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# THEISM, PLATONISM, AND THE METAPHYSICS OF MATHEMATICS

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In a previous paper, Thomas V. Morris and I sketched a view on which abstract objects, in particular, properties, relations, and propositions (**PRPs**), are created by God no less than contingent, concrete objects. In this paper I suggest a way of extending this account to cover mathematical objects as well. Drawing on some recent work in logic and metaphysics, I also develop a more detailed account of the structure of **PRPs** in answer to the paradoxes that arise on a naive understanding of the structure of the abstract universe.

## §1 *The Dilemma of the Theistic Platonist*

Theists generally hold that God is the creator of all there is distinct from himself. Traditionally, the scope of God's creative activity has extended across two disjoint realms: The physical, whose chief exemplars are ordinary middle-size objects, and the mental or spiritual, encompassing such things as angels and souls. Now, many theists are also platonists, or "metaphysical realists." That is, in addition to the physical and mental realms, some theists also acknowledge the existence of a realm of *abstract* objects like numbers, sets, properties, and propositions. For such theists, the traditional understanding of creation presents a philosophical and theological dilemma. On the platonist conception, most, if not all, abstract objects are thought to exist necessarily. One can either locate these entities outside the scope of God's creative activity or not. If the former, then it seems the believer must compromise his view of God: rather than the sovereign creator and lord of all things visible and invisible, God turns out to be just one more entity among many in a vast constellation of necessary beings existing independently of his creative power. If the latter, the believer is faced with the problem of what it could possibly mean for God to *create* an object that is both necessary and abstract.

Though not utterly incompatible with a robust theism, the first horn of the dilemma seems undesirable for the theistic platonist. God remains the greatest possible being, since, presumably, it is not possible for any being to exert any influence over abstract entities; but the platonist must put severe, indeed embarrassing, qualifications on the scope of God's creative activity and on his status as the source of all existence. This leaves the platonist with the second horn,

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*viz.*, making sense of the idea of creation with respect to necessary, abstract objects. In a separate paper, Tom Morris and I have argued that sense can indeed be made of this idea.[1]

An early obstacle to avoid is the (roughly) deistic model of the creation. On this model, God's creation of an object is conceived of as a distinct, isolable, temporally located event of (at most) finite duration, and after its occurrence the object continues to exist of its own ontological momentum. This model is no doubt incompatible with the thesis at hand, since there could be no time at which God could have brought a necessary, hence eternal, being into existence. But there is just no *a priori* reason whatever to rule out the possibility of eternal creatures. Hence, a more subtle model is needed.

A model that is both philosophically and theologically sounder on this point is that of continuous creation. In its most fundamental sense, to create something is to cause it to be, to play a direct causal role in its existing.[2] It is this broader conception of creation that is incorporated in the model in question. On that model, God is always playing a direct causal role in the existence of his creatures; his creative activity is essential to a creature's existence at *all* times throughout its temporal career, irrespective of whether or not there happens to be a specific time at which he *begins* to cause it to exist.[3] This then provides us with a framework in which it can be coherently claimed that God creates absolutely all objects, necessary or not: one simply holds that, necessarily, for any object *a* (other than God himself) and time *t*, God plays a direct causal role in *a's* existing at *t*.[4]

With that obstacle removed, the next step is to give some account of the sort of causal relation there could be between God and the abstract objects we want to claim he creates. Here Morris and I essentially just reclothe the venerable doctrine of divine ideas in contemporary garb. We call the refurbished doctrine "theistic activism." Very briefly, the idea is this. We take properties, relations, and propositions (**PRP**s, for short) to be the contents of a certain kind of divine intellective activity in which God, by his nature, is essentially engaged. To grasp a **PRP**, then, whether by abstracting from perceptual experience, or perhaps by "combining" **PRP**s already grasped, is to grasp a product of the divine intellect, a divine idea. This divine activity is thus causally efficacious: **PRP**s, as abstract products of God's "mental life," exist at any given moment *because* God is thinking them;[5] which is just to say that he creates them.

## §2 *Activism, Numbers, and Sets*

With regard to **PRP**s then, the activist model provides a coherent, substantive (if programmatic) account of the sort of activity in which God is engaged that gives rise to abstract objects. On the face of it, though, a substantial, indeed perhaps the most important, portion of the abstract universe is not obviously accounted for, *viz.*, the abstract ontology of mathematics. In particular, how do numbers and sets fit into the picture? Traditionally, these entities are not thought of as **PRP**s of any kind, and hence they find no clear place in the story thus far.

I want to argue that a place can be found, though the search is going to lead us into some fairly deep waters. Let's begin with numbers. A natural view of the numbers, as I will argue below, is that they are properties. For the past century, however, the dominant view of the numbers has been that they are abstract *particulars* of one kind or another. The philosophical roots of this view go back to Frege, in particular, to one of his most distinctive doctrines: that there is an inviolable ontological divide between the denotations (*Bedeutungen*) of predicates, or concepts (*Begriffe*), and the denotations of singular terms, or objects (*Gegenstände*). Taking 'property' to be a loose synonym for 'concept', this doctrine entails that no property can be denoted by a singular term. Since in mathematics the numbers are in fact denoted by singular terms, e.g., most saliently, the numerals, it follows that the numbers are not properties but objects, or, loosely once again, particulars.

Now, while few would dispute the inviolability of the distinction between properties and particulars, there are well known and notorious difficulties with the idea that this is simultaneously a distinction between the semantic values of predicates and the semantic values of singular terms.[6] We needn't rehearse these in detail here. For present purposes, let's just note first that our ordinary usage itself doesn't easily square with Frege's doctrine. For there exist a prodigious number of singular terms in natural language that, to all appearances, refer straightforwardly to properties: abstract singular terms ('wisdom', 'redness'), infinitives ('to dance', 'to raise chickens'), gerunds ('being faster than Willie Gault', 'running for president'), etc. Now, there are several non-Fregean semantic theories in which this fact of natural language is preserved, i.e., theories that are *type-free* in the sense that difference in ontological type (property, particular, etc.) is not reflected in a difference of semantic type (referent of predicate, referent of singular term, etc.) that prevents singular terms from referring to properties.[7] Hence, we can consistently maintain, *pace* Frege, that at least some properties are the semantic values of both predicates and singular terms. There are thus good reasons for rejecting Frege's semantic doctrine, and hence no cogent reasons for rejecting the idea that numbers are properties, *a priori* on Fregean grounds.

There are two plausible accounts that identify the numbers with properties, both of which trace their origins back to the beginnings of contemporary mathematical logic. The first extends back to Frege himself. Frege clearly saw that statements of number typically involve the predication of a numerical property of some kind. For Frege, what is involved is the predication of such a property to a *concept*. Thus, for example, the statement "There are four moons of Jupiter" is the predication of the property *having four instances* (which is expressed by the quantifier 'There are four') of the concept *moons of Jupiter*. Frege's concept/object doctrine however prevented him from taking this property itself to be the number four, assigning that function rather to its extension.[8] As we've just seen, though, one needn't follow Frege here. In the absence of this doctrine, one is free to make the identification in question, and hence, in general, to take the number *n* to be the property of having *n* instances.[9]

Cantor suggested a different though related view in his (often entirely opaque) discussions of the nature of number. Cantor's insight was that the notion of number is understood most clearly in terms of a special relation between *sets*: we associate the same number with two sets just in case they are "equivalent," i.e., just in case a one-to-one correspondence can be established between the members of the sets. In assigning the same number to two sets then, we are isolating a common property the two sets share; such properties (roughly) Cantor identifies as the cardinal numbers. In his words, the cardinal number of a given set **M** is

> the general concept under which fall all and only those sets which are equivalent to the given set.[10]

More precisely, the number *n* is a property common to all and only *n* membered sets, or more simply, the property of having *n* members.[11] Russell presents essentially the same account in *The Principles of Mathematics*, though he ultimately rejects it (unnecessarily, as it happens) because of problems he finds with its nonextensional character.[12]

There are thus at least two ways of understanding the numbers to be properties, both of which are natural and appealing. The first, quasi-Fregean account has a semantical bent, focussing in particular on the predicative nature of statements of number, while the Cantorian account emphasizes the intuitive connection between number and relative size in the more general, abstract form of one-to-one correspondence between sets. Both, however, provide us with good reasons for thinking that numbers are properties of some kind. If so, we have found room for numbers within our theistic framework.

Now, of course, if we adopt the Cantorian view of number, then it obviously remains to explain just how *sets* fit into our picture. We could avoid this question by choosing instead the quasi-Fregean picture, since it makes no appeal to sets. We will not so choose, however, for two reasons. First, for reasons I will not go into here, I think the Cantorian view is the superior of the two accounts. Second, perhaps more importantly, we want to be able to give our picture the broadest possible scope, and hence we want it to encompass all manner of abstract flora and fauna whose existence platonism might endorse.

So what then are we to say about sets? Are they too assimilable into our framework as it stands? Formally, yes. Both George Bealer and Michael Jubien, for instance, have developed theories in which sets are identified with certain "set-like" properties; roughly, a set $\{a,b,\ldots\}$ is taken to be the property **being identical with *a* or being identical with *b* or** … .[13] Given sufficiently powerful property theoretic axioms, one can then show that analogues of the usual axioms of Zermelo-Fraenkel set theory hold for these set-like properties, and hence that one loses none of the mathematical power of standard set theory.

But this is not an altogether happy move. For instance, as Cocchiarella has noted, intuitively, sets are just not the same sort of thing as properties. Sets are generally thought of as being wholly constituted by their members; a set, "has its being in the objects which belong to it."[14] This conception is deeply at odds with the view of **PRP**s that underlies metaphysical realism, according to which **PRP**s "are in no sense to be thought of as having their being in the objects which are their instances."[15] Since however it is this very idea of sets as having their being in their members that motivates the axioms of **ZF**, and since this is an inappropriate conception of properties, there seems to be no adequate motivation for a **ZF** style property theory.

So there is at least ground for suspicion of the thesis that sets are just a species of property. It would be desirable, then, if our account could respect the intuitive distinction between the two sorts of entity. In particular, we should like to trace the origin of sets to a different, and more appropriate, sort of divine activity than that to which we've traced the origin of **PRP**s. But what sort? Here we have a fairly rich (though often unduly obscure) line of thought to draw upon from the philosophy of set theory. A common idea one often encounters in expositions of the notion of set is that sets are "built up" or "constructed" in some way out of previously given objects.[16] Though generally taken to be no more than a helpful metaphor in explaining the contemporary iterative conception of set, a number of thinkers seem to have endorsed the idea at a somewhat more literal level. Specifically, their writings suggest that sets are the upshots of a certain sort of constructive *mental* activity.

Adumbrations of this idea can be seen in the very origins of set theory. Cantor himself held that the existence of a set was a matter of *thinking* of a plurality as a unity.[17] His distinguished mentor Dedekind is plausibly taken to be embracing a similar line when he writes that

> [i]t very frequently happens that different things . . . can be considered from a common point of view, can be associated in the mind, and we say they form a *system* S . . . . Such a system (an aggregate, a manifold, a totality) as an object of our thought is likewise a thing.[18]

Comparable thoughts are expressed by Hausdorff, Fraenkel, and more recently by Schoenfield, Rucker, and Wang.[19] Of these it is Wang who seems to develop the idea most extensively. He writes:

> It is a basic feature of reality that there are many things. When a multitude of given objects can be collected together, we arrive at a set. For example, there are two tables in this room. We are ready to view them as given both separately and as a unity, and justify this by pointing to them or looking at them or thinking about them either one after the other or simultaneously. Somehow the viewing of certain objects together suggests a loose link which ties the objects together in our intuition.[20]

I interpret this passage in the following way. Wang here is keying on a basic feature of our cognitive capacities: the ability to selectively direct one's attention to certain objects and collect or gather them together mentally, to view them in such a way as to "tie them together in our intuition;" in Cantorian terms, to think of them as a unity. Wang stresses the particular manifestation of this capacity in *perception*, one of a number of related human perceptual capacities emphasized especially by the early Gestalt psychologists.[21] It is best illustrated for our purposes by a simple example. Consider the following array:

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* * *
* * *
* * *
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Think of the asterisks as being numbered left to right from 1 to 9, beginning at the upper left hand corner. While focussing on the middle dot 5, it is possible to vary at will which dots in the array stand out in one's visual field (with perhaps the exception of 5 itself), e.g., [1,5,9], [2,4,5,6,8], or even [1,5,8,9]. The dots thus picked out, I take Wang to be saying, are to be understood as the elements of a small "set" existing in the mind of the perceiver.

The account obviously won't do as it stands. The axiom of extensionality, for instance, seems not to hold on this picture: if you and I direct our attention to the same dots, then each of us has his own "set", despite the fact that they have the same members. More importantly, human cognitive limitations put a severe restriction on the number and size of sets there can be. Wang is well aware of this, and hence builds an account of the nature of sets based on an *idealized* notion of collecting. Irrespective of the success of Wang's efforts, such an idealization, I believe, can be of use to us here in developing a model of divine activity that works sets into our activist framework.

The idea is simple enough: we take sets to be the products of a collecting activity on God's part which we model on our own perceptual collecting capacities. Consider first all the things that are not sets in this sense. While the number of these "first order" objects that we can apprehend at any given time is extremely limited, presumably God suffers from no such limitations; all of them fall under his purview. Furthermore, we can suppose that his awareness is not composed of more or less discrete experiential episodes the way ours is, and hence that he is capable of generating, not just one collection of first order objects at a time, but all *possible* collections of them simultaneously.[22] We suppose next that, once generated, the "second order" products of this collecting activity on first order objects are themselves candidates for "membership" in further collectings, and hence that God can produce also all possible "third order" collections that can be generated out of all the objects of the first two orders. The same of course ought to hold for these latter collections, and for the collections generated from them, and so on, for all finite orders. Finally, in a speculative application of the doctrine of divine infinitude, we postulate that there are no determinate bounds on God's collecting activity, and hence that it extends unbounded through the Cantorian infinite.[23]

Identifying sets with the products of God's collecting activity, then, and supposing that God in fact does all the collecting it is possible for him to do, what we have is a full set theoretic cumulative hierarchy as rich as in any platonic vision. In this way we locate not just the platonist's **PRP**s, but the entire ontology of sets as well firmly within the mind of God.

## §3 *A Volatile Ontology*

Alas, but as is so often wont to be the case in matters such as this, things are more complicated than they appear. The lessons of the last hundred years are clear that caution is to be enjoined in constructing an ontology that includes sets and **PRP**s. Too easily the abstract scientist, eager to exploit the philosophical power of a platonic ontology, finds himself engaged unwittingly in a metaphysical alchemy in which the rich ore of platonism is transmuted into the worthless dross of inconsistency. Our account thus far is a laboratory ripe for such a transmutation. There are several paradoxes that, on natural assumptions, are generated in the account as it stands.

The first traces its lineage back to Russell, who reported a related paradox (distinct from the one bearing his name) in §500 of the *Principles*.[24] In the cumulative or "iterative" picture of sets we've developed here, all the things that are not sets form the basic stuff on top of which the cumulative hierarchy is constructed. Now, the chief intuition behind the iterative picture, one implicit in our theistic model above, is that any available objects can be collected into a set; the "available" objects at any stage form the basis of new sets in the next stage. Thus, since propositions are not sets, all the propositions there are are among the atoms of the cumulative hierarchy; and since all the atoms are available for collecting (God, after all, apprehends them all "prior" to his collecting), there is a set **S** of all propositions. I take it as a basic logical principle that for any entities $x$ and $y$ and any property $P$,

(*) if $x \neq y$, then $[\lambda\ Px] \neq [\lambda\ Py]$,[25]

i.e., that if $x$ is not identical with $y$, then the proposition $[\lambda\ Px]$ that $x$ is $P$ is not identical with the proposition $[\lambda\ Py]$ that $y$ is $P$.[26] Consider then any property you please; the property **SET** of being a set, say. Then by (*), there is a one-to-one correspondence between Pow(S) (the power set of **S**) and the set $\mathbf{T} = \{[\lambda\ \mathbf{SET}(s)] : s \in \text{Pow(S)}\}$. But $\mathbf{T} \subseteq \mathbf{S}$, since **T** is a set of propositions, hence Pow(S) $\succcurlyeq$ S,[27] contradicting Cantor's theorem.

A strictly analogous paradox arises for properties (and, in general, relations as well). This is true in particular if one holds that every object *a* has an *essence*, i.e., the property **being a**, or perhaps **being identical with *a***.[28] For the same reasons we gave in the case of propositions, properties (and relations) are also among the atoms of the cumulative hierarchy, and hence there is a set **M** of all properties. Essences being what they are, we have, for any $x$ and $y$, that

(**) if $x \neq y$, *then* $E_x \neq E_y$,

*where* $\mathbf{E}_z$ is the essence of $z$. Consider then Pow(M). By (**) there is a one-to-one correspondence between Pow(M) and the set $\mathbf{E} = \{E_z : z \in \text{Pow(M)}\}$. But $\mathbf{E} \subseteq \mathbf{M}$, since **E** is a set of properties, hence Pow(**M**) $\leqslant$ **M**, contradicting Cantor once again.[29]

There are three quick replies to these related paradoxes to consider. The first is to question the fine-grainedness principles (*) and (**). Certainly there are views of PRPs on which this would be appropriate. Possible worlds theorists in the tradition of Montague, for example, define **PRP**s such that they are identical if necessarily coextensional, a "coarse-grained" view incompatible with the fine-grained view we are advocating here. Similarly, views that might be broadly classified as "Aristotelian" hold that properties and relations exist first and foremost "in" the objects that have them, not separate from them, and are "abstracted" somehow by the mind. Such views rarely find any need for **PRP**s any more fine-grained than are needed to distinguish one state of an object, or one connection between several objects, from another. Whatever the appeal of these alternatives, the problem is that they are out of keeping with our activist model. If we are pushing the idea that **PRP**s are literally the products of God's conceiving activity, then it would seem that properties which intuitively differ in content, i.e., which are such that grasping one does not entail simultaneously grasping the other, could not be the products of exactly the same intellective activity and hence must be distinct. This is especially pronounced in the cases of singular propositions and essences that "involve" distinct individuals, such as those with which we are concerned in (*) and (**); it is just not plausible that, e.g., singular propositions "about" distinct individuals could nonetheless be the products of the same activity. To abandon these principles in the context of our present framework, then, would be unpalatable.

The second reply is simply to deny the power set axiom. After all, one might argue, many set theorists find the axiom dubious; so why suppose it is true in general, and in the arguments at hand in particular?

The power set axiom has indeed been called into question by mathematical logicians and philosophers of mathematics over the years. The root cause of this disaffection, however, has always been the radically nonconstructive character of the axiom—mathematicians are not in general able to specify any sort of general property or procedure that will enable them to pick out every arbitrary subset of a given set. In this sense, it is the platonic axiom *par excellence*, declaring sets to exist in utter spite of any human capacity to grasp or "construct" them.

It should be clear that any sort of objection on these grounds, as with the previous objection, is just out of place here. For obviously we are far from supposing that set existence has anything whatever to do with human cognitive capacities. Quite the contrary; on our model, the puzzle would rather be how the power set axiom could *not* be true. For supposing that God has collected some set $s$, since each of its members falls under his purview just as the elements of some small finite collection of our own construction fall under ours, how could he not be capable of generating all possible collections that can be formed from members of $s$ as well? So this response to the paradoxes is ineffective.[30]

The third reply is that, since there are at least as many propositions (and properties) as there are sets, it is evident that there is no set **S** of all propositions any more than there is a set of all sets; there are just "too many" of them. Hence the argument above breaks down. The same goes for properties, so the second paradox fares no better.

Briefly, the problem with this reply is that how many of a given sort of thing there are in and of itself has nothing whatever to do with whether or not there is a set of those things.[31] The reason there is no set of all sets is not because there are "too many" of them, but rather because there is no "top" to the cumulative hierarchy, no definite point at which no further sets can be constructed. On our model as it stands, however, as nonsets, the propositions and properties there are exist "prior" (in a conceptual sense) to the construction of all the sets. Hence, they are all equally available for membership. But if so, there seems no reason for denying the existence of the sets **S** and **M**.[32] So an appeal to how many **PRP**s there are won't turn back the arguments.

§4 *A Russell-Type Solution*

Though always discomfiting, the discovery of paradox needn't necessarily spell disaster. As in the case of set theory, it may rather be an occasion for insight and clarification. Russell's original paradox of naive set theory was grounded in a mistaken conception of the structure of sets that was uncovered with the development of the iterative picture. Perhaps, in the same way, the paradoxes here have taken root in a similar misconception about **PRP**s. There are two avenues to explore.

The final paragraph in the last section uncovers a crucial assumption at work in the paradoxes: that all **PRP**s are conceptually prior to the construction of the sets; or again, that all the **PRP**s there are are among the atoms of the hierarchy. The Russellian will challenge this. He will argue that one cannot so cavalierly divide the world into an ordered hierarchy of sets on the one hand and a logically unstructured domain of nonsets on the other. For although they are not sets, the nonsets too fall into a natural hierarchy of logical *types*. More specifically, in the simple theory of types, concrete and abstract particulars, or "individuals", are the entities of the lowest type, usually designated '$i$'. Then, recursively, where $t_1,\ldots, t_n$ are types, let $(t_1,\ldots, t_n)$ be the type of $n$-place relation that takes entities of these $n$ types as its arguments. So, for example, a property of individuals would be of type $(i)$; a 2-place relation between individuals and properties of individuals would be of type $(i,(i))$; and so on. The type of any entity is thus, in an easily definable sense, higher than the type of any of its possible arguments.[33] By dividing entities whose types are the same height into disjoint levels we arrive at a hierarchy of properties and relations analogous to, but rather more complicated than, the (finite) levels of the cumulative hierarchy.

To wed this conception with our current model we propose that *both* sets and **PRP**s are built up *together* in the divine intellect so that we have God *both* constructing new sets *and* conceiving new **PRP**s in every level of the resulting hierarchy. Thus, at the most basic level are individuals; at the next level God constructs all sets of individuals and conceives all properties and relations that take individuals as arguments; at the next level he constructs all sets of entities of the first two levels and conceives all properties and relations that take entities of the previous (and perhaps both previous) level(s) as arguments; and so on. Thus, since there are new **PRP**s at every level, it is evident that there will be no level at which there occurs, e.g., the set of *all* properties, and hence it seems that the paradoxes above can be explained in much the same way as Russell's original paradox.[34]

Easier said than done. Serious impediments stand in the way of implementing these ideas. First of all, there are several well known objections to type theory that are no less cogent here than in other contexts. For example, on a typed conception of **PRPs**, there can be no universal properties, such as the property of being self-identical, since no properties have all entities in their "range of significance."[35] The closest approximation to them are properties true of everything of a given type. But, thinking in terms of our model, even if many **PRPs** *are* typed, there seems no reason why God shouldn't also be able to conceive properties whose extensions, and hence whose ranges of significance, include all entities whatsoever.

Along these same lines, type theory also prevents any property from falling within its own range of significance, and in particular it rules out the possibility of self-exemplification. Thus, for example, there can be no such thing as the property of being a property, or of being abstract, but only anemic, typed images of these more robust properties at each level, true only of the properties or abstract entities of the previous level.

Standard problems aside, much more serious problems remain. In many simple type theories, including our brief account above, propositions are omitted altogether. Those that make room for them[36] lump them all together in a single type (quite rightly, in the context of simple type theory). This clearly won't do on the current proposal since the entities of any given type are all at the same level and hence form a set at the next level, thus allowing in sufficient air to revive our first paradox.

A related difficulty is that this proposal is still vulnerable to a modified version of the second paradox as well. Consider any relation that holds between individuals $u$ and sets $s$ of properties of individuals, e.g., the relation $I$ that holds between $u$ and $s$ just in case $u$ exemplifies some member of $s$. Let $\mathbf{A}$ be the set of all properties of individuals. For each $s \in \mathrm{Pow}(\mathbf{A})$, we have the property $[\lambda x\, Ixs]$ of bearing $I$ to $s$. By a generalization of the fine-grainedness schemas (*) and (**) (cf. note 29), for all $s,s' \in \mathrm{Pow}(\mathbf{A})$ we have that

(***) if $s \neq s'$, then $[\lambda x\, Ixs] \neq [\lambda x\, Isx']$.

Consider now the set $\mathbf{I}^* = \{[\lambda x\, Ixs] : s \in \mathrm{Pow(A)}\}$. By (***) there is a one-to-one correspondence between Pow(A) and $\mathbf{I}^*$. But $\mathbf{I}^* \subseteq \mathbf{A}$, since $\mathbf{I}^*$ is a set of properties of individuals. Hence, $\mathrm{Pow}(\mathbf{A}) \succcurlyeq \mathbf{A}$, contradicting Cantor's theorem.

A little reflection reveals a feature common to both paradoxes that seems to lie at the heart of the difficulty. First, we need an intuitive fix on the idea of (the existence of) one entity "presupposing the availability of" another. The idea we're after is simple: for God to create (i.e., construct or conceive) certain entities, he must have "already" created certain others; the former, that is to say, presuppose the availability of the latter. For sets this is clear. Say that an entity $e$ is a *constituent* of a set $s$ just in case it is a member of the transitive closure of s.[37] Then we can say that a set $s$ presupposes the availability of some entity $e$ just in case $e$ is a constituent of $s$. For **PRPs** we need to say a little more. As suggested above, there seems a clear sense in which **PRP**s, like sets, can be said to have constituents. Thus, a set-like "singleton" property such as $[\lambda x\, x =$ Kripke$]$ contains Kripke as a constituent. But not just Kripke; for the identity relation too is a part of the property's make-up, or "internal structure;" it is, one might say, a structured composite of those two entities. (We will develop this idea in somewhat more detail shortly.) Combining the two notions of constituency (one for sets, one for **PRP**s), we can generalize the concept of presupposition to both sets and **PRP**s: one entity $e$ presupposes the availability of another $e^*$ just in case $e^*$ is a constituent of $e$.

Now, even though the properties $[\lambda x\, Ixs]$ are properties of individuals, if we look at their internal structure, we see that many of these properties presuppose the availability of entities that themselves presuppose the availability of those very properties, to wit, those properties $P = [\lambda x\, Ixs]$ such that $P \in s$. (Analogously for those propositions $p = [\lambda\ \mathbf{SET}(s)]$ such that $p \in s$.) Call such properties *self-presupposing*; this notion alone, independent of the power set axiom and our fine-grainedness principles, is sufficient for generating Russell-type paradoxes.[38] Conjoined with power set and fine-grainedness, the possibility of self-presupposing properties can be held responsible for the sort of unrestrained proliferation of **PRP**s of (in general) any type that fuels the Cantor-style paradoxes as well.

The source of all our paradoxes, then, in broader terms, lies in a failure so far adequately to capture the dependence of complex **PRP**s on their internal constituents. What we want, then, is a model that is appropriately sensitive to internal structure, but which at the same time does not run afoul of the standard problems of type theory.

## §5 *A Constructive Solution*

Let's review. Our excursion into type theory was prompted by doubts over the idea that **PRP**s are conceptually prior to the construction of sets. Type theory suggested an alternative: **PRP**s themselves form a hierarchy analogous to the cumulative hierarchy of sets such that a **PRP**'s place in the hierarchy depends on the kind of arguments it can sensibly take. The idea then was to join the two sorts of hierarchy into one. However, even overlooking the standard problems of type theory, we found that the resulting activist model (-sketch) was still subject to paradox. Our analysis of these paradoxes led us to see that our problems stemmed from the fact that our models were insensitive to the dependence of **PRP**s on their internal constituents.

How, then, do we capture this dependence? Here we can draw on some recent ideas in logic and metaphysics. Logically complex **PRP**s are naturally thought of as being "built up" from simpler entities by the application of a variety of logical operations. For example, any two **PRP**s can be seen as the primary constituents of a further **PRP**, their conjunction, which is the result of a *conjoining* operation. Thus, in particular, the *conjunction* of two properties $P$ and $Q$ can be thought of as the relation $[\lambda xy\ Px\ \&\ Qy]$ that $a$ bears to $b$ just in case $Pa$ and $Qb$. A further operation, *reflection*, can be understood to act so as to transform this relation into the property $[\lambda x\ Px\ \&\ Qx]$ of having $P$ and $Q$. Related operations can be taken to yield complements (e.g., $[\lambda x\ \neg Px]$), generalizations (e.g., $[\lambda\ (\exists x)(Px)]$), and **PRP**s that are directly "about" other objects such as our set-like property $[\lambda x\ x = \text{Kripke}]$, or the "singular" proposition $[\lambda\ PHL(\text{Kaplan})]$ that Kaplan is a philosopher.[39]

On this view, then, the constituents of a complex **PRP** are simply those entities that are needed to construct the **PRP** by means of the logical operations, just as the constituents of a set are those entities that are needed to construct the set. It is in this sense that a complex **PRP** is dependent on its constituents. This then suggests that, analogous to sets on the iterative conception, **PRP**s are best viewed as internally "well-founded," or at least, noncircular, in the sense that a **PRP** cannot be one its own constituents.[40]

This picture of **PRP**s is especially amenable to activism. For as with set construction, the activist can take the logical operations that yield complex **PRP**s to be quite literally activities of the divine intellect. This leads us to a further, more adequate model of the creation of abstract entities. At the logically most basic level of creation we find concrete objects and logically simple properties and relations (whatever those may be). The next level consists of (i) all the objects of the previous level (this will make the levels cumulative), (ii) all sets that can be formed from those objects, and (iii) all new **PRP**s that can be formed by applying the logical operations to those objects. The third level is formed in the same manner from the second. Similarly for all succeeding finite levels. As in our initial models, there seems no reason to think this activity cannot continue into the transfinite. Accordingly, we postulate a "limit" level that contains all the objects created in the finite levels, which itself forms the basis of new, infinite levels. And so it continues on through the Cantorian transfinite.

Now, how do things stand with respect to our paradoxes? As we should hope, they cannot arise on the current model. Consider the first paradox. Since there are new propositions formed at every level of the hierarchy, there cannot be a set of all propositions any more than there can be a set of all nonselfmembered sets. Similarly for the second paradox: since essences (as depicted above) contain the objects that exemplify them in their internal structure and hence do not appear to be simple, they too occur arbitrarily high up in the hierarchy and hence also are never collected into a set. What about the two new paradoxes above? The first of these is just a type-theoretic variant on the original paradoxes, and so poses no additional difficulty. And although the concept of self-presupposition can be reconstructed in our type-free framework, the corresponding paradox still cannot arise since there can be no set of all non-self-presupposing properties as the paradox requires.[41] We seem at last to have found our way out of this dense thicket of Cantorian and Russellian paradoxes.

But our task is still not quite complete. Recall that one of our first orders of business was to work (cardinal) numbers into the activist framework. We opted for the Cantorian-inspired view that the numbers are properties shared by equinumerous sets. But just where do they fit into our somewhat more developed picture? Intuitively, numbers seem to be logically simple; they do not appear to be, e.g., conjunctions or generalizations of other **PRP**s. Hence, they seem to belong down at the bottom of our hierarchy. It follows that there is a set **C** of all numbers at the next level, according to our model. But this supposition, of course, assuming the truth of the axioms of **ZF**, leads to paradox in a number of ways. For example, one can use the axiom of replacement on **C** to prove that there is a set of all von Neumann cardinals. I've argued elsewhere[42] that, on certain conceptions of the abstract universe that might allow "overly large" sets, it is appropriate to restrict this axiom to sufficiently "small" sets, and such a restriction would not permit its use here. This would still not redeem the situation, though. On our model, I think we must hold that for every set there exists a definite property which is its cardinal number. For it seems quite impossible that God should construct a set without also conceiving its cardinality, the property it shares with any other set that can be put into one-to-one correspondence with it. Hence, the set **C** of all numbers must itself have a cardinality $k$, and so $k \in$ **C**. But it is easy to show (with only unexceptionable uses of replacement) that $k$ is strictly greater than every member of **C**,[43] and hence that $k>k$. Once more we have to confront paradox.

Happily, there is a simple and intuitive solution to this paradox. How many numbers must we say there are? Given our understanding of the numbers as properties of sets, and our reasoning in the previous paragraph, if we divide up the universe of sets according to size, then there must be as many numbers as there are divisions. Numbers are thus in a certain sense dependent on sets in a way that other sorts of properties are not. This suggests a natural way of fitting numbers into our hierarchy in such a way as to avoid paradox: a given number is not introduced into the hierarchy until a set is constructed whose cardinality is that number. The number is then introduced at the next level; God, we might say, doesn't conceive the number until he "has to." Since there are larger and larger sets at every new stage in our hierarchy, there will be no point after which new numbers are no longer introduced, and thus there can be no set of them. Hence, our numerical paradox above cannot get started.

Since our model is informal, the only rigorous way of demonstrating that it is indeed paradox-free is to formalize the picture of the abstract universe it yields and then to prove the consistency of the resulting theory. This can be done. The universe of the activist model can be formalized in a first-order theory that includes all of **ZFC** and a rich logic of **PRP**s that embodies all the fine-grainedness principles above; and this theory is provably consistent relative to **ZF**.

## §8 *Loose Ends*

Many difficult and important issues remain, of course. Perhaps the most pressing are those having to do with modality. For instance, a natural question facing activism is whether God could have created more, fewer, or other **PRP**s than the ones he in fact created. Morris and I argued in "Absolute Creation" that there is a relatively straightforward answer to this question, but the hierarchical picture developed here suggests that the question is somewhat more subtle. In particular, does it not seem possible that God could have continued his collecting activity and generated a set that contains all the objects in the actual universe? Couldn't he then have generated new **PRP**s that would have contained that set as a constituent? If so, then it seems there could at least have been more **PRP**s than those that exist in fact. A second issue is engendered by the fact that the constructive nature of complex **PRP**s seems to entail the doctrine known as *existentialism*, i.e., the doctrine that **PRP**s are ontologically dependent on their constituents.

An adequate treatment of these issues will require a clear account of the truth conditions of modal propositions. This in turn raises the question of how modal propositions, and modal **PRP**s generally, are to be worked into the activist universe. Important issues all, deserving much further exploration; for the time being we will rest content with the ones we've managed to address thus far.[44]

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## NOTES

1. "Absolute Creation," *American Philosophical Quarterly*, 23 (1986), 353-362; reprinted in Thomas V. Morris, *Anselmian Explorations* (Notre Dame: University of Notre Dame Press, 1987).

2. "Direct" as opposed to merely causing the thing to exist by bringing about some *other* series of events that are causally sufficient for its existence. We take the notion of causation here to be primitive; in particular, we cannot analyze it counterfactually (cf. note 5).

3. This doctrine is strongly suggested by St. Paul (Colossians 1:16) and the writer of the Hebrews (1:3), and is at least implicit in St. Thomas (e.g., *Summa Theologica* I, Q. 45, Art. 3). Its most overt expression in the modern period is found in Descartes (*Principles of Philosophy*, 1, XXI).

The doctrine has been defended most recently by J. Kvanvig and H. McCann in their paper "Divine Conservation and the Persistence of the World," in Thomas V. Morris (ed.), *Divine and Human Action* (Ithaca: Cornell University Press, forthcoming).

4. Note that it is compatible with this view that other persons *collaborate* with God in acts of creation. Our view only stipulates explicitly that God necessarily plays a direct causal role in the existence of every object at every time; other persons might also play such roles with respect to some objects.

5. Note that we can't analyze the causal relation here counterfactually as simply the claim that if God hadn't thought abstract objects, they wouldn't have existed, since (on the existing semantics for counterfactuals) it is equally true that if abstract objects hadn't existed, God wouldn't have. Despite this *logical* symmetry between God and abstract objects, we claim that there is a causal asymmetry.

6. Not least of these is Frege's own well known puzzle of the concept *horse*. See "On Concept and Object," in *Translations from the Philosophical Writings of Gottlob Frege*, translated and edited by P. Geach and M. Black (Oxford: Basil Blackwell, 1952), 42-55.

7. Cf., e.g., G. Chierchia, *Topics in the Syntax and Semantics of Infinitives and Gerunds*, Ph.D. dissertation, Dept. of Linguistics, Univ. of Massachusetts (1984); R. Turner, "A Theory of Properties," *Journal of Symbolic Logic*, forthcoming; G. Bealer, *Quality and Concept* (Oxford: Oxford University Press, 1982).

8. G. Frege, *The Foundations of Arithmetic*, translated by J. L. Austin (Evanston: Northwestern University Press, 1980). See also T. Burge, "Frege on the Extensions of Concepts, from 1884 to 1903," *The Philosophical Review* 93 (1984), 03-34.

9. This of course can be understood noncircularly in the usual Fregean/Russellian way. This is essentially the analysis of number developed in Bealer, *Quality and Concept*, ch. 6.

10. Quoted in M. Hallett, *Cantorian Set Theory and the Limitation of Size* (Oxford: Oxford University Press, 1984), 122. Hallett, I should note, disputes the idea that Cantor held that cardinal numbers are concepts or properties.

11. For a recent development and defense of this position, see P. Maddy, "Sets and Numbers," *Nous* 15 (1981), 495-511.

12. B. Russell, *The Principles of Mathematics* (New York: W. W. Norton, 1937), 112ff.

13. See Bealer, *Quality and Concept*, ch. 5, and Jubien, "Models of Property Theory," unpublished.

14. Cocchiarella, "Review of Bealer, *Quality and Concept*," *Journal of Symbolic Logic*, 51 (1983).

15. *Ibid.*

16. Cf., e.g., G. Boolos, "The Iterative Conception of Set," *Journal of Philosophy*, 68 (1971), 215-231.

17. G. Cantor, *Gesammelte Abhandlungen* (Berlin: Springer, 1932), 204.

18. R. Dedekind, *Essays in the Theory of Numbers*, trans. by W. W. Beman (New York: Dover, 1963), 45.

19. F. Hausdorff, *Grundzuge der Mengenlehre* (Leipzig, von Veit, 1914); A. Fraenkel, *Abstract Set Theory* (Amsterdam: North-Holland, 1961); J. Schoenfield, *Mathematical Logic* (Reading: Addison-Wesley, 1967); R. Rucker, *Infinity and the Mind* (Boston: Birkhauser, 1982); H. Wang, *From Mathematics to Philosophy* (London: Routledge & Kegan Paul, 1974).

20. *From Mathematics to Philosophy*, 182.

21. Cf., e.g., W. Köhler, *Gestalt Psychology* (New York: The New American Library, 1947).

22. That is, possible in the sense of all the collections he can form from all the nonsets there happen to be; I don't want to suggest that there could be sets containing "merely possible" objects, of which I think there are none.

23. Let me register my awareness that this is no trivial postulate, and that it deserves much further discussion for which there is simply no room here.

24. A very similar paradox arises in connection with the notion of a possible world. See M. Loux (ed.), *The Possible and the Actual* (Ithaca: Cornell University Press, 1979), 52-03; P. Grim, "There Is No Set of All Truths," *Analysis* 44 (1984), 206-08; S. Bringsjord, "Are There Set-theoretic Possible Worlds?" *Analysis* 45 (1985), 64; C. Menzel, "On Set Theoretic Possible Worlds," *Analysis* 46 (1986), 68-72; and P. Grim, "On Sets and Worlds: A Reply to Menzel," *Analysis* 46 (1986), 186-191.

25. This is what can be called a "fine-grainedness" principle that is natural to those conceptions of **PRP**s that tend to individuate them on psychological grounds, e.g., that properties are identical if it is not possible to conceive of the one without conceiving of the other. Cf., e.g., Chisholm, *The First Person* (Minneapolis: University of Minnesota Press, 1981), ch. 1; Plantinga, *The Nature of Necessity* (Oxford: Oxford University Press, 1974). For more formal developments of this conception, cf. Bealer, *Quality and Concept*, and C. Menzel, "A Complete Type-free 'Second-order' Logic and Its Philosophical Foundations," Report No. CSLI-86-40, Center for the Study of Language and Information, Stanford University, 1986.

26. A general notational remark: $[\lambda x_1 \ldots x_n \mathbf{A}]$, where **A** is a (for our purposes) first order formula is the n-place relation that holds among entities $a_1, \ldots, a_n$ just in case $\mathbf{A}(x_i/a_i)$, $0<i<n+1$. Where $n = 0$, this is just the proposition that **A**.

27. Where $\mathbf{A} \succeq \mathbf{B}$ means, in essence, that **A** is in one-to-one correspondence with a subset of **B**.

28. Cf., e.g., Plantinga, *Nature of Necessity*, ch.5.

29. Both of these arguments are essentially just special cases of an argument schema that is generally applicable to all $n$-place relations, $n \succeq 0$ (where 0-place relations are propositions). Let **A** be the set of all $n$-place relations, for some given $n$. For any $(n+1)$-place relation $R$, we have the fine-grainedness principle that, for any $x$ and $y$,

$$(**_n) \text{ if } x \neq y, \text{ then } [\lambda x_1 \ldots x_n Rx_1 \ldots x_n x] \neq [\lambda x_1 \ldots x_n Rx_1 \ldots x_n y],$$

where the complex terms here signify the $n$-place relations that result from "plugging" $x$ and $y$ respectively into the $n+1^{st}$ argument place of $R$. Assuming that any object can be so plugged into our relation $R$, it follows from $(**_n)$ that there is a one-to-one correspondence between Pow(A) and the set $\mathbf{B} = \{[\lambda x_1 \ldots x_n Rx_1 \ldots x_n z : z \in \text{Pow(A)}\} \subseteq \mathbf{A}$, and hence that $\text{Pow(A)} \succeq \mathbf{A}$.

30. As we'll see below, we can even give the objector the power set axiom and the fine-grainedness principles, since there are other paradoxes that still arise in the current picture. But I prefer to meet the objector head on here to defend the legitimacy and appropriateness of both fine-grainedness and power set in our activist framework.

31. I have argued this at length in "On the Iterative Explanation of the Paradoxes," *Philosophical Studies* 49 (1986), 37-61.

32. Of course, the paradoxes above aside, in such a picture the axiom of replacement would have to be restricted in some way, else replacing on, e.g., the set $\{[\lambda\ \mathbf{SET}(x)] : x \text{ is a set}\}$ would yield the set of all sets. Cf. Menzel, "On the Iterative Explanation."

33. Specifically, let the *order* ord($i$) of the basic type $i$ be 0; and if $t$ is the type $(t_1, \ldots, t_n)$, let

ord($t$) = max(ord($t_1$), . . . , ord($t_n$)); then we can say that one type $t$ is *higher than* another $t'$ just in case ord($t$) > ord($t'$).

34. Note that to pull this off in any sort of formal detail one would have to assign types to sets as well. Since sets on the cumulative picture would be able to contain entities of all finite types, we would also have to move to a transfinite type theory. However, as we'll see, the issue is moot.

35. To use Russell's term; see his "Mathematical Logic as Based on the Theory of Types," in J. van Heijenoort (ed.), *From Frege to Gödel* (Cambridge: Harvard University Press, 1967), 161.

36. E.g., the theory in E. Zalta, *Abstract Objects* (Dordrecht: D. Reidel, 1983), ch. 5.

37. I.e., just in case it is a member of $s$, or a member of a member of $s$, or a member of a member of a member of $s$, or ... .

38. More exactly, and more generally, for every formula $\mathbf{A}(s)$, where '$s$' is a variable occurring free in $\mathbf{A}$ that ranges over sets of properties of individuals, we define the condition of being *self-presupposing with respect to* $\mathbf{A}$ ($\mathbf{SP}_{\mathrm{A}}$) such that, for any property $P$ of individuals, $\mathbf{SP}_{\mathrm{A}}(P)$ iff for some $s$, $P = [\lambda x\ \mathbf{A}(s)]$ and $P \in s$. Consider now the set $s^* = \{P : \neg\mathbf{SP}_{\mathrm{A}}(P)$ (which exists by the axiom of separation). Let $P^* = [\lambda x \mathbf{A}(\mathrm{sx})]$, and let's ask: $\mathbf{SP}_{\mathrm{A}}(\mathbf{P}^*)$? I leave it to the reader to show that $\mathbf{SP}_{\mathrm{A}}(P^*) \equiv \neg\, \mathbf{SP}_{\mathrm{A}}(P^*)$.

A similar paradox that doesn't rely on power set or fine-grainedness is found in Grim, "On Sets and Worlds."

39. These are ideas with syntactic roots in Quine, "Variables Explained Away," reprinted in his *Selected Logic Papers* (New York: Random House, 1966), and Bernays, "Über eine natürliche Erweiterung des Relationenskalkuls," in A. Heyting (ed.), *Constructivity in Mathematics* (Amsterdam: North-Holland, 1959), 01-14, and have close algebraic ties to Henkin, Monk, and Tarski, *Cylindrical Algebras* (Amsterdam: North Holland, 1971). For fuller development within the context of metaphysical realism, cf. Bealer, *Quality and Concept*, Zalta, *Abstract Objects*, and Menzel, "Type-free 'Second-order' Logic." Several categories of logical operations have been omitted here to simplify exposition.

40. This claim is severely called into question in J. Barwise and J. Etchemendy, *The Liar: An Essay on Truth and Circularity*, (Oxford: Oxford University Press, 1987). A full defense of the claim would have to deal at length with their challenge.

41. In type-free terms, for every formula $\mathbf{A}(z)$ where '$z$' occurs free in $\mathbf{A}$, we define the condition $\mathbf{TSP}_{\mathrm{A}}$ such that $\mathbf{TSP}_{\mathrm{A}}(y)$ iff for some $z$, $y = [\lambda x\ \mathbf{A}(z)]$ and $y \in z$. It is easy to see that, on the present model, nothing—in particular, no property—satisfies this condition. For any property of the form $[\lambda x\ \mathbf{A}(s)]$, where $s$ is a set, contains $s$ as an internal constituent, and hence on our current model can only have been constructed *after* the construction of $s$. Thus, there is no set $s^* = \{y : \neg\mathbf{TSP}_{\mathrm{A}}(y)\}$ (since this would be the universal set) and so no property $y^* = [\lambda x\ \mathbf{A}(s^*)]$.

42. In "On the Iterative Explanation of the Paradoxes."

43. One can, for example, use the theorem (which requires only replacement on $\omega$) that there are arbitrarily large fixed points in the mapping $\aleph$ of (von Neumann) ordinals onto (von Neumann) cardinals. This entails that for any $n \in \mathrm{C}$ there is an $m > n$ such that $|\{j \in \mathrm{C} : j < m\}| = \mathrm{m}$. But $\{j \in \mathrm{C} : j < m\} \subseteq \mathrm{C}$, hence $|\mathrm{C}| = k \geqslant m > n$.

44. I am indebted to Jon Kvanvig, Hugh McCann, Tom Morris, Rick Otte, Al Plantinga, Del Ratzsch, and Nick Wolterstorff for encouragement, discussion, and critical comments