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Universal Central Extensions of Direct Limits of Hom-Lie Superalgebras

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UNIVERSAL CENTRAL EXTENSIONS OF DIRECT LIMITS OF

HOM-LIE SUPERALGEBRAS

By

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A THESIS

Submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

In Mathematical Sciences

MICHIGAN TECHNOLOGICAL UNIVERSITY

2019

This thesis has been approved in partial fulfillment of the requirements for the Degree of MASTER OF SCIENCE in Mathematical Sciences.

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Contents

Abstract

It was shown by Neher and Sun in [12] that, for perfect Lie superalgebras L_i, the universal central extension of the direct limit of L_i is isomorphic to the direct limit of the universal central extensions of L_i. In this thesis, we extend the result to Hom-Lie superalgebras, first introduced by Ammar and Makhlouf in [1], and construct the universal central extension of a perfect Hom-Lie superalgebra by defining a uce functor on the category of Hom-Lie superalgebras. In Theorem 4.2.3, we show that if a Hom-Lie superalgebra L is perfect, then $\mathfrak{uce}(L)$ is a universal central extension of L. In Theorem 4.3.2, we show that the universal central extension of the direct limit of perfect Hom-Lie superalgebras L_i is isomorphic to the direct limit of the universal central extensions of L_i.

Chapter 1

Introduction

In [1], Ammar and Makhlouf introduced Hom-Lie superalgebras and constructed several nontrivial examples, including a q-deformed Witt superalgebra and a deformation of the orthosymplectic Lie superalgebra $\mathfrak{osp}(1,2)$. In the same paper, a Hom-Lie superalgebra is defined as a triple $(L, [\cdot, \cdot], \alpha)$ consisting of a supermodule L over a base superring S, a product $[\cdot, \cdot]$ satisfying super skew-symmetry and the super Hom-Jacobi identity, and an even supermodule homomorphism α.

Hom-Lie superalgebras are a generalization of the notions of Hom-Lie algebras and Lie superalgebras. Both are well-studied and find applications in mathematical physics. For example, Lie superalgebras are essential for describing the mathematics of supersymmetry in particle physics [13] and Hom-Lie algebras have been used to generalize

the Yang-Baxter equation [14].

Central extensions of Hom-Lie superalgebras yield new Hom-Lie superalgebras with more interesting representations than the original ones. When possible, determining the universal central extension of a Hom-Lie superalgebra gives "all" of the central extensions. It is well-known that the universal central extension exists for a perfect Lie superalgebra [11]. In this thesis, we show in Theorem 4.2.3 that the universal central extension of a perfect Hom-Lie superalgebra can be constructed by defining a uce functor on the category of Hom-Lie superalgebras.

Taking the direct limit of an indexed family of Hom-Lie superalgebras is a valuable way to create new Hom-Lie superalgebras by means of a universal property. It was shown in [12] that this operation commutes with the operation of taking the universal central extensions of a family of perfect Lie superalgebras. In this thesis, we extend the result to the category of Hom-Lie superalgebras in Theorem 4.3.2.

The structure of this thesis is as follows. In chapter 2, we review some category theoretical definitions and introduce the notion of the direct limit of an indexed family of objects within a category. In chapter 3, we define Hom-Lie superalgebras and provide a non-trivial example [1], constructed from the Lie superalgebra $\mathfrak{osp}(1,2)$. In chapter 4, we give the main results of this thesis. We prove that the universal central extension of the direct limit of perfect Hom-Lie superalgebras L_i is isomorphic to the direct limit of the universal central extensions of L_i.

Chapter 2

Categories and Functors

A *category* $\mathcal{C}$ is a set of objects $\mathrm{ob}(\mathcal{C})$, together with a set $\mathrm{hom}(\mathcal{C})$ of arrows, or morphisms, between the objects (For these classical definitions, see, for example, [10]). We say that an arrow f from a to b for $a, b \in \mathrm{ob}(\mathcal{C})$ has source a, target b, and we write $f : a \to b$. Furthermore, these arrows may be composed. That is, for $a, b, c \in \mathrm{ob}(\mathcal{C})$ and $f \in \mathrm{hom}(b, c)$, $g \in \mathrm{hom}(a, b)$,

$$a \xrightarrow{\ g\ } b \xrightarrow{\ f\ } c$$

there exists a composition map $\mathrm{hom}(b, c) \times \mathrm{hom}(a, b) \to \mathrm{hom}(a, c)$ sending (f, g) to $f \circ g$.

We require that axioms of associativity and the existence of an identity hold in a

category. Equivalently, in a category $\mathcal{C}$, the following properties are satisfied:

1. For every $a \in \mathrm{ob}(\mathcal{C})$, there exists a morphism $\mathrm{id}_a \in \mathrm{hom}(a, a)$ such that $\mathrm{id}_a \circ f = f$ and $g \circ \mathrm{id}_a = g$ for $f \in \mathrm{hom}(b, a)$ and $g \in \mathrm{hom}(a, c)$, where $a, b, c \in \mathrm{ob}(\mathcal{C})$.

2. Composition is associative, i.e., $(f \circ g) \circ h = f \circ (g \circ h)$ for $f \in \mathrm{hom}(c, d)$, $g \in \mathrm{hom}(b, c)$, and $h \in \mathrm{hom}(a, b)$, where $a, b, c, d \in \mathrm{ob}(\mathcal{C})$.

One example of a category is the class of all sets taken together with functions between them as morphisms. In this category, **Set**, the objects are the sets and composition of arrows is performed as the familiar composition of functions. Clearly, arrow composition is associative in this category and we always have the identity arrow.

Similarly, we may also consider the class of finite-dimensional associative algebras over a field F as a category where the arrows are homomorphisms between these F-algebras.

Naturally, we would like to define mappings on categories. Such mappings are called functors. A *covariant functor* $D : \mathcal{A} \to \mathcal{B}$ is a mapping between categories $\mathcal{A}$ and $\mathcal{B}$ that respects the arrows of $\mathcal{A}$. In other words, $D : \mathcal{A} \to \mathcal{B}$ is a map $D : \mathrm{ob}(\mathcal{A}) \to \mathrm{ob}(\mathcal{B})$ and, for every $x, y \in \mathrm{ob}(\mathcal{A})$, a map $D : \mathrm{hom}_{\mathcal{A}}(x, y) \to \mathrm{hom}_{\mathcal{B}}(D(x), D(y))$ exists such that the following axioms are satisfied.

1. D respects the identity morphisms. i.e., $D(\mathrm{id}_x) = \mathrm{id}_{D(x)}$ for $x \in \mathrm{ob}(\mathcal{A})$.

2. D respects composition. i.e., $D(\phi \circ \psi) = D(\phi) \circ D(\psi)$ for $\phi \in \hom_{\mathcal{A}}(y, z), \psi \in$

$\hom_{\mathcal{A}}(x, y)$.

A *contravariant functor* satisfies all of the axioms of a covariant functor, but is "arrow

reversing." That is, a contravariant functor $D : \mathcal{A} \to \mathcal{B}$ is a map $D : \mathrm{ob}(\mathcal{A}) \to \mathrm{ob}(\mathcal{B})$

and for every $x, y \in \mathrm{ob}(\mathcal{A})$, a map $D : \hom_{\mathcal{A}}(x, y) \to \hom_{\mathcal{B}}(D(y), D(x))$ exists such

that D respects composition, i.e., $D(\phi \circ \psi) = D(\psi) \circ D(\phi)$, and the identity morphisms.

Relevant to the topic of this thesis is the concept of the direct limit in a category $\mathcal{C}$.

We start by defining a directed system. Let $(I, \leq)$ be a directed set and (L_i, f_{ij}) be a

set of objects $L_i \in \mathrm{ob}(\mathcal{C})$ indexed by I and f_{ij} a morphism $f_{ij} : L_i \to L_j$ for all $i \leq j$.

Then we say (L_i, f_{ij}) is a *directed system* if

1. f_{ii} is the identity of L_i.

2. $f_{jk} \circ f_{ij} = f_{ik}$ for all $i \leq j \leq k$.

Then a *direct limit* of the directed system (L_i, f_{ij}) is an object L along with morphisms

$\phi_i : L_i \to L$ satisfying $\phi_i = \phi_j \circ f_{ij}$. We additionally require the existence of a unique

morphism $\phi : L \to L'$ for any other pair (L', ϕ_i') satisfying $\phi_i' = \phi_j' \circ f_{ij}$, such that the

following diagram commutes.

$$
\begin{array}{ccc}
L_i & \xrightarrow{\quad f_{ij} \quad} & L_j
\end{array}
$$
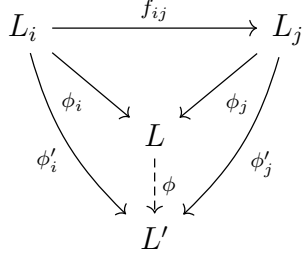
We denote a direct limit of (L_i, f_{ij}) by $\varinjlim L_i$. The ϕ_i are called the *canonical maps*.

As noted in [10], a direct limit in a category C may be considered as a functor. A direct limit is a special case of a "colimit," a more general structure defined on categories as a special functor $D : \mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is an "index category" (see [10], page 67).

By the universal property, a direct limit of a directed system, if it exists, is unique up to a unique isomorphism. The direct limit of the directed system (L_i, f_{ij}) can be constructed as the disjoint union of the L_i modulo an equivalence relation:

$$
\varinjlim L_i = \bigsqcup_i L_i / \sim .
$$

The equivalence relation $\sim$ is given as follows. If $x_i \in L_i$ and $x_j \in L_j$, then $x_i \sim x_j$ if and only if there exists $k \in I$, with $i \leq k$ and $j \leq k$ such that $f_{ik}(x_i) = f_{jk}(x_j)$. That is, $x_i \sim x_j$ in $\varinjlim L_i$ if they "eventually become equal" in the directed system.

It is important to note that there may exist a certain relationship between directed systems. If (K_i, g_{ij}) and (L_i, f_{ij}) are two directed systems in $\mathcal{C}$, both indexed by the

directed set I, then a *morphism* from (K_i, g_{ij}) to (L_i, f_{ij}) is an indexed set $\{h_i : i \in I\}$ of morphisms $h_i : K_i \to L_i$ such that for all pairs (i, j) with $i \leq j$ the following diagram commutes.

$$
\begin{array}{ccc}
K_i & \xrightarrow{\ g_{ij}\ } & K_j \\
\downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle h_j} \\
L_i & \xrightarrow{\ f_{ij}\ } & L_j
\end{array}
$$

A morphism from (K_i, g_{ij}) to (L_i, f_{ij}) gives rise to a unique morphism

$$
h = \varinjlim h_i : \varinjlim K_i \to \varinjlim L_i
$$

such that $h \circ \phi_i = \psi_i \circ h_i$ for all $i \in I$ where $\phi_i : K_i \to \varinjlim K_i$ and $\psi_i : L_i \to \varinjlim L_i$ are the canonical maps.

Example 2.1.1. Consider the group $\mathrm{GL}(n, F)$ consisting of invertible $n \times n$ matrices over a field F with the operation given by matrix multiplication. Note that we have a group homomorphism $\mathrm{GL}(n, F) \to \mathrm{GL}(n + 1, F)$ given by padding matrices in $\mathrm{GL}(n, F)$ with zeros along the last row and column and placing a 1 in the bottom right corner. Thus, inductively, we have group homomorphisms $\phi_{m,n} : \mathrm{GL}(m, F) \to \mathrm{GL}(n, F)$ for $m \leq n$. The direct limit of the directed system $(\mathrm{GL}(m, F), \phi_{m,n})$ is the general linear group of F, denoted by $\mathrm{GL}(F)$, consisting of the set of invertible infinite matrices differing from the infinite identity matrix in only finitely many places.

Chapter 3

Hom-Lie Superalgebras

3.1   Lie Algebras

An *algebra* A over a field F is a vector space over F equipped with a bilinear oper-

ation $A \times A \to A$, denoted here by $\cdot$, such that the operation obeys left and right

distributivity and is compatible with scalar multiplication. For $a, b, c \in A; \alpha, \beta \in F$,

we have

1. $(\alpha a + \beta b) \cdot c = \alpha(a \cdot c) + \beta(b \cdot c)$;

2. $c \cdot (\alpha a + \beta b) = \alpha(c \cdot a) + \beta(c \cdot b)$.

This operation is usually called the *product* or *multiplication* on A.

Note that we are guaranteed almost nothing with respect to the product on A - we are told nothing about the commutativity, existence of a multipliative identity, or even associativity of the multiplication. For example, we may consider the algebra formed by the vector space $\mathbb{R}^3$ with multiplication given by the cross product. This structure clearly satisfies all of the conditions necessary to be an algebra, but has no multiplicative identity, is not associative or commutative, and does not even satisfy the cancellation law. We generalize this example to give the definition of a Lie algebra. These are well-studied structures and the following definitions can be found in many resources. For example, see [7].

An algebra L over a field F with bilinear product $[\cdot, \cdot] : L \times L \to L$, is called a *Lie algebra* over F if the following properties hold:

1. (*Skew Symmetry*) $[x, x] = 0$ for all $x \in L$;

2. (*Jacobi Identity*) $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$ for all $x, y, z \in L$.

If L satisfies the definition of a Lie algebra, then $[\cdot, \cdot]$ is called the *Lie bracket* of L. It is important to note that the Lie bracket is anticommutative, that is $[x, y] = -[y, x]$ for $x, y \in L$.

As an example, consider a finite dimensional vector space V over a field F and denote

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by $\operatorname{End} V$ the set of linear transformations from V to V. As a vector space over F, $\operatorname{End} V$ has dimension n^2 where $n = \dim V$ and, furthermore, $\operatorname{End} V$ is a ring relative to function composition and pointwise addition. Define a new operation $[x, y] = xy - yx$ for $x, y \in \operatorname{End} V$ called the *bracket* of x and y. With this operation, $\operatorname{End} V$ becomes a Lie algebra over F, denoted by $\mathfrak{gl}(V)$.

A *Lie algebra homomorphism* between Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ such that $\phi([x, y]_{\mathfrak{g}}) = [\phi(x), \phi(y)]_{\mathfrak{h}}$ for $x, y \in \mathfrak{g}$. We say that $\mathfrak{g}$ and $\mathfrak{h}$ are *isomorphic* if ϕ is a bijection. Finally, a subspace $\mathfrak{f}$ of $\mathfrak{g}$ is a *Lie subalgebra* if it is closed under the Lie bracket on $\mathfrak{g}$.

Example 3.1.1. Consider the *special linear Lie algebra* of rank $n - 1$, denoted by $\mathfrak{sl}_n(F)$. This Lie algebra is composed of all $n \times n$ matrices over a field F with trace zero and with the Lie bracket of $x, y \in \mathfrak{sl}_n(F)$ given by the commutator of x and y. That is, $[x, y] = xy - yx$.

Then, we may define a map $\phi : \mathfrak{sl}_n(F) \to \mathfrak{sl}_{n+1}(F)$ by sending a matrix x in $\mathfrak{sl}_n(F)$ to the matrix in $\mathfrak{sl}_{n+1}(F)$ formed by padding $x = (a_{i,j})$ with zeros on the bottom and

rightmost side. Explicitly,

$$\phi : \mathfrak{sl}_n(F) \to \mathfrak{sl}_{n+1}(F)$$

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \mapsto \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} & 0 \\ \vdots & \ddots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

Clearly, ϕ is a Lie algebra homomorphism and $\operatorname{Im}\phi$ is a Lie subalgebra of $\mathfrak{sl}_{n+1}(F)$.

As a final note in this section, we will also eventually use generalizations of the *derived algebra* of $\mathfrak{g}$, denoted by $[\mathfrak{g}, \mathfrak{g}]$ and defined as the ideal of $\mathfrak{g}$ generated by all elements $[a, b]$ with $a, b \in \mathfrak{g}$. If $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, we call $\mathfrak{g}$ a *perfect* Lie algebra.

3.2   Lie Superalgebras

It was discovered in the 1970s that certain supersymmetry properties held in quantum fields and that these properties could be well described using $\mathbb{Z}_2$-graded Lie algebras. In this section, we work towards a formulation of these $\mathbb{Z}_2$-graded Lie algebras, called Lie superalgebras and first investigated formally by V. Kac in 1977 [8].

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We start by recalling some definitions. For reference, see [11]. A ring S is called $\mathbb{Z}_2$-*graded* if $S = S_{\bar{0}} \oplus S_{\bar{1}}$ as an abelian group and $S_{\bar{\alpha}} S_{\bar{\beta}} \subset S_{\bar{\alpha}+\bar{\beta}}$ for $\bar{\alpha}, \bar{\beta} \in \mathbb{Z}_2$. The elements of S_i, where $i = \{\bar{0}, \bar{1}\}$, are called *homogeneous* and of parity i. We denote the parity of a such a homogeneous element $x \in S_i$ by $|x|$.

If S has a multiplicative identity, we call S *unital*. If S is unital, then $1 \in S_{\bar{0}}$ since $1 \cdot 1 = 1$ and $S_{\bar{1}} S_{\bar{1}} \subset S_{\bar{0}}$. Furthermore, we say that S is *commutative* if

$$st = (-1)^{|s||t|} ts \quad \text{for } s, t \in S \text{ and} \quad s_{\bar{1}}^2 = 0 \quad \text{for } s_{\bar{1}} \in S_{\bar{1}}.$$

If S is unital, associative, and commutative, S is a *base superring*.

Furthermore, we define an S-*supermodule* to be a left module M over a base superring S that is $\mathbb{Z}_2$-graded ($M = M_{\bar{0}} \oplus M_{\bar{1}}$) as an abelian group and $S_{\bar{\alpha}} M_{\bar{\beta}} \subset M_{\bar{\alpha}+\bar{\beta}}$ for $\bar{\alpha}, \bar{\beta} \in \mathbb{Z}_2$. Note that we can also treat M as an S-bimodule, defining the right action as $ms = (-1)^{|s||m|} sm$ for $s \in S$ and $m \in M$.

We can also define homomorphisms between supermodules. That is, if we let M and N be two S-supermodules, and $\bar{\alpha} \in \mathbb{Z}_2$, a *homomorphism of degree $\bar{\alpha}$* from M to N is a map $f : M \to N$ satisfying the following axioms:

1. $f(M_{\bar{\beta}}) \subset N_{\bar{\alpha}+\bar{\beta}}$ for all $\bar{\beta} \in \mathbb{Z}_2$;

2. f is additive and $f(ms) = f(m)s$ for $m \in M$ and $s \in S$.

We say that a homomorphism f is *even* if f is of degree $\bar{0}$ and *odd* if f is of degree $\bar{1}$.

If M, N, and P are S-supermodules, an *S-bilinear map of degree* $\bar{\gamma}$ is a map $b :$ $M \times N \to P$ satisfying

1. $b(M_{\bar{\alpha}}, N_{\bar{\beta}}) \subset P_{\bar{\alpha}+\bar{\beta}+\bar{\gamma}}$ for all $\bar{\alpha}, \bar{\beta} \in \mathbb{Z}_2$;

2. b is additive in each argument;

3. $b(ms, n) = b(m, sn)$ and $b(m, ns) = b(m, n)s$ for all $m \in M, n \in N$, and $s \in S$.

An *S-superalgebra* is an S-supermodule $A = A_{\bar{0}} \oplus A_{\bar{1}}$ together with some S-bilinear map $A \times A \to A$, of degree $\bar{0}$, called the *product*.

Assume $\frac{1}{2} \in S$. An S-superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ with product $[\cdot, \cdot]$ is a *Lie S-superalgebra* $(L, [\cdot, \cdot])$ if, for all $x, y, z \in L$, L satisfies the following properties:

1. *Super skew-symmetry*:
$$[x, y] = -(-1)^{|x||y|}[y, x];$$

2. *Super Jacobi Identity*:

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|z||y|}[z, [x, y]] + (-1)^{|y||x|}[y, [z, x]] = 0.$$

We define homomorphisms between Lie S-superalgebras $(L, [\cdot, \cdot]_L$ and $(M, [\cdot, \cdot]_M)$ as

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bracket-respecting even homomorphisms between L and M as supermodules. That is, an even supermodule homomorphsim $\phi : L \to M$ is a *Lie superalgebra homomorphism* if $\phi([x, y]_L) = [\phi(x), \phi(y)]_M$ for $x, y \in L$.

Remark 3.2.1. Note that if, for $L = L_{\bar{0}} \oplus L_{\bar{1}}$, $L_{\bar{1}} = \emptyset$, L is an ordinary Lie algebra. However, there are many examples of Lie superalgebras that are not Lie algebras. For instance, we may form a Lie superalgebra from an associative superalgebra A by defining a bracket on A as $[x, y] = xy - (-1)^{|x||y|}yx$ for $x, y \in A$. Then it can be shown that $(A, [\cdot, \cdot])$ is a Lie superalgebra. Let $M(m, n; A)$ be the $(m + n) \times (m + n)$ matrix superalgebra with coefficients in an associative superalgebra A. Then $\mathfrak{gl}(m, n; A)$ is the Lie superalgebra associated to $M(m, n; A)$. For $m + n \geq 3$, the subsuperalgebra of $\mathfrak{gl}(m, n; A)$ generated by $E_{ij}(a)$, $1 \leq i \neq j \leq m + n$, $a \in A$, is called the *special linear Lie superalgebra* $\mathfrak{sl}(m, n; A)$ (see [5]).

3.3   Hom-Lie Superalgebras

First appearing in [1], a Hom-Lie superalgebra is a Lie superalgebra "with a twist." We extend the definition of a Hom-Lie superalgebra over a field of characteristic 0 in [1] to a base superring. Let L be an S-supermodule over a base superring S containing $\frac{1}{2}$. Then a *Hom-Lie superalgebra* is a triple $(L, [\cdot, \cdot], \alpha)$ consisting of a supermodule L, a product $[\cdot, \cdot]$, and an even supermodule homomorphism $\alpha : L \to L$ satisfying

1. Super Skew-Symmetry:

$$[x, y] = -(-1)^{|x||y|}[y, x];$$

2. Super Hom-Jacobi Identity:

$$(-1)^{|x||z|}[\alpha(x), [y, z]] + (-1)^{|z||y|}[\alpha(z), [x, y]] + (-1)^{|y||x|}[\alpha(y), [z, x]] = 0,$$

for all homogeneous elements $x, y, z \in L$.

Let $(L, [\cdot, \cdot], \alpha)$ and $(L', [\cdot, \cdot]', \alpha')$ be two Hom-Lie superalgebras. An even supermodule homomorphism $f : L \to L'$ is said to be a *homomorphism of Hom-Lie superalgebras* if

$$[f(x), f(y)]' = f([x, y]) \quad \forall x, y \in L;$$

$$f \circ \alpha = \alpha' \circ f.$$

Note that every Lie superalgebra is automatically a Hom-Lie superalgebra by defining the map α as the identity. However, non-trivial examples of Hom-Lie superalgebras exist. One such example, appearing in [1], may be constructed from the orthosymplectic Lie superalgebra $\mathfrak{osp}(1, 2)$.

Example 3.3.1. Let $\mathfrak{osp}(1, 2) = L_{\bar{0}} \oplus L_{\bar{1}}$ be the orthosymplectic Lie superalgebra

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over the base field $\mathbb{R}$, where $L_{\bar{0}}$ is generated by

$$
H = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad
X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad
Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}
$$

and $L_{\bar{1}}$ is generated by

$$
F = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad
G = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Furthermore, we impose the following relations.

$$
[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H,
$$

$$
[Y, G] = F, \quad [X, F] = G, \quad [H, F] = -F, \quad [H, G] = G,
$$

$$
[G, F] = H, \quad [G, G] = -2X, \quad [F, F] = 2Y.
$$

Now, for $\lambda \in \mathbb{R}^*$, we define $\alpha_\lambda : \mathfrak{osp}(1,2) \to \mathfrak{osp}(1,2)$ as the following linear mapping.

$$
\alpha_\lambda(X) = \lambda^2 X, \quad \alpha_\lambda(Y) = \frac{1}{\lambda^2} Y, \quad \alpha_\lambda(H) = H, \quad \alpha_\lambda(F) = \frac{1}{\lambda} F, \quad \alpha_\lambda(G) = \lambda G.
$$

Then, $(\mathfrak{osp}(1,2), [\cdot,\cdot]_{\alpha_\lambda}, \alpha_\lambda)$ is a Hom-Lie superalgebra where the Hom-Lie superalgebra bracket $[\cdot,\cdot]_{\alpha_\lambda}$ is defined on the basis elements, for $\lambda \neq 0$ by

$$[H,X]_{\alpha_\lambda} = 2\lambda^2 X, \quad [H,Y]_{\alpha_\lambda} = -\frac{2}{\lambda^2}Y, \quad [X,Y]_{\alpha_\lambda} = H,$$

$$[Y,G]_{\alpha_\lambda} = \frac{1}{\lambda}F, \quad [X,F]_{\alpha_\lambda} = \lambda G, \quad [H,F]_{\alpha_\lambda} = -\frac{1}{\lambda}F, \quad [H,G]_{\alpha_\lambda} = \lambda G,$$

$$[G,F]_{\alpha_\lambda} = H, \quad [G,G]_{\alpha_\lambda} = -2\lambda^2 X, \quad [F,F]_{\alpha_\lambda} = \frac{2}{\lambda^2}Y.$$

As long as $\lambda \neq 1$, these Hom-Lie superalgebras are not Lie superalgebras.

Chapter 4

Central Extensions

4.1   Generalities of Central Extensions

The concept of central extensions in the category of Lie superalgebras is defined in [11]. An *extension* of a Lie superalgebra L is a short exact sequence of Lie superalgebras

$$0 \longrightarrow J \xrightarrow{g} K \xrightarrow{f} L \longrightarrow 0,$$

i.e., K and J are Lie superalgebras and f and g are Lie superalgebra homomorphisms such that $\operatorname{Im} g = \operatorname{Ker} f$. We may say that K is an extension of L by J.

Recall that the *center* of a ring is the set of elements that commute with all other

elements of the ring with respect to the product. In a Lie superalgebra L, the center of L is exactly the set of elements $x \in L$ such that $[x, y] = 0$ for all $y \in L$. If, in the above extension, it happens to be the case that the kernel of f is contained in the center of K, denoted $\mathfrak{z}(K)$, we say that the extension is a *central extension*.

Example 4.1.1. Consider the Heisenberg Lie algebra [3]. Let $A = \mathbb{R}[t_1, ..., t_n]$ be the polynomial ring over $\mathbb{R}$. Define l_i $(1 \leq i \leq n)$, δ_i $(1 \leq i \leq n)$, $1 \in \text{End}(A)$ as follows:

$$l_i : f \mapsto ft_i$$

$$\delta_i : f \mapsto \frac{\partial f}{\partial t_i}$$

$$1 : f \mapsto f.$$

These span the $(1 + 2n)$-dimensional subspace $\mathcal{H} \subset \text{End}(A)$. $\mathcal{H}$ is closed under the Lie bracket of $\mathfrak{gl}(A)$. All commutators are trivial except for $[\delta_i, l_i] = 1$. The center of $\mathcal{H}$ is $\mathbb{R}1$. The Lie algebra $\mathcal{H}$ is called the Heisenberg Lie algebra. It is a central extension of the trivial (commutative) Lie algebra $\mathbb{R}^{2n}$ by $\mathbb{R}$, i.e., we have a short exact sequence of Lie algebras

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{H} \longrightarrow \mathbb{R}^{2n} \longrightarrow 0,$$

and $\mathbb{R}$ is the center of $\mathcal{H}$.

Furthermore, a central extension $0 \to J \xrightarrow{g} K \xrightarrow{f} L \to 0$ is said to be a *universal*

central extension if for every other central extension $0 \to J' \xrightarrow{g'} K' \xrightarrow{f'} L \to 0$ there exists a unique homomorphism of Lie superalgebras $\pi : K \to K'$ and a unique homomorphism of Lie superalgebras $\tau : J \to J'$ such that $f' \circ \pi = f$ and $\pi \circ g = g' \circ \tau$. Or, equivalently, the following diagram commutes.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J & \xrightarrow{g} & K & \xrightarrow{f} & L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\
0 & \longrightarrow & J' & \xrightarrow{g'} & K' & \xrightarrow{f'} & L & \longrightarrow & 0
\end{array}
$$

We may say that K is a universal central extension of L by J. Note that two universal central extensions of L are isomorphic as Lie superalgebras by the universal property.

It was shown in [11] that the universal central extension of a Lie superalgebra L exists if and only if it is perfect, i.e., $L = [L, L]$. Additionally, in the same paper, it was shown that, given a Lie superalgebra L over a base superring S containing $\frac{1}{2}$, we may form the S-supermodule

$$\mathfrak{uce}(L) = (L \otimes L)/\mathcal{B}$$

where $\mathcal{B}$ is the S-submodule of $L \otimes L$ spanned by all elements

$$x \otimes y + (-1)^{|x||y|} y \otimes x,$$

$$(-1)^{|x||z|} x \otimes [y, z] + (-1)^{|y||x|} y \otimes [z, x] + (-1)^{|z||y|} z \otimes [x, y].$$

Denote $x \otimes y + \mathcal{B}$ in $\mathfrak{uce}(L)$ by $\langle x, y \rangle$. The S-supermodule $\mathfrak{uce}(L)$ satisfies all of the

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conditions of a Lie superalgebra with respect to the product

$$[\langle l_1, l_2 \rangle, \langle l_3, l_4 \rangle] = \langle [l_1, l_2], [l_3, l_4] \rangle \qquad \text{for } l_i \in L,$$

and, furthermore, the map

$$\mathfrak{u} : \mathfrak{uce}(L) \to L$$

$$\langle x, y \rangle \mapsto [x, y]$$

is a universal central extension of L.

As an example, we may consider the special linear Lie superalgebra $\mathfrak{sl}(m, n; A)$. This Lie superalgebra is perfect and has the universal central extension $\mathfrak{st}(m, n; A)$, for $m + n \geq 5$, the corresponding *Steinberg Lie superalgebra* (see [5]).

Central extensions in the category of Hom-Lie superalgebras are defined in [2]. A *central extension* of a Hom-Lie superalgebra $(K, [\cdot, \cdot]_K, \alpha_K)$ is a short exact sequence of Hom-Lie superalgebras

$$0 \longrightarrow (J, [\cdot, \cdot]_J, \alpha_J) \xrightarrow{g} (K, [\cdot, \cdot]_K, \alpha_K) \xrightarrow{f} (L, [\cdot, \cdot]_L, \alpha_L) \longrightarrow 0$$

such that $\mathrm{Ker}(f)$ is contained in the center of K, i.e., $[K, g(J)]_K = 0$. The universal central extensions of Hom-Lie superalgebras are defined similarly as in the case of Lie

superalgebras.

4.2 The functor uce for Hom-Lie superalgebras

Universal central extensions of Lie superalgebras and Hom-Lie algebras are constructed in [11] and [4]. In this section, we will construct universal central extensions of Hom-Lie superalgebras.

Let S be a base superring containing $\frac{1}{2}$. Let L be a Hom-Lie Superalgebra over S with an even supermodule homomorphism α (i.e., a homomorphism that respects the $\mathbb{Z}_2$-grading). Consider the supermodule $L \otimes L$ over S. Now, define $\mathcal{B}$ to be the S-submodule of $L \otimes_S L$ spanned by elements

$$x \otimes y + (-1)^{|x||y|} y \otimes x,$$

$$(-1)^{|x||z|}\alpha(x) \otimes [y, z] + (-1)^{|z||y|}\alpha(z) \otimes [x, y] + (-1)^{|y||x|}\alpha(y) \otimes [z, x],$$

where $x, y, z \in L$. Define the supermodule

$$\mathrm{uce}(L) = (L \otimes_S L)/\mathcal{B}$$

and let

$$\langle x, y \rangle = x \otimes y + \mathcal{B} \in \mathrm{uce}(L).$$

We claim that the supermodule $\mathfrak{uce}(L)$ is a Hom-Lie superalgebra over S.

Proposition 4.2.1. Let S be a base superring containing $\frac{1}{2}$. Let L be a Hom-Lie superalgebra over S with an even homomorphism α. Then the supermodule $\mathfrak{uce}(L)$ is a Hom-Lie superalgebra over S with respect to the product

$$[\cdot, \cdot]_{\mathfrak{uce}(L)} : \mathfrak{uce}(L) \times \mathfrak{uce}(L) \to \mathfrak{uce}(L)$$

$$(\langle v_1, v_2 \rangle, \langle v_3, v_4 \rangle) \mapsto \langle [v_1, v_2], [v_3, v_4] \rangle,$$

and the even supermodule homomorphism

$$\overline{\alpha} : \mathfrak{uce}(L) \to \mathfrak{uce}(L),$$

$$\langle v_1, v_2 \rangle \mapsto \langle \alpha(v_1), \alpha(v_2) \rangle,$$

where $v_i \in L$ and $[\cdot, \cdot]$ is the product in L.

Proof.

We need to check that $\mathfrak{uce}(L)$ satisfies the super skew-symmetry and super Hom-Jacobi identity that guarantee that $\mathfrak{uce}(L)$ is a Hom-Lie superalgebra.

First, we want to show that $[x, y]_{\mathfrak{uce}(L)} + (-1)^{|x||y|}[y, x]_{\mathfrak{uce}(L)} = 0$ where $x, y \in \mathfrak{uce}(L)$. Let $x = \langle v_1, v_2 \rangle$ and $y = \langle w_1, w_2 \rangle$ for some $v_i, w_i \in L$. For brevity, we set $\overline{L} = \mathfrak{uce}(L)$.

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Note that $|\langle v_1, v_2 \rangle| = |[v_1, v_2]|$. Then

$$[x, y]_{\overline{L}} + (-1)^{|x||y|}[y, x]_{\overline{L}} = [\langle v_1, v_2 \rangle, \langle w_1, w_2 \rangle]_{\overline{L}} + (-1)^{|\langle v_1, v_2 \rangle||\langle w_1, w_2 \rangle|}[\langle w_1, w_2 \rangle, \langle v_1, v_2 \rangle]_{\overline{L}}$$

$$= \langle [v_1, v_2], [w_1, w_2] \rangle + (-1)^{|[v_1, v_2]||[w_1, w_2]|}\langle [w_1, w_2], [v_1, v_2] \rangle$$

$$= [v_1, v_2] \otimes [w_1, w_2] + \mathcal{B} + (-1)^{|[v_1, v_2]||[w_1, w_2]|}[w_1, w_2] \otimes [v_1, v_2] + \mathcal{B}$$

$$= 0 \quad \text{in } \mathfrak{uce}(L).$$

Second, we want to show that

$$(-1)^{|x||z|}[\overline{\alpha}(x), [y, z]_{\overline{L}}]_{\overline{L}} + (-1)^{|z||y|}[\overline{\alpha}(z), [x, y]_{\overline{L}}]_{\overline{L}} + (-1)^{|y||x|}[\overline{\alpha}(y), [z, x]_{\overline{L}}]_{\overline{L}} = 0$$

for $x, y, z \in \mathfrak{uce}(L)$. Let $x = \langle a_1, a_2 \rangle, y = \langle b_1, b_2 \rangle, z = \langle c_1, c_2 \rangle$ for some $a_i, b_i, c_i \in L$.

Then

$$(-1)^{|x||z|}[\overline{\alpha}(x), [y, z]_{\overline{L}}]_{\overline{L}} = (-1)^{|\langle a_1, a_2 \rangle||\langle c_1, c_2 \rangle|}[\overline{\alpha}(\langle a_1, a_2 \rangle), [\langle b_1, b_2 \rangle, \langle c_1, c_2 \rangle]_{\overline{L}}]_{\overline{L}}$$

$$= (-1)^{|[a_1, a_2]||[c_1, c_2]|}[\langle \alpha(a_1), \alpha(a_2) \rangle, \langle [b_1, b_2], [c_1, c_2] \rangle]_{\overline{L}}$$

$$= (-1)^{|[a_1, a_2]||[c_1, c_2]|}\langle [\alpha(a_1), \alpha(a_2)], [[b_1, b_2], [c_1, c_2]] \rangle$$

$$= (-1)^{|[a_1, a_2]||[c_1, c_2]|}[\alpha(a_1), \alpha(a_2)] \otimes [[b_1, b_2], [c_1, c_2]] + \mathcal{B}$$

$$= (-1)^{|[a_1, a_2]||[c_1, c_2]|}\alpha([a_1, a_2]) \otimes [[b_1, b_2], [c_1, c_2]] + \mathcal{B}$$

Hence,

$$(-1)^{|x||z|}[[\overline{\alpha}(x), [y,z]_{\overline{L}}]_{\overline{L}} + (-1)^{|z||y|}[[\overline{\alpha}(z), [x,y]_{\overline{L}}]_{\overline{L}} + (-1)^{|y||x|}[[\overline{\alpha}(y), [z,x]_{\overline{L}}]_{\overline{L}}$$

$$=\circlearrowleft_{[a_1,a_2],[b_1,b_2],[c_1,c_2]} (-1)^{|[a_1,a_2]||[c_1,c_2]|}\alpha([a_1,a_2]) \otimes [[b_1,b_2],[c_1,c_2]] + \mathcal{B}$$

$$= 0 \quad \text{in } \mathfrak{uce}(L).$$

∎

Next, we will show that the construction $\mathfrak{uce}(L)$ gives a central extension of L.

Proposition 4.2.2. Assume a Hom-Lie superalgebra L is perfect. Then the even S-linear map

$$\mathfrak{u} : \mathfrak{uce}(L) \to L$$

$$\langle x, y \rangle \mapsto [x, y]$$

is a surjective Hom-Lie superalgebra homomorphism with kernel $\operatorname{Ker}\mathfrak{u} \subset \mathfrak{z}(\mathfrak{uce}(L))$. Equivalently, $\mathfrak{uce}(L)$ is a central extension of L.

Proof. First, we want to show that $\mathfrak{u}$ is, in fact, a Hom-Lie superalgebra homomorphism. In order to do so, we need to check two conditions.

1. First, we want to show $[\mathfrak{u}(x), \mathfrak{u}(y)] = \mathfrak{u}([x,y])$ where $x, y \in \mathfrak{uce}(L)$. Let $x =$

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$\langle x_1, x_2 \rangle$, $y = \langle y_1, y_2 \rangle$ for some $x_i, y_i \in L$. Then

$$[\mathfrak{u}(x), \mathfrak{u}(y)] = [\mathfrak{u}(\langle x_1, x_2 \rangle), \mathfrak{u}(\langle y_1, y_2 \rangle)]$$

$$= [[x_1, x_2], [y_1, y_2]]$$

$$= \mathfrak{u}(\langle [x_1, x_2], [y_1, y_2] \rangle)$$

$$= \mathfrak{u}([\langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle]_{\overline{L}})$$

$$= \mathfrak{u}([x, y]_{\overline{L}})$$

2. Secondly, we want to show $\alpha(\mathfrak{u}(x)) = \mathfrak{u}(\overline{\alpha}(x))$ for any $x = \langle x_1, x_2 \rangle \in \mathfrak{uce}(L)$. In

fact,

$$\alpha(\mathfrak{u}(\langle x_1, x_2 \rangle)) = \alpha([x_1, x_2])$$

$$= [\alpha(x_1), \alpha(x_2)]$$

$$= \mathfrak{u}(\langle \alpha(x_1), \alpha(x_2) \rangle)$$

$$= \mathfrak{u}(\overline{\alpha}(\langle x_1, x_2 \rangle)).$$

Next, we need to check that $\mathfrak{u}$ is a central extension of L. Let $\gamma = \langle \gamma_1, \gamma_2 \rangle \in \operatorname{Ker} \mathfrak{u}$

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and $l = \langle l_1, l_2 \rangle \in \mathfrak{uce}(L)$ for $\gamma_1, \gamma_2, l_1, l_2 \in L$. We want to show that $[\gamma, l]_{\overline{L}} = 0$.

$$[\gamma, l]_{\overline{L}} = [\langle \gamma_1, \gamma_2 \rangle, \langle l_1, l_2 \rangle]_{\overline{L}}$$

$$= \langle [\gamma_1, \gamma_2], [l_1, l_2] \rangle$$

$$= \langle \mathfrak{u}(\gamma), \mathfrak{u}(l) \rangle$$

$$= \langle 0, [l_1, l_2] \rangle$$

$$= [0, \langle l_1, l_2 \rangle]_{\overline{L}}$$

$$= 0.$$

Finally, $\operatorname{Im} \mathfrak{u} = [L, L]$. Because L is perfect, $\operatorname{Im} \mathfrak{u} = L$. Hence, $\mathfrak{u}$ is surjective. Therefore, $\mathfrak{uce}(L)$ is a central extension of L. ∎

The goal of the rest of this section is to show that if L is perfect, then $\mathfrak{uce}(L)$ gives a model for the universal central extension of L and, furthermore, $\mathfrak{uce}$ defines a covariant functor on the category of perfect Hom-Lie superalgebras. A similar result was proven in [4] for hom-Lie algebras and in [11] for Lie superalgebras. We will be following their methods to show the result for Hom-Lie superalgebras.

Theorem 4.2.3. Assume (L, α_L) is a perfect Hom-Lie superalgebra. Then $(\mathfrak{uce}(L), \overline{\alpha})$ is a universal central extension of (L, α_L).

Proof.

Assume L is perfect and suppose there exists another central extension

$$0 \to (M, \alpha_M) \xrightarrow{i} (K, \alpha_K) \xrightarrow{\pi} (L, \alpha_L) \to 0$$

In order to show that $\mathfrak{uce}(L)$ is a universal central extension of L, we must show that there exists a unique pair of homomorphisms (ϕ, ψ) such that the following diagram commutes (equivalently, $\pi \circ \phi = \mathfrak{u}$ and $\phi \circ i_1 = i_2 \circ \psi$).

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (\operatorname{Ker} \mathfrak{u}, \overline{\alpha}) & \xrightarrow{\; i_1 \;} & (\mathfrak{uce}(L), \overline{\alpha}) & \xrightarrow{\; \mathfrak{u} \;} & (L, \alpha) & \longrightarrow & 0 \\
& & \downarrow{\psi} & & \downarrow{\phi} & & \downarrow{\operatorname{Id}} & & \\
0 & \longrightarrow & (M, \alpha_M) & \xrightarrow{\; i_2 \;} & (K, \alpha_K) & \xrightarrow{\; \pi \;} & (L, \alpha) & \longrightarrow & 0
\end{array}
$$

Since L is perfect, we know $\mathfrak{uce}(L)$ is perfect. Therefore there exists at most one homomorphism from $\mathfrak{uce}(L)$ to K. Thus, we only need to show that a homomorphism $\phi : (\mathfrak{uce}(L), \overline{\alpha}) \to (K, \alpha_K)$ such that $\pi \circ \phi = \mathfrak{u}$ exists.

We claim that the mapping ϕ defined as follows meets these requirements. For $l_1, l_2 \in L$, define

$$\phi : (\mathfrak{uce}(L), \overline{\alpha}) \to (K, \alpha_K)$$

$$\langle l_1, l_2 \rangle \mapsto [k_1, k_2]$$

where $\pi(k_1) = l_1$ and $\pi(k_2) = l_2$.

First, we can check that ϕ is well-defined. Suppose there exist $k_1, k_2, j_1, j_2 \in (K, \alpha_K)$ such that for $l_1, l_2 \in L$, $\pi(k_1) = l_1, \pi(j_1) = l_1, \pi(k_2) = l_2, \pi(j_2) = l_2$.

Note that

$$[k_1 - j_1, k_2 + j_2] = [k_1, k_2 + j_2] - [j_1, k_2 + j_2]$$

$$= [k_1, k_2] + [k_1, j_2] - [j_1, k_2] - [j_1, j_2].$$

Hence,

$$[k_1 - j_1, k_2 + j_2] - [k_1, j_2] + [j_1, k_2] = [k_1, k_2] - [j_1, j_2].$$

Because $k_1 - j_1 \in \operatorname{Ker} \pi$ and K is a central extension of L, $[k_1 - j_1, k_2 + j_2] = 0$.

Therefore,

$$[k_1, k_2] - [j_1, j_2] = [j_1, k_2] - [k_1, j_2]. \tag{4.1}$$

Similarly,

$$[k_1 + j_1, k_2 - j_2] = [k_1, k_2] - [k_1, j_2] + [j_1, k_2] - [j_1, j_2],$$

and, because $k_2 - j_2 \in \operatorname{Ker} \pi$ and K is a central extension of L, $[k_1 + j_1, k_2 - j_2] = 0$.

So we have

$$[k_1, k_2] - [j_1, j_2] = [k_1, j_2] - [j_1, k_2]. \tag{4.2}$$

Combining equations (4.1) and (4.2) yields

$$[k_1, k_2] - [j_1, j_2] = - ([k_1, k_2] - [j_1, j_2]),$$

which implies that $[k_1, k_2] = [j_1, j_2] = 0$. So ϕ is well-defined.

Next, we will check that ϕ respects the product. Let $x, y \in \mathfrak{uce}(L)$ where $x = \langle x_1, x_2 \rangle, y = \langle y_1, y_2 \rangle$ and $\pi(x_1) = k_1, \pi(x_2) = k_2, \pi(y_1) = k_3, \pi(y_2) = k_4$. Then

$$
\begin{aligned}
\phi([x, y]_{\overline{L}}) &= \phi([\langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle]_{\overline{L}}) \\
&= \phi(\langle [x_1, x_2], [y_1, y_2] \rangle) \\
&= [[k_1, k_2], [k_3, k_4]] \\
&= [\phi(\langle x_1, x_2 \rangle), \phi(\langle y_1, y_2 \rangle)].
\end{aligned}
$$

Furthermore, we need to check that $\phi \circ \overline{\alpha} = \alpha_k \circ \phi$. Let $\langle l_1, l_2 \rangle \in \mathfrak{uce}(L)$. Then

$$
\begin{aligned}
(\phi \circ \overline{\alpha})(\langle l_1, l_2 \rangle) &= \phi(\langle \alpha_L(l_1), \alpha_L(l_2) \rangle) \\
&= [k_1, k_2],
\end{aligned}
$$

where $\pi(k_1) = \alpha_L(l_1)$ and $\pi(k_2) = \alpha_L(l_2)$. Additionally,

$$(\alpha_K \circ \phi)(\langle l_1, l_2 \rangle) = \alpha_K([k_1', k_2'])$$

$$= [\alpha_K(k_1'), \alpha_K(k_2')],$$

where $\pi(k_1') = l_1$ and $\pi(k_2') = l_2$. Note that $k_i - \alpha_K(k_i') \in \operatorname{Ker} \pi$ for $i = 1, 2$ since

$$\pi(k_i - \alpha_K(k_i')) = \pi(k_i) - \pi(\alpha_K(k_i'))$$

$$= \pi(k_i) - \alpha_L(\pi(k_i'))$$

$$= \alpha_L(l_i) - \alpha_L(l_i)$$

$$= 0.$$

Because K is a central extension of L,

$$0 = [k_1 - \alpha_K(k_1'), k_2] + [k_1 - \alpha_K(k_1'), \alpha_K(k_2')]$$

$$= [k_1, k_2] - [\alpha_K(k_1'), k_2] + [k_1, \alpha_K(k_2')] - [\alpha_K(k_1'), \alpha_K(k_2')].$$

Hence,

$$[k_1, k_2] - [\alpha_K(k_1'), \alpha_K(k_2')] = [\alpha_K(k_1'), k_2] - [k_1, \alpha_K(k_2')].$$

Similarly,

$$0 = [k_1, k_2 - \alpha(k_2')] + [\alpha_K(k_1'), k_2 - \alpha(k_2')],$$

which implies

$$[k_1, k_2] - [\alpha_K(k_1'), \alpha_K(k_2')] = [k_1, \alpha_K(k_2')] - [\alpha_K(k_1'), k_2].$$

Because $[k_1, k_2] - [\alpha_K(k_1'), \alpha_K(k_2')]$ is equal to its additive inverse, we have shown that

$$[k_1, k_2] - [\alpha_K(k_1'), \alpha_K(k_2')] = 0.$$

Therefore, ϕ is a Hom-Lie superalgebra homomorphism.

Finally, we can check that $\pi \circ \phi = \mathfrak{u}$ since

$$\pi \circ \phi(\langle x_1, x_2 \rangle) = \pi([k_1, k_2])$$
$$= [\pi(k_1), \pi(k_2)]$$
$$= [x_1, x_2]$$
$$= \mathfrak{u}(\langle x_1, x_2 \rangle),$$

where $\langle x_1, x_2 \rangle \in \mathfrak{uce}(L)$ and $\pi(x_1) = k_1, \pi(x_2) = k_2$.

We can obtain the map ψ by restricting ϕ to Ker $\mathfrak{u}$. We can easily check that $\phi \circ i_1 = i_2 \circ \psi$. So $(\mathfrak{uce}(L), \overline{\alpha})$ is a universal central extension of (L, α_L). $\blacksquare$

In order to consider the $\mathfrak{uce}$ construction as a functor on the categories of Hom-Lie

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superalgebras, we need one more result. Namely, we need to prove that a morphism of Hom-Lie superalgebras $f : L \to M$ with even homomorphisms α and β, respectively, implies a morphism of Hom-Lie superalgebras

$$\mathfrak{uce}(f) : \mathfrak{uce}(L) \to \mathfrak{uce}(M) : \quad \langle l_1, l_2 \rangle \mapsto \langle f(l_1), f(l_2) \rangle$$

with even homomorphisms $\overline{\alpha}$ and $\overline{\beta}$, respectively.

Proposition 4.2.4. The construction $\mathfrak{uce}$ is a covariant functor on the category of Hom-Lie superalgebras.

Proof.

First, we want to show that $[\mathfrak{uce}(f)(x), \mathfrak{uce}(f)(y)]_{\overline{M}} = \mathfrak{uce}(f)([x, y]_{\overline{L}})$ where $x = \langle x_1, x_2 \rangle$, $y = \langle y_1, y_2 \rangle$ for $x_1, x_2, y_1, y_2 \in L$. For brevity, we write $\overline{M}$ and $\overline{L}$ in place of

$\mathsf{uce}(M)$ and $\mathsf{uce}(L)$, respectively. Then

$$[\mathsf{uce}(f)(x), \mathsf{uce}(f)(y)]_{\overline{M}} = [\mathsf{uce}(f)(\langle x_1, x_2\rangle), \mathsf{uce}(f)(\langle y_1, y_2\rangle)]_{\overline{M}}$$

$$= [\langle f(x_1), f(x_2)\rangle, \langle f(y_1), f(y_2)\rangle]_{\overline{M}}$$

$$= \langle [f(x_1), f(x_2)]_M, [f(y_1), f(y_2)]_M\rangle$$

$$= \langle f([x_1, x_2]_L), f([y_1, y_2]_L)\rangle$$

$$= \mathsf{uce}(f)(\langle [x_1, x_2]_L, [y_1, y_2]_L\rangle)$$

$$= \mathsf{uce}(f)([\langle x_1, x_2\rangle, \langle y_1, y_2\rangle]_{\overline{L}})$$

$$= \mathsf{uce}([x, y]_{\overline{L}}).$$

Second, we want to show $\mathsf{uce}(f) \circ \overline{\alpha} = \overline{\beta} \circ \mathsf{uce}(f)$. Let $x_1, x_2 \in L$.

$$\mathsf{uce}(f)(\overline{\alpha}(\langle x_1, x_2\rangle)) = \mathsf{uce}(f)(\langle \alpha(x_1), \alpha(x_2)\rangle)$$

$$= \langle f(\alpha(x_1)), f(\alpha(x_2))\rangle$$

$$= \langle \beta(f(x_1)), \beta(f(x_2))\rangle$$

$$= \overline{\beta}(\langle f(x_1), f(x_2)\rangle)$$

$$= \overline{\beta}(\mathsf{uce}(f)(\langle x_1, x_2\rangle)).$$

We can check that uce respects composition and the identity morphisms. Therefore, the uce mapping defines a functor on the category of perfect Hom-Lie superalgebras.

∎

4.3 Direct Limits of Hom-Lie Superalgebras

The uce functor commutes with the direct limit functor for the category of Lie superalgebras. This result was proven in [12]. In this section, we generalize the result in [12] to the category of Hom-Lie superalgebras. In particular, our results in this section recover the results in [9], which deals with the special case of Hom-Lie algebras.

Let (L_i, f_{ij}) be a directed system of perfect Hom-Lie superalgebras indexed by the directed set I with direct limit $\varinjlim L_i$ and let $\mathfrak{u}_i : \mathfrak{uce}(L_i) \to L_i$ be the construction of the universal central extension of L_i defined in section 4.2. Since the functor uce is covariant, we obtain another directed system $(\mathfrak{uce}(L_i), \widehat{f_{ij}})$, where $\widehat{f_{ij}}$ denotes the image of f_{ij} under the functor uce. That is, the following diagrams commute.

$$L_i \xrightarrow{\quad f_{ij} \quad} L_j$$
$$\phi_i \qquad \phi_j$$
$$\varinjlim L_i$$

$$\mathfrak{uce}(L_i) \xrightarrow{\quad \widehat{f_{ij}} \quad} \mathfrak{uce}(L_j)$$
$$\widetilde{\phi_i} \qquad \widetilde{\phi_j}$$
$$\varinjlim \mathfrak{uce}(L_i)$$

The ultimate goal of this section is to prove that $\varinjlim \mathfrak{uce}(L_i) \cong \mathfrak{uce}\left(\varinjlim L_i\right)$, i.e., the direct limit of the universal central extensions of L_i is isomorphic to the universal central extension of a direct limit of perfect Hom-Lie superalgebras L_i.

Note that the family $\mathfrak{u}_i$ $(i \in I)$ defines a family of morphisms on the category of perfect Hom-Lie superalgebras from the directed system $(\mathfrak{uce}(L_i), \widehat{f_{ij}})$, with canonical maps $\widetilde{\phi_i}$, to the directed system (L_i, f_{ij}), with canonical maps ϕ_i.

Therefore, there exists a morphism $\varinjlim \mathfrak{u}_i : \varinjlim \mathfrak{uce}(L_i) \to \varinjlim L_i$. We claim that $\varinjlim \mathfrak{uce}(L_i)$ is a central extension of $\varinjlim L_i$.

Proposition 4.3.1. Let (L_i, f_{ij}) be a directed system of perfect Hom-Lie superalgebras. Then the morphism

$$\varinjlim \mathfrak{u}_i : \varinjlim \mathfrak{uce}(L_i) \to \varinjlim L_i$$

is a central extension of $\varinjlim L_i$.

Proof. If all L_i are perfect, then all $\mathfrak{u}_i$ are surjective. Hence, we only need to show

that $\varinjlim \mathfrak{u}_i$ has a central kernel.

Let $x \in \mathrm{Ker}(\varinjlim \mathfrak{u}_i)$. Then there exists $x_j \in \mathfrak{uce}(L_j)$ for sufficiently large j such that

$x = \widetilde{\phi}_j(x_j)$. Furthermore, since $x \in \mathrm{Ker}(\varinjlim \mathfrak{u}_i)$, it must be true that $0 = \varinjlim \mathfrak{u}_i(x) =$

$\phi_j(\mathfrak{u}_j(x_j))$ in the Hom-Lie superalgebra $\varinjlim L_i$. The equivalence relation in the direct

limit implies that $\mathfrak{u}_j(x_j) \sim 0$ in $\varinjlim L_i$ if and only if there exists some $k \geq j$ such

that $f_{jk}(\mathfrak{u}_j(x_j)) = f_{jk}(0)$ in L_k. But $f_{jk}(0) = 0$, so there exists $k \geq j$ such that

$f_{jk}(\mathfrak{u}_j(x_j)) = 0$ in L_k. Also note that $\phi_k(f_{jk}(\mathfrak{u}_j(x_j))) = \phi_k(0) = 0$ in $\varinjlim L_i$.

We need to show that, for any $y \in \varinjlim \mathfrak{uce}(L_i)$, we have $[x,y] = 0$. Note that $y = \widetilde{\phi}_p(y_p)$

for some p where $y_p \in \mathfrak{uce}(L_p)$. Now, choose q such that $q \geq k \geq j$ and $q \geq p$. Note

that $f_{jq}(\mathfrak{u}_j(x_j)) = (f_{kq} \circ f_{jk})(\mathfrak{u}_j(x_j)) = 0$ in L_q and that, by construction, we have

the following commutative diagram.

$$
\begin{array}{ccc}
\mathfrak{uce}(L_i) & \xrightarrow{\widehat{f_{ij}}} & \mathfrak{uce}(L_j) \\
\downarrow{\scriptstyle u_i} & & \downarrow{\scriptstyle u_j} \\
L_i & \xrightarrow{\;\;f_{ij}\;\;} & L_j
\end{array}
$$

Thus, $0 = f_{jq}(\mathfrak{u}_j(x_j)) = \mathfrak{u}_q(\widehat{f_{jq}}(x_j))$, which implies $\widehat{f_{jq}}(x_j) \in \mathrm{Ker}(\mathfrak{u}_q)$. Because $\mathfrak{uce}(L_q)$

is a central extension of L_q, $\mathrm{Ker}(\mathfrak{u}_q)$ is contained in the center of L_q. Therefore,

$[\widehat{f_{jq}}(x_j), \widehat{f_{pq}}(y_p)]_{\mathfrak{uce}(L_q)} = 0$. Hence,

$$[x, y]_{\varinjlim \mathfrak{uce}(L_i)} = [\widetilde{\phi}_j(x_j), \widetilde{\phi}_p(y_p)]_{\varinjlim \mathfrak{uce}(L_i)}$$

$$= \widetilde{\phi}_q([\widehat{f}_{jq}(x_j), \widehat{f}_{pq}(y_p)]_{\mathfrak{uce}(L_q)}) = 0.$$

∎

Suppose that (L_i, f_{ij}) is a directed system of perfect Hom-Lie superalgebras. Then $\varinjlim L_i$ is also perfect and, by Theorem 4.2.3, $\mathfrak{U} : \mathfrak{uce}(\varinjlim L_i) \to \varinjlim L_i$ is a universal central extension of $\varinjlim L_i$. Then we obtain a unique Hom-Lie superalgebra homomorphism $\widehat{\phi}_i := \mathfrak{uce}(\phi_i)$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\mathfrak{uce}(L_i) & \xrightarrow{\widehat{\phi}_i} & \mathfrak{uce}(\varinjlim L_i) \\
\downarrow{\scriptstyle u_i} & & \downarrow{\scriptstyle \mathfrak{U}} \\
L_i & \xrightarrow{\phi_i} & \varinjlim L_i
\end{array}
$$

Because $\widehat{\phi}_i = \mathfrak{uce}(\phi_i) = \mathfrak{uce}(\phi_j \circ f_{ij}) = \mathfrak{uce}(\phi_j) \circ \mathfrak{uce}(f_{ij}) = \widehat{\phi}_j \circ \mathfrak{uce}(f_{ij})$, the outer triangle in the diagram below commutes.

Therefore, by the universal property of the direct limit, $\varinjlim \mathfrak{uce}(L_i)$, there exists a

unique Hom-Lie superalgebra morphism $\phi : \varinjlim \mathfrak{uce}(L_i) \to \mathfrak{uce}(\varinjlim L_i)$ such that the above diagram is commutative.

This brings us to the main result of this thesis.

Theorem 4.3.2. Suppose that (L_i, f_{ij}) is a directed system of perfect Hom-Lie superalgebras. Then the map

$$\phi : \varinjlim \mathfrak{uce}(L_i) \to \mathfrak{uce}(\varinjlim L_i)$$

is an isomorphism of Hom-Lie Superalgebras and $\varinjlim u_i : \varinjlim \mathfrak{uce}(L_i) \to \varinjlim L_i$ is a universal central extension of $\varinjlim L_i$.

Proof.

Because $\varinjlim L_i$ is perfect, we have a universal central extension $\mathfrak{U} : \mathfrak{uce}(\varinjlim L_i) \to$
$\varinjlim L_i$. By proposition 4.3.1, we also have a central extension $\varinjlim \mathfrak{u}_i : \varinjlim \mathfrak{uce}(L_i) \to$
$\varinjlim L_i$. Therefore, by the universal property of the universal central extension $\mathfrak{U}$,
there exists a unique Hom-Lie superalgebra homomorphism ψ such that the following
diagram commutes.

$$
\begin{array}{ccc}
\mathfrak{uce}(\varinjlim L_i) & \xrightarrow{\quad\psi\quad} & \varinjlim \mathfrak{uce}(L_i) \\
 & \searrow^{\mathfrak{U}} \quad \downarrow^{\varinjlim \mathfrak{u}_i} \swarrow & \\
 & \varinjlim L_i &
\end{array}
$$

We claim that $\psi \circ \phi = \mathrm{Id}_{\varinjlim \mathfrak{uce}(L_i)}$ and $\phi \circ \psi = \mathrm{Id}_{\mathfrak{uce}(\varinjlim L_i)}$. For the proof of both of
these claims, we use the following diagram.

$$
\begin{array}{ccccc}
 & & \xrightarrow{\quad\widehat{\phi}_i\quad} & & \\
\mathfrak{uce}(L_i) & \xrightarrow{\phi_i} & \varinjlim \mathfrak{uce}(L_i) & \xleftarrow[\phi]{\psi} & \mathfrak{uce}(\varinjlim L_i) \\
\downarrow^{\mathfrak{u}_i} & & & \searrow^{\varinjlim \mathfrak{u}_i} & \downarrow^{\mathfrak{U}} \\
L_i & \xrightarrow{\qquad\qquad\phi_i\qquad\qquad} & & & \varinjlim L_i
\end{array}
$$

Note that $\mathfrak{U}$ is a homomorphism from $\mathfrak{uce}(\varinjlim L_i)$ to $\varinjlim L_i$, and so is $\mathfrak{U} \circ (\phi \circ \psi)$. By
the universal property of $\mathfrak{uce}(\varinjlim L_i)$, if we can show that $\mathfrak{U} \circ (\phi \circ \psi) = \mathfrak{U}$, we will have
shown that $\phi \circ \psi = \mathrm{Id}_{\mathfrak{uce}(\varinjlim L_i)}$. Because $\mathfrak{U} = \varinjlim \mathfrak{u}_i \circ \psi$, we only need to show that

$\mathfrak{U} \circ \phi = \varinjlim \mathfrak{u}_i$. This follows from the fact that $\mathfrak{U} \circ \phi \circ \widetilde{\phi}_i = \mathfrak{U} \circ \widehat{\phi}_i = \phi_i \circ \mathfrak{u}_i = \varinjlim \mathfrak{u}_i \circ \widetilde{\phi}_i$.

To check that $\psi \circ \phi = \mathrm{Id}_{\varinjlim \mathfrak{uce}(L_i)}$, by the universal property of $\varinjlim \mathfrak{uce}(L_i)$, it is sufficient to check that $(\psi \circ \phi) \circ \widetilde{\phi}_i = \widetilde{\phi}_i$. We've already shown that $\phi \circ \widetilde{\phi}_i = \widehat{\phi}_i$, thus, we only need to show that $\psi \circ \widehat{\phi}_i = \widetilde{\phi}_i$. This is true since $\varinjlim \mathfrak{u}_i \circ \psi \circ \widehat{\phi}_i = \mathfrak{U} \circ \widehat{\phi}_i = \phi_i \circ \mathfrak{u}_i = \varinjlim \mathfrak{u}_i \circ \widetilde{\phi}_i$ and the perfectness of $\mathfrak{uce}(L_i)$ implies that $\psi \circ \widehat{\phi}_i = \widetilde{\phi}_i$.

Therefore, $\varinjlim \mathfrak{uce}(L_i) \cong \mathfrak{uce}(\varinjlim L_i)$ as Hom-Lie superalgebras and hence $\varinjlim \mathfrak{u}_i : \varinjlim \mathfrak{uce}(L_i) \to \varinjlim L_i$ is a universal central extension of $\varinjlim L_i$. $\blacksquare$

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