \mu)$ is a perfect square. If N is a subgroup of G such that* $\gcd(|N|, |G| / |N|) = 1$ *and $|G| / |N|$ is odd, then $D_1 = D \cap N$ is a (not necessarily non-trivial) regular $(v_1, k_1, \lambda_1, \mu_1)$-PDS with*

$$|D_1| = \frac{1}{2}\left[|N| + \beta_1 \pm \sqrt{(|N| + \beta_1)^2 - (\Delta_1 - \beta_1^2)(|N| - 1)}\right].$$

Here $\Delta_1 = \pi^2$ with $\pi =$ $\gcd(|N|, \sqrt{\Delta})$ and $\beta_1 = \beta - 2\theta\pi$ where $\beta = \lambda - \mu$ and θ is the integer satisfying $(2\theta - 1)\pi \leq \beta < (2\theta + 1)\pi$.

S. Ma used these propositions when creating his necessary conditions in generating his table in 1994 [15]. We find many of these results very useful in Chapter 4 where we work on restricting parameter sets for cases of order p^k, p prime.

Proposition 2.2.15 *Suppose there exists a regular (v, k, λ, μ)-PDS D in a group G. Note the parameters $\beta = \mu - \lambda$ and $\Delta = (\lambda - \mu)^2 + 4(k - \mu)$.*

(a) *If $D \neq \emptyset$ and $D \neq G \setminus \{e\}$, then $0 \leq \lambda \leq k - 1$ and $0 \leq \mu \leq k - 1$.*

(b) *The parameters β and Δ have the same parity.*

(c) *The PDS D is nontrivial if and only if $-\sqrt{\Delta} < \beta < \sqrt{\Delta} - 2$. Also, if $D \neq G \setminus \{e\}$, then D is nontrivial if and only if $1 \leq \mu \leq k - 1$.*

(d) *If Δ is not a square, then $(v,\ k,\ \lambda,\ \mu)$=$(4t+1,\ 2t,\ t-1,\ t)$ for some positive integer t; furthermore, if G is abelian, then $v = p^{2s+1}$ for some prime $p \equiv 1 \pmod 4$.*

(e) *If G is abelian and $D \neq \emptyset$ and $D \neq G \setminus \{e\}$, then $v^2 \equiv (2k-\beta)^2 \equiv 0 \pmod{\Delta}$; furthermore, if D is nontrivial, then v, Δ, and v^2/Δ have the same prime divisors.*

(f) *The set $(G \setminus D) \setminus \{e\}$ is a PDS with parameters $(v',\ k',\ \lambda',\ \mu')$=$(v,\ v-k-1,\ v-2k-2+\mu,\ v-2k+\lambda)$ called the complement of D.*

(g) *If D is nontrivial, then there exists a nontrivial regular $(v, k^+, \lambda^+, \mu^+)$-PDS D^+ (in an abelian group of order v) with $\Delta^+ = (\lambda^+ - \mu^+)^2 + 4(k^+ - \mu^+) = v^2/\Delta$. (The PDS D^+ is called the dual of D.)*

One common technique we use to demonstrate the nonexistence of certain PDS relies on the fact that the variance of any set of real numbers must be nonnegative. We show this in the following theorem:

Theorem 2.2.16 *Let $S = \{x_1, x_2, ..., x_n\}$ be a set of real numbers. The variance of S is nonnegative; that is $n \cdot \sum_{i=1}^{n}(x_i)^2 - \left(\sum_{i=1}^{n} x_i\right)^2 \geq 0$.*

Proof. We can expand $\left(\sum_{i=1}^{n} x_i\right)^2 = \sum_{i=1}^{n}(x_i)^2 + \sum_{1\leq i<j\leq n}(2x_ix_j)$. Thus, we want to show:

$$(n-1)\sum_{i=1}^{n}(x_i)^2 - \sum_{1\leq i<j\leq n}(2x_ix_j) \geq 0.$$

However, we notice the second summand in this expression contains each x_i $n-1$ times, exactly once with each x_j such that $i \neq j$. Our inequality becomes:

$$\sum_{1 \leq i < j \leq n} x_i^2 - 2x_ix_j + x_j^2 = \sum_{1 \leq i < j \leq n} (x_i - x_j)^2 \geq 0,$$

but the sum of any number of squares is nonnegative (and in fact 0 if and only if each $x_i = x_j$).

Chapter 3 uses this variance fact in much the same way as [6], but in order to show nonexistence for the cases of order 216, we impose a projective plane with weighted points on the structure. In order to do so, we give a taste of the underlying design theory which leads up to projective planes by introducing balanced incomplete block designs (BIBDs), symmetric BIBDs, and projective planes. For a more complete reference on these combinatorial designs, one can refer to, for example, [21].

Definition 2.2.17 *A balanced incomplete block design, abbreviated BIBD, with parameters (v, k, λ, r, b) is a set of b blocks of size k which are subsets of a set of v points, such that each pair of points is contained in exactly λ blocks and each point appears in exactly r blocks.*

Some counting arguments dealing with BIBDs yield quickly that $bk = vr$, $\lambda(v-1) = r(k-1)$, and $b \geq v$. The first two facts imply that (v, k, λ) fully define the BIBD, and we can simply compute $r = \frac{\lambda(v-1)}{k-1}$ and $b = \frac{vr}{k} = \frac{\lambda v(v-1)}{k(k-1)}$. Naturally, we are interested in the cases where equality holds between v and b, and we define these as the symmetric designs.

Definition 2.2.18 *A symmetric BIBD is a BIBD where $v = b$ (the number of points equals the number of blocks), or equivalently, $k = r$ (the size of each block equals the number of blocks each point appears in).*

One special case of the symmetric BIBD is the projective plane, which proves to be useful in Chapter 3 to show the nonexistence of two potential parameter sets of partial difference sets.

Definition 2.2.19 *A BIBD with $v = n^2 + n + 1$, $k = n + 1$, and $\lambda = 1$ with $n \geq 2$ is called the projective plane of order n.*

For any prime power $q = p^n$, there exists a projective plane of order q. No non-prime power projective planes have been constructed, but the nonexistence of projective planes of non-prime power order in general is a problem of great interest in combinatorial design theory.

We also will discuss some results relating partial difference sets to partial geometries of rigid type.

Definition 2.2.20 *A proper partial geometry, denoted $pg(s, t, \alpha)$ is a set of points P, lines L, and incidences $I \subset P \times L$ such that:*

- *Each line contains $s + 1$ points.*
- *Each point is incident to (or on) $t + 1$ lines.*
- *Given any point $x \in P$, any line $l \in L$ such that $(x, l) \notin I$, there are α lines incident to x which are also incident to points on l.*
- *Each pair of lines intersects in at most one point. (Consequently, each pair of points can be on at most one line.)*
- $\alpha < min(s, t)$.

Definition 2.2.21 *The point graph $G = (V, E)$ of a partial geometry is the graph with V the point set of the partial geometry, and E containing the edge between two points if and only if those two points share a line.*

It turns out that a $pg(s,t,\alpha)$ with an abelian Singer group induces a $\big((s+1)(\frac{st+\alpha}{\alpha}), s(t+1), s-1+t(\alpha-1), \alpha(t+1)\big) - PDS$, which can be determined by observing the point graph of the partial geometry.

As shown by Dr. De Winter [4], if $\mathcal{S}$ is a partial geometry, $pg(s,t,\alpha)$, and G is an abelian Singer group acting on $\mathcal{S}$ (that is, G acts sharply transitively on $\mathcal{S}$), then the stabilizer of any line of $\mathcal{S}$ has size 1 or $s+1$. This dichotomy lead to the classification of partial geometries with an abelian Singer group into 3 categories:

Definition 2.2.22 *Let $\mathcal{S}$ be a partial geometry, $pg(s,t,\alpha)$, and G be an abelian Singer group acting on $\mathcal{S}$. We call the pair $(\mathcal{S}, G)$:*

- *spread type if $|Stab_G(L)| = s+1$ for every line of $\mathcal{S}$,*
- *rigid type if $|Stab_G(L)| = 1$ for every line of $\mathcal{S}$, and*
- *mixed type otherwise.*

2.3 Overview of results

There were 3 main results of this thesis: the proof of nonexistence of two longstanding cases of particular partial difference sets on abelian groups of small order; some restrictions of the parameters k and $\Delta = (\lambda - \mu)^2 + 4(k - \mu)$, as first defined in

2.2.9, on the class of infinite partial difference sets on groups of order p^n, p prime; and the proof of nonexistence of a number of partial geometries on rigid type partial geometries with an abelian Singer group.

In his 1994 survey paper [15], S. Ma generated all the possible parameter sets of PDS on abelian groups with $k \leq 100$. In particular, he checked every possible parameter set with $2 \leq k \leq 100$ to see if it satisfied Proposition 2.2.15 parts (a), (b), (d), and (e), as well as $k \leq \frac{v-1}{2}$ and $\Delta \leq v$ (otherwise, such a PDS would be the compliment or the dual of a PDS in the list, the two special PDS defined in parts (f) and (g) of Proposition 2.2.15). 187 such parameter sets were found from this parameter search. 100 of them are PCP type (which are a subclass of Latin square type PDS defined in Definition 2.2.8), which are based on partial congruence partitions. 27 of these parameter sets are Paley type, which occur for each prime power $q \equiv 1 (\text{mod } 4)$, these being $(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$-PDS. There are 12 parameter sets which were constructed by other assorted methods, and 14 parameter sets which were shown not to exist. This left 32 parameter sets which were undetermined. 13 of the remaining cases were shown not to exist by S. Ma in [16], and the (512, 73, 12, 10)-PDS, was shown to exist in 1998 [10], leaving 18 undetermined cases until 2016. At this point, Proposition 2.2.9 was shown, and using this result, Theorem 2.2.16, and Proposition2.2.14, S. De Winter, E. Kamischke, and Z. Wang [6] showed the nonexistence of 16 of these 18 cases. The remaining cases, both of order $216 = 2^3 \cdot 3^3$, were left open after this paper; the numerical parameters all worked using the methods of [6].

In Chapter 3, we show the nonexistence of these two remaining cases. We do so by determining the possible distribution of elements in $\mathcal{D}$, where $\mathcal{D}$ is assumed to be either a (216, 40, 4, 8)-PDS or (216, 43, 10, 8)-PDS on an abelian group G of order 216. Particularly, we look at the elements of order 3 in G, and look at the variance of subsets of G contained in $\mathcal{D}$ which "contain" that element (that is, it is a product of that element of order 3 with a lower-order element). This divides $\mathcal{D}$ into 26 disjoint subsets. We can make an observation which allow us to pair these sets, giving us 13 distinct subsets which we can solve for potential sizes of. Since we have 13 subsets, we can map these to the projective plane of order 3 (which has 13 points). The lines of this projective plane must have certain weights, which are imposed by the structure of the partial difference set, and we show that none of the valid variance solutions satisfy these allowable weight distributions. This result completes the classification of PDS on abelian groups with $k \leq 100$ by existence.

In Chapter 4, we try to find some general results on PDS of order p^n on abelian group G, where p is an odd prime and n is odd. Recently, S. De Winter and Z. Wang showed the nonexistence of all PDS on abelian groups of order $8p^3$ and fully classified PDS on abelian groups of order $4p^2$ (on which the only examples are PCP type or the (36, 14, 4, 6)-PDS) [8] [9]. Very few cases of non-Paley type PDS of order p^n are known when both p and n are odd and p is prime, and with the previous success on infinite classes S. De Winter and Z. Wang had, these prime power cases seemed very natural to study. Working up to the compliment of the PDS, as S. Ma assumed in generating his table for $k \leq 100$, so $k < \frac{v}{2}$, we are able to bound k slightly lower, with the bound being tighter the smaller Δ is. This restriction on k was accomplished by solving for the parameters in terms of a quadratic in λ, then a quadratic in k, looking particularly at the sign of the discriminant. This result allowed us to show that $\Delta = p^d$, where d is even and $\frac{d}{n} > \frac{2}{3}$ using some of the previously known integrality conditions with our new k bound. Specifically when $n = 5$, we show that $\Delta = p^4$; by brute force, these conditions allowed us to show that there are only 5 possible parameter sets remaining with $p \leq 250$. We predict that the tightening bound leads to the number of parameter sets to become increasingly sparse with increasing p.

In Chapter 5, we work on rigid type partial geometries with an abelian Singer group. The vast majority of known partial geometries are of spread type. It has been shown that every pg with $\alpha = 1$ which admits an abelian Singer group is of spread-type

[5], while the only pg with $\alpha = 2$ of rigid type is the pg(5, 5, 2), the van Lint-Schrijver partial geometry [4], [22]. The majority of work in the field had been done on cases where $\alpha = 2$. S. De Winter showed that there are no other rigid type partial geometries with a Singer group when $\alpha = 2$, and much work has been done in attempt to show that there are no mixed typed partial geometries with $\alpha = 2$ [13]. It is also worth noting that there are no known partial geometries of mixed type, but these have been much more difficult to study.

Ellen Kamischke further studied the necessary conditions for the existence of rigid type partial geometries in her master's thesis for larger values of α, resulting in 12 possible parameter sets with $\alpha \leq 8$, including the $pg(5, 5, 2)$ which we know to exist [11]. In chapter 5, this work is extended by showing the nonexistence of 10 of the 11 undetermined cases from that thesis, leaving only the potential pg(11, 23, 3) generating a PDS of order 2^{10}. The project then expanded to $\alpha \leq 200$ using a slightly different computer search method. The nonexistence of an infinite class of rigid type partial geometries which pass the necessary conditions, which have a point graph imposing a negative Latin square type partial difference set, is shown. This class accounts for 6 of our 20 cases which pass our new necessary conditions with $\alpha \leq 200$.

Chapter 3

Nonexistence of two partial difference sets of order 216

The results presented in this chapter were published in September 2017 in Discrete Mathematics, see [7]. We will expand on the details of the proof provided in that paper in this chapter to give a full picture of the argument.

In [15] Ma presented a table of parameters for which the existence of a regular PDS with $k \leq 100$ in an abelian group was known or could not be excluded. In particular, the list contained 32 cases where (non)-existence was not known. In [16] Ma excluded the existence of a PDS in 13 of these 32 cases. In [10] and [12], existence of the (512, 73, 12, 10)-PDS was shown, and De Winter, Kamischke and Wang [6] showed nonexistence of 16 of the 18 remaining cases. This left only the existence of a $(216, 40, 4, 8)$-PDS and $(216, 43, 10, 8)$-PDS on an abelian uncertain. This chapter will show nonexistence of these two cases, completing the classification of PDS with $k \leq 100$ in an abelian group based on existence. We proceed by using the LMT and variance methods, Theorems 2.2.9 and 2.2.16, in the manner of [6] combined with a new argument based on weighing points and lines in the projective plane of order 3 to show our result.

Theorem 3.0.1 *Neither a* $(216, 40, 4, 8)$*-PDS, nor a* $(216, 43, 10, 8)$*-PDS, can exist in an abelian group of order 216.*

Proof. We handle each parameter set separately. First, assume $\mathcal{D}$ is a $(216, 40, 4, 8)$-PDS in an abelian group G of order 216. By Proposition 2.2.10, we know that $G \cong \mathbb{Z}_2^3 \times \mathbb{Z}_3^3$.

Let $g_1, g_2, \dots, g_{26}$ be all elements of order 3 in G, and let $\mathcal{B}_{g_i} = \{ag_i \,|\, o(a) = 1 \text{ or } 2,\ ag_i \in D\}$, and $B_i = |\mathcal{B}_{g_i}|$, $i = 1, 2, \dots, 26$. That is, B_i equals the number of elements in $\mathcal{D}$ whose fourth power equals g_i.

Now observe that the LMT implies that raising elements to the fifth power provides a bijection between $\mathcal{B}_{g_i}$ and $\mathcal{B}_{g_i^2}$, since $(ag_i)^5 = a^5 g_i^5 = ag_i^2$. Hence $|\mathcal{B}_{g_i}| = |\mathcal{B}_{g_i^2}|$.

Let N be the Sylow-2-subgroup of G. We want to use Proposition 2.2.14, and can check that $gcd(|N|, |G/N|) = gcd(8, 27) = 1$, and $|G/N|$ is odd. We compute that $\Delta = (\lambda - \mu) + 4(k - \mu) = 144$, so $\pi = gcd(|N|, \sqrt{\Delta}) = 4$, $\Delta_1 = 16$, and $\beta = -4$. Solving our inequality for θ, we get that $\theta = 0$ and $\beta_1 = -4$. This finally implies that $|N \cap D| = |D_1| = 0$ or 4, or that there are 0 or 4 elements of order 2 in $\mathcal{D}$.

We work each of these two cases separately. First assume that D contains no elements of order 2, that is, $|D_1| = 0$. We see that $\Sigma_i B_i = k - |D_1| = 40$. Counting the differences of order 2 in $\mathcal{D}$ in two ways, we obtain that $\Sigma_i B_i(B_i - 1) = \lambda(|D_1|) + \mu(7 - |D_1|) = 56$; adding these two equations gives that $\Sigma_i B_i^2 = 96$.

By relabeling the g_i if necessary, we may assume that $C_j := B_{2j-1} = B_{2j}$, for $j =$

$1, 2, \ldots, 13$, and $C_1 \geq C_2 \geq \cdots \geq C_{13}$ because of our bijection observation. This cuts the B_i sums in half, and we simplify to

$$\Sigma_j C_j = 20 \quad \text{and} \quad \Sigma_j C_j^2 = 48. \tag{3.1}$$

We can solve the system (3.1) to find exactly the following nonnegative integer solutions, listed as 13 tuples $(C_1, C_2, \ldots, C_{13})$:

$(5, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$, $(5, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 0)$,

$(4, 4, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 0)$, $(4, 3, 3, 2, 2, 1, 1, 1, 1, 1, 1, 0, 0)$,

$(4, 3, 2, 2, 2, 2, 2, 1, 1, 1, 0, 0, 0)$, $(4, 2, 2, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0)$,

$(3, 3, 3, 3, 2, 2, 1, 1, 1, 1, 0, 0, 0)$, $(3, 3, 3, 2, 2, 2, 2, 2, 1, 0, 0, 0, 0)$.

Secondly, assume that D contains 4 elements of order 2. It follows that $\Sigma_i B_i = 40 - 4 = 36$. By counting the number of ways elements of order 2 can be written as differences of elements of D, we obtain that $\Sigma_i B_i(B_i - 1) + 4 \cdot 3 = 4 \cdot 4 + 3 \cdot 8$, or $\Sigma_i B_i(B_i - 1) = 28$. Using the same labeling as above, we now obtain

$$\Sigma_j C_j = 18 \quad \text{and} \quad \Sigma_j C_j^2 = 32. \tag{3.2}$$

Once again, we solve the system of equations (3.2) and find the following nonnegative integer solutions:

$(3, 3, 2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$, $(3, 2, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 0)$,

$(2, 2, 2, 2, 2, 2, 2, 1, 1, 1, 1, 0, 0)$.

Recall that N is the unique subgroup isomorphic to $\mathbb{Z}_2^3$ in G. Let $P_1, \ldots, P_{13}$ be the 13 subgroups of G isomorphic to $\mathbb{Z}_3$, and let $L_1, \ldots, L_{13}$ be the 13 subgroups of G isomorphic to $\mathbb{Z}_3^2$. Now consider the incidence structure $\mathcal{P}$ with points the subgroups $P_i \times N$, $i = 1, \ldots, 13$, of G, with blocks the subgroups $L_i \times N$, $i = 1, \ldots, 13$, of G, and with containment as incidence. It follows that $\mathcal{P}$ is a $2-(13, 4, 1)$ design, or equivalently, the unique projective plane of order 3. We next assign a weight to each point of $\mathcal{P}$ in the following way: if point p corresponds to subgroup $P_i \times N$ then the weight of p is $\frac{1}{2}|((P_i \times N) \setminus N) \cap \mathcal{D}|$. In this way the weights of the 13 points of $\mathcal{P}$ correspond to the 13 values $C_1, C_2, \ldots, C_{13}$, that is, half of the number of elements of order 3 or 6 from $\mathcal{D}$ in the subgroup underlying the given point. Without loss of generality we may assume the labeling is such that point $P_i \times N$ has weight C_i. The weight of a block will simply be the sum of the weights of the points in that block.

We next count how many elements of order 3 or 6 from $\mathcal{D}$ a specific subgroup of the form $L_i \times N$ can contain. Assume that $|(L_i \times N) \cap \mathcal{D}| = m$. Let ag and bh be two distinct elements from $\mathcal{D}$, with $a^2 = b^2 = g^3 = h^3 = e$. Then $agh^{-1}b^{-1}$ belongs to $L_i \times N$ if and only if $gh^{-1} \in L_i$. It follows that if $g \in L_i$ there are $m-1$ possibilities for bh such that $gh^{-1} \in L_i$, whereas if $g \notin L_i$ there are $\frac{|\mathcal{D}|-m-2}{2}$ possibilities for bh such that $gh^{-1} \in L_i$.

Counting the number of differences of elements of $\mathcal{D}$ that are in $L_i \times N$ in two ways, we obtain

$$m(m-1) + (k-m)(\frac{k-m-2}{2}) = \lambda m + \mu(71-m), \tag{3.3}$$

where $(k, \lambda, \mu) = (40, 4, 8)$. This yields that $m = 8$ or 16.

Now define $m' := \frac{1}{2}|((L_i \times N) \setminus N) \cap \mathcal{D}|$. We obtain the following table:

Case 1: $(216, 40, 4, 8)$-PDS	D contains 0 elements of order 2	$m' = 4$ or 8
Case 2: $(216, 40, 4, 8)$-PDS	D contains 4 elements of order 2	$m' = 2$ or 6

We now note that the values m' must be the weights of the blocks of $\mathcal{P}$, and that in both cases these weights are even. We first show that no value C_i can be odd. Assume by way of contradiction that C_i is odd for some i. Let the weight of the four blocks that contain $P_i \times N$ be $n_1, \ldots, n_4$ respectively. Then

$$\sum_{j=1}^{13} C_j = C_i + \sum_{t=1}^{4} (n_t - C_i).$$

The left hand side here is simply the sum of all the points' weights, where the right hand side is the sum of all the block weights excluding C_i, plus the weight of C_i. Since each block has even weight, we deduct an odd weight an even number of times, then add an odd number on the right hand side, this implies $\sum_{j=1}^{13} C_j$ is odd, contradicting that $\sum_{j=1}^{13} C_j = 20$ or 18.

This leaves us with only the possibility $(4, 2, 2, 2, 2, 2, 2, 2, 2, 0, 0, 0, 0)$ for $(C_1 \ldots, C_{13})$ in case 1. In this case, by considering the four blocks through a point with weight 2 it follows that it is not possible to distribute the thirteen given weights in such a way that every block has weight 4 or 8 (exactly one of the points in the block containing 4 and this 2 must be 0, but the rest of the blocks must have an even number of 0s in it; however, there are an even number of 0s, a contradiction). This concludes the proof for the (216, 40, 8, 12) case.

Now, let $\mathcal{D}$ be a (216, 43, 10, 8)-PDS on an abelian group G. By Proposition 2.2.10, we again know that $G \cong \mathbb{Z}_2^3 \times \mathbb{Z}_3^3$. Call the values B_i in the same way as before.

Applying Proposition 2.2.14, we can find $\Delta = 144$ again, $\theta = 0$, and $\beta = \beta_1 = 2$.

With this change, we can find that $\mathcal{D}_1 = 3$ or 7. From these, we can follow the same computation procedure for the $\Sigma_i B_i$, $\Sigma_i B_i^2$, $\Sigma_i C_i$, and $\Sigma_i C_i^2$ to find the values are the same. That is, when $\mathcal{D}_1 = 3$, we obtain Equation 3.1 and when $\mathcal{D}_1 = 7$, we obtain Equation 3.2, and so our solution sets are the same as in the (216, 40, 8, 12)-PDS case.

Apply the same projective plane construction to $\mathcal{D}$. Using Equation 3.3 with $(k, \lambda, \mu) = (43, 10, 8)$, we can obtain that $m = 11$ or 19. We have the following table:

Case 3: $(216, 43, 10, 8)$-PDS	D contains 3 elements of order 2	$m' = 4$ or 8
Case 4: $(216, 43, 10, 8)$-PDS	D contains 7 elements of order 2	$m' = 2$ or 6

But these are the same possibilities from Case 1 and 2 respectively. Since the point weight sets and allowable block weights are the same, we have a contradiction.

Chapter 4

Parameter bounds for regular PDS in abelian groups of order p^n

In this chapter, more strict bounds on k and $\Delta = (\lambda - \mu)^2 + 4(k - \mu)$ are obtained for PDS in an abelian group with order p^n, where p is prime. This study was motivated by the general lack of non-Paley type PDS on abelian groups of this type with n odd, although the results do apply if n is even as well. These restrictions were proved using S.L. Ma's results from [16], particularly those in Proposition 2.2.15, studying the discriminant of quadratic equations obtained from identities of strongly regular graphs, integrality of the parameters, and observation of series expansions. This work was done primarily to study cases of PDS in abelian groups of order p^n where p is prime and n is odd, due to the rarity of these cases in the literature. To observe this more closely, the chapter ends with a computer search for parameter sets of order p^5 with p an odd prime.

Throughout this chapter we will assume that p is a prime with $p \geq 5$. The reason for this is twofold: the case $p = 3$ already was dealt with in [7], and some of our arguments are only valid if $p \geq 5$. Furthermore, we will always assume that $\mathcal{D}$ is a nontrivial regular PDS in an abelian group.

4.1 Parameter Restrictions

4.1.1 Restriction on k

By part (f) of Proposition 2.2.15 we may assume that $k \leq v/2$. By part (g) of Proposition 2.2.15, we may assume that $\Delta \leq v$. Since the Cayley graph of a $(v,\ k,\ \lambda,\ \mu)$–PDS is a $(v,\ k,\ \lambda,\ \mu)$-strongly regular graph, we have

$$k(k-\lambda-1) = \mu(v-k-1). \tag{4.1}$$

By substituting $\frac{k(k-\lambda-1)}{v-k-1}$ for μ into $\Delta = (\lambda-\mu)^2 + 4(k-\mu)$, we get

$$\Delta = \left(\lambda - \frac{k(k-\lambda-1)}{v-k-1}\right)^2 + 4\left(k - \frac{k(k-\lambda-1)}{v-k-1}\right). \tag{4.2}$$

Lemma 4.1.1 *If a non-trivial (v, k, λ, μ)-PDS exists in an abelian group with $v = p^n$, $p \geq 3$ a prime, $\Delta = p^d$ and $k \leq \frac{v}{2}$, then we have $k \leq \frac{1}{2}(p^n-1)\left(1-\sqrt{1-\frac{p^d}{p^n}}\right)$.*

Proof. Setting $\Delta = p^d$ and $v = p^n$ in Equation (4.2), and solving the obtained quadratic equation for λ, we get

$$\lambda = \frac{(1+k-p^n)\big(3k + \frac{k^2(p^n-2)}{1+k-p^n} \pm \sqrt{4k^2p^n - 4kp^n(p^n-1) + p^d(p^n-1)^2}\big)}{(p^n-1)^2}.$$

As λ is an integer, the discriminant must be nonnegative, hence

$$4k^2p^n - 4kp^n(p^n-1) - p^d(p^n-1)^2 \geq 0. \tag{4.3}$$

Solving the quadratic equation $4k^2p^n - 4kp^n(p^n-1) - p^d(p^n-1)^2 = 0$ for k yields

$$k = \frac{1}{2}\left(p^n - 1 \pm p^{-n}\sqrt{(p^n-1)^2(p^n-p^d)p^n}\right).$$

The bigger root is greater than $p^n/2 = v/2$, but we already know $k \leq v/2$. This means that for Inequality (4.3) to hold, we have

$$k \leq \frac{1}{2}(p^n - 1 - p^{-n}\sqrt{(p^n-1)^2(p^n-p^d)p^n} = \frac{1}{2}\left(p^n-1\right)\left(1 - \sqrt{1-\frac{p^d}{p^n}}\right).$$

□

4.1.2 Restriction on Δ

From this bound on k we can obtain obtain a bound on Δ:

Lemma 4.1.2 *If a non-trivial (v,k,λ,μ)-PDS exists in an abelian group with $v = p^n$, $p \geq 3$ a prime, $\Delta = p^d$ and $k \leq \frac{v}{2}$, then $d > \frac{2}{3}n$.*

Proof. Set $\Delta = p^d$ and $v = p^n$, and note that by Proposition 2.2.15 part (e), we have $2k - \beta \equiv 0 \pmod{p^{\frac{d}{2}}}$. We obtain $k = \frac{\beta + xp^{\frac{d}{2}}}{2}$ for some integer x. By Equation 4.1 combined with the fact that $\Delta = (\lambda-\mu)^2 + 4(k-\mu)$, we obtain $\mu = \frac{x^2-1}{4p^{n-d}}$, so we know either $2p^{n-d}|(x-1)$ or $2p^{n-d}|(x+1)$. We can thus write $x = 2tp^{n-d} \pm 1$, t a positive integer. It then follows:

$$k = \frac{2tp^{n-\frac{d}{2}} \pm p^{\frac{d}{2}} + \beta}{2}, \qquad \mu = \frac{(2tp^{n-d} \pm 1)^2 - 1}{4p^{n-d}} = t^2p^{n-d} + t.$$

By Proposition 2.2.15 part (c), we have $-p^{d/2} < \beta < p^{d/2} - 2$, so we obtain

$$k_{min} > \frac{2tp^{n-\frac{d}{2}} - 2p^{\frac{d}{2}}}{2} = tp^{n-\frac{d}{2}} - p^{\frac{d}{2}}.$$

From Lemma 4.1.1 and performing a series expansion on the square root term:

$$k \leq (p^n + 1)\left(\frac{1}{4p^{(n-d)}} + \frac{1}{16p^{2(n-d)}} + \frac{1}{32p^{3(n-d)}} + \frac{5}{256p^{4(n-d)}} + \cdots\right).$$

Combining these two inequalities gives us:

$$t < \frac{1}{p^n} + p^{\frac{d}{2}}\left(\frac{1}{4p^{(n-d)}} + \frac{1}{16p^{2(n-d)}} + \frac{1}{32p^{3(n-d)}} + \frac{5}{256p^{4(n-d)}} + \cdots\right)$$

However, if $\frac{d}{2} \leq n - d$, this implies $t < 1$, contradicting the requirement t is a positive integer. Thus,

$$\frac{d}{2} > n - d \implies d > \frac{2}{3}n.$$

□

4.2 Computer search for non-Paley type PDS of order p^5 in an abelian group

The above restrictions severely restrict the number of possibilities for Δ for any non-Paley type PDS in a prime power order abelian group. In particular, when $n = 5$, we have that $\Delta = p^4$:

Lemma 4.2.1 *If $\mathcal{D}$ is a regular (v, k, λ, μ)-PDS in an abelian group with $v = p^5$ and $\mathcal{D}$ is not a Paley type PDS, $k \leq v/2$ and $\Delta \leq v$ then $\Delta = p^4$.*

Proof. Since $\mathcal{D}$ is not a Paley type PDS, by Ma's Proposition 2.2.15 (d) and (e), Δ must be a square, and v, Δ, v^2/Δ have the same prime divisors. It follows that $\Delta = p^2$ or $\Delta = p^4$. Now, by Lemma 4.1.2, $\Delta \neq p^2$, so it must be true that $\Delta = p^4$. □

Using these new restrictions, a program was written to generate cases with $p < 250$ which satisfy these necessary conditions, listed in Table 4.1, and generated with the following Mathematica code. Only 5 of the 54 primes less than 250 yielded a parameter set which pass these conditions.

Table 4.1: Cases which pass the necessary conditions for a PDS of order p^5, p prime, $p < 250$.

Case No.	p	$v = p^5$	k	λ	μ	$\Delta = p^4$
1	3	243	22	1	2	81
2	19	2476099	27180	149	300	130321
3	31	28629151	207900	1199	1512	923521
4	113	18424351793	38963232	79661	82404	163047361
5	191	254194901951	271720900	274815	290472	1330863361

Code for generating cases of order p^5 with $p < 250$, p prime:

```
Clear[V, K, L, p, a, b, c, d, v, x, y, z, \[Lambda], \[Mu], r, t]

V[x_] := x^5
M[x_, y_, z_] := x*(x - y - 1)/(z - x - 1)
del[a_, b_, c_] := (b - c)^2 + 4*(a - c)
Eig1[a_, b_, c_] := ((b - c) + a)/2
mult1[a_, b_, c_, d_] := (d - 1)/2 - (2*a + (d - 1)*(b - c))/(2*p^2)

L[x_, y_, u_, v_] :=
 1/(u - 1)^2 ((1 + y - u) (3 y + (y^2 (u - 2))/(1 + y - u) +
      x*Sqrt[4*y^2*u - 4*y*u (u - 1) + p^v*(u - 1)^2]))


For[p = 2, p < 250, p++,
 If[IntegerQ[p/10], Print[p]];
 If[PrimeQ[p],
  v = V[p];
  For[ a = 1, a < 3, a++,
   For[K = Ceiling[N[Sqrt[v + 1], 1]] ,
    K < 1/2 (v - 1) (1 - N[Sqrt[1 - 1/p], 50]), K = K + 1,
    If[Mod[K, p - 1 ] == 0,
     \[Lambda] = L[(-1)^a, K, v, 4];
     \[Mu] = M[K, \[Lambda], v];
     If[ del[K, \[Lambda], \[Mu]] == p^4 &&
       IntegerQ[\[Lambda]] &&  \[Lambda] > 0 &&
       IntegerQ[ \[Mu]] && \[Mu] > 0 &&
       IntegerQ[mult1[K, \[Lambda], \[Mu], v]] ,
      Print["p = ", p, ":  (", v, ",", K , ",", \[Lambda],
       ",", \[Mu], ")    Delta = ", p^4]
      ]
     ]
    ]
   ]
  ]
 ]
```

Chapter 5

Results on rigid type partial geometries with an abelian Singer group

This chapter builds on the work done in 2013 by E. Kamischke in her master's thesis on rigid type partial geometries with abelian Singer groups [11]. This chapter expands on that prior work in three major ways. First, it shows nonexistence for 10 of the 11 undetermined parameter sets listed in E. Kamischke's thesis. This is accomplished using the combination of a result from K. Leung, S.L. Ma and B. Schmidt from 2008 [13] and techniques similar to those in Chapter 3 which show the PDS associated with the partial geometry's point graph doesn't exist. The only remaining undetermined case with $\alpha \leq 8$ is of prime power order, the $pg(11, 23, 3)$, which has $1024 = 2^{10}$ points.

Second, this chapter uses the K. Leung, S.L. Ma and B. Schmidt result and a reworked form of the necessary conditions from E. Kamischke's thesis in order to expand the computer search for parameter sets from $\alpha \leq 8$ to $\alpha \leq 200$. This search yielded only 20 parameter sets for which a rigid type partial geometry with an abelian Singer group could exist. Observing these cases also led to the discovery of an infinite class of cases which pass our necessary conditions and yield a point graph which has the parameters of a negative Latin square type PDS, that is, it satisfies Definition 2.2.8.

Finally, the third part of this chapter proves the nonexistence of this infinite class using a generalized form of the variance method used in Chapter 3 and shows the nonexistence of 8 cases found in the $\alpha \leq 200$ computer search. 6 of the cases from the table are of this negative Latin square type and are eliminated by the general nonexistence proof, 2 cases are shown not to exist by other ad-hoc methods, and 1 case is known to exist (the so-called van Lint-Schrijver partial geometry, see [22]), leaving only 11 undetermined cases of partial geometries with abelian Singer groups and $\alpha \leq 200$.

5.1 Some necessary conditions

We begin this section by deriving some necessary conditions for parameter sets of rigid-type partial geometries with an abelian Singer group. First, we cite a Benson-type theorem from S. De Winter [4].

Theorem 5.1.1 *Let $\mathcal{S}$ be a partial geometry $pg(s,t,\alpha)$ and let θ be any automorphism of $\mathcal{S}$. Let f be the number of fixed points of $\mathcal{S}$ under θ and g be the number of points x of $\mathcal{S}$ for which x is collinear with x^θ , where x^θ denotes the image of x under θ. Then,* $(1+t)f + g \equiv (1+s)(1+t) \pmod{s+t-\alpha+1}$.

Let $\mathcal{S}$ be a rigid-type partial geometry with an abelian Singer group G. Since G is sharply transitive, for any nonidentity element, there are no fixed points. Furthermore, there exists an automorphism which maps each point to non-collinear points, so we obtain $f = g = 0$ in Theorem 5.1.1. This implies that $(s+1)(t+1) \equiv 0 \pmod{s+t-\alpha+1}$ for rigid type partial geometries with an abelian Singer group.

Second, we can show that $t+1 = x(s+1)$ for all rigid type partial geometries with an abelian Singer group. Let S be a partial geometry $pg(s,t,\alpha)$ and G an abelian Singer group of S. Suppose that the pair (S, G) is of rigid type. Pick any point p_0 and any line L containing p_0 from S. Assume that p_0, p_1, $\cdots$, p_s are the $s+1$ points on L. Since G acts sharply transitively on S, for any two collinear points p_0 and p_i, there exists a unique $g_i \in G$ such that $p_i^{g_i} = p_0$. Since $|Stab_G(L)| = 1$, $\{L, L^{g_1}, L^{g_2}, \cdots, L^{g_s}\}$ is a set of $s+1$ lines through the point p_0. As there are $t+1$ lines passing through the point p_0, we have $t+1 = x(s+1)$ for some positive integer x.

Combining $(s+1)(t+1) \equiv 0 \pmod{s+t-\alpha+1}$ and $t+1 = x(s+1)$ for rigid type pgs, we have $x(s+1)^2 \equiv 0 \pmod{s+t-\alpha+1}$. Rewriting t in the modulo gives:

$$x(s+1)^2 \equiv 0(\bmod\ (s+1)(x+1)-(\alpha+1)).$$

Multiplying by $(x+1)^2$ and subtracting $x(\alpha+1)^2$:

$$x((x+1)^2(s+1)^2-(\alpha+1)^2) \equiv -x(\alpha+1)^2 \pmod{(x+1)(s+1)-(\alpha+1)}, \text{ so}$$

$$x\Big(\big((x+1)(s+1)-(\alpha+1)\big)\big((x+1)(s+1)+(\alpha+1)\big)\Big) \equiv -x(\alpha+1)^2$$
$$\pmod{(x+1)(s+1)-(\alpha+1)}.$$

But now the left hand side is equivalent to 0, and we have $x(\alpha+1)^2 \equiv 0 \pmod{(x+1)(s+1)-(\alpha+1)}$. Since $x(\alpha+1)^2$ is positive, we obtain $x(\alpha+1)^2 \geq (x+1)(s+$

$1) - (\alpha + 1)$. Subtracting x and using the fact that $\alpha < s$ in any proper partial geometry, we can simplify our inequality to find $s < (\alpha + 1)^2 - 1$. These facts were used to compute the possible parameter sets for $\alpha \leq 8$ by a computer search on the parameters s, x, and α by E. Kamischke in her thesis [11].

5.2 Cases with $\alpha \leq 8$

One of the goals of this project was to expand on the research in Ellen Kamischke's master thesis [11], which worked on rigid type partial geometries. In that thesis, the existence of rigid type partial geometries with an abelian Singer group was explored, and parameter sets were computed using necessary conditions in a program for $\alpha \leq 8$. First, we show nonexistence of all but one undetermined case from her thesis. Table 5.2 shows the details of these remaining cases; throughout this section we refer to this table to label these 12 cases.

5.2.1 Direct nonexistence by Ma's proposition

Henceforth, assume $\mathcal{D}$ is a (v, k, λ, μ)-PDS in an abelian group G, with $|G| = v$. By Proposition 2.2.10, if v is not itself prime, we cannot have any primes to the first power in the decomposition of v (otherwise, the corresponding Sylow-p group would be cyclic of order p). This eliminates cases 3, 6, 7, 10, and 11 immediately, as they contain cyclic Sylow-2, Sylow-2, Sylow-5, Sylow-5, and Sylow-2 subgroups respectively.

Table 5.1: Cases for rigid type partial geometries with an abelian Singer group which pass the necessary conditions for $\alpha \leq 8$, from [11].

Case number	s	t	α	x	v	k	λ	μ
1	5	5	2	1	$81 = 3^4$	30	9	12
2	11	23	3	2	$1024 = 2^{10}$	264	56	72
3	14	14	4	1	$750 = 2 \cdot 3 \cdot 5^3$	210	55	60
4	19	59	4	3	$5625 = 3^2 \cdot 5^4$	1140	195	240
5	29	119	5	4	$20736 = 2^8 \cdot 3^4$	3480	504	600
6	27	27	6	1	$3430 = 2 \cdot 5 \cdot 7^3$	756	161	168
7	34	69	6	2	$13720 = 2^3 \cdot 5 \cdot 7^3$	2380	378	420
8	41	209	6	5	$60025 = 5^2 \cdot 7^4$	8610	1085	1260
9	55	335	7	6	$147456 = 2^{14} \cdot 3^2$	18480	2064	2352
10	44	44	8	1	$10935 = 3^7 \cdot 5$	1980	351	360
11	62	188	8	3	$91854 = 2 \cdot 3^8 \cdot 7$	11718	1377	1512
12	71	503	8	7	$321489 = 3^8 \cdot 7^2$	35784	3591	4032

5.2.2 Application of Leung/Ma/Schmidt corollary

We have an immediately useful result relating $\sqrt{\Delta} = \delta = \sqrt{(\lambda - \mu)^2 + 4(k - \mu)} = s + t - \alpha + 1$ to s:

Proposition 5.2.1 *[13] Suppose $srg(s, t, \alpha)$ is of rigid type, and $s > 2\alpha - 1$. Then, every prime divisor of δ also divides $s + 1$.*

We can show non-existence 4 more cases by applying Proposition 5.2.1 :

- In case 4, we have $s = 19, t = 59, \alpha = 4$, and $\delta = s + t - \alpha + 1 = 75$. But, $3 \mid 75$ and since $s + 1 = 20$ and $3 \nmid 20$, this case can be excluded.

- In case 8, we have $s = 41, t = 209, \alpha = 6$, and $\delta = 245$. But, $5 \mid 245$ and since $s + 1 = 42$ and $5 \nmid 42$, this case can be excluded.

- In case 9, we have $s = 55, t = 335, \alpha = 7$, and $\delta = 384$. But, $3 \mid 384$ and since $s + 1 = 56$ and $3 \nmid 56$, this case can be excluded.

- In case 12, we have $s = 71, t = 503, \alpha = 8$, and $\delta = 567$. But, $7 \mid 567$ and since $s + 1 = 72$ and $7 \nmid 72$, this case can be excluded.

5.2.3 Case 5: A variance method

The methods described above do not eliminate the possibility of case 5 existing, so we use a new method. We know that the variance of any set must be positive, by Theorem 2.2.16. If we can guarantee that any PDS with the given parameters will break the elements of $\mathcal{D}$ into subsets which together have a negative variance of differences of a particular type generated, we can exclude the existence of such a PDS. In this case, we will show that the differences which have order 2^i will be broken up into subsets generating a negative variance if the parameters of the PDS are to hold.

Assume $\mathcal{D}$ is a $(2^8{\cdot}3^4, 3480, 504, 600)$-PDS on an abelian group G, as would be required for the existence of Case 5. We begin analyzing the structure of $\mathcal{D}$ using Proposition 2.2.14 to determine the number of elements of order 2^i in $\mathcal{D}$. Let N be the Sylow-2 subgroup of G; we have $|G|/|N| = 81$ and $|N| = 256$. Since $\delta = 144$, we have $\pi = gcd(144, 256) = 16$. Thus $\Delta_1 = 16^2 = 256$, and since $\beta = -96$, we can obtain $\theta = -3$, as $-7\pi = -112 \leq -96 < -80 = -5\pi$. This gives us $\beta_1 = 0$ and $|\mathcal{D}_1| = 120$ or 136, that is, $\mathcal{D}$ contains either 120 or 136 elements of N.

We now make an observation on how elements $g, h \in G$ can have $o(gh^{-1}) = 2^i$. Denote the elements of order 3 or 9 in G as $g_1, g_2, ..., g_{80}$. Denote the sets:

$$\mathcal{B}_i = \{g_i x \big| (g_i x)^{2^8 \cdot 7} = g_i, g_i x \in \mathcal{D}\}$$

Call $|\mathcal{B}_i| = B_i$. We break into two cases now:

Case 5.1: $|\mathcal{D}_1| = 120$

We know that there are k elements in $\mathcal{D}$, so, there are $k - 120$ elements of order not equal to a power of 2. Thus:

$$\textstyle\sum_{i=1}^{80} B_i = 3480 - 120 = 3360.$$

The only way to get a difference with order a power of 2 is to use differences of two elements of order 2^i or from the same $\mathcal{B}_i$ subset of $\mathcal{D}$. Thus:

$$\textstyle\sum_{i=1}^{80} B_i(B_i - 1) + 120 * 119 = \lambda(120) + \mu(255 - 120),$$

$$\textstyle\sum_{i=1}^{80} B_i(B_i - 1) = 127200$$

Adding these two, we obtain:

$$\textstyle\sum_{i=1}^{80} {B_i}^2 = 130560.$$

However, we observe the variance is negative:

$$80 \textstyle\sum_{i=1}^{80} {B_i}^2 - \left(\sum_{i=1}^{80} B_i\right)^2 = -844800, \text{ a contradiction.}$$

Case 5.2: $|\mathcal{D}_1| = 136$

This is analogous to the previous case:

$$\sum_{i=1}^{80} B_i = 3480 - 136 = 3344.$$
$$\sum_{i=1}^{80} B_i(B_i - 1) + 136 * 135 = \lambda(136) + \mu(255 - 136) = 121584,$$
$$\sum_{i=1}^{80} B_i(B_i - 1) = 121584.$$

Adding these two, we obtain:

$$\sum_{i=1}^{80} {B_i}^2 = 124928.$$

However, we observe the variance is negative:

$$80 \sum_{i=1}^{80} {B_i}^2 - \left(\sum_{i=1}^{80} B_i\right)^2 = -1188096, \text{ a contradiction.}$$

5.2.4 Final conclusions

From van Lint and Schrijver, we know that case 1 does, in fact, exist, see [22]. Thus, we only have case 2, the $pg(11, 23, 3)$ remaining undetermined, whether a pg(11, 23, 3) with an abelian Singer group exists:

Theorem 5.2.2 *If a rigid type partial geometry permits an abelian Singer group with $2 \leq \alpha \leq 8$, then either it is a $pg(5, 5, 2)$, which exists, or it is a $pg(11, 23, 3)$, for which existence is still uncertain.*

It is worth noting that two abelian $(2^{10}, 264, 56, 72) - PDS$ have been constructed by J. Polhill in the groups $\mathbb{Z}_2^{10}$ and $(\mathbb{Z}_2)^6 \times (\mathbb{Z}_4)^2$, see [18], [19]. However, we cannot construct a $pg(11, 23, 3)$ with an abelian Singer group from these PDS because they contain an element of order 2.

Theorem 5.2.3 *Let G be an abelian group and $\mathcal{D}$ be a regular partial difference set in G. Let the Cayley graph of $\mathcal{D}$ be a pseudo-geometric $(s,\ t,\ \alpha)$-graph, and $(s+1)$ doesn't divide $|G|$. If $\mathcal{D}$ contains an element of order 2, then the Cayley graph of $\mathcal{D}$ is not geometric, that is, the Cayley graph of $\mathcal{D}$ is not the point graph of a $pg(s, t, \alpha)$.*

Proof: We will prove it by contradiction. Assume on the contrary that $\mathcal{D}$ contains an element g of order 2, and the Cayley graph is the point graph of a partial geometry S with parameters $(s,\ t,\ \alpha)$. Since $\mathcal{D}$ is a regular partial difference set in G, it follows that G acts sharply transitively on the Cayley graph of $\mathcal{D}$ by mapping d to gd, $g \in G$, and thus G acts sharply transitively on S.

Since $g^2 = e$, as an automorphism, g maps the point g to $g^2 = e$ and maps the point e to g. As there is a unique line L_0 through g and e, as an automorphism, g stabilizes L_0, thus $|Stab_G(L_0)| > 1$. On the other hand, since a partial geometry with an abelian Singer group either has $|Stab_G(L)| = 1$ or $|Stab_G(L)| = s+1$, we have $|Stab_G(L)| = s+1$. Since $Stab_G(L)$ is a subgroup of G, we have $|Stab_G(L)| = s+1$ divides $|G|$, contradicting the assumptions. □

Now, observe the two abelian $(2^{10}, 264, 56, 72) - PDS$ constructed by J. Polhill, and note that they both contain an element of order 2. If the Cayley graph of an abelian $(2^{10}, 264, 56, 72)$ partial difference difference is a pseudo-geometric $(s,\ t,\ \alpha)$-graph, then we have:

$$(s+1)(\frac{st+\alpha}{\alpha}) = 2^{10},\ s(t+1) = 264,\ s-1+t(\alpha-1) = 56,\ \alpha(t+1) = 72.$$

As $s/\alpha = 264/72 = 11/3$, we know that α must be a multiple of 3, and we can verify that $\alpha = 3$ because $t \geq s$. Thus the Cayley graph of a $(2^{10}, 264, 56, 72)$ is a pseudo-geometric $(11, 23, 3)$ graph. In this case, $s+1 = 12$, and 12 doesn't divide 2^{10}, and $\mathcal{D}$ contains elements of order 2, by Theorem 5.2.3 the Cayley graph of $\mathcal{D}$ with parameters $(2^{10}, 264, 56, 72)$ constructed by Polhill is not geometric.

5.3 Expanding to $\alpha \leq 200$

After showing nonexistence for all but one undetermined case in her table (which has shown to be difficult due to it being of prime order), this project set the goal to expand the program to $\alpha \leq 200$. In order to do so, we explored the necessary conditions further to try and reduce them to a more easily computable form with fewer cases to check. We begin from the previously derived necessary condition for a rigid type partial geometry, $(\alpha+1)^2 x \equiv 0 \pmod{(x+1)(s+1)-(\alpha+1)}$, implying for some positive integer c,:

$$(\alpha+1)^2 x = c((x+1)(s+1)-(\alpha+1)).$$

Solving for c and using the condition that $\alpha < s$ for any proper partial geometry, this gives us

$$c = \frac{x(\alpha+1)^2}{(x+1)(s+1)-(\alpha+1)} < \frac{x(s+1)^2}{(x+1)(s+1)-(s+1)} = \frac{x(s+1)^2}{x(s+1)} = s+1$$

.

We can also solve for x using the same expression to find $x = \frac{c(s-\alpha)}{(\alpha+1)^2 - cs - c}$. This allows us to use the following Mathematica code to determine possible parameter sets of rigid type partial geometries (which also satisfy Proposition 5.2.1):

```
For[\[Alpha] = 2, \[Alpha] < 200, \[Alpha]++,
 For[s = \[Alpha] + 1, s < (\[Alpha] + 1)^2 - 1, s++,
  For[c = 1, c < s + 1, c++,
   If[(\[Alpha] + 1)^2 - c*s - c > 0,
    x = (c (s - \[Alpha]))/((\[Alpha] + 1)^2 - c*s - c);
    If[x == Ceiling[x],
     If[Mod[(\[Alpha] + 1)^2*x, (s + 1)*x + s - \[Alpha]] == 0,
      t = (s + 1)*x - 1;
      If[Mod[(s + 1)*(t + 1), s + t - \[Alpha] + 1] == 0,
       v = (s + 1)*(s*t + \[Alpha])/\[Alpha];
       \[Delta] = s + t - \[Alpha] + 1;
       If[Mod[v, \[Delta]] == 0,

        If[Min[Last /@ FactorInteger[v]] > 1 ,
         If[
          s <= 2*\[Alpha] ||
           SubsetQ[First /@ FactorInteger[s + 1],
            First /@ FactorInteger[\[Delta]]],
          k = s*(t + 1);
          \[Lambda] = s - 1 + t*(\[Alpha] - 1);
          \[Mu] = \[Alpha]*(t + 1);

          Print[MatrixForm[{{s, t, \[Alpha], x, v,
              k, \[Lambda], \[Mu]}}]];

          If[PrimeNu[v] == 1,
           Print[v = Superscript @@@ FactorInteger[v]],

           Print[v =
             CenterDot @@ (Superscript @@@
                FactorInteger[v])]]]]]]]]]]]]
Print["done"]
```

This code produces the list of cases shown in Table 5.3.

Table 5.2: Cases for rigid type partial geometries with abelian Singer group which pass the necessary conditions for $\alpha \leq 200$.

Case No.	s	t	α	x	v	k	λ	μ
1	5	5	2	1	3^4	30	9	12
2	11	23	3	2	2^{10}	264	56	72
3	29	119	5	4	$2^8 \cdot 3^4$	3480	504	600
4	89	719	9	8	$2^{10} \cdot 5^4$	64080	5840	6480
5	119	119	14	1	$2^2 \cdot 3^5 \cdot 5^2$	14280	1665	1680
6	39	39	15	1	2^{12}	1560	584	600
7	305	4895	17	16	$2^{12} \cdot 3^8$	1493280	78624	83232
8	1121	35903	33	32	$2^{14} \cdot 17^4$	40248384	1150016	1184832
9	4289	274559	65	64	$2^{16} \cdot 3^4 \cdot 11^4$	1177587840	17576064	17846400
10	839	1679	69	2	$2^4 \cdot 5^5 \cdot 7^3$	1409520	115010	115920
11	254	1274	84	5	$2^2 \cdot 5^2 \cdot 17^3$	323850	105995	107100
12	455	1367	95	3	$2^{12} \cdot 3^6$	622440	128952	129960
13	4752	4752	96	1	$5^2 \cdot 7^2 \cdot 97^3$	22586256	456191	456288
14	272	272	104	1	$3^4 \cdot 7^4$	74256	28287	28392
15	944	2834	104	3	$3^4 \cdot 5^2 \cdot 7^4$	2676240	292845	294840
16	16769	2146559	129	128	$2^{18} \cdot 5^4 \cdot 13^4$	35995664640	274776320	276906240
17	234	2114	140	9	$2^3 \cdot 47^2$	494910	294079	296100
18	3407	6815	141	2	$2^5 \cdot 7^2 \cdot 71^2$	23222112	957506	961056
19	373	747	153	2	$2^9 \cdot 11^2$	279004	113916	114444
20	3023	9071	188	3	$2^2 \cdot 3^8 \cdot 7^5$	27424656	1699299	1705536

5.4 Nonexistence of a family of negative Latin square PDSs and their associated pgs

When this project began with the goal to expand Table 5.2, the program used to generate the Table 5.3 did not use Proposition 5.2.1, and along with the cases that remain on the table, there was a case for each value of *alpha* which imposed a negative Latin square type partial difference set. This trend can be seen for small values of α in Table 5.2 by observing cases 1, 2, 4, 5, 8, 9, and 12. To further observe why these cases existed, the parameters for a negative Latin square PDS with $n = (\alpha+1)^2(\alpha-1)$ and $r = \alpha^2 - 1$ (as observed in the table) were equated to the parameters of the PDS implied by the point graph of the $pg(s, t, \alpha)$ with an abelian Singer group. Our goal is to show that these yield a $pg(\alpha(\alpha + 1) - 1, \alpha(\alpha(\alpha + 1) - 1), \alpha)$:

$$\frac{(s+1)(s*t+\alpha)}{\alpha} = n^2 = (\alpha+1)^4(\alpha-1)^2 \tag{5.1}$$

$$s(t+1) = r(n+1) = (\alpha^2-1)((\alpha+1)^2(\alpha-1)+1) \tag{5.2}$$

$$s-1+t(\alpha-1) = -n+r^2+3r = -((\alpha+1)^2(\alpha-1)_+(\alpha^2-1)^2+3(\alpha^2-1) \tag{5.3}$$

$$\alpha(t+1) = r^2+r = (\alpha^2-1)^2+\alpha^2-1 \tag{5.4}$$

Using Equation 5.4, we can obtain that $t + 1 = \alpha(\alpha^2 - 1)$. Substituting this result in, from Equation 5.2 we obtain that $s+1 = \alpha(\alpha+1)$, as we wanted. Further, notice that for $\alpha > 1$, $s > 2\alpha - 1$, so we can use Proposition 5.2.1. We can compute as well that $\sqrt{\Delta} = \delta = s+t-\alpha+1 = (\alpha+1)^2(\alpha-1) = n$. By Proposition 5.2.1, every prime divisor of $(\alpha - 1)(\alpha + 1)^2$ must also divide $s + 1 = \alpha(\alpha + 1)$. Notice that $2|\alpha(\alpha + 1)$ for all α, but for any larger prime $p|\alpha - 1$, $p \not|\alpha(\alpha + 1)$, so if such a partial geometry with an abelian Singer group exists, $\alpha - 1 = 2^k$ for some nonnegative integer k. We now show that these potential parameter sets cannot allow such a geometry if $k \geq 2$.

Theorem 5.4.1 *Let $\alpha = 2^n + 1$ and $n \geq 2$. The negative Latin square type PDS imposed by $m = (\alpha + 1)^2(\alpha - 1)$ and $r = \alpha^2 + 1$ does not exist on an abelian group, and thus there is no $pg(\alpha(\alpha+1)-1, (\alpha-1)(\alpha)(\alpha+1)-1, \alpha)$ with an abelian Singer group.*

Proof. Assume $\mathcal{D}$ is an $(m^2, r(m+1), -m+r^2+3r, r^2+r)$-PDS on an abelian group G, where $m = (\alpha+1)^2(\alpha-1)$, $r = \alpha^2-1$, $\alpha = 2^n+1$, and $n \geq 2$. We want to use Proposition 2.2.14 to determine $|\mathcal{D}_1|$, the number of elements of the Sylow-2 subgroup in $\mathcal{D}$. Let N be the Sylow-2 subgroup of G. Notice that this gives $|N| = 2^{2n+4}$ and $|G/N| = (2^{n-1}+1)^4$. This also gives us $gcd(|N|, |G/N|) = 1$ so long as $n > 1$. We can compute directly that $\delta = 2^{n+2}(2^{n-1}+1)^2$, $\pi = 2^{n+2}$, $\pi^2 = 2^{2n+4}$, and $\beta = -2^{3n} - 2^{2n+1}$.

To calculate θ, we notice that $(2\theta-1)\pi \leq \beta < (2\theta+1)\pi$ implies that $\theta - \frac{1}{2} \leq -2^{n-2} - 2^{2n-3} < \theta + \frac{1}{2}$. But since the inside of this inequality is an integer, θ must also be that integer. Therefore, we obtain $\theta = -2^{n-2} - 2^{2n-3}$, and we can again calculate $\beta_1 = 0$, and thus $|\mathcal{D}_1| = 2^{2n+3} \pm 2^{n+1}$, that is, there are $2^{2n+3} + 2^{n+1}$ or $2^{2n+3} - 2^{n+1}$ elements of the Sylow-2 subgroup of G contained in $\mathcal{D}$.

Label the nonidentity elements of G/N as $g_1, g_2, ..., g_{(2^{n-1}+1)^4-1}$. Let c be a solution to the congruence $1 \equiv c \cdot 2^{2n+4} \pmod{(2^{n-1}+1)^4}$, (which exists by the Chinese Remainder theorem since $|N|$ and $|G/N|$ are relatively prime), and define:

$$\mathcal{B}_i = \{g_i x \big| (g_i x)^{c \cdot 2^{2n+4}} = g_i, g_i x \in \mathcal{D}\}$$

Call the cardinalities $|\mathcal{B}_i| = B_i$. We want to show that the variance of these cardinalities is negative, and thus that the partial difference set cannot exist. Note that all summations in these cases are across all the B_i, that is they range from $i = 1$ to $i = (2^{n-1}+1)^4 - 1$. We compute the variance in general, then apply our two $|\mathcal{D}_1|$ values.

Since the set $\mathcal{D}$ contains k elements and $|\mathcal{D}_1|$ are from the Sylow-2 subgroup,

$$\sum B_i = k - |\mathcal{D}_1|.$$

Double counting the number of differences of order a power of 2 which occur in the

PDS, we find:

$$\sum B_i(B_i - 1) + (|\mathcal{D}_1|)(|\mathcal{D}_1| - 1) = (|\mathcal{D}_1|) \cdot \lambda + (|N| - 1 - |\mathcal{D}_1|) \cdot \mu.$$

By adding these two expressions and simplifying, we can find:

$$\sum B_i^2 = \lambda(|\mathcal{D}_1|) + \mu(|N| - 1 - |\mathcal{D}_1|) - |\mathcal{D}_1|^2 + k.$$

By Theorem 2.2.16, we know the variance of the B_i is $V(B_i) = (|G/N| - 1)\sum B_i^2 - (\sum B_i)^2$. In each case for the possible values of $\mathcal{D}_1$, we can compute using $|N| = 2^{2n+4}$, $|G/N| = (2^{n-1}+1)^4$, $|D_1| = 2^{2n+3}+2^{n+1}$ or $2^{2n+3}-2^{n+1}$ in the two cases respectively, and using the k, λ, and μ definitions, we compute in the two cases:

$$V(B_j) = -2^{2n+4}(2^n + 2)(7 \cdot 2^{2n} - 3 \cdot 2^{n+1} - 8)(2^{4n} + 15 \cdot 2^{3n} + 2^{2n+5} + 3 \cdot 2^{n+2} - 16);$$

$$V(B_j) = -2^{4n-4}(2^n + 2)(7 \cdot 2^n - 2)(2^{3n} + 15 \cdot 2^{2n} + 9 \cdot 2^{n+2} + 28).$$

We can observe further that the first variance is negative for all values of $n > 0$, and the second is negative for all values of $n > 0.67$. Since the variance is forced to be negative for all $n \geq 2$, we have a contradiction, and this type of PDS (and thus partial geometry) cannot exist. □

The argument shows the nonexistence of an infinite class of cases which originally passed the necessary conditions, and specifically 6 cases from Table 5.3 (cases 3, 4, 7, 8, 9, and 16). Now, we show nonexistence for some of the remaining individual cases.

5.5 Other cases with $\alpha \leq 200$

5.5.1 Case 12: $(2^{12} \cdot 3^6, 622400, 188925, 129960)$-PDS

For this we use the same method as Case 5 in the $\alpha \leq 8$ case.

Assume $\mathcal{D}$ is a $(2^{12} \cdot 3^6, 622440, 128952, 129960)$-PDS on an abelian group G. We begin analyzing the structure of $\mathcal{D}$ using Proposition 2.2.14 to determine the number of elements of order 2^i in $\mathcal{D}$. Let N be the Sylow-2 subgroup of G; we have $|G|/|N| = 729$ and $|N| = 4096$. Since $\delta = 1728$, we have $\pi = gcd(4096, 1728) = 64$. Thus $\Delta_1 = 64^2 = 4096$, and since $\beta = -1008$, we can obtain $\theta = -8$, as $-17\pi = -1088 \leq -1008 < -960 = -15\pi$. This gives us $\beta_1 = 16$ and $|\mathcal{D}_1| = 1512$ or 2600, that is, $\mathcal{D}$ contains either 1512 or 2600 elements of N.

Denote the sets (since $2^{12} * 118 \equiv 1 \pmod{3^6}$):

$$\mathcal{B}_i = \{g_i x \big| (g_i x)^{2^{12} \cdot 118} = g_i, g_i x \in \mathcal{D}\}$$

Call $|\mathcal{B}_i| = B_i$. We break into two cases now:

Case 12.1: $|\mathcal{D}_1| = 1512$

All sums will be over the range from 1 to 728, as there are 728 elements which of order $3^k, k \geq 1$. We know that there are k elements in $\mathcal{D}$, so, there are $k - 1512$ elements of order not equal to a power of 2. Thus:

$\sum B_i = 622400 - 1512 = 620888.$

The only way to get a difference with order a power of 2 is to use differences of two elements of order 2^i or from the same $\mathcal{B}_i$ subset of $\mathcal{D}$. Thus:

$\sum B_i(B_i - 1) + 1512 * 1511 = \lambda(1512) + \mu(4095 - 1512)$, so

$\sum B_i(B_i - 1) = 528377472$

Adding these two, we obtain:

$\sum {B_i}^2 = 528998360.$

We observe a negative variance:

$728 \sum {B_i}^2 - (\sum B_i)^2 = -391102464$, a contradiction by Theorem 2.2.16.

Case 5.2: $|\mathcal{D}_1| = 2600$

Following the same arguments, we see

$\sum B_i = 622400 - 2600 = 619800.$

$\sum B_i(B_i - 1) + 2600 * 2599 = \lambda(2600) + \mu(4095 - 2600)$, so

$\sum B_i(B_i - 1) = 522808000$

Adding these two, we obtain:

$\sum {B_i}^2 = 523427800.$

However, we observe the variance is negative:

$728 \sum {B_i}^2 - (\sum B_i)^2 = -3096601600$, a contradiction. Thus, there is no PDS with this parameter set, and equivalently no $pg(455, 1367, 95)$.

5.5.2 Case 14: $(3^4 \cdot 7^4, 74256, 28287, 28392)$-PDS

For this case, we use Propositions 2.2.12 and 2.2.13. Assume $\mathcal{D}$ is a $(3^4{\cdot}7^4, 74256, 28287, 28392)$-PDS. We find $\nu_2 = 168$, $\nu_3 = -273$, $m_2 = 120224$, and $m_3 = 74256$. Using the note after Proposition 2.2.13 we can obtain that $a_1 = 17408$ and $a_1' = 17030$. Therefore, we have (where s_7 is the number of elements of order a power of 7 in $\mathcal{D}$):

$$120224 + 17408s_7 + 17030(2400 - s_7) = 2401x + 343(120224 - x), \text{ implying}$$

$$s_7 = \frac{49x + 5824}{9}$$

This implies $s_7 \equiv 2 \pmod 9$, but the LMT, Proposition 2.2.9, implies $s_7 \equiv 0 \pmod 6$, a contradiction.

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