3-31-2015

# The Catalan case of Armstrong's conjecture on simultaneous core partitions Read More: https://epubs.siam.org/doi/10.1137/130950318

Richard Stanley
*MIT*

Fabrizio Zanello
*Michigan Technological University*



Follow this and additional works at: https://digitalcommons.mtu.edu/math-fp
Part of the Mathematics Commons

# THE CATALAN CASE OF ARMSTRONG'S CONJECTURE ON SIMULTANEOUS CORE PARTITIONS

RICHARD P. STANLEY AND FABRIZIO ZANELLO

Abstract. A beautiful recent conjecture of D. Armstrong predicts the average size of a partition that is simultaneously an $s$-core and a $t$-core, where $s$ and $t$ are coprime. Our goal is to prove this conjecture when $t = s + 1$. These simultaneous $(s, s+1)$-core partitions, which are enumerated by Catalan numbers, have average size $\binom{s+1}{3}/2$.



## 1. Introduction and some simple cases

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ be a *partition of size* $n$, i.e., the $\lambda_i$ are weakly decreasing positive integers summing to $n$. We can represent $\lambda$ by means of its *Young* (or *Ferrers*) *diagram*, which consists of a collection of left-justified rows where row $i$ contains $\lambda_i$ cells. To each of these cells $B$ one associates its *hook length*, that is, the number of cells in the Young diagram of $\lambda$ that are directly to the right or below $B$ (including $B$ itself). Figure 1 represents the Young diagram of the partition $\lambda = (5, 3, 3, 2)$ of size 13; the number inside each cell represents its hook length.

Let $s$ be a positive integer. We say that $\lambda$ is an *$s$-core* if $\lambda$ has no hook of length equal to $s$ (or equivalently, equal to a multiple of $s$). For instance, from Figure 1 we can see that $\lambda = (5, 3, 3, 2)$ is an $s$-core for $s = 6$ and for all $s \geq 9$. Finally, $\lambda$ is an *$(s,t)$-core* if it is simultaneously an $s$-core and a $t$-core.

The theory of $(s,t)$-cores has been the focus of much interesting research in recent years (see [5, 6, 8] for some of the main results). In particular, when $s$ and $t$ are coprime, there exists only a finite number of $(s,t)$-core partitions. In fact, there are exactly $\binom{s+t}{s}/(s+t)$ such cores (see [5]), the largest of which has size $(s^2-1)(t^2-1)/24$ [8]. More generally, a nice result of J. Anderson [5] provides a bijective correspondence between $(s,t)$-cores and order ideals of the poset of the positive integers that are not contained in the numerical semigroup generated by $s$ and $t$, which we write as $P_{(s,t)}$. The partial order on $P_{(s,t)}$ is determined by specifying that $a$ covers $b$ whenever $a - b$ equals either $s$ or $t$. (Our poset terminology follows [9, Chap. 3].)

For instance, let $s = 3$ and $t = 5$. Then $P_{(3,5)} = \{1, 2, 4, 7\}$, where $7 > 4 > 1$ and $7 > 2$. Figure 2 represents the Hasse diagram of the poset $P_{(3,5)}$, rotated 45° counterclockwise from

2010 *Mathematics Subject Classification.* Primary: 05A15; Secondary: 05A17, 05A19, 20M99.

*Key words and phrases.* Integer partition; core partition; Catalan number; numerical semigroup; Ferrers diagram; hook length.

| | | | | |
|---|---|---|---|---|
| 8 | 7 | 5 | 2 | 1 |
| 5 | 4 | 2 | | |
| 4 | 3 | 1 | | |
| 2 | 1 | | | |

FIGURE 1. The Young diagram of $\lambda = (5, 3, 3, 2)$. The number inside each cell indicates its hook length.

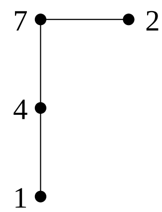


FIGURE 2. The Hasse diagram of the poset $P_{(3,5)}$.

the usual convention. The order ideals of $P_{(3,5)}$ are the following $\frac{1}{3+5}\binom{3+5}{3} = 7$ subsets: $\emptyset, \{1\}, \{2\}, \{2, 1\}$, $\{4, 1\}$, $\{4, 2, 1\}$, and $\{7, 4, 2, 1\}$. Notice that from this diagram it is clear that if an element $a$ of $P_{(3,5)}$ belongs to a given order ideal $I$, then all elements immediately to the right or below $a$ also belong to $I$.

Anderson's result then gives that $(s, t)$-cores correspond bijectively to the order ideals of $P_{(s,t)}$ by associating the ideal $\{a_1, \dots, a_j\}$, where $a_1 > \cdots > a_j$, to the $(s, t)$-core partition $(a_1 - (j - 1), a_2 - (j - 2), \dots, a_{j-1} - 1, a_j)$. For instance, the $(3, 5)$-cores are the following seven partitions: $\emptyset$ (corresponding to the order ideal $\emptyset$ of $P_{(3,5)}$), $(1)$ (corresponding to $\{1\}$), $(2)$ (corresponding to $\{2\}$), $(1, 1)$ (corresponding to $\{2, 1\}$), $(3, 1)$ (corresponding to $\{4, 1\}$), $(2, 1, 1)$ (corresponding to $\{4, 2, 1\}$), and $(4, 2, 1, 1)$ (corresponding to $\{7, 4, 2, 1\}$).

The following conjecture of D. Armstrong, informally stated sometime in 2011 and then recently published in [6], predicts, for any $s$ and $t$ coprime, a surprisingly simple formula for the average size of an $(s, t)$-core.

**Conjecture 1.1.** *For any coprime positive integers $s$ and $t$, the average size of an $(s, t)$-core is $(s + t + 1)(s - 1)(t - 1)/24$. Equivalently, the sum of the sizes of all $(s, t)$-cores is*

$$\frac{(s + t + 1)(s - 1)(t - 1)}{24(s + t)}\binom{s + t}{s}. \tag{1}$$

For instance, the seven $(3, 5)$-cores computed above are of size $0, 1, 2, 2, 4, 4$, and $8$, with average size 3, as predicted by Armstrong's conjecture.

One of the interesting aspects of this conjecture, besides the partition-theoretic result that it predicts, is the extra combinatorial information that it would imply on numerical semigroups generated by two elements. In fact, even though one would generally expect these semigroups to be very well understood, Armstrong's conjecture had until now resisted all attempts of significant progress.

The main goal of this paper is to show the conjecture in what is probably its most interesting case, namely that of $(s, s+1)$-cores. The number of these cores is the *Catalan number* $C_s := \frac{1}{s+1}\binom{2s}{s}$, and the corresponding posets $P_{(s,s+1)}$ present a particularly nice structure, which will allow us to use induction in the proof.

We now wrap up this first section by briefly discussing Armstrong's conjecture in a few initial cases. For any given $s$, in principle the conjecture can be verified computationally for all $(s, t)$-cores, given how explicitly one can determine these cores by means of Anderson's bijection. In fact, the authors of [6] indicate that C. Hanusa has verified the conjecture for small values of $s$, though they provide no details in the paper. (We thank C. Hanusa for subsequently informing us that he had checked the conjecture on Mathematica for all $(s, ms+1)$-cores and $(s, ms-1)$-cores, when $s \leq 10$.) We also wish to remark here that, since this paper has appeared as a preprint, lots of work has already been done that has applied or extended our ideas. For instance, for a nice proof of the case $(s, ms+1)$ of Armstrong's conjecture, for arbitrary $s$ and $m \geq 1$, see [3], while for two interesting works on multiple simultaneous cores, see [4, 11]. Instead, for a complete proof of the analogous of Armstrong's conjecture for the case of self-conjugate $(s, t)$-cores (also stated in [6]), see [7].

In this section, we will just present a short proof of the case $(3, t)$ of Armstrong's conjecture (the case $(2, t)$ being trivial), which also gives us the opportunity to state a simple but useful fact on arbitrary $(s, t)$-cores that seems to have not yet been recorded in the literature. We will provide this lemma without proof, since the argument is analogous to the classical proof that if $\lambda$ is an $s$-core, then it is also an $ms$-core, for all $m \geq 1$ (see e.g. the first author's [10], Exercise 7.60 and its solution on pp. 518–519). In principle, the use of this lemma would considerably simplify a "brute-force" proof for any given $s$, and indeed the case $s = 4$ is still relatively quick to prove along the same lines; nonetheless, for higher values of $s$ the computations remain extremely unpleasant.

**Lemma 1.2.** *If a partition $\lambda$ is an $(s, t)$-core, then it is also an $(s, s+t)$-core.*

**Proposition 1.3.** *Armstrong's conjecture holds for all $(3, t)$-cores.*

*Proof.* Let $s = 3$. We will show Formula (1) for $t = 3n-2$, the case $t = 3n-1$ being entirely similar. Notice that by Lemma 1.2 all $(3, 3n-2)$-cores are $(3, 3n+1)$-cores. Thus by induction, proving the result is now equivalent to showing that the sum of the sizes of the $(3, 3n+1)$-cores that are *not* also $(3, 3n-2)$-cores is the difference between the two total sums predicted by Armstrong's conjecture, namely

$$\Delta(n) = \frac{(3n+5)\cdot 2\cdot 3n}{24(3n+4)}\binom{3n+4}{3} - \frac{(3n+2)\cdot 2\cdot(3n-3)}{24(3n+1)}\binom{3n+1}{3} = \binom{3n+2}{3}.$$

Figure 3 represents the Hasse diagrams of $P_{(3,10)}$ and $P_{(3,13)}$. From these diagrams, we can see that the order ideals of $P_{(3,13)}$ that are not also in $P_{(3,10)}$ are exactly the six principal

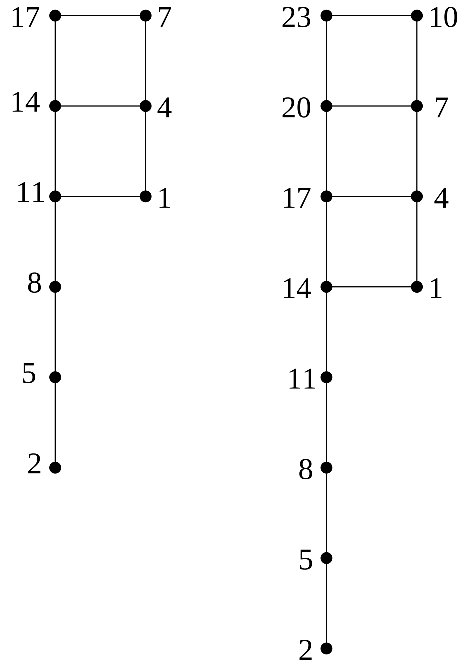


FIGURE 3. The Hasse diagrams of $P_{(3,10)}$ (on the left) and $P_{(3,13)}$.

ideals generated by 11, 14, 17, 20, 23, and 10, plus the seven ideals generated by $\{2,10\}$, $\{5,10\}$, $\{8,10\}$, $\{11,10\}$, $\{14,10\}$, $\{17,10\}$, and $\{20,10\}$.

In a similar fashion, it can be seen that the order ideals of $P_{(3,3n+1)}$ but not of $P_{(3,3n-2)}$ are exactly the $n+2$ principal ideals generated by $3n-1$, $3n+2$, ..., $6n-1$, and $3n-2$, and the $2n-1$ ideals generated by $\{2,3n-2\}$, $\{5,3n-2\}$, ..., $\{6n-4,3n-2\}$.

A standard computation now gives that $\Delta(n)$, i.e., the sum of the elements of the above order ideals $I$ minus $\binom{\#I}{2}$, where $\#I$ denotes the cardinality of $I$, is given by

$$\Delta(n) = (2+5+\cdots+(3n-1)) + \sum_{i=n}^{2n-1}[(2+5+\cdots+(3i+2))+(1+4+\cdots+3(i-n)+1)]$$

$$+2n(1+4+\cdots+(3n-2)) + \sum_{i=0}^{2n-2}(2+5+\cdots+(3i+2))$$

$$-\left[\binom{n}{2} + \sum_{i=n}^{2n-1}\binom{2i-n+2}{2} + \sum_{i=0}^{2n-1}\binom{n+i}{2}\right].$$

Showing now that the right-hand-side is equal to $\binom{3n+2}{3}$ is a routine task that we omit. This completes the proof. □

We only remark here that using Lemma 1.2, Armstrong's conjecture can also be verified relatively quickly for $s=4$, i.e., for all $(4,2n+1)$-cores (though the computations are of course already much more tedious than for $s=3$). In fact, by Formula (1), in this case one has to show that the sum of the sizes of all $(4,2n+1)$-cores equals $S(n) := (4n+6)\binom{n+3}{4}$.

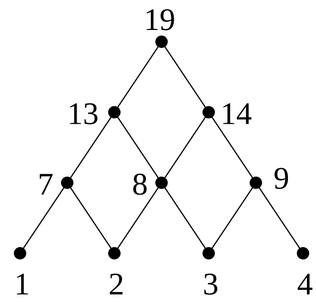


FIGURE 4. The Hasse diagram of $T_5$.

It is easy to check (see also [1]) that, for all $n \geq 7$, the sequence $S(n)$ satisfies the following curious recursive relation:

$$\sum_{i=0}^{6} (-1)^i \binom{6}{i} S(n-i) = 0.$$

It would be very interesting to combinatorially explain this identity in the context of $(4, 2n+1)$-cores, and thus give an elegant proof of Armstrong's conjecture for $s = 4$.

## 2. The Catalan case

The goal of this section is to show Armstrong's conjecture for $(s, s+1)$-cores. We denote by $T_s := P_{(s,s+1)}$ the corresponding poset according to Anderson's bijection [5]. For simplicity, we will draw the Hasse diagram of $T_s$ from top to bottom; thus, each element of $T_s$ covers the two elements immediately below, and the elements increase by $s$ at each step up and to the left, and by $s+1$ at each step up and to the right. (See Figure 4 for the Hasse diagram of $T_5$.)

Let us define the functions

$$\begin{aligned} g_j &:= \frac{j(j-1)}{12}\binom{2j}{j}, \\ f_j &:= \frac{j^2+5j+2}{8j+4}\binom{2j+2}{j+1} - 4^j, \\ h_j &:= 2^{2j-1} - \binom{2j+1}{j} + \binom{2j-1}{j-1}, \end{aligned}$$

where by convention we set $h_0 := 0$. We need the following two identities. We thank Henry Cohn for verifying them for us on Maple.

**Lemma 2.1.**

$$f_s = \sum_{i=1}^{s} C_{s-i}(2f_{i-1} + h_{i-1}).$$

*Proof.* This is the Maple code that verifies the identity (it gives 0 as output):

```
g := j -> binomial(2*j,j)*j*(j-1)/12;
f := j -> binomial(2*j+2,j+1)*(j^2+5*j+2)/(8*j+4)-4^j;
```

```
h := j -> 2^(2*j-1)-binomial(2*j+1,j)+binomial(2*j-1,j-1);
C := j -> binomial(2*j,j)/(j+1);
simplify(sum(C(s-i)*(2*f(i-1)+h(i-1)),i=2..s)-f(s));
```
□

**Lemma 2.2.**

$$g_s = \sum_{i=1}^{s} 2C_{s-i}g_{i-1} + 2(s-i+1)C_{s-i}f_{i-1} + (s-i+3)C_{s-i}h_{i-1} + (i-1)C_{s-i}C_{i-1} - h_{s-i}h_{i-1}.$$

*Proof.* This is the Maple code that verifies the identity (it gives 0 as output):

```
g := j -> binomial(2*j,j)*j*(j-1)/12;
f := j -> binomial(2*j+2,j+1)*(j^2+5*j+2)/(8*j+4)-4^j;
h := j -> 2^(2*j-1)-binomial(2*j+1,j)+binomial(2*j-1,j-1);
C := j -> binomial(2*j,j)/(j+1);
simplify(sum(2*C(s-i)*g(i-1)+2*(s-i+1)*C(s-i)*f(i-1)+(s-i+3)*C(s-i)*h(i-1) + (i-1)*C(s-i)*C(i-
1)-h(s-i)*h(i-1),i=2..s-1) + 2*C(0)*g(s-1)+2*(1)*C(0)*f(s-1)+(3)*C(0)*h(s-1) + (s-1)*C(0)*C(s-
1) - g(s));
```
□

**Theorem 2.3.** *Armstrong's conjecture holds for all $(s, s+1)$-cores.*

*Proof.* For any given $s$, and for any weight function $w : T_s \to \mathbb{Z}$, define the two functions

$$f_s(w) := \sum_{I \in J(T_s)} \sum_{a \in I} w(a),$$

$$g_s(w) := \sum_{I \in J(T_s)} \left( \sum_{a \in I} w(a) - \binom{\#I}{2} \right) = f_s(w) - \sum_{I \in J(T_s)} \binom{\#I}{2},$$

where as usual $J(P)$ denotes the set of order ideals of a poset $P$.

We consider three weight functions on $T_s$. The weight $\sigma$ is the "standard weight" that associates, to each element of $T_s$, itself as a weight; i.e., $\sigma(a) = a$, for all $a \in T_s$. The weight $\tau$ is identically 1; i.e., $\tau(a) = 1$, for all $a \in T_s$. Finally, $\rho$ records the ranks of the elements of $T_s$, when we see this latter as a ranked poset whose minimal elements have rank 0. Figure 5 represents the Hasse diagrams of $T_5$, where the elements are being weighted according to $\tau$ and $\rho$.

Showing Armstrong's conjecture for $(s, s+1)$-cores in the form of Formula (1) is tantamount to proving that

$$g_s(\sigma) = g_s = \frac{s(s-1)}{12}\binom{2s}{s}.$$

Notice that the elements of rank 0 of $T_s$ are $1, 2, \ldots, s-1$. We can partition $J(T_s)$ as $J(T_s) = \bigcup_i J_i(T_s)$, where $J_i(T_s)$ is the set of those order ideals of $T_s$ whose least element that they do *not* contain is $i$. Notice that either $1 \le i \le s-1$, or we are considering order ideals

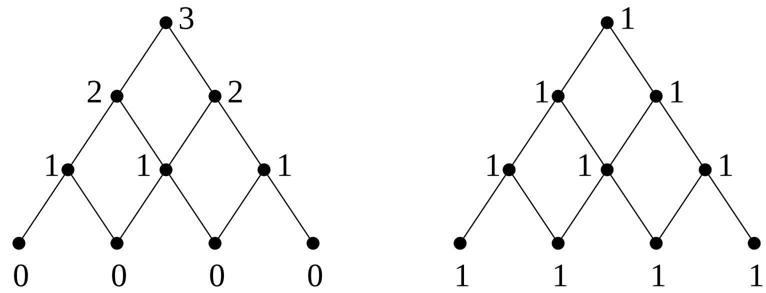


FIGURE 5. The poset $T_5$ with weights $\rho(a)$ on the left and $\tau(a)$ on the right.

whose least missing element $i$ (if any) has positive rank. With some abuse of notation, in this latter case we set by convention $i := s$, so that we can write

$$J(T_s) = \bigcup_{i=1}^{s} J_i(T_s).$$

Notice that, given $i$, the elements $I$ of $J_i(T_s)$ *must* contain all of $1, 2, \ldots, i-1$, *cannot* contain any element covering $i$ (this is an empty condition for $i = s$), and *may or may not* contain any other element. Figure 6 gives the Hasse diagram of $T_{10}$; for $i = 5$, it indicates by squares the elements of $T_{10}$ that must belong to any given order ideal $I \in J_5(T_{10})$, by open circles the elements that cannot be in $I$, and by solid circles the elements that may or may not be in $I$.

It follows that any given order ideal $I \in J_i(T_s)$ can be partitioned into the disjoint union of two order ideals, say $I_1$ and $I_2$, plus the elements $1, 2, \ldots, i-1$. Notice that, in the Hasse diagram of $T_s$, $I_1$ belongs to a poset that is isomorphic to $T_{i-1}$ and sits to the left of $i$ (starting in rank one), and $I_2$ belongs to a poset that is isomorphic to $T_{s-i}$ and sits to the right of $i$. The posets $T_1$ and $T_0$, which arise when $i = 1$, $i = 2$, $i = s-1$, or $i = s$, are empty. (See again Figure 6 for the case $n = 10$ and $i = 5$.)

Given this, it is a simple exercise to show that the sum of the elements of $T_s$ that belong to a given order ideal $I = I_1 \cup I_2 \cup \{1, 2, \ldots, i-1\} \in J(T_s)$ is given by:

$$\sum_{a \in I} \sigma(a) = \sum_{a \in I_1} w(a) + \sum_{a \in I_2} w(a) + \binom{i}{2},$$

where the weight function $w$ is defined as

$$w := \sigma + (s+1)\tau + (s-i+1)\rho$$

over $I_1$, and by

$$w := \sigma + i\tau + i\rho$$

over $I_2$. Further, notice that, given $i$, we can choose the order ideals $I_1 \in J(T_{i-1})$ and $I_2 \in J(T_{s-i})$ independently. Therefore, the elements $a \in I_1$ will appear a total of $C_{s-i}$ times in the order ideals $I$ of $T_s$, and similarly, the elements $a \in I_2$ will appear a total of $C_{i-1}$ times in the order ideals $I$ of $T_s$.

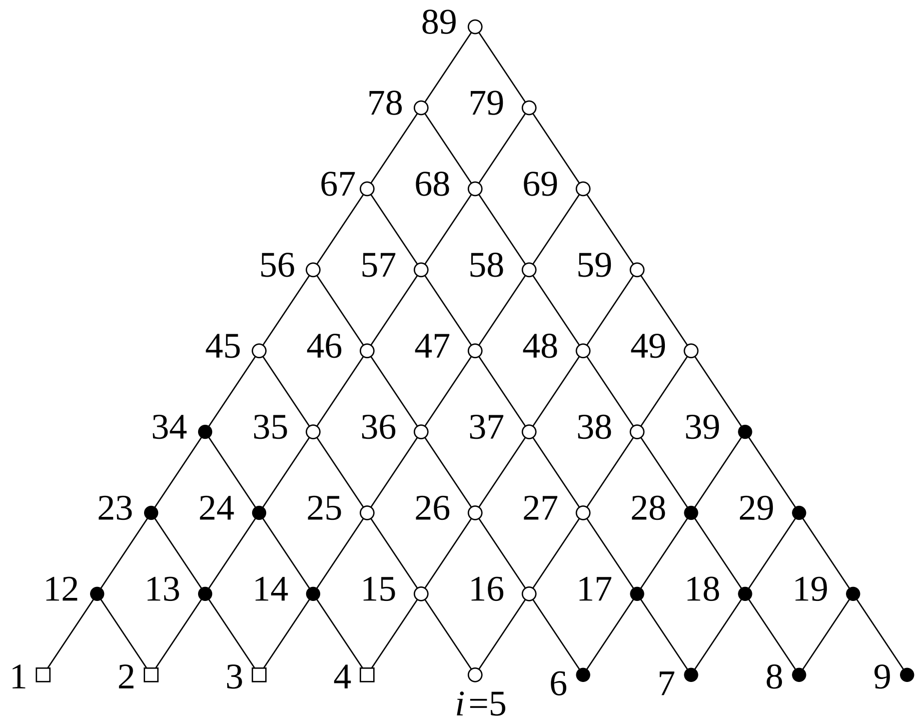


FIGURE 6. The possible elements of the order ideals $I \in J_5(T_{10})$. Elements that must appear in $I$ are indicated by squares, that cannot appear by open circles, and that may or may not appear by solid circles.

Therefore, the contribution of any given $i$ to the desired function $g_s(\sigma)$ is given by

$$m(i,s) - \sum_{I_1 \in J(T_{i-1}), I_2 \in J(T_{s-i})} \binom{\#I_1 + \#I_2 + i - 1}{2}, \tag{2}$$

where we have

$$\begin{aligned} m(i,s) :=& \sum_{I_1 \in J(T_{i-1})} C_{s-i} \left( \sum_{a \in I_1} w(a) + \binom{i}{2} \right) + \sum_{I_2 \in J(T_{s-i})} C_{i-1} \sum_{a \in I_2} w(a) \\ =& \, C_{s-i}(f_{i-1}(\sigma) + (s+1) f_{i-1}(\tau) + (s-i+1) f_{i-1}(\rho)) \\ &+ C_{s-i} C_{i-1} \binom{i}{2} + C_{i-1}(f_{s-i}(\sigma) + i f_{s-i}(\tau) + i f_{s-i}(\rho)). \end{aligned}$$

Let us now consider, again for a fixed $i$, the term that is being subtracted in Formula (2). Notice that

$$\binom{\#I_1 + \#I_2 + i - 1}{2} = \binom{\#I_1}{2} + \binom{\#I_2}{2} + (i-1)\#I_1 + (i-1)\#I_2 + (\#I_1)(\#I_2) + \binom{i-1}{2}.$$

Thus, once we sum over all $I_1$ and $I_2$, similar considerations to the above on the number of such order ideals give us that

$$\sum_{I_1\in J(T_{i-1}),I_2\in J(T_{s-i})}\binom{\#I_1+\#I_2+i-1}{2}=\sum_{I_1\in J(T_{i-1})}C_{s-i}\left(\binom{\#I_1}{2}+(i-1)\#I_1\right)$$

$$+\sum_{I_2\in J(T_{s-i})}C_{i-1}\left(\binom{\#I_2}{2}+(i-1)\#I_2\right)+\left(\sum_{I_1\in J(T_{i-1})}\#I_1\right)\left(\sum_{I_2\in J(T_{s-i})}\#I_2\right)+C_{s-i}C_{i-1}\binom{i-1}{2}.$$

Essentially by definition, we have $\sum_{I_1\in J(T_{i-1})}\#I_1=f_{i-1}(\tau)$, and likewise, $\sum_{I_2\in J(T_{s-i})}\#I_2=f_{s-i}(\tau)$. Also, it is a known fact (see e.g. [2]) that the function $f_j(\tau)$ appearing in the above formula for $m(i,s)$ satisfies

$$f_j(\tau)=2^{2j-1}-\binom{2j+1}{j}+\binom{2j-1}{j-1}.$$

As for determining $f_j(\rho)$, by employing the above decomposition of the order ideals $I$ and summing over all $i$, with a similar argument we can see that:

$$f_s(\rho)=\sum_{i=1}^{s}C_{s-i}(f_{i-1}(\rho)+f_{i-1}(\tau))+C_{i-1}f_{s-i}(\rho),$$

which, by rearranging the indices, yields:

$$f_s(\rho)=\sum_{i=1}^{s}C_{s-i}(2f_{i-1}(\rho)+f_{i-1}(\tau)).$$

Therefore, by induction, if we apply Lemma 2.1 with $f_j=f_j(\rho)$ and $h_j=f_j(\tau)$, we promptly get the following formula for $f_j(\rho)$:

$$f_j(\rho)=\frac{j^2+5j+2}{8j+4}\binom{2j+2}{j+1}-4^j.$$

Finally, notice that $g_{i-1}(\sigma)=f_{i-1}(\sigma)-\sum_{I_1\in J(T_{i-1})}\binom{\#I_1}{2}$, and similarly for $g_{s-i}(\sigma)$.

Therefore, by Formula (2) and the subsequent formula for $m(i,s)$, if we sum over $i=1,2,\ldots,s$, after some tedious but routine computations (that include rearranging the indices where necessary) we obtain:

$$g_s(\sigma)=\sum_{i=1}^{s}[2C_{s-i}g_{i-1}(\sigma)+2(s-i+1)C_{s-i}f_{i-1}(\rho)$$

$$+(s-i+3)C_{s-i}f_{i-1}(\tau)+(i-1)C_{s-i}C_{i-1}-f_{i-1}(\tau)f_{s-i}(\tau)].$$

The theorem now follows by induction on $s$, if we apply Lemma 2.2 with $f_j=f_j(\rho)$, $g_j=g_j(\sigma)$, and $h_j=f_j(\tau)$. $\square$

## 3. Acknowledgements

The second author warmly thanks the first for his hospitality during calendar year 2013. The two authors wish to thank the referee for a careful reading of their manuscript and several comments that improved the presentation, and Henry Cohn for verifying the identities of Lemmas 2.1 and 2.2 on the computer package Maple. The present work was done while the first author was partially supported by the National Science Foundation under Grant No. DMS-1068625, and the second author by a Simons Foundation grant (#274577).

## References

[1] A005585, On-Line Encyclopedia of Integer Sequences, http://oeis.org/A005585.

[2] A006419, On-Line Encyclopedia of Integer Sequences, http://oeis.org/A006419.

[3] A. Aggarwal: *Armstrong's conjecture for* $(k, mk+1)$*-core partitions*, preprint. Available on the arXiv.

[4] T. Amdeberhan and E. Leven: *Multi-cores, posets, and lattice paths*, preprint. Available on the arXiv.

[5] J. Anderson: *Partitions which are simultaneously* $t_1$*- and* $t_2$*-core*, Discrete Math. **248** (2002), 237–243.

[6] D. Armstrong, C.R.H. Hanusa, and B. Jones: *Results and conjectures on simultaneous core partitions*, European J. Combin. **41** (2014), 205–220.

[7] W.Y.C. Chen, H.H.Y. Huang, and L.X.W. Wang: *Average size of a self-conjugate* $(s,t)$*-core partition*, preprint. Available on the arXiv.

[8] J. Olsson and D. Stanton: *Block inclusions and cores of partitions*, Aequationes Math. **74** (2007), 90–110.

[9] R. P. Stanley, "Enumerative Combinatorics", vol. 1, second ed., Cambridge University Press, Cambridge (2012).

[10] R. P. Stanley, "Enumerative Combinatorics", vol. 2, Cambridge University Press, Cambridge (1999).

[11] J.Y.X. Yang, M.X.X. Zhong, and R.D.P. Zhou: *On the enumeration of* $(s, s+1, s+2)$*-core partitions*, preprint. Available on the arXiv.

Department of Mathematics, MIT, Cambridge, MA 02139-4307
*E-mail address*: rstan@math.mit.edu

Department of Mathematical Sciences, Michigan Tech, Houghton, MI 49931-1295
*E-mail address*: zanello@mtu.edu