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Enumeration of (16,4,16,4) Relative Difference Sets

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Abstract

A complete enumeration of relative difference sets (RDS) with parameters $(16, 4, 16, 4)$ in a group of order 64 with a normal subgroup N of order 4 is given. If $N = Z_4$, three of the eleven abelian groups of order 64, and 23 of the 256 nonabelian groups of order 64 contain $(16, 4, 16, 4)$ RDSs. If $N = Z_2 \times Z_2$, six of the abelian groups and 194 of the non-abelian groups of order 64 contain $(16, 4, 16, 4)$ RDSs.

Keywords: Relative difference set; symmetric net.

1 Introduction

A relative difference set (RDS) with parameters (m, n, k, λ) in a finite group G of order mn relative to a normal subgroup N of order n is a k-subset R of G such that every element of $g \in G \setminus N$ appears exactly λ times in the multiset $S = \{ab^{-1} \mid a, b \in R, a \neq b\}$, and no element of N appears in S [1]. An RDS is called *abelian* if G is abelian, and *nonabelian* otherwise.

Relative difference sets are closely related to difference sets, group-divisible designs, generalized Hadamard matrices, symmetric nets, and finite geometry [1], [4], [6]. A comprehensive survey on RDS is the paper by Pott [5]. The existence problem of (p^a, p^b, p^a, p^{a-b}) RDSs is considered to be one of the most important questions concerning RDSs [5].

In [7], Schmidt studied the existence of abelian (p^a, p^b, p^a, p^{a-b}) RDS, and settled the existence problem of abelian $(16, 4, 16, 4)$ RDS completely.

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In this paper, a complete enumeration of $(16, 4, 16, 4)$ RDSs is given, for all groups, abelian and nonabelian, of order 64. In summary, RDSs exist in 6 of the 11 abelian groups of order 64, as well as in 195 of the 256 nonabelian groups of order 64. If $N = Z_4$, three of the eleven abelian groups of order 64, and 23 of the 256 non-abelian groups of order 64 contain $(16, 4, 16, 4)$ RDSs. If $N = Z_2 \times Z_2$, six of the abelian groups and 194 of the non-abelian groups of order 64 contain $(16, 4, 16, 4)$ RDSs. The computer algebra package Magma [2] was used in the computations.

2 RDS and symmetric nets

Our approach to the enumeration of $(16, 4, 16, 4)$ RDSs is based on their link to incidence structures known as symmetric $(4, 4)$-nets.

A *symmetric* $(4,4)$*-net*[1] is an incidence structure $I = (X, \mathcal{B})$ consisting of a set X of 64 *points* and a collection $\mathcal{B}$ of 64 *blocks*, each block being a subset of 16 points of X, having the following properties:

- Each point belongs to 16 blocks.
- There exists a partition $\mathcal{P}$ of the point set X into 16 subsets of size 4, called *groups*, so that every two points belonging to different groups appear together in exactly 4 blocks, while every two points belonging to the same group do not appear together in any block.
- The 64 blocks are partitioned into 16 *parallel classes*, each class consisting of 4 pairwise disjoint blocks, so that every two blocks belonging to different parallel classes share exactly 4 points.

Other terms used for a structure with the above properties are group-divisible design, or a transversal design [1].

An *automorphism* of an incidence structure I is any permutation of the point set which preserves the collection of blocks. The set of all automorphisms of I form a group, called the *full* automorphism group, $Aut(I)$, of I. The subgroups of $Aut(I)$ are called automorphism groups.

A symmetric $(4, 4)$-net is *class-regular* if it admits an automorphism group N of order 4 which acts transitively on each group of points and each parallel class of blocks. The group N is then called a group of *bitranslations*.

If R is a $(16, 4, 16, 4)$ RDS in a group G of order 64, relative to a normal subgroup $N \leqslant G$ of order 4, one can associate with R a class-regular $(4, 4)$-net I with point set G and blocks being the subsets $B_g \subseteq G$ of the form

$$B_g = \{Rg \mid g \in G\}.$$

[1] More generally, a net is defined as a resolvable 1-design, and a symmetric net is a net with equal number of points and blocks. For more definitions concerning designs see [1].

The partition $\mathcal{P}$ of the points into subsets of size 4 is defined as the partition of G into cosets of N. Consequently, G acts as an automorphism group of I, and the subgroup N acts transitively on each point group and each parallel class.

Thus, any $(16, 4, 16, 4)$ RDS corresponds to a class-regular symmetric $(4, 4)$-net which admits a regular automorphism group.

All nonisomorphic class-regular symmetric $(4, 4)$-nets were enumerated by Harada, Lam and Tonchev in [4], and, implicitly, by Gibbons and Mathon in [3] (two incidence structures are isomorphic if there is an incidence preserving bijection between their point sets). Up to isomorphism, there are exactly 226 nets with group of bitranslations $N = Z_2 \times Z_2$, and 13 nets with $N = Z_4$.

These results reduce the enumeration of $(16, 4, 16, 4)$ RDSs to finding sharply transitive regular subgroups G of the full automorphism groups of those class-regular symmetric $(4, 4)$-nets which admit automorphism groups acting transitively on the points, such that N is a normal subgroup of G. We used Magma to find the conjugacy classes of sharply transitive regular subgroups.

There are 267 groups of order 64, of which 11 are abelian and 256 are nonabelian. Among the 226 nets with group of bitranslations $Z_2 \times Z_2$, only 200 nonisomorphic regular subgroups of order 64 appeared within the automorphism groups of those nets, of which 6 were abelian and 194 were nonabelian. Among the 13 nets with group of bitranslations Z_4, only 26 nonisomorphic regular subgroups of order 64 appeared within the automorphism groups of those nets, of which 3 were abelian and 23 were nonabelian.

3 The results

Tables 1 and 2 list the nets with automorphism groups which admit regular subgroups with normal subgroup N. Each entry is as follows:

- #: The index of the net within the list of nets with a group of bitranslations $N = \mathbb{Z}_2 \times \mathbb{Z}_2$ available at http://www.math.mtu.edu/~tonchev/Z2Z2nets and at http://www.math.mtu.edu/~tonchev/Z4nets for the nets with $N = \mathbb{Z}_4$. Missing indices indicate that the corresponding nets do not have transitive automorphism groups.

- *Order:* The order of the automorphism group.

- *2-Rank:* The 2-rank of the incidence matrix of the net.

- *Total:* In the format x/y, x indicates the total number of conjugacy classes of regular subgroups containing the group of bitranslations, found within the automorphism group of each net, while y is the number of nonisomorphic regular subgroups.

- *Abelian* and *Nonabelian:* In the format x/y, x indicates the number of conjugacy classes of each type of subgroup found within the net's automorphism group, while y is the number of nonisomorphic subgroups.

- *List of indices:* Below the previous data is a list of the indices of the regular subgroups of order 64 found in each automorphism group according to Magma's list of the groups of order 64. Entries of the form $x(y)$ indicate that groups isomorphic to group x appeared in y distinct conjugacy classes. Entries marked with an asterisk are abelian, and all others are nonabelian.

Tables 3 and 4 give details about the structure of the regular abelian subgroups of order 64 found. Only such subgroups containing the relevant group N of bitranslations were considered. Groups with the following indices had a single regular abelian subgroup isomorphic to $\mathbb{Z}_2^6$, and are not listed in Table 3: 6, 99, 100, 103, 104, 105, 107, 111, 113, 120, 121, 127, 128, 131, 142, 143, 144, 145, 146, 147, 148, 149, 150, 151, 181, 182, 192, 193, 194, 195, 196, 197, 198, 212, 214, 221, 224, 225, 226.

Tables 5 and 6 summarize the structures of the regular abelian subgroups of order 64 found within the nets.

Table 7 gives the structures of all abelian groups of order 64 which do not appear as a regular subgroup in any net. Consequently, those are the groups which do not support any $(16, 4, 16, 4)$ RDS.

Table 1: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
1	1105920	16	*173/57*	*6/2*	*167/55*
	4(2), 9(2), 18(2), 20(2), 23(5), 25(2), 32(6), 33(2), 34, 35(3), 36, 37(3), 56(8), 60(2), 74, 77, 80, 88(4), 90(5), 92(2), 100(2), 102(2), 132, 165, 192(4)*, 193(4), 194, 198(2), 199(3), 200(2), 202(3), 206(3), 207(3), 210(4), 214(3), 215(3), 217(3), 219(4), 223(8), 224(5), 226(2), 227(4), 229(2), 230(2), 232(10), 236, 237(6), 239(4), 240, 241(2), 242(9), 243, 244(5), 245, 262(3), 264(2), 267(2)*				
2	13824	18	*8/7*	*2/2*	*6/5*
	60(2), 88, 132, 192*, 193, 262, 267*				
4	4608	18	*8/5*	*0/0*	*8/5*
	56(2), 88(2), 100, 132, 262(2)				
5	18432	16	*61/21*	*6/2*	*55/19*
	4(2), 9(2), 20, 23, 32(4), 35(2), 56(14), 60(2), 74(3), 88(3), 90, 92, 100, 132(2), 192(5)*, 193(5), 202, 207(2), 242(5), 262(3), 267*				
6	4608	20	*28/19*	*1/1*	*27/18*
	20(2), 23(2), 58, 60(6), 69(2), 71, 74, 75, 78, 90, 92, 100, 109, 193, 202(2), 204, 207, 247, 267*				
8	384	20	*4/3*	*0/0*	*4/3*
	88, 132(2), 262				
9	4608	18	*8/7*	*2/2*	*6/5*
	60(2), 88, 132, 192*, 193, 262, 267*				

Table 1: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
22	6144	16	*84/25*	*6/2*	*78/23*
	4(2), 9(2), 20, 23, 32(2), 56(29), 60, 66, 67(2), 69(3), 71(2), 73(2), 75(3), 78(2), 88(3), 90, 92, 109, 131, 164, 192(5)*, 193(8), 194(8), 260*, 261				
24	1536	20	*48/35*	*0/0*	*48/35*
	18, 22, 25(2), 88(2), 93(2), 97, 100(5), 102, 111, 114, 116, 117, 120, 122, 132(5), 133, 143, 145, 149, 151, 156, 158, 163, 166(3), 168, 170, 200, 206, 217, 229, 230, 249, 252, 255, 262				
25	1536	18	*9/5*	*0/0*	*9/5*
	56(3), 88(2), 100, 132, 262(2)				
30	1536	20	*50/42*	*0/0*	*50/42*
	18, 22, 25(2), 33, 36, 91, 93(2), 97(2), 99, 100(2), 105, 111, 119, 120, 121, 122, 129, 130, 131, 132(2), 133, 142, 143, 144, 145(3), 146, 148, 149, 151, 156, 158, 166(2), 168, 170, 176, 178, 200, 249, 251, 252, 254, 255				
31	1536	20	*45/26*	*1/1*	*44/25*
	23, 24, 32(6), 35(4), 58, 59, 60, 61(2), 66, 67(2), 69, 72(2), 74, 75, 78, 85, 90(2), 101(2), 136(3), 138(2), 202(2), 207, 212, 242(2), 255(2), 260*				
32	128	20	*4/4*	*1/1*	*3/3*
	146, 148, 206, 246*				
33	512	20	*66/44*	*1/1*	*65/43*
	18, 25, 32(4), 33(4), 35(2), 86(2), 88, 91(2), 97(2), 99, 100, 102(2), 105, 108(3), 115, 116, 119, 122, 132, 133, 135, 138, 139(3), 145, 146(2), 148(2), 149, 151, 155, 158(2), 163, 166(3), 170, 179, 206(2), 217, 223(2), 227, 232, 237, 242, 246*, 251, 254				
36	73728	16	*1158/85*	*6/3*	*1152/82*
	4(2), 9(2), 18(6), 20(6), 23(13), 25(6), 32(14), 33(10), 34(5), 35(7), 36(5), 37(7), 56(13), 60(3), 62, 74(4), 77(2), 80, 88(6), 90(11), 92(3), 93, 99(2), 100(4), 102(6), 132(2), 192(4)*, 193(6), 194(2), 195(5), 196(5), 197(3), 198(13), 199(15), 200(5), 201(6), 202(7), 203(6), 204(6), 205(7), 206(29), 207(7), 209(8), 210(52), 211(3), 212(3), 213(12), 214(17), 215(16), 216(12), 217(19), 218(9), 219(56), 220(44), 221(18), 222(26), 223(68), 224(12), 225(13), 226(15), 227(58), 228(19), 229(22), 230(11), 231(5), 232(69), 233(46), 234(38), 235(23), 236(20), 237(33), 238(9), 239(5), 240(18), 241(27), 242(15), 243(24), 244(31), 260*, 261, 262(4), 263, 264(5), 265, 267*				
78	6144	20	*28/24*	*0/0*	*28/24*
	4, 5, 9(4), 57, 62, 67, 68, 69(2), 70, 73, 74, 77, 78, 79, 81, 82, 87, 112, 131, 132, 164, 165, 208, 263				

Table 1: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
99	1152	20	*4/2*	*1/1*	*3/1*
	60(3), 267*				
100	1152	22	*4/2*	*1/1*	*3/1*
	60(3), 267*				
103	384	20	*4/2*	*1/1*	*3/1*
	60(3), 267*				
104	384	22	*4/2*	*1/1*	*3/1*
	60(3), 267*				
105	1152	18	*4/2*	*1/1*	*3/1*
	60(3), 267*				
107	6912	20	*5/3*	*1/1*	*4/2*
	60(3), 193, 267*				
109	384	22	*4/4*	*0/0*	*4/4*
	100, 109, 204, 247				
111	512	20	*52/19*	*1/1*	*51/18*
	20(2), 23(2), 58(2), 60(10), 69(6), 71(2), 74(3), 75(3), 78(3), 90, 92(2), 100, 109, 193(2), 202(6), 204(2), 207(2), 247, 267*				
113	1536	20	*48/19*	*1/1*	*47/18*
	20(2), 23(2), 58(2), 60(10), 69(6), 71(2), 74(3), 75(3), 78(3), 90, 92, 100, 109, 193(2), 202(3), 204(2), 207(2), 247, 267*				
115	256	20	*4/2*	*0/0*	*4/2*
	9(2), 132(2)				
116	768	20	*4/2*	*0/0*	*4/2*
	9(2), 132(2)				
117	128	21	*0/0*	*0/0*	*0/0*
120	1536	20	*31/19*	*1/1*	*30/18*
	20(2), 23(2), 58, 60(6), 69(2), 71, 74, 75, 78, 90, 92(2), 100, 109, 193, 202(4), 204, 207, 247, 267*				
121	384	22	*4/2*	*1/1*	*3/1*
	60(3), 267*				
122	128	22	*4/4*	*0/0*	*4/4*
	100, 109, 204, 247				
123	192	22	*1/1*	*0/0*	*1/1*
	20				
124	64	22	*1/1*	*0/0*	*1/1*
	23				

Table 1: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
125	64	22	*1/1*	*0/0*	*1/1*
	132				
126	192	22	*1/1*	*0/0*	*1/1*
	132				
127	768	20	*6/3*	*1/1*	*5/2*
	60(4), 193, 267*				
128	256	20	*7/3*	*1/1*	*6/2*
	60(4), 193(2), 267*				
131	768	20	*6/3*	*1/1*	*5/2*
	60(4), 193, 267*				
132	512	20	*52/32*	*1/1*	*51/31*
	23, 24, 32, 33(2), 58, 59(2), 61, 62, 63, 65(2), 66, 67(3), 68(3), 69(7), 70, 71, 72, 75, 77(3), 78(2), 80, 81(2), 85, 90(2), 98(2), 101(2), 139, 192*, 197, 204, 205, 212				
134	256	20	*64/21*	*0/0*	*64/21*
	58, 59(2), 61, 63, 65, 68(2), 69(10), 70(5), 71(4), 72, 73(2), 75(9), 77(4), 78(7), 79(3), 80(3), 81(2), 195(2), 196, 197, 204(2)				
136	512	20	*52/28*	*1/1*	*51/27*
	23, 24, 32, 35(2), 55*, 58(2), 59, 61, 63(3), 66(2), 67(4), 69(8), 71, 72(4), 74(3), 75, 76, 78(2), 81(2), 90(2), 98, 101(2), 104, 136, 197, 208, 255, 262				
137	768	20	*24/13*	*0/0*	*24/13*
	57, 59, 69(4), 70(3), 72, 77(3), 78(2), 79(2), 80(3), 81, 82, 197, 212				
138	1536	20	*45/30*	*1/1*	*44/29*
	23, 24, 32(2), 33(4), 34(4), 55*, 58, 61, 62(2), 65, 67, 69, 70, 71, 75, 77, 79, 90(2), 98, 101(2), 104, 134(2), 139(3), 194, 196, 203, 205(2), 209, 241(2), 254				
139	512	21	*52/40*	*0/0*	*52/40*
	18, 20, 25(2), 89, 91(2), 92, 95, 97, 98, 100(3), 102(2), 105, 112, 115, 117, 119, 120, 129, 131(2), 132, 133(2), 145(2), 148, 151, 159, 160, 163, 164, 165, 166, 169(2), 198, 200, 207(2), 217(2), 227(2), 229, 252, 255, 264				
140	512	21	*66/44*	*0/0*	*66/44*
	18, 25, 32(2), 33(6), 35(2), 86(2), 88, 91(2), 97, 100(3), 102(2), 105, 108(2), 109, 116, 117, 120, 122, 132(2), 137, 138, 139(3), 143, 148(2), 151(4), 156, 158, 160, 165, 166(3), 168, 182, 204, 206, 217, 222, 225, 230, 232, 233, 244, 247, 252, 255				

Table 1: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
142	512	19	*52/19*	*1/1*	*51/18*
	20(2), 23(2), 32(2), 60(10), 66(2), 67(2), 69(4), 71(3), 77(4), 78(4), 88, 90, 92, 109(2), 193(2), 195(2), 202(5), 205(2), 267*				
143	6144	20	*45/33*	*1/1*	*44/32*
	5, 9(2), 20, 23(2), 32, 57, 58, 60(5), 61, 62, 66(2), 67, 68, 69(2), 70, 71, 77(2), 78, 79, 81, 88, 90(4), 92, 109, 112, 164, 165, 193, 195, 202, 205, 208, 267*				
144	128	20	*4/2*	*1/1*	*3/1*
	60(3), 267*				
145	256	20	*64/6*	*1/1*	*63/5*
	60(21), 69(21), 77(7), 80(7), 209(7), 267*				
146	64	22	*1/1*	*1/1*	*0/0*
	267*				
147	128	20	*4/2*	*1/1*	*3/1*
	60(3), 267*				
148	128	22	*4/2*	*1/1*	*3/1*
	60(3), 267*				
149	128	22	*4/2*	*1/1*	*3/1*
	60(3), 267*				
150	384	20	*4/2*	*1/1*	*3/1*
	60(3), 267*				
151	1152	21	*2/2*	*1/1*	*1/1*
	60, 267*				
152	512	21	*66/43*	*0/0*	*66/43*
	18(2), 25(2), 32(2), 33(2), 34(3), 35(3), 91(6), 97(2), 99, 100(3), 105(2), 115(2), 117(2), 118, 119(2), 120, 129, 130, 132, 133(3), 144, 145, 146, 148, 149, 151, 159, 160, 164, 165, 200(2), 227, 229, 232, 233, 234, 236, 241, 244, 251, 252, 254, 255				
156	256	25	*8/7*	*0/0*	*8/7*
	35, 84, 91, 97, 170, 172(2), 214				
157	256	21	*68/48*	*0/0*	*68/48*
	86, 89, 95, 97, 98, 99, 105, 109(3), 116(2), 117(2), 118, 119, 122, 123, 128, 130, 133(2), 141, 144, 146(4), 147(2), 149(3), 150(2), 151, 155, 157(2), 159, 160(2), 162, 163(4), 164(2), 166, 170, 171, 179, 180, 216, 218, 219, 221(2), 223, 226, 227(2), 228, 232, 233, 248, 254				

Table 1: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
158	256	21	*68/47*	*0/0*	*68/47*
	85, 88, 96(2), 97, 100, 104, 106, 108, 109, 115(2), 116(2), 120(2), 121, 122, 129, 132(2), 133, 143, 145, 148(2), 149(3), 151(7), 158, 159, 160(4), 162, 163, 165(2), 166(4), 169, 172, 180, 182, 201, 204, 206, 210, 214, 220, 228, 229, 230, 232, 233, 235, 247, 252				
166	1024	20	*52/33*	*1/1*	*51/32*
	9(2), 20(2), 23(2), 33, 35, 58, 62(3), 63, 65, 66, 67(3), 68(2), 69(2), 72(3), 74, 75(2), 77, 85, 88(2), 90(2), 92(2), 93(4), 100, 104, 109, 114, 192*, 195(2), 204, 205, 212, 217, 262				
167	512	21	*52/40*	*1/1*	*51/39*
	18(3), 20(2), 22, 25(2), 33, 36, 83*, 85, 92, 100, 105(3), 111, 112, 114, 119, 120, 121, 122, 131, 132, 145(2), 146, 148, 149, 151, 155, 156(2), 157, 158(2), 166(2), 169(2), 180(2), 199, 200, 207, 210, 214, 223, 232, 237				
168	512	19	*66/33*	*0/0*	*66/33*
	20(2), 23(2), 32(6), 33(2), 35(2), 58, 59, 62(2), 66(5), 67(4), 68(3), 69(2), 70(2), 71, 72, 75(3), 76, 78(2), 79(2), 90, 91(2), 93, 94(2), 100, 109, 194, 195(2), 196, 204(2), 205(2), 212, 232(4), 247				
169	512	20	*48/21*	*0/0*	*48/21*
	5(2), 9(4), 59(5), 63, 68(2), 69(8), 70, 72(2), 74, 76, 77(2), 78(4), 79(4), 80, 81, 113(2), 132, 164(2), 165, 197(2), 212				
175	512	20	*2/2*	*0/0*	*2/2*
	4, 87				
177	64	22	*1/1*	*0/0*	*1/1*
	132				
178	64	22	*1/1*	*0/0*	*1/1*
	233				
181	384	21	*4/2*	*1/1*	*3/1*
	60(3), 267*				
182	1536	19	*76/29*	*1/1*	*75/28*
	20(2), 23(2), 32(4), 33(4), 34(2), 35(2), 60(3), 88, 90, 92, 99, 100, 193(2), 195, 196, 202(7), 203, 204, 205, 207, 209(2), 211, 216(4), 219(8), 227(8), 232(6), 241(6), 263, 267*				
183	512	21	*52/39*	*0/0*	*52/39*
	25(2), 86(2), 91(2), 99, 100(3), 102(2), 108(2), 119, 120, 121, 122, 129, 131, 132(2), 135, 137, 139(2), 142, 143, 145(2), 146, 148, 149, 151, 156, 158, 159, 160, 163, 166(3), 168, 169, 178, 182, 217(2), 251, 252, 254, 255				

Table 1: $(16,4,16,4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
184	256	21	*64/38*	*0/0*	*64/38*
	96(2), 98, 99, 106, 109, 119(2), 120(2), 121(2), 122, 123, 129(3), 131(2), 132, 133(2), 142, 143, 144(2), 145(4), 146(2), 147(2), 151(4), 157, 158, 159(2), 160(4), 161(2), 162, 165(3), 166(2), 169(2), 170, 172, 176, 178, 180, 182, 252, 254				
185	512	19	*52/32*	*0/0*	*52/32*
	20(2), 23(2), 33, 35, 58, 59, 62(2), 66(3), 67(3), 68(3), 69(4), 70(2), 71(2), 72, 74, 75(5), 76, 77(2), 79(2), 80, 90, 91, 93, 94, 100, 109, 194, 195, 204, 205, 212, 247				
186	512	21	*52/31*	*1/1*	*51/30*
	18, 19, 83*, 85, 86(2), 88(2), 89(3), 91, 94, 97, 101, 102, 104(2), 105(2), 108, 110, 112(3), 114(3), 115(2), 116(4), 117(6), 126(2), 127(2), 135, 137, 138, 139, 206, 247, 248, 256				
187	512	20	*48/23*	*0/0*	*48/23*
	5(2), 9(4), 58, 59, 61, 62(2), 63, 66(4), 67(2), 69(2), 70(3), 72(2), 73, 75(5), 77, 78(3), 79(4), 113(2), 131, 164(3), 196, 204, 212				
190	1536	20	*28/11*	*0/0*	*28/11*
	4(2), 9(4), 62(3), 67(4), 73(2), 74(3), 82(2), 87(2), 131(2), 132(2), 263(2)				
192	128	21	*4/2*	*1/1*	*3/1*
	60(3), 267*				
193	128	22	*4/2*	*1/1*	*3/1*
	60(3), 267*				
194	384	20	*4/2*	*1/1*	*3/1*
	60(3), 267*				
195	128	21	*4/2*	*1/1*	*3/1*
	60(3), 267*				
196	640	22	*4/2*	*1/1*	*3/1*
	60(3), 267*				
197	384	22	*2/2*	*1/1*	*1/1*
	60, 267*				
198	384	20	*4/2*	*1/1*	*3/1*
	60(3), 267*				
201	64	22	*1/1*	*0/0*	*1/1*
	20				
202	64	22	*1/1*	*0/0*	*1/1*
	23				
203	128	22	*4/4*	*0/0*	*4/4*
	97, 108, 206, 247				

Table 1: $(16,4,16,4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
204	64 20	22	*1/1*	*0/0*	*1/1*
206	64 9	22	*1/1*	*0/0*	*1/1*
207	128 90(2)	22	*2/1*	*0/0*	*2/1*
208	64 109	22	*1/1*	*0/0*	*1/1*
209	64 233	22	*1/1*	*0/0*	*1/1*
211	30720 18(4), 20(4), 23(4), 25(4), 32(8), 33(8), 34(4), 35(4), 36(4), 37(4), 60, 62, 88(2), 90(2), 92, 93, 99(2), 100(2), 102(4), 192*, 193(2), 194, 195(9), 196(17), 197(5), 198(22), 199(17), 200(7), 201(22), 202(7), 203(17), 204(15), 205(17), 206(62), 207(7), 208(9), 209(10), 210(128), 211(2), 212(4), 213(32), 214(34), 215(29), 216(32), 217(36), 218(31), 219(132), 220(124), 221(66), 222(66), 223(116), 224(19), 225(31), 226(34), 227(122), 228(65), 229(46), 230(19), 231(19), 232(135), 233(122), 234(122), 235(65), 236(34), 237(65), 238(19), 239(11), 240(34), 241(47), 242(15), 243(63), 244(66), 245(5), 262(2), 263(4), 264(6), 265(2), 267*	18	*2316/78*	*2/2*	*2314/76*
212	2048 5, 9(2), 20, 23(4), 32, 58(3), 59, 60(10), 61(3), 62(3), 64, 66(6), 67(3), 68(5), 69(6), 70, 71(2), 74, 76, 77(4), 78(3), 79(2), 80, 81, 88, 90(8), 92, 109, 112, 132, 164, 193(2), 195(2), 197, 202(2), 205(3), 208, 267*	20	*92/38*	*1/1*	*91/37*
213	2048 4, 5, 9(4), 59, 62(2), 64, 67(3), 68(5), 69(6), 70, 73, 74(3), 76, 77, 78(3), 79(2), 80, 81, 82, 87, 112, 131, 132(2), 164, 197, 208, 263	20	*48/27*	*0/0*	*48/27*
214	1024 4, 9(2), 24(2), 55*, 57, 59, 66(2), 68, 69(2), 71(3), 72, 75(2), 77, 80, 87, 88, 89, 197, 207, 208(2), 212(2), 222, 247	20	*32/23*	*1/1*	*31/22*
221	768 60(7), 69(7), 77(3), 80(3), 209(3), 267*	19	*24/6*	*1/1*	*23/5*
222	256 35, 90, 114, 131, 145, 169, 178, 222	22	*8/8*	*0/0*	*8/8*

Table 1: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_2 \times \mathbb{Z}_2$

#	Order	2-Rank	Total	Abelian	Nonabelian
223	192	22	*1/1*	*0/0*	*1/1*
	20				
224	768	20	*24/6*	*1/1*	*23/5*
	60(7), 69(7), 77(3), 80(3), 209(3), 267*				
225	256	20	*64/6*	*1/1*	*63/5*
	60(21), 69(21), 77(7), 80(7), 209(7), 267*				
226	320	21	*1/1*	*1/1*	*0/0*
	267*				

Table 2: $(16, 4, 16, 4)$ RDS with $N = \mathbb{Z}_4$

#	Order	2-Rank	Total	Abelian	Nonabelian
1	512	22	*14/9*	*0/0*	*14/9*
	58, 59(2), 61(2), 72(4), 85, 175, 208, 212, 262				
5	256	22	*16/10*	*0/0*	*16/10*
	59, 61, 63, 65, 70(4), 72(2), 195, 197(2), 204(2), 212				
6	512	22	*14/13*	*0/0*	*14/13*
	58, 59, 61, 63, 65, 70(2), 72, 85, 168, 194, 197, 204, 212				
8	512	22	*14/10*	*2/2*	*12/8*
	55*, 58(2), 59(3), 61, 70, 104, 142, 192*, 195, 196(2)				
9	7680	22	*9/8*	*2/2*	*7/6*
	55*, 57, 58(2), 61, 104, 143, 197, 260*				
10	128	22	*4/4*	*0/0*	*4/4*
	59, 72, 197, 212				
11	384	22	*2/2*	*1/1*	*1/1*
	55*, 70				
12	64	24	*1/1*	*0/0*	*1/1*
	238				
13	384	22	*2/2*	*0/0*	*2/2*
	57, 212				

Table 3: Structures of abelian regular subgroups in $\mathbb{Z}_2 \times \mathbb{Z}_2$ nets

#	Order	Abelian	Group structure
1	1105920	*6/2*	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2(4), \mathbb{Z}_2^6(2)$
2	13824	*2/2*	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2, \mathbb{Z}_2^6$
5	18432	*6/2*	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2(5), \mathbb{Z}_2^6$
9	4608	*2/2*	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2, \mathbb{Z}_2^6$
22	6144	*6/2*	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2(5), \mathbb{Z}_2^4 \times \mathbb{Z}_4$
31	1536	*1/1*	$\mathbb{Z}_2^4 \times \mathbb{Z}_4$
32	128	*1/1*	$\mathbb{Z}_2^3 \times \mathbb{Z}_8$
33	512	*1/1*	$\mathbb{Z}_2^3 \times \mathbb{Z}_8$
36	73728	*6/3*	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2(4), \mathbb{Z}_2^4 \times \mathbb{Z}_4, \mathbb{Z}_2^6$
132	512	*1/1*	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$
136	512	*1/1*	$\mathbb{Z}_4^3$
138	1536	*1/1*	$\mathbb{Z}_4^3$
166	1024	*1/1*	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$
167	512	*1/1*	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$
186	512	*1/1*	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$
211	30720	*2/2*	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2, \mathbb{Z}_2^6$
214	1024	*1/1*	$\mathbb{Z}_4^3$

Table 4: Structures of abelian regular subgroups in $\mathbb{Z}_4$ nets.

#	Order	Abelian	Group structure
8	512	*2 / 2*	$\mathbb{Z}_4^3, \mathbb{Z}_2^2 \times \mathbb{Z}_4^2$
9	7680	*2 / 2*	$\mathbb{Z}_4^3, \mathbb{Z}_2^4 \times \mathbb{Z}_4$
11	384	*1 / 1*	$\mathbb{Z}_4^3$

Table 5: All regular abelian subgroups of order 64 appearing in GDDs with bitranslation group $\mathbb{Z}_4$.

Group	Structure
55	$\mathbb{Z}_4^3$
192	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$
260	$\mathbb{Z}_2^4 \times \mathbb{Z}_4$

Table 6: All regular abelian subgroups of order 64 appearing in GDDs with bi-translation group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Group	Structure
55	$\mathbb{Z}_4^3$
83	$\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_8$
192	$\mathbb{Z}_2^2 \times \mathbb{Z}_4^2$
246	$\mathbb{Z}_2^3 \times \mathbb{Z}_8$
260	$\mathbb{Z}_2^4 \times \mathbb{Z}_4$
267	$\mathbb{Z}_2^6$

Table 7: Abelian groups which do not contain any $(16, 4, 16, 4)$ RDS

Group	Structure
1	$\mathbb{Z}_{64}$
2	$\mathbb{Z}_8 \times \mathbb{Z}_8$
26	$\mathbb{Z}_4 \times \mathbb{Z}_{16}$
50	$\mathbb{Z}_2 \times \mathbb{Z}_{32}$
183	$\mathbb{Z}_2^2 \times \mathbb{Z}_{16}$

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