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The University of San Francisco

INVESTIGATION OF THE MISCONCEPTIONS RELATED TO THE CONCEPTS OF EQUIVALENCE AND LITERAL SYMBOLS HELD BY UNDERPREPARED COMMUNITY COLLEGE STUDENTS

A Dissertation Presented
to
The Faculty of the School of Education
Learning and Instruction Department

In Partial Fulfillment
of the Requirements for the Degree
Doctor of Education

by
Terrence Maguire
San Francisco
May 2012

THE UNIVERSITY OF SAN FRANCISCO

Dissertation Abstract

Investigation of the Misconceptions Related to the Concepts of Equivalence and Literal Symbols Held by Underprepared Community College Students

Many students struggle to learn mathematics in K-8 grades. Research has shown that lower grade students often misconceive equivalence as an operation rather than a relation, and that students also form various misconceptions of literal symbols. Many students arrive at college seriously underprepared in mathematics, but there is scant research on the difficulties and misconceptions of these college students. The purpose of this research was to learn if underprepared community college students harbor misconceptions of equivalence and of literal symbols similar to K-8 students.

For this study, 191 underprepared college students were surveyed for misconceptions by a questionnaire of 43 items selected from the established suite of effective items. The items for each concept were further partitioned into the definition, properties, and applications of each concept.

Many students (84%) were expert regarding the definition of equivalence. An additional 13% of the students also demonstrated knowledge of the concept, although they did not always take advantage of it. Similarly, over 40% of the students demonstrated expert understanding of the properties of equivalence, but an additional 53% demonstrated a restricted understanding of the concept. Only 5% of the students

were considered expert with the fundamental applications of equivalence and less than 60% demonstrated a basic knowledge of the applications.

Few students (33%) were knowledgeable of the definition of literal symbols, and fewer (< 5%) demonstrated knowledge of the properties of the literal symbol. Consequent to their minimal knowledge of the concept, very few students were able to demonstrate knowledge of literal symbol applications.

Community college students underprepared in mathematics are generally aware of the relational definition of equivalence, but many are not fluent in its use. Most attention needs to be directed to the applications of equivalence. The same students are generally not aware of the concept of literal symbols and much attention needs to be directed not only at the applications of literal symbols, but also at their definition and properties.

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This dissertation, written under the direction of the candidate's dissertation committee and approved by the members of the committee, has been presented to and accepted by the Faculty of the School of Education in partial fulfillment of the requirements for the degree of Doctor of Education. The content and research methodologies presented in this work represent the work of the candidate alone.

_________Terrence J. Maguire_______ _____5 / 9 / 2012___
Candidate Date

Dissertation Committee

________Dr. M. Mitchell___________ ____7 / 31 / 2012___
Chairperson

________Dr. X. Apedoe____________ _____5 / 9 / 2012___

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CHAPTER I

THE RESEARCH PROBLEM

Statement of the Problem

Degree and certification programs at universities and colleges in the United States all contain core requirements in reading, writing, and mathematics. The core requirements for mathematics are different among different programs and degrees, but as a minimum, most require a course in college algebra, which is generally accepted as basic knowledge in the modern world. Schield (2008) stated, “Even non-STEM [Science, Technology, Engineering, and Mathematics] students need to be quantitatively literate to excel in their fields and to be capable citizens in a modern data-based democracy where most social and political issues involve quantitative reasoning” (p. 87). Some mathematical skill is necessary to cope, for example, with modern living in the commercial, banking, and real estate environments. Just how much mathematical skill, and what kinds, varies among individuals, making it unrealistic to set standardized skill levels for adults of any age.

At the college level, the general minimum requirement of college algebra was established generations ago, and although adults’ needs have changed since then, there is no concerted movement today to revisit the core mathematical requirements from a content point of view. Just a few visionaries (e.g., Schield, 2008) have suggested content revisions in opinion papers. For adult students, algebra is universally agreed to be a “gateway” course to a career (Kaput, 1998; Knuth, Stephens, McNeil & Alibali, 2006; McGlaughlin, Knoop & Holliday, 2005; Phillps, 1998). Algebra is, for some, their only education in abstract reasoning and in problem solving (Nathan, 2002). Some accommodations for students majoring in the liberal arts, where mathematical analysis is not a significant part of the curriculum, were made in the form of courses in mathematics designed to cover topics in applied mathematics, such as trigonometry, probability and statistics, but some of these topics are themselves algebra-based. As a result, for most programs, even for liberal arts programs, algebra is a requisite part of the college curriculum.

Many students, however, exit high school crucially underprepared for college-level studies in reading, writing, or, most commonly, mathematics and find it necessary to make up the difference. Nonetheless, they are not inhibited from attempting to obtain a college education (Provasnik & Planty, 2008). In addition to this group of recent high school graduates, it is reasonable to expect that there are nearly as many reentrant adult students who have forgotten, or never learned, much basic mathematics. For instance, in 2007-2008, of the freshman ranks in both 2-year and 4-yearl institutions who took at least one remedial class, 22.2% was less than 24 years old and 15.5% was over 29 years old (Snyder & Dillow, 2011).

The annual influx of underprepared freshmen has driven colleges and universities to provide remedial instruction and extra support for their students with subpar preparation in the areas of reading, writing, and mathematics. In the case of mathematics, colleges and universities are attempting to develop effective remedial programs, which include fundamental remedial courses and support services. The four most common remedial courses are Basic Mathematics, Prealgebra, Beginning Algebra, and Intermediate Algebra. Basic Mathematics and Prealgebra are thorough reviews of arithmetic, including a bridge to algebra for the underprepared students. Beginning and

Intermediate Algebra are similar in content to high school algebra. Some four-year institutions have developed more efficient remedial programs. For example, Middle Tennessee State University developed an alternate sequence of two courses for their underprepared students to reduce the cost and time of remedial education, both for the student and for the University (Lucas & McCormick, 2007). In addition, many colleges have found it necessary to provide multiple levels of supplementary tutoring services and special study programs for their underprepared students in the forms of intensive summer mathematics programs (e.g., Turner, 2008), very highly structured mathematics courses (e.g., Siadat, Musial, & Sagher, 2008), and study skills workshops to accompany the remedial mathematics classes.

The diverse causes of the misalignment between the abilities of high school graduates and the expectations of colleges are neither obvious nor simple. Internally, for example, students may have cognitive problems: memory limitations or a failure to develop appropriate schemata (Cirino, Morris, & Morris, 2002). Externally, the students may have been hampered by institutional features: inadequately trained teachers, limited resources, and inadequate facilities (Ladson-Billings, 1997; McNeil, Grandau, Knuth, Alibali, Stephens, Hattikudur, & Krill, 2006). In the case of mathematics, research at the middle school and early high school levels has shown that many students have experienced difficulty learning concepts.

One type of difficulty that students have experienced has been the *misconception*. Misconceptions in mathematics are incomplete or incorrect understandings of mathematical constructs (Champagne & Klopfer, 1983; Clement, Brown, & Zeitsman 1989; Confrey, 1990). Furthermore, Champagne and Klopfer, Clement et al., Knuth, Alibali, Hattikudur, McNeil, and Stephens (2008), and Molina and Ambrose (2006) agreed that such misconceptions can be very robust, or resilient to external and internal remediation.

Among the reasons for being underprepared are mathematical difficulties that have roots in the arithmetic curricula of elementary school (Christou, Vosniado, & Vamvakoussi, 2007); other difficulties arise in middle school (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; McNeil et al., 2006) or in early high school (Asquith, Stephens, Knuth, & Alibali, 2007), where they typically encounter algebraic thinking for the first time. For some students, the difficulties often persist to some degree through repeated attempts to learn algebra through high school and into college (Knuth et al., 2006; McNeil & Alibali, 2005).

The two concepts in mathematics that are prone to misconceptions and that are most fundamental to progress in arithmetic and algebra are the concepts of equivalence and of the literal symbol (Knuth et al., 2005). Each concept is described in terms of its core definition, its common properties, and its most basic applications.

In mathematics, *equality* is defined as the relationship between the two quantities wherein the two quantities have the same value and are therefore interchangeable. An example of a numerical statement of equality is the statement $3+4+5=10+2$, where the equal symbol, =, separates the two quantities of the relation. Each quantity has the value 12 and the quantities are therefore equal and interchangeable. An example of an algebraic statement of equality is the statement $\mathrm{p}+\mathrm{p}=2\mathrm{p}$, where each side of the statement has the value of 2p and the two sides are therefore equal and interchangeable. *Equivalence (equivalent)* is a term that has replaced *equality (equal)* in the literature. The definitions of equality and of equivalence, which subsumes the definition of equality, each describe the same-value relation between the quantities and both are generally referred to as the *relational* definition of equivalence (equality) in the literature.

The relationship of equivalence exhibits three properties that are true for all expressions of real numbers: (a) reflexive, meaning that any number equals itself, or $n = n$; (b) symmetric, meaning that if one expression equals a second expression, the second expression also equals the first expression, or, if $n = m$, then $m = n$; and (c) transitive, meaning that if one expression equals a second expression, and if the second expression equals a third, the first and third expressions are also equal, or if $m = n$ and $n = p$, then $m = p$. In contrast, the *greater than* relation, >, is neither reflexive nor symmetric, but is transitive. Students need to be knowledgeable of all three properties of equivalence (Baroody & Ginsberg, 1983).

The applications of equivalence are countless in mathematics, but the three applications encountered first by students learning algebra are the arithmetic identity, arithmetically equivalent expressions, and the assignment statement. The identity is a strong statement of equivalence, a declaration of a mathematical truism. The identity may be arithmetic, such as $5 + 3 = 8$, or it may be algebraic, if it contains literal symbols, such as $n * 1 = n$, which is true for any value of n.

One expression is equivalent to another expression if each represents the same quantity, but are structurally different. For instance, the expression $457 + 325$ can be converted to an equivalent expression $450 + 7 + 325$, which is of different structure, because the identity $457 = 450 + 7$ has been applied to a part of the original expression. The second expression is more complex because it contains three terms, but it may be easier for a student to calculate. In another way, the expression 38 + 42 can be converted to an equivalent expression by moving 2 units from the group of 42 to the group of 38. The expression is then 40 + 40, which is easier to add. The application of equivalent expressions is used primarily to simplify calculations.

An assignment statement establishes or changes the value given to a literal symbol. In describing how to calculate the sales tax, say 9%, for a purchase, a general statement can be made from the definition of a sales tax, that the tax due is .09* P, where P represents the purchase price. At the store, one could assign P the value of a particular purchase, say $50.00, and then calculate the tax due for that purchase. Like the algebraic identity, the assignment statement invokes both the concept of equivalence and the concept of the literal symbol.

Many students in elementary school develop a simplistic interpretation of equivalence, which some students carry over to middle school, high school, and college. The students learn, and are often encouraged, to think of an equivalence statement, or the equal sign, as an instruction to carry out the operations contained in the equation and to record the result. For example, by this simple definition of equality, the equal sign in the statement 6 + 2 = __ directs the student to perform the indicated addition and to place the result (8) on the indicated blank, which is the correct response. This simple definition of an equality statement always serves for equivalence problems of the canonical format a + b = c, where c is blank. Students can become very satisfied with this definition (Singer & Revenson, 1996). This view of equality has been dubbed the *operational definition* of equality in the literature. At the same time, it is the predominant misconception related to equality or equivalence because it is drastically limited in applicability and is a severe handicap for students attempting to learn algebra (Stephens, 2006).

Students who are constrained to the operational definition of equality are understandably confused about what to do with all the noncanonical problem formats, where there are more than three terms, where there is more than one blank, or where the blank is in other positions in the statement. This constraint is mathematically fatal for students (McNeil & Alibali, 2005). Unfortunately, the relational definition and its applications have not been topics relevant to the arithmetic curriculum (Warren, 2003b), and are topics considered unnecessary in traditional elementary algebra courses (Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007).

In algebra, a literal symbol is typically a single Roman letter, either upper or lower case. In all instances, a literal symbol is defined as a representation of a numerical quantity and has two properties. A single literal symbol can be used like a blank, a box, or a question mark to represent an unidentified number in problems. One property of a literal symbol is that a literal symbol that appears in more than one place in an expression or equation represents the same quantity in each place. For instance, the identity that the product of any number and 1 is the original number can be expressed in symbolic form as $n * 1 = n$, where n represents any number. In this situation, the repeated use of the literal n to represent a number in two places requires that the same number appears in both places. The other property of a literal symbol is that different literal symbols imply unidentified numbers that may or may not be the same. An example of two literal symbols in the same expression is $n * m = m * n$, which states that the product of any two numbers, which could be different, is the same regardless of the order of multiplication.

Many distinct applications of a literal symbol abound in algebra, but three of them are particularly pertinent to beginning algebra. The three ways a literal may be applied in beginning algebra are as (a) a *generalized number* in a statement of a property of numbers where the literal represents any number, but where the use of a particular number would obscure the general significance of the statement; (b) a *variable* in a relationship with other variables, where the associated value(s) may be assigned by the user; or (c) an *unknown* quantity, where the value(s) associated with the literal are initially not known and need to be determined.

Many students establish the first application of a literal they encounter as the universal definition of a literal, and that narrow view is the predominant misconception related to the use of literal symbols in algebra. Self-constrained to a single definition, a student becomes stymied upon encountering a situation that employs a different application of a literal (Kieran, 1991).

In each of these three applications, the outward appearances of the statements are subtly different and not easily discriminated by the inexperienced student, whereas, the applications of the literals are distinctly different. As in the case of the equal sign, it is crucial that students learning algebra be fluent with the three fundamental applications of the literal because the applications are used nimbly and can only be identified by their context.

The research directed at obstacles to learning algebra has been grouped by grade levels. Most of the research on obstacles to learning algebra was initially focused on the middle school grades and was concerned with recognizing and reversing obstacles to learning algebra acquired in elementary school (e.g. MacGregor & Stacey, 1997; Warren, 2003a). Research on algebra-related topics at the elementary school level began in the mid-1990s (Kieran, 2006), and was directed at identifying the practices of arithmetic education that encouraged the development of obstacles to learning algebra (e.g. Baroudi, 2006; Falkner, Levi, & Carpenter, 1999; Warren, 2003b). Research at the college level has been predominately directed at higher level algebraic constructs: ratios, percentages, and functions, (e.g., Dubec, & Underwood-Gregg, 2002), or to college level mathematics courses (e.g., Bogomolny, 2006; Hegarty, Mayer, & Monk, 1995; Herman, 2007).

The research related to learning difficulties in mathematics spans all grade levels from pre-school to graduate school, but has almost ignored the entire population of underprepared students in the lower college levels (Stigler, Givvin & Thompson, 2010). Stigler et al. studied the results of a placement and diagnostic test designed to identify a freshman's readiness for taking an algebra course. They identified 13 test items that were answered incorrectly most frequently by the students in developmental mathematics classes. Nine of the 13 test items were arithmetic questions (no literal symbols), but were not questions on topics as fundamental as equivalence (operations with fractions and percentage problems, for example), which may have been the root cause in some cases.

What other research of underprepared college students exists has been directed at classroom techniques (Reynolds & Uptegrove, 2006) or has been limited to students identified with innate learning disabilities (Cirino et al., 2002), neither of which is within the bounds of this study. The algebra-related learning obstacles of the underprepared students are clearly not the same as those of other college students, but may be similar to those of students at lower grade levels, as suggested by the persistence of the obstacles found in studies of middle school and high school students (e.g., Kieran, 1981; Knuth et al., 2005).

In summary, to progress in arithmetic beyond manipulation of basic number facts and to begin the study of algebra, it is crucial that students have a firm comprehension of the two fundamental concepts of equivalence and literal symbol, a comprehension based on knowledge not only of their basic definitions, but also of their properties and of their elementary applications.

Because the research related to students' difficulties in learning algebra has a gap between early high school and college, the difficulties experienced by freshmen students are not well known. This study begins the general study of freshmen's difficulties by investigating the existence of the basic error (misconceptions) of the most fundamental concepts (equivalence and the literal symbol) as suggested by research at lower grade levels.

Purpose of the Study

All colleges and universities have core requirements for reading, writing, and mathematics, but not all freshmen are academically prepared to undertake college level credit courses in the three core subject areas. Nearly all four-year colleges and universities, and virtually all community colleges, offer remedial, non-credit courses to support their underprepared students. This is especially true in the area of mathematics.

Underprepared college freshmen are a subset of college students who have unique difficulties with core subjects. In particular, the difficulties that the underprepared freshmen have with mathematics have been only lightly studied and are not well understood. However, research in middle schools and in high schools has suggested that difficulties such as misconceptions of fundamental concepts are robust and may persist through high school and in to college.

The purpose of this study was to estimate the proportion of college students in remedial mathematics classes who are highly knowledgeable of the definitions, properties, and applications of the two concepts of equivalence and the literal symbol. To estimate the proportions of students with lesser knowledge, this study examined the breadth and depth of their conceptual knowledge by their ability to apply the concepts to problems of increasing complexity.

Significance of the Study

This study showed that a notable fraction of the students are arriving at college without a firm grasp of the crucial concepts of equivalence and the literal symbol (Knuth et al., 2005; McNeil & Alibali, 2005) in spite of the research and interventions that has been developed in the past 30 years. There were two reasons why this study was important.

The first reason this study was important is because it provided an indication that it is necessary at the high school and college freshman levels to consider concepts as fundamental as equivalence and the literal symbol as possible root causes of student's difficulties, as implied by the research at the middle school and high school levels. Equivalence, for example, is not discussed after middle school (Alibali, et al., 2007; McNeil & Alibali, 2005). Mathematics, by nature, is a highly sequential subject. New concepts are constructed from previous concepts; procedures are constructed from previous concepts and previous procedures. A student can fail to understand a higher level concept or procedure, such as fractions or equation solving as pointed out by Stigler et al. (2010), but attempting to remediate a student's understanding of a higher concept or procedure may be treating the symptom rather than the cause (Thornton, 1982), which is never a cure, and may lead to much frustration on the part of the student and the professor. This study showed that the root cause can be a more fundamental concept, such as equivalence or the literal symbol. It is critical to identify the root cause of their misunderstanding to make an effectual intervention.

The second reason this study was important was that this study resolved some uncertainties resulting from the gap in the research related to difficulties experienced by students in learning algebra. The research related to the most fundamental concepts of mathematics has been centered on middle school and elementary school, with some studies in early high school. Few such studies have been conducted at the college freshman level (McGlaughlin et al., 2005), creating a gap in the research. The studies at the lower grade levels discussed in detail the results of questionnaires and the amount of achievement (always less than 100%) resulting from interventions (e.g., Asquith et al., 2007; Falkner et al., 1999; Knuth et al., 2005; Knuth et al., 2006; Macgregor & Stacey, 1997). Not discussed in these studies were the indications that many students at the lower levels still were falling short to some degree of being at the expected level of performance in mathematics, which left some uncertainty about the readiness of college freshmen. It was presumed that the students would have absorbed the missing critical concepts in the remaining years of high school. Some students may have acquired the critical concepts; some may not have (Kieran, 1981). Those who did not are those who come to college underprepared in mathematics. College curricula, even remedial curricula, are based on the same presumption that the fundamental concepts have been learned. This study provided indications of how widespread are the fatal misconceptions of equivalence and the literal symbol among underprepared freshmen students: students are generally familiar with the concept of equivalence, but many students need some help to become expert in the knowledge and need much help in its applications. On the other hand, this study also showed that most of the underprepared students are very weak in knowledge of the literal symbol and need much detailed instruction on this concept.

Theoretical Framework

The theoretical basis for this study is a combination of (a) Piaget's (Piaget & Inhelder, 1966/2000) theory of conception formation (Constructivism) in children and (b) the organization of mathematical conceptions as described by Jacobs, Franke, Carpenter, Levi, and Battey, 2007. A core element in Piaget's theory is the development of concepts throughout childhood. The work by researchers after Piaget extended the knowledge to adolescents and adults (e.g. Posner, Strike, Hewson, & Gertzog, 1982; Thornton, 1982). Jacobs et al. tied the concepts associated with equivalence to relational thinking, which is the basis of the second part of the framework. In the second part, the model described by Jacobs et al. is refined and further extrapolated to abstract thinking and to the concept of literal symbols.

The theoretical underpinning for this study is the theory of cognitive development by Piaget and Inhelder (1966/2000) as further developed following Piaget by Babai (2006), Baroody and Ginsberg (1983), Case (1996), DeVries (2000), Hunting (1986), and Jones (2007). Piaget's theory focused on the process of concept development and accumulation in children from infancy through adolescence. Among other advances, post-Piagetian theory extends Piagetian theory by including post-adolescents as subjects and complements Piagetian theory by including the related branch of the development and remediation of misconceptions

Prior to the 1970s, research on the development of conceptions in children and adults was flourishing. It was during this period that Piaget was observing the cognitive development of children and was developing the constructivist theory (Easley, 1977; Piaget & Inhelder, 2000; Sinclair, 1987; Singer & Revenson, 1996; van Glasersfeld, 1982, 1989).

During the 1950s and 1960s, a new model of cognitive development in children was spearheaded by Jean Piaget (e.g., Piaget & Inhelder, 1966/2000), gradually replacing the reigning behaviorist theory. Piaget probed children's cognitive processes focusing on what conceptions were learned, what structures were formed, and in what sequence they were formed. He noticed that the process was progressive: a continuous accumulation of knowledge gradually built upon the child's prior knowledge. Furthermore, he observed that the process was time-phased in stages that correlated well to the biological stages of the child's growth from infancy through adolescence. Building on these observations, Piaget pioneered the constructivist paradigm of cognitive development, a distinct, but not contradictory, shift from the prevailing behaviorist paradigm. Without a causal relation between biological development and cognitive development, the cognitive development process would not be universally regimented: some allowances were necessary to accommodate for social and scholastic differences in children's environments. As a result, the end sets of conceptions and skills among adolescents would vary.

According to Piaget and Inhelder (1966/2000), as children build their knowledge and skills structures over time, they often develop incorrect conceptions, or incomplete conceptions. Although the children's concepts may have been adequate for the tasks then at hand, they were not the same conceptions held by experts, and they would eventually require adaptation to be more broadly applicable. Piaget did not label these immature conceptions as misconceptions, but claimed that adaptation of the concepts by assimilation of additional experiences and accommodation of the conceptions to new experiences would eventually produce the accepted concept. For post-adolescents who hold misconceptions, however, there is less time remaining for adaptation to occur and intervention is more urgent, lest the young adult remains burdened with a misconception.

In the post-Piagetian period, one thread of the research was in the direction of systematic errors, or misconceptions, in mathematics as well as in computer science. The recognition of misconceptions in the domain of mathematics is relatively new, not appearing in the literature until the middle 1970s, e.g. Behr, Erlwanger, and Nichols (1976), although the phenomenon was well recognized in the natural sciences (Resnick, 2006). The *misconception* label itself began to appear in the 1980s, (e.g. Clement et al., 1989). Misconceptions in mathematics may be procedural errors in the operational algorithms that students learn to accomplish mathematical operations, such as addition or long division, or they may be fundamental misunderstandings of the basic concepts of addition or division themselves (Rittle-Johnson & Alibali, 1999). In either case, these misconceptions are systems-level misconceptions that manifest themselves consistently.

The research on misconceptions from the 1980s to today has focused on the details of children's conceptions and misconceptions at all levels, but primarily centers on the misconceptions of basic constructs, such as equivalence and the literal symbol, as exhibited by middle school and early high school students. The focus has been on diagnosing the most popular misconceptions and, sometimes, their remediation. This study carries on where current research leaves off: investigating possible basic misconceptions held by college freshmen.

Researchers have converged on just two terms to convey the idea of understanding: *concept* and *conception*. The terms are not specially defined in the current literature; they follow dictionary definitions and are consistent with the early definitions by Tall and Vinner (1981). Today, a *concept* is the explanation, presumed correct, of some physical phenomenon or some mathematical property that is generally accepted by experts in the field. A *conception* is an individual's internalized understanding of the phenomenon or property, however incomplete, however incorrect. A conception is acquired incrementally, synthesized from the individual's personal experiences and from external vectors of instruction. A conception is a work in progress and may not be complete or correct at any point in time (Piaget & Inhelder, 1969/2000). In mathematics, for example, a child's early, incomplete conception of a literal symbol used in a mathematical expression or equation might be that any literal may be assigned any convenient value (Küchemann, 1981a), which is only sometimes true.

Misconceptions in students' minds can occur at the concept level and at the application level. At the concept level, misconceptions of the core meaning or of the properties of the concept may exist. At the application level, misconceptions will often be procedural errors, but may have conceptual underpinnings. The purpose of this study is intended to detect the existence of misconceptions at the two levels for the two concepts of equivalence and the literal symbol.

The second part of the theoretical basis of this study is the organization of conceptions based on a model suggested by Jacobs et al. (2007) that lends an order to the concepts of equivalence and literal symbols. The literature describes the concepts of equivalence and the literal symbol as multi-faceted concepts because they each appear to have many conflicting meanings (e.g., Freudenthal, 1983; Kieran, 2006; Molina, Castro, & Castro, 2009; Nie, Cai, & Moyer, 2009; Usiskin, 1999). Sometimes an equivalence statement is used to state an identity; at other times an equivalence statement is used to calculate an unknown quantity. Similarly, literal symbols often refer to a specific number; on other occasions, a literal symbol is used to represent a general number. Because the assorted meanings of each concept bear little resemblance to one another, students who are being exposed to several meanings of either concept over a short period, often without explanation, can easily become confused and frustrated (Harel, Fuller, & Rabin, 2008). Jacobs et al. (2007) began to organize the meanings of equivalence by grouping several meanings of equivalence as *applications* of relational thinking. Their main point was that relational thinking was a way of thinking that had broad applicability and was immensely powerful. In addition, however, the notion of applications is itself valuable.

Jacobs et al. (2007) defined *relational thinking* as thinking of an arithmetic expression or equation not as a calculation to be performed, but as a structure of related numbers and operations. Jacobs et al. described one application of relational thinking as the establishment of the relational interpretation of the equal sign, or equivalence, as opposed to the operational view.

A second application of relational thinking described by Jacobs et al. (2007) was the use of strategic identities within an expression in an open statement to solve the equivalence statements with a minimum of calculation. This application depends on the relational view of equivalence and therefore is better described as an application of the relational view of equivalence, rather than a direct application of the relational thinking. The example used by Jacobs et al. was the equation $57 + 36 = \_\_ + 34$, which could be solved alternatively by applying the identity $36 = 2 + 34$ and convert the left side of the equation to $57 + 2 + 34$. From there, it is clear that the 59 must be put in the blank.

The third application of relational thinking described by Jacobs et al. (2007) is the establishment of some basic arithmetic identities (relationships containing no literal symbols, such as the basic number facts). As the students mature mathematically, their repertoire of basic identities grows. Jacobs et al. pointed out that a full appreciation of identities, and most of their uses, is only obtained when the application is extended to include algebraic identities (relationships containing literal symbols). For example, most elementary students learn early that *any number plus 0 is the same number,* which can be expressed algebraically as $n + 0 = n$, where n represents any number. Making and using these generalizations invokes the relational understanding of equivalence and is therefore also better considered an application of the relational view of equivalence.

The model of relational thinking described by Jacobs et al. (2007) begins with the concept of relational thinking supporting three applications: the relational view of equivalence, arithmetic equivalent structures, and arithmetic identities. Because the relational view of equivalence is a prerequisite for the other two applications, the model is more accurate when the other two applications are considered as applications of the relational view of equivalence. The advantage of the model suggested by Jacobs et al., with the mentioned modifications, lies in the separation of the concept definition from its applications. The core definition of the concept stands alone and supports each application; each application puts the core definition to a particular use, but does not conflict with the definition. The confusions caused by multiple meanings of each concept thus vanish. The left portion of Figure 1 is a diagram of Jacobs el al.'s model, as modified here.

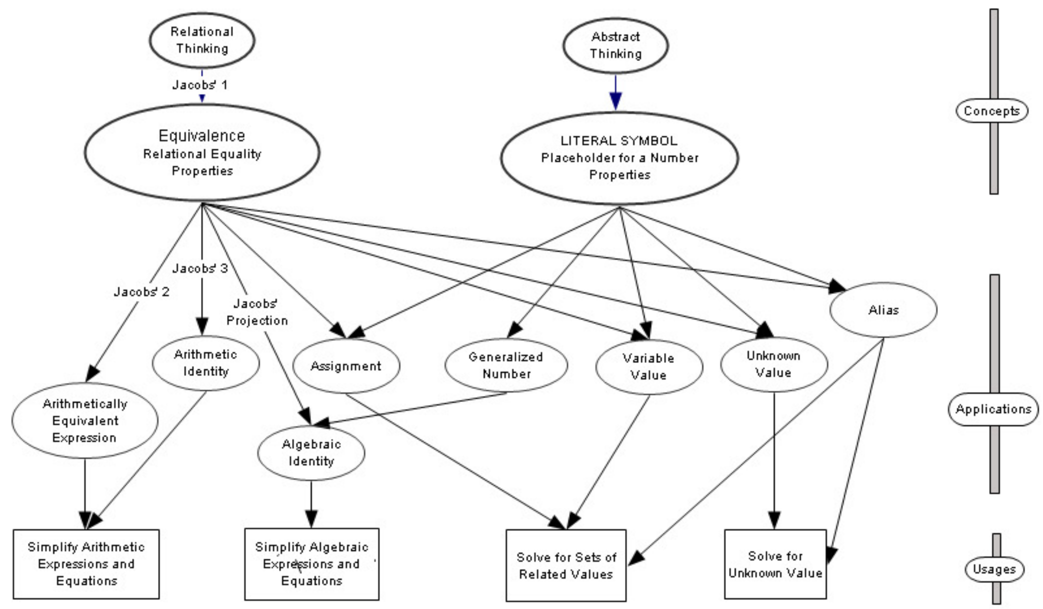

Figure 1. Expanded diagram of the applications and usages associated with the concepts of relational equivalence and the literal symbol, showing the interdependence of the two concepts.

The link from relational thinking to relational equivalence points to the first application discussed by Jacobs et al. (2007) and is labeled as *Jacobs' 1.* Similarly, the links pointing to the other two applications discussed by Jacobs et al., arithmetic equivalent structures and arithmetic identity are labeled as *Jacobs' 2* and *Jacobs' 3*, although their direct links from relational thinking have been changed to links from relational equivalence. Jacobs et al. also explained that identity applications come to full power when algebraic identities are included, which is linked from the application of generalized number on the literal symbol side of the figure. The linkage between relational equality and algebraic identity is labeled as *Jacobs' Projection*.

The same diagramming approach can be applied to the literal symbol concept, creating a parallel model of a well-defined concept of the literal symbol supporting several applications. The two concepts of equivalence and the literal symbol are not isolated from one another: the applications of the literal symbol are highly dependent on the concept of equivalence because the applications almost always are stated in the form of equations. The concept of equivalence, however, is independent of the concept of the literal symbol, with the one exception identified by Jacobs et al. (2007): the application of algebraic identities.

The combined diagram, shown in Figure 1, shows the basic concepts, their applications, and usages. Also made clear in the diagram is the unequal interdependence of the two concepts. The upper area of the figure is the concepts area where the concepts of relational equivalence and the literal symbol are shown. The middle area of the figure contains the applications that are pertinent to beginning algebra.

The lower area of Figure 1 shows the links between each application and its primary usages. The linkages between applications and usages are not always one-to-one. In some cases, applications support more than one usage; in other cases, more than one application is necessary to support a usage.

The arrows in the diagram indicate parent-child relations. For example, the Assignment application of Equivalence is a child of the concepts of both Equivalence and the Literal Symbol because knowledge Equivalence and the Literal Symbol are necessary for complete knowledge of the Assignment application.

Background and Need

Some mathematics is required to manage the business of life all through adulthood and is considered a core requirement of any college education plan. In addition, many college degree and certificate programs require more advanced mathematics based on the particular mathematical requirements of the career. Within the higher education systems, however, no common standards by academic level have been published, but the minimum requirements in mathematics are agreed by consensus: college algebra plus, often, one other course, such as statistics, trigonometry, or business calculus. At the community college level, Blair (2006) set general guidelines for the mathematical curricula, including specific guidelines for developmental mathematics. Blair’s general guidelines for curricula implied a college level algebra course or a statistics course. Blair’s three standards specific to developmental mathematics include: (a) the ability to apply strategies to manage mathematical anxiety, (b) the ability to develop mathematical skills needed to complete other courses successfully, and (c) confidence in doing mathematics and solving real-world problems.

Algebra, in particular, is a core requirement because algebra is a basis for higher mathematics for some students and an introduction to abstract thinking and to problem-solving methods that go beyond mathematics itself for everybody (Nathan, 2002). Williams and Molina (1997) summarized the need when they stated that “ … every student needs to understand how quantities depend on one another, how a change in one quantity affects the other, and how to make decisions based on these relationships” (p. 41).

To demonstrate how mathematics pervades the business of life, Steen (2001) suggested over six examples of mathematical applications for each of seven areas of general living: a) education, b) personal finance, c) personal health, d) management, e) work, f) citizenship, and g) culture. For example, Steen offered 10 applications of mathematics under the area of personal finance: a) understanding depreciation and its effects on the purchase of cars and computer equipment, b) comparing credit card offers with different interest rates for different periods of time, c) understanding the relation of risk to return in retirement investments, d) understanding the investment benefits of diversification and income averaging, e) calculating income tax and understanding the tax implications of financial decisions, f) estimating the long-term costs of making lower monthly credit card payments, g) understanding interactions among different factors affecting a mortgage, h) using the Internet to make decisions about travel plans, i) understanding that there are no schemes for winning lotteries, and j) choosing insurance plans, retirement plans, or finance plans for buying a house. Not everyone needs to be fluent in all the areas, but Steen clearly demonstrated that mathematical thinking beyond arithmetic problems is required just to manage the daily living experiences almost everyone encounters.

Colleges and universities, from their position in the education system, base their primary mathematics curriculum on the assumption that students enter college with a basic understanding of algebra and are prepared to expand that knowledge with a college-level algebra course. Unfortunately, not all students meet those ideal requirements. The mismatch between college students' preparation in mathematics and the expectations of the colleges is not a localized phenomenon; it is both broad in its reach and deep in its effect. The breadth of the mismatch is indicated by the extent to which colleges and universities across the nation have responded to the situation, which was once outside of the scope of their charters but now expected. The depth of the mismatch is indicated by the proportion of college freshmen who find it necessary to enroll in remedial courses.

A majority of colleges and universities across the United States have responded to the annual influx of underprepared freshmen by providing remedial instruction and extra support for their students with subpar preparation in the areas of reading, writing, and mathematics. In 2007, for example, the proportion of all public degree-granting four-year institutions nationwide offering remedial courses was 74.1%, gradually declining from 85% in 1996 (Snyder, Dillow, & Hoffman, 2009). Over that period, four-year institutions have been shedding the responsibility for remedial education, transferring the responsibility to community colleges. At the same time, 99.5% of public two-year institutions were offering remedial education classes, continuing their participation in remedial services at greater than 98% since 1996 (Snyder et al., 2009). Although these figures are for remedial education in mathematics, reading, and writing combined, Parsad, Lewis and Greene (2003) reported that in 2000, when 76% of all public two-year and four-year institutions were offering remedial courses, 71% of the same institutions were offering remedial mathematics courses, indicating that remedial courses in mathematics were in wide demand. The level and persistence of the number of institutions that offer remedial courses in mathematics define the breadth of the mismatch between freshmen knowledge and college expectations in mathematics.

To gauge the depth of the mismatch between freshmen’s mathematical skills and college expectations, it is necessary to look at the number of students needing remedial courses in mathematics. Estimates of the proportion of entering freshmen nationwide who are not ready for college level mathematics range from 22% to 47% across the nation (Attewell, Lavin, Domina, & Levey 2006; Lutzer, Rodi, Kirkman & Maxwell, 2007; Parsad et al., 2003; Snyder et al., 2009). In California, in 2000, the proportion of entering freshmen who enrolled in the most basic arithmetic level courses in community colleges was approximately 40% (California Community Colleges Chancellor's Office, 2009; California Postsecondary Education Commission, 2010). In light of the progressively technical nature in the workplace and of life in general, subpopulations of students of these magnitudes should not be ignored. Doing so would jeopardize the career objectives of a large number of students.

At most colleges and universities, freshmen undergo placement testing, at least in mathematics, which identifies those students who are underprepared for college mathematics. In the year 2003, for example, 29% of the freshman students at community colleges and 19% at public four-year institutions self-reported taking remedial classes (Provasnik & Planty, 2008). Although the lack of mathematical understanding exhibited by many students as they enter college is thus made very apparent, the reasons for the lack of understanding were not assessed. Research of students' mathematical difficulties is very strong at the middle school and early high school levels, but it stops at mid-high school. The research at those lower levels has shown, usually not intentionally, that not all students achieve fluency by mid-high school (Christou et al., 2007; Kücheman, 1981; MacGregor, & Stacey, 1997). Assessments of 12^{th}-grade students' mathematical skills consistently show a large proportion of students who fail to meet minimum requirements. The Nation's Report Card for 2009 (National Center for Education Statistics, 2010) showed that 36% of 12th-grade students were performing below the ”basic” level in mathematics, on a 3-level scale with “proficient” and “advanced”. Nonetheless, the reasons for poor performance at the 12th grade level have not been studied.

The possible reasons for students’ mathematical difficulties are numerous, ranging from internal cognitive problems (Rourke, 1993) to external inhibitors such as constrained curricula (Baroody & Ginsburg, 1983). To begin the study of difficulties at the freshman level, the simplest premise to adopt would be that students entering college underprepared in mathematics resemble the students at the middle school and high school levels, retaining the same difficulties they may have had earlier.

If some freshmen who have difficulties learning mathematics suffer, to some degree, from the same obstacles to learning algebra as middle school students, these obstacles may either be ripe for repair due to the college students’ accumulated experience, increased motivation, and advanced maturity or their obstacles may be more entrenched and therefore more difficult to remediate. Other freshmen who have difficulties learning mathematics may suffer more severe obstacles, perhaps some not yet identified, which are in need of investigation, but are beyond the scope of this investigation.

The dominant difficulty nationally observed and studied in the K-12 grades has been the field of misconceptions, which is an expected phenomenon in cognitive development, as noted by Piaget and Inhelder (2000). Some students hold misconceptions regarding place value in the decimal number system and others hold misconceptions regarding fractions, for example. The most popular misconceptions in the middle school and high school grades, however, are misconceptions concerning equivalence and misconceptions concerning the literal symbol, both of which are devastating to the learning of algebra.

If their misconceptions are not corrected in high school, the students will not be prepared for college mathematics (Kieran, 1981). This study estimated the proportions of college freshmen who adhere to the common misconceptions of the definitions and properties related to equivalence and to the literal symbol. This study also estimated the limit of the student's ability to apply the concepts to questions of increasing difficulty, which indicates the strength of their knowledge of the concepts, or indicates that some other misconceptions are at work.

Key research related to equivalence or the equal sign.

Misconceptions of equivalence can be related to the definition or to the properties of equivalence. They can also affect a student's ability to apply the concepts to questions of increasing complexity. The relationships of definition, properties, and applications are shown in Figure 1. One of the earliest studies of misconceptions related to equivalence was conducted by Behr et al. (1976) who explored elementary and middle school students' understanding of the definition of the equal sign (i.e., sameness and its corollary, interchangeability, of two expressions). Behr et al. interviewed the students, asking them to complete 3-term statements of equivalence in the canonical format, 2 + 4 = __, and in its symmetrical format, __ = 2 + 4. This research showed that elementary and middle school students have an entrenched concept about the equal sign as a "do something" indicator (the operational definition), leading to an immense difficulty in understanding equivalence statements with no plus symbol (+), with more than one plus symbol, and other noncanonical statements, all of which are incomprehensible without the more general relational definition of equivalence. Armed only with the operational definition of equality, the students would be at a loss because they could not determine which operation to perform.

Falkner et al. (1999) used the 4-term problem format: $8 + 4 = \_\_ + 5$ to differentiate first- to sixth-grade students who held the operational definition of the equal sign from those who held the more general relational definition. Students relying on the operational definition would insert 12, the sum of 8 and 4, in the blank, ignoring the + 5 term, or would insert 17, the sum of 8, 4, and 5; students holding the relational definition would quickly recognize that 7 needs to be inserted in the blank so that each side of the equation is equivalent to 12, making the two quantities equivalent to each other. The point was made that a more sophisticated definition of the equal sign is learnable at the lowest grades, but only by a few students: understanding equivalence appears to be age-, or stage-, dependent.

Freiman and Lee (2004) conducted an experiment with kindergarten students, 3rd grade students, and 6th grade students to obtain a profile of the level of sophistication of the students' understanding of the equal sign across the elementary grades. Freiman and Lee gave the students a questionnaire of three questions in the four-term format of $a = b + c$, in which the blank was in the first position (a), second position (b), or third position (c). These questions did not match the canonical arithmetic format of $a + b = \_\_$, nor did they match Falkner's four-term format of $a + b = \_\_ + c$ (Falkner et al., 1999). Freiman and Lee also provided the students with a questionnaire of 3 questions in the four-term format of $a + b = c + d$, in which the blank appeared in the second, third, or fourth position. A blank in the third position corresponded to Falkner's format. The results of Freiman and Lee's study and Falkner et al.'s study are the same for the kindergarten class, but Freiman and Lee's students in the 3rd and 6th grades performed strikingly better. Freeman and Lee explained that their students had the advantage of a regular mathematics enrichment program that supplemented the standard mathematics curriculum. The enrichment program was not described, but the implication was that more sophisticated concepts of the equal sign can be mastered in the elementary grades with intervention.

A different approach to assessing the understanding of the equal sign among middle school students was taken by Alibali et al. (2007). A questionnaire containing the equivalence statement "3 + 4 = 7," with an arrow pointing to the equal sign, and three questions, was given to each student. The first question was "The arrow points to a symbol. What is the name of the symbol?" The second question was "What does the symbol mean?" The purpose of the first question was to discourage a simple response of "equal" to the second question, with the expectation of receiving a more descriptive response to the second question. The third question was "Can the symbol mean anything else? If yes, please explain." Previous experience had suggested that, given the opportunity, students often offer more than one interpretation. Written responses to these questions may reveal more or different information about the students' knowledge of the equal sign than do simple numerical responses. The questionnaire was distributed to the students four times over a 3-year period covering grades 6, 7, and 8. The results showed a steady increase in the number of students exhibiting a relational understanding of equivalence, but at the end of grade 8, the proportion of those students was still less than 60%. Some of the remaining 40% of the students may acquire the relational understanding in high school if the curricula encourage it, but some may enter college still clinging to the operational definition of the equal sign.

The properties of equivalence have not been studied as thoroughly as the basic definition of equivalence; test items bearing on the properties, if asked, were always included within a questionnaire bearing primarily on the definition of equivalence. For example, Behr et al. (1976), in their study of students, understanding of equivalence, asked elementary students to evaluate the truthfulness of statements of equivalence in the formats 3 = 3, 3 = 5, 2 + 3 = 3 + 2, which are items related to the reflective and symmetrical properties of equivalence. No statistics were provided, but the quoted interview suggested that not all students in the first two grades recognized the basic properties of equivalence.

Molina and Ambrose (2006), also, found that third graders originally did not hold the relational definition of equivalence. Two months after an intervention directed at the relational definition, Molina and Ambrose discovered that not all of the students retained the relational view of equivalence and its properties. For instance, only 5 of 15 students knew that the reflective statement 3 = 3 was true and that 10 students knew that the statement 7 = 12 was false. Intervention at the elementary level appears to have some effect, but it is fragile. Interventions at the freshman level, if necessary, may have a more lasting effect because of the greater experience of the students.

Jacobs et al. (2007) included, in their study of equivalence, test items that they referred to as “targeted-computation” items, which were questions most easily solved by taking advantage of the application of numerically equivalent expressions. One example was 25 + 59 – 59 = __, which can be solved by left-to-right calculations, but is readily solved if the student looks at the whole structure and notices that 59-59 = 0 by itself, leaving 25 as the answer. Another example was 54 + 37 – 36 = __. Recognizing that 37 – 36 = 1, the answer 55 becomes readily apparent. These are examples of how the application of numerically equivalent expressions can be used to simplify expressions and simplify the equation-solving process. Jacobs et al. reasoned that students with a relational view of equivalence would more often synthesize a strategic equivalent structure to simplify the solution than would a student using the operational definition. Other test items with less obvious equivalent expressions were also included. For instance, the item 98 + 62 + 2 = __ could be solved by jumping into a left-to-right calculation, but is more easily solved by noticing that 98 + 2 = 100 by itself, which simplifies the item considerably. A less obvious equivalent expression could similarly be used to solve the item 46 + 27 = __ + 28. In this instance, the student would have to observe that 1 could be moved from the 46 (leaving 45) and be added to the 27, making it 28. The resulting equivalent expression on the left side of the equation becomes 45 + 28, indicating clearly that 45 must be put in the blank. Jacobs et al. provided a short group of test items that spanned a range of easy to difficult problems in which the best strategic equivalent expression ranged from obvious to obscure.

Alibali et al. (2007) assessed the use of the relational definition of equivalence of students through grades 6 to 8. The assessment also included two test items that were aimed at the application of equivalent expressions. One item consisted of two equations: 2 * __ + 15 = 31 and 2 * __ + 15 – 9 = 31 – 9. The student was asked only if the same number would be put into the blank in each equation. The question could be answered by algebraic methods and calculations, but it would be easier to answer if the student observed that the group 2 * __ + 15 appears in the second equation and is equivalent and interchangeable with 31 by the first equation. Inserting 31 in place of the group 2 * __ + 15 in the second equation, which assumes that the quantity in the blanks are the same, makes the second equation obviously true, independent of the quantity placed in the blanks. In the second test item related to equivalent expressions, the students were given the equation __ + 18 = 35 and were told that the number in the blank was 17. The students were than asked if that fact could be used to solve a second equation: __ + 18 + 27 = 35 + 27. Noticing that the first equation, with 17 in the position of the blank, can be used to replace the 35 in the second equation, making the two sides of the second equation identical except the left side contains a blank and the right side contains 17 in the same relative position. Both test items assess a student's recognition and use of the application of numerical equivalent expressions to solve equations, but the second item is less complex and more obvious than the first because the number of terms involved was fewer. Alibali, et al. found that fewer than 50% of the students were able to answer both questions correctly and held the relational view of equivalence, but nearly 40% of the students could also answer both questions correctly and did not hold the relational view. Alibali et al, showed that a relational view helps but is not necessary for solving equations, at least not numeric equations where calculations can be used to obtain the solution, although less efficiently. The full power of the equivalent expressions technique will become apparent to students when the algebraic equivalent expressions application is added to the menu of applications.

Küchemann (1981a) included some items in the CSMS test related to the application of assignment. One item consisted of a given statement in the form $x \rightarrow x + 2$

followed by an incomplete statement that the student was asked to complete: $6 \rightarrow$ __ . To answer this item, the student assigned the value 6 to the literal and calculated the value of "6" + 2 as 8. Küchemann also included other, more difficult, items in the test, which involved the multiplication operation or algebraic assignments.

Falkner, Levi, and Carpenter (1999) included in their assessment of elementary students' definition of equivalence some questions related to numerical identities, all of which were based on the basic number facts. The students had no difficulty recognizing that statements like $4 + 5 = 9$ (true) and $12 – 9 = 5$ (false), but when presented with statements such as $7 = 3 + 4$ and $7 + 4 = 15 – 4$ (both true) the majority of students were disturbed and insisted that the statements were false, but could not explain why. Falkner et al. found that even numerical identities, when presented in unfamiliar formats, were difficult for elementary students to accept. College students may be more familiar with alternate formats and may not have the same difficulties.

In summary, students have many misconceptions in mathematics, but misconceptions about equivalence, which is very fundamental to mathematical understanding, are especially common. The most widely held misconception is the fixation on the operational definition, which severely limits the student's progress in mathematics. Research has shed much light on the misconception and many improvements have been made, yet many students fail to recognize the relational definition of equivalence in the K-12 educational system. Following the combined diagram of equivalence and the literal symbol (Figure 1), students who do not recognize the relational definition of equivalence also find it difficult to fathom its properties and its applications. Because no studies of students' misconceptions after mid-high school exist, some freshmen college students may very likely retain such misconceptions, as this study confirmed.

Key research related to the use of literal symbols.

Mathematical research in the 1980s and beyond included a focus on identifying and understanding the assorted misconceptions that were held by students of beginning algebra (Kieran, 2006). The literal symbol, new to students in beginning algebra, is ripe for misconceptions. Following the combined concept diagram shown in Figure 1, misconceptions of the literal symbol can be associated with the definition and the properties of the literal symbol. These misconceptions limit a student's ability to apply the definitions and properties to problems of increasing complexity. Three misconceptions that students have can be related to the definition of the literal symbol (i.e., a literal symbol represents a quantity). Many beginning algebra students confuse a literal symbol associated with an object as representing the object itself rather than as representing a quantity associated with the object (the pencils rather than the cost of the pencils, for example) which creates considerable internal confusion in a problem-solving situation (Küchemann, 1981a). A second common misconception related to the definition of a literal symbol is the tendency of many beginning algebra students to equate the first application of a literal symbol that they encounter as the definition of a literal symbol. For those students, the definition of a literal symbol may be an unknown value that needs to be determined. An unknown is one application of a literal symbol, but there are many others. Upon encountering a different application, such as a variable quantity, the student becomes confused by the conflict between his understanding of a literal symbol as an unknown and its use in the context of a variable.

The algebra test that Küchemann (1981a) analyzed was developed by a team under the Concepts in Secondary Mathematics and Science (CSMS) research program. The test covered 10 mathematics topics relevant to the students of ages 11 to 16 in the United Kingdom in 1976. Topics included Algebra, the source of most Literal Symbol-related items in this study, as well as Measurement, Fractions, and Number Operations. The sample for the Algebra test consisted of volunteer classes in seven schools, 960 students total (Hart, 1981). Küchemann was a member of the development team and wrote the analysis and discussions for the results of the Algebra topic for 14 year-old students.

Across the entire test, the philosophy guiding the composition of test items was to test understanding of concepts, not skill levels. The aim was to have a range of difficulties represented for each concept (Küchemann, 1981b). The original all-encompassing list of items was critically culled to fit the test to a one-hour period. A pilot test was conducted in which a sample of students attempted the test in an interview with a researcher. The intent of the pilot test was to limit the test to two factors: difficulty and the meanings ascribed to literal symbols (Hart, 1981).

Küchemann (1981a) developed a 4-level scale of difficulty for the Algebra test. At level 1, the least difficult, some problems were numeric; others with literals could be solved while misconceiving of the literals as representing objects. At level 2, the problems were more complex, and literals could still be misconstrued as objects. At level 3, literals could only be interpreted as generalized numbers and did not result in numeric answers, which was disconcerting to some students (lack of closure). At level 4, literals could also be interpreted as unknowns and variables. After the test, the success rates for each item were in general agreement with the pre-assigned difficulties, but there were some exceptions, which gave rise to a more detailed understanding of what makes a problem complex. Küchemann (1981a) observed that item difficulty is made up of two components: (a) the inherent complexity of the involved concept, and (b) the context or structural complexity of the item. An example of different inherent concept complexities is solving open arithmetic statements compared to solving algebraic equations. An example of a hierarchy of structural complexities is a set of open sentence problems with small whole numbers compared to similar sentences using large numbers, fractions, or negative numbers.

Küchemann‘s (1981a) analysis of the errors made by the students on the selected questions from the test revealed six common interpretations regarding the literal symbol, listed in order of increasing sophistication: (a) a literal could be assigned a value immediately; (b) the literal could be ignored; (c) the literal was interpreted as an alias for a physical object, or, as an object in itself; (d) the literal represented a specific, but unknown, number; (e) the literal was interpreted as a generalized number; and (f) the literal was interpreted as a variable quantity. The first three interpretations are misconceptions; the last three are three different applications of a literal, which are the same three identified in this study as the most basic to a beginning algebra student. Several test items were designed to assess each property and application. The students found the items within a group to be progressively difficult, due to the relative complexity of the problem statements. The responses were used to sort the students into four levels of understanding. Küchemann (1981a) established that misconceptions of the literal symbol are common in high school students; 65% of his 14-year old students (early high school) were classified as level two in understanding or below, on a scale of one to four. He described level one and two students as being unable to "…cope consistently with items that can properly be called algebra at all, i.e., items where the use of letters as unknown numbers cannot be avoided." (p. 118) The data suggest that remediation of these misconceptions may not be complete by the time the students graduate from high school and enter college.

In his analysis of the data related to the understanding of literal symbols from the algebra portion of a large scale mathematics test of high school students, Küchemann (1981a) also focused on one set of questions that asked students to express the perimeter of a polygon whose sides were marked with a number or a literal. The perimeter of a polygon is defined as the sum of all sides of the polygon or. . $P = s_1 + s_2 + \ldots$. In this example, P and each s are variables. For example, one simple polygon was a triangle whose sides were marked as e, e and e, implying the sides were each e units in length. The expression for the perimeter of this triangle would be 3e or e+e+e. Ninety-four percent of the students were able to answer this question correctly. The success proportion dropped to 38% when the polygon was not completely drawn, but the problem stated that there were n sides to the figure. Three of the sides were marked with a 2, meant to imply that each side was 2 units in length. The expected perimeter expression was 2n. For this question, complexity of the figure (number of sides) clearly contributed to the difficulty of the problem.

In the CSMS test of 14-year old students, Küchemann (1981a) included test items related to the application of a literal symbol as an unknown quantity. One of the more difficult items asked the student to solve the equation b + 2 = 2b. (answer: b = 2). Less difficult questions of the application of unknowns are included in the questionnaire for this study.

A different misconception related to the definition of a literal symbol is the format of the numbers that the literal symbol can represent. The definition of a literal symbol does not restrict the format of the numbers, but beginning algebra students are reluctant to assign fractional values, decimal numbers, signed values, or other algebraic values as quantities represented by a literal symbol (Christou et al., 2007; Weinberg, Stephens, McNeil, Krill, Knuth, & Alibali 2004).

Weinberg et al. (2004) sought some insight into the understanding of literal symbols by middle school students using a questionnaire, followed by individual interviews. The first set of questions was related to the expression 2n + 3, where an arrow pointed to the literal n. The questions were (a) The arrow above points to a symbol. What does the symbol stand for?; (b) Could the symbol stand for the number 4? Please explain; (c) Could the symbol stand for the number 37? Please explain; and (d) Could the symbol stand for the expression 3r + 2? Please explain. Question (a) was meant to elicit a student's own definition of a literal symbol. Questions (b), (c), and (d) were questions related to the possible values that can be assigned to a literal for which the answers are each "yes." These three answers demonstrated a progressively sophisticated understanding of the definition and properties of a literal symbol. Weinberg et al. found that there were fewer correct answers for question (c) than for question (b), and fewer yet for question (d) at each of three grade levels, and more correct answers for each question across grade levels, but never more than 90%. Other questions probed for the misconception of interpreting a literal as a representation of an object rather than a numerical value, such as the unit cost of some set of objects. Performance on these questions increased across grade levels but did not exceed 60% at the 8th grade level. Weinberg et al. showed that misconceptions related to the definition and properties of the literal symbol are common and fade only slowly, possibly persisting to the college level.

Some students develop misconceptions regarding the properties of a literal symbol (i.e., like literals represent same quantity; different literals represent quantities that may be the same or different). Some students allow the same literal in an expression to have multiple values at the same time; some students refuse to allow different literals to take on the same value (Steinle, Gvozdenko, Price, Stacey, & Pierce, 2006).

Steinle et al. (2006) investigated misconceptions about the values that may be assigned to same and different variables in an expression. In their study, 327 middle school students were provided two written multiple choice questions. One asked for a set of values which could be applied to the literals in the equation $x + x + x = 12$, for which the choices were (a) 2, 5, 5; (b) 10, 1, 1; and (c) 4, 4, 4 (correct). The other question asked for possible sets of values for the equation $x + y = 16$, for which the choices were (a) (6, 10); (b) (9, 7); and (c) (8, 8), all of which were correct. Only 23% of the students were able to answer both questions correctly indicating an expert ability to apply the properties of a literal symbol. The results showed that some misconceptions abound in middle school and improve only slightly from grade 7 to grade 8, which suggests that the misconceptions are robust and that some students may have residual misconceptions about the properties of a literal symbol when they enter college.

Literal symbols are used in algebra to succinctly represent unspecified numbers in an algebraic expression of some numerical identity applicable to all numbers. For instance, the identity that any number multiplied by zero is zero can be expressed as $n * 0 = 0$, for any number n. Jacobs et al. (2007) in their study of equivalence knowledge in elementary school students included test items of this application of generalized numbers. The students were asked if the statements were always true or not always true. Sample items were $c + b = b + c$ and $c - c = 0$. The items were found to be unreliable and the results were inconclusive. The same items might be more meaningful with students more familiar with the use of literal symbols.

Some of the research of misconceptions related to the use of literal symbols focuses on the genesis of misconceptions. MacGregor and Stacey (1997) found that middle school students with no prior instruction in algebra based their early understandings on intuition and analogies to other symbols or possibly on misleading teaching materials, but the first eight weeks of algebra instruction resulted in significant achievement. Misconceptions were found to develop in subsequent years from teaching approaches, from teaching materials, and from the learning environment. MacGregor and Stacey have shown that misconceptions are a subtle, ongoing threat to learning and may result in lingering misconceptions after several years of instruction and intervention through high school and college.

In summary, research over several decades has shown that misconceptions of the literal symbol, its properties and its applications, as well as the equivalence relation, are common up to middle school, resilient to remediation, and come in assorted types. National assessments of 12^{th} grade students have shown a profound deficiency in students' mathematical skills, suggesting that misconceptions may well continue to plague students at the college level, but it has not yet been studied beyond early high school. This study will assess the prevalence of knowledge related to the relational definition of equivalence and the definition of a literal symbol, as well as their properties and applications, in a contemporary group of freshmen in developmental mathematics classes.

Research Questions

Six research questions were examined in this study: three questions related to the concept of equivalence and three corresponding questions related to the concept of the literal symbol.

1. What proportion of students in remedial mathematics classes recognizes the relational definition of equivalence?
2. What proportion of students in remedial mathematics classes recognizes the reflective and symmetrical properties of equivalence?
3. What proportion of students in remedial mathematics classes recognizes and can apply the three applications of equivalence: arithmetic identity, arithmetic equivalent expressions, and assignment?
4. What proportion of students in remedial mathematics classes recognizes the quantitative definition of a literal symbol?
5. What proportion of students in remedial mathematics classes recognizes the two properties of a literal symbol that same literals represent the same quantities and that different literals represent the same or different quantities?

6. What proportion of students in remedial mathematics classes recognizes and can apply the three application of a literal symbol: an unknown, a variable and a generalized number?

Definition of Terms

Accommodation: A Piagetian term referring to modification of an internal concept to incorporate a new experience.

Adaptation: A Pigetian term referring to either assimilation or accommodation

Alias: An application of the literal symbol as a convenient representation of an awkward, but well-known and available, number. An alias is sometimes referred to as a *constant* in the literature.

Example: The Greek literal π is used to represent the nonterminating decimal number 3.14… in the formula for the circumference of a circle.

Application: A particular manner of using a concept.

Example: a literal symbol could be used to represent a quantity that is can be specified and changed by the user (the variable application).

Synonyms: meaning, interpretation, use, usage, definition, facet,

Assignment: The application of equivalence that specifies or alters the value represented by a literal symbol.

Example: (Let) $x = 6$

Assimilation: A Piagetian term referring to the association of a new experience to an internal concept.

Concatenation: The adjoining, without punctuation, of two separate expressions.

Example: $4x$, which signifies 4 times the quantity represented by x;

Example: LW, which signifies the product of the quantities represented by L and by W.

Corollary: A proposition that is deduced simply, but not obviously, from an accepted proposition and is sufficiently different to be recognized individually.

Example: Given, for quantities a, b and c, that if $a = b$ and $b = c$, then $a = c$ (transitivity). A corollary of that statement would be that if $a = b$ and $a = c$, then $b = c$.

Equality: The relationship between two expressions of quantity of having the same value.

Example: The expressions $8 + 4$ and $10 + 2$ are equal because they each have a value of 12.

Equation: A statement that two expressions separated by the equal sign (=) are equivalent, but often presented in studies of mathematics as a proposal that the student is asked to deduce if true or false.

Example: $5+ 4 = 4 + 5$

Equivalence: The relationship between two expressions of quantity of having the same value. Additionally, the relationship is reflexive, symmetrical and transitive.

Corollary: expressions that are equivalent are interchangeable.

Example: Given that $4x + 9 = 17$, then the expression $(4x +9)$ can be replaced by 17, and vice versa.

Expression: A way of writing a quantity in terms of literals, numbers and operations, which may be regarded holistically as the intended quantity or as the set of instructions to obtain the quantity.

Example: $2x + 4$

Generalized number: The application of a literal symbol to comprehensively express a relation that is true for any number in a specified range.

Example: the relation that any number multiplied by one is equal to the original number can be expressed by using n to represent a generic number: for any real number n, $n * 1 = n$.

Identity: An application of the concept of equivalence: a statement of equivalence between two expressions that is unconditionally true for any value of the literals in the expressions, if any.

Example: $7 + 4 = 11$

Example: $(a + b)^2 = a^2 + b^2 + 2ab$

Literal: A nonnumeric character that is used to represent an unspecified number. See also "literal symbol".

Literal symbol: A single nonnumeric character that is used to represent an unspecified number. See also "literal".

Example: x, representing the cost of one apple.

Reflexive: The property of equivalence that declares an expression is equivalent to itself.

Example: $5 = 5$

Symmetrical: The property of equivalence that declares one expression is equivalent to its mirror image.

Example: $x + 3 = 3 + x$

Transitive: The property of equivalence that declares if one expression is equivalent to a second expression and if the second expression is equivalent to a third expression, then the first and third expressions are necessarily equivalent.

Example If Abe's age is the same as Bob's age, and Bob's age is the same as Clara's age, then Abe's age is the same as Clara's age.

Unknown: The application of a literal symbol where a literal represents a particular quantity, but the quantity has not yet been determined. An equation provides the clue needed to determine the quantity.

Example: t is the unknown sales tax on a particular purchase of $50. The user may determine the unknown sales tax by using the equation $t = 50 * 0.095$

Variable: The application of a literal symbol in which the literal symbol may be assigned values at the discretion of the user.

Example: If V is the volume of a rectangular box of length L, width W, and height H, the volume of the box is defined as $V = LWH$. The literals V. L, W, and H are variables because values of any three of the literal symbols can be any positive numbers the user selects. The user can calculate the volume of a particular box, or may create a list of volumes for boxes of various dimensions.

CHAPTER II

LITERATURE REVIEW

Restatement of the Problem

The focus of this study is the prevalence among college freshmen of misconceptions related to the concepts of equivalence and the literal symbol. The research of misconceptions in mathematics is strong in the elementary grades and middle school grades, but stops in early high school. At that point, the literature shows large proportions of students with residual misconceptions. This suggests that some students may carry their misconceptions on to college. Knowing the concepts of which students harbor misconceptions is a valuable tool in recognizing and remediating students' difficulties in learning mathematics.

Overview

This section is a review of the literature pertinent to concepts and misconceptions in the domain of mathematics. Some literature is psychology-based and is not grade specific, and some of the literature is directed to specific school levels. The literature related to college level mathematics learning is directed at more advanced algebraic concepts and higher mathematics; the literature related specifically to college freshmen is scarce. Confrey (1990) conducted a review of 173 publications related to conceptions studies in mathematics, science, and programming. Of these, 34 publications focused on college level students, none of which focused on college freshmen students in developmental courses. Prior to 1980, the research strongly favored the elementary and middle school students and classrooms (McGlaughlin, et al., 2005). The pertinent literature for this study encompasses studies of adults, as well as studies of children that seem generalizable to college students.

Today, there is some literature directed at the high school level of mathematics, and abundant literature exists at the levels of middle school and elementary school. Because this study concerns the mathematical misconceptions developed early and possibly retained by college freshmen in developmental mathematics classes, the literature through 12th grade is most pertinent.

This literature review will cover four areas, followed by a summary. The first part will review the literature related to the formation of mathematical conceptions. Topics covered will be how and when conceptions are formed, and how the research has evolved over time. The second part will review the literature related to misconceptions. Topics will include how misconceptions are formed, and how the research has evolved over time.

The third part of this section will review the literature related to equivalence. Literature related to erroneous conceptions of equality, as well as incomplete understandings of equality, will be included. Topics will include the relational definition of equivalence and the basic applications of equivalence that are crucial to the learning of algebra: assignments, identities, and equivalent expressions. The fourth part will focus on the literature related to the use of literal symbols in mathematics. This literature review will cover the definition of a literal symbol and three of its applications: as a generalized number, as an unknown number, and as a variable number.

Research Related to Concepts

Research on the formation of concepts in children has a rich legacy. Research may be found in the literature dating to the 1950s (e.g. Bruner, 1960/1977). Then, the focus was on *structure*, i.e., a system of related conceptions assembled for a specific purpose. The term was not precisely defined, but Bruner's example from the domain of mathematics was that

> … algebra is a way of arranging knowns and unknowns in equations so that the unknowns are made knowable. The three fundamentals [conceptions] involved with these equations are commutation, distribution, and association.… Whether the student knows the formal names of these operations is less important for transfer than whether he is able to use them [to solve for an unknown quantity by operating on the equation using the three concepts, i.e., structure]. (pp.7- 8)

In the early research, a detailed breakdown of terms related to mathematical concepts was provided by Tall and Vinner (1981). Tall and Vinner described a *concept image* (conception) as the words and mental pictures that an individual has collected over time to describe the total cognitive structure of his or her understanding of a concept. At any time, an individual's concept image, his *evoked concept image* (conception) may be limited in application or incorrect to some degree. A *concept* is defined as the form of words that specify the expert concept, which may be formal, if a formal definition exists, or personally developed language that explains the concept.

Tall and Vinner (1981) used the example of functions to demonstrate how instruction can mold and sometimes interfere with the orderly construction of a conception. A formal definition of functions may be stated in simple form as a relation between two sets of quantities, A and B, in which each quantity in the set A is precisely related in some way to one quantity in set B. For example, any radius in a set of possible radii of circles is related to precisely one circumference among a set of possible circumferences. The circumference of the circle is then said to be a function of its radius. This concept image of a function looks at a function as two related lists. A function may also be viewed, however, as a formula relating two variables ($C = 2 * \pi * r$, in this example), or as a two dimensional graph, of which each axis represents the values of one variable. As Tall and Vinner pointed out, however, the emphasis of instruction may not be evenly distributed over all three views.

> All or none of these aspects may be in an individual's concept image. But a teacher may give the formal definition and work with the general notion for a short while before spending long periods in which all examples are given by formulae. In such a case the concept image may develop into a more restricted notion, only involving formulae, whilst the concept definition is largely inactive in the cognitive structure. Initially the student in this position can operate quite happily with his restricted notion adequate in its restricted context. … Later, when he meets functions defined in a broader context he may be unable to cope. Yet the teaching programme itself has been responsible for this unhappy situation. (p. 3)

Tall and Vinner (1981) are referring to an individual's experience of learning about the concept of functions, but the same scenario takes place when elementary students learn about the concept of equivalence (Woodward & Howard, 1994). In the beginning, a teacher may define equality as "being the same quantity," but soon spends enormous periods of time drilling the students on problems in the form of $3 + 4 =$ __ , or $8 – 5 =$ __, attempting to foster automaticity with number facts. During this time, the students lose sight of the broader picture of the concept of equivalence and thereby form a restricted concept image, which they are unable to apply to problems in other formats; in other words, the students become victims of the teaching program.

Prior to the 1970s, research on the development of conceptions in children and adults was flourishing. Under the behaviorist paradigm, studies were designed as clinical experiments, and the concept to be inductively learned by each subject was an artificial problem: given a stimulus, often a visual or audible image, the subject was to decipher through trial and error which key to press to produce a reward. Such artificial concepts were used to eliminate prior experience or learning as a variable in the experiment. Early behaviorist researchers included Howard Kendler, Tracy Kendler, A. Karasik, S. Glucksberg, and R. Keston. These researchers together showed that conception development was dependent upon maturity, as corroborated by Piaget and Inhelder (1969/2000), and that college students were capable of quickly learning and accommodating complex conceptions. Kendler and Karasik (1958) studied the formation of conceptions in college students. They showed that forming a reliable conception required discrimination among stimuli that predicted no reward, as well as stimuli that predicted a reward. Kendler, Glucksberg, and Keston (1961) also designed more complex conception tests that were intended to measure the time required for the college student to relearn a conception (to adapt his conception) after the roles of the stimuli were changed. They found that a significant relearning time was required only if a visual stimulus was changed. Kendler and Kendler (1962) studied conception formation in kindergartners and third grade children using simpler stimulus-response experiments. They found that age, or development, produced an improvement. During the same period, an alternative model of cognitive development in children was spearheaded by Jean Piaget (e.g., Piaget & Inhelder, 1966/2000).

Jean Piaget's clinical method was based on experimentation, interviews, and observation (Singer & Revenson, 1996). It was the interview process that was revolutionary at the time. Piaget's method of interviewing the children was criticized as being too subjective and unworthy of the weight that Piaget assigned to it (Case, 1998).

Piaget's experiments consisted of showing the child subjects some physical demonstration, followed by a conversation with the child including probing questions of what the child subject was thinking. Singer and Revenson (1996) provided the following example of a Piaget experiment designed to investigate the concept of conservation of number. Piaget would arrange ten buttons in two equally-spaced rows of five each. Once the child, of ages 2 to 7 years, had verified that each row contained the same number of buttons, perhaps by counting, or by visual alignment, the experimenter spread the buttons of one row, making that row longer than the other. Now the child would point to the longer row when asked which row had more buttons, even when they would count the buttons. Other arrangements confirmed that the child was associating length with number of buttons, not realizing that the quantity of buttons was unchanged (conserved) by rearrangement. Based on many years of generalizing from similar experiments and observations, Piaget developed the constructivist theory of cognitive development.

The 1970s marked the beginning of an immense interest in research related to the cognitive development of children and adults. The period from 1970 through 1988 was marked by research in three different directions: (a) in the tradition of Piaget, (b) in the direction of conceptual change, and (c) in the direction of systematic errors (Confrey, 1990). The first direction was further development and refinement of Piaget's initial research. For example, von Glasersfeld (1982) argued against a misinterpretation of Piaget's theory that had arisen over the previous years: that it was necessary to promptly correct young children when they demonstrate immature conceptions. Immature conceptions, according to Piaget and Von Glasersfeld, were to be expected, and the children should be left to acquire, by assimilation and accommodation, an expert conception in due time. Piaget and Inhelder (1966/2000) were primarily interested in the development of children, but others (e.g. Clement et al.,1981; Goodson-Espy, 1995; Posner et al. 1982; Thornton, 1982) have since found that constructivism is equally applicable to adult education.

Goodson-Espy (1995) studied university students' internal transition from arithmetic thinking and problem-solving to algebraic thinking and problem-solving. Each student, in an unstructured interview setting, was presented with seven inequality problems, intentionally a type of mathematical problem with which they were unfamiliar. Each problem could be solved using only arithmetic methods, but could also be solved more efficiently using algebraic methods. A sample problem was:

> You can rent a 15 foot moving truck from I-haul rental for $100 per day plus 10 cents per mile or you can rent a comparable truck from Spyder rental for $75 per day plus 20 cents per mile. How many miles would you have to drive the truck during one moving day for it to be cheaper to rent from I-Haul? (p. 3)

Each problem presented the students with two models for renting a vehicle or two models for reporting business expenses. Each model consisted of a fixed amount plus a variable amount that depended on miles driven. The students were asked to determine at what value of the variable (how many miles) the total cost for one model became less than the total cost for the other model in the problem. Problems one and two were arithmetically easier than the others because only one model in each problem included a variable component. The eighth item was the direction to re-solve problems solved originally with arithmetic methods, but re-solve them using algebraic methods. This eighth item was in some cases the prompt that instigated algebraic thinking.

The problems were similar to other problems they were familiar with, but not quite, and the question was very different. The problems created an intentional cognitive conflict in the students and the purpose of the study was to observe the differences in approaches between those students who were successful in discovering the algebraic approach and those who were not. Goodson-Espy (1995) described the ideal approach to make the transition would follow a pattern of making an initial representation of the problem and subsequently making abstractions of that representation, ending with a symbolic representation of the problem. Not all students were successful in making the transition.

Goodson-Espy (1995) divided the complete process of developing an algebraic approach into four levels: recognition, re-presentation, structural abstractions, and structural awareness. A student who successfully made the transition first recognized the arithmetic solution, which had been proven to work for other problems, but could not foresee difficulties that may arise. As the student reflected about the problem, he or she would then reduce the solution method to an efficient model (re-presentation), which he or she could use to anticipate difficulties. Further reflection would bring the student to the third level, structural abstraction, at which he or she could consider possible other uses of the solution or improvements to prior solutions. At the fourth level of the process, structural awareness, the solution would be complete and usable with a high degree of self-confidence. Some students in Goodson-Espy's study did reach this plateau; others did not. Goodson-Espy wanted to know how their transition trajectories differed.

There were three categories of solutions that the students used. One was pure arithmetic methods, systematic trial and error until arriving at a sufficiently close answer. The second method was an improvement over the first, in which a table was developed for different values of the variable (miles). The value of the variable that marked the transition from higher model to lower model was apparent in the table. The third solution category was an all-algebraic method.

Two case studies represented the extremes of the students in the study. The first student did not attain full abstraction. He was able to solve all the problems arithmetically, advancing to the use of tables in the third problem. At the sixth problem, he advanced to the re-presentation level, exercising some thought about what would happen when he used a table of different values of the variable. At test item eight, he was asked to think of the variable and solve the problem again. The student succeeded in creating the abstracted model but was not able to use it to solve the problem. Goodson-Espy (1995) attributed this gap as a failure to recognize the expression for the total cost of a model as a quantity that could be equated to another quantity. For this student, the expression was no more than a set of instructions to calculate the total cost once the number of miles was assigned. He could create a table of values with this view, but not solve the problem with an abstract number of miles.

The second case study from Goodson-Espy's (1995) study was a student (solver #4) who had succeed in developing the algebraic methods of solving the problems. When asked to solve the same problem using algebraic techniques, he found that the unknown value (x) was necessary in two places in the problem, which was problematic for him and he would need a different approach. His first adaptation failed, but he then discovered the table approach. With the table approach, he was able to solve all the remaining problems. At problem eight, when he was prompted to use an abstract variable, he constructed an algebraic method, with which he was able to resolve all the problems.

Goodson-Espy (1995) concluded that it was necessary to prompt students to reflect on their methods and that it was necessary for the students to be comfortable with the dual nature of an expression: as a process and as an object. She also observed that students who succeeded in developing algebraic methods also had robust conceptions of equivalence and the variable application of a literal symbol.

Research Related to Misconceptions

The recognition of misconceptions in the domain of mathematics is relatively new, not appearing in the literature until the middle 1970s (e.g. Behr et al., 1976), although the phenomenon was well recognized in the natural sciences (Resnick, 2006). The *misconception* label itself began to appear in the 1980s (e.g. Clement et al., 1989). Brown and Burton (1978) categorized errors as either *careless* or as *procedural.* The *bugs* described by Brown and Burton were procedural errors that students commit consistently as they solve similar problems. The systematic errors are built on a misunderstanding of an algorithm or the theory supporting the algorithm, some of which may be subtle and difficult to discern. Some systematic problems may be obvious to a teacher who is attuned to errors that are common and with which he or she is familiar. However, not all systematic errors are common or obvious, especially those that do not always result in an error in the problem's answer. Uncommon systematic errors may go unnoticed and unresolved. One approach to better misconception diagnoses has been to assemble a thorough computer-based diagnostic assessment, which has led to several computer-based diagnostic programs (e.g., Brown & Burton, 1978; Russell, O'Dwyer, & Miranda, 2009; Sleeman, 1984).

Because subtle and unexpected systematic errors may not always result in an incorrect response, a diagnostic analysis often requires the analysis of a sequence of exercises (Brown & Burton, 1978). For example, a student answers the following addition problems (correct answers are in parentheses) 41 + 9 = 50 (50), 328 + 917 = 1345 (1245), 989 + 52 = 1141 (1041), 66 + 887 = 1053 (953), 216 + 13 = 229 (229). Some answers are correct; others are not. Is the error random or systematic? Scrutinizing the answers, an evaluator may uncover the faulty procedure: the student accumulates the carry amount to each column. If the ones column produces a carry of one, it is added correctly to the tens column, but if the tens column also produces a carry of one, the student adds two (both carried quantities) to the hundreds column. This faulty algorithm produces an erroneous answer only when more than one column produces a carry term. Many other scenarios of systematic errors exist, each requiring a unique set of diagnostic problems. Detection of subtle and uncommon errors becomes a tedious task and is best performed in a one-on-one setting between the student and the evaluator. Such detection depends strongly on the evaluator’s encyclopedic knowledge of possible misconceptions and appropriate probing questions. To reduce the reliance on an omniscient evaluator, one could develop a computerized diagnostic testing program, consisting of a large computer data base of all possible misconceptions that could be matched to another data base of signature problems and leading questions. A program that diagnoses a student’s misconceptions, as described, is different from programs that screen a student’s overall mathematical skill levels, which are more appropriate for mass placement purposes, often part of college admissions procedures.

Brown and Burton (1978) developed an extensive computerized network of arithmetic knowledge designed to illuminate systematic errors in arithmetic. Each operation was described as a complex network of every small procedural step and option. For example, a portion of the network for addition is shown in Figure 2, used here with permission (Appendix A). The initial application of the computer program was a game (Buggy) for student teachers in which the computer played the role of a student with a single faulty algorithm. Buggy would provide a few sample solutions to the teachers. The teachers would then provide additional strategic problems for the computer to solve with the same faulty algorithm, thus leading the teachers to an understanding of the fault.

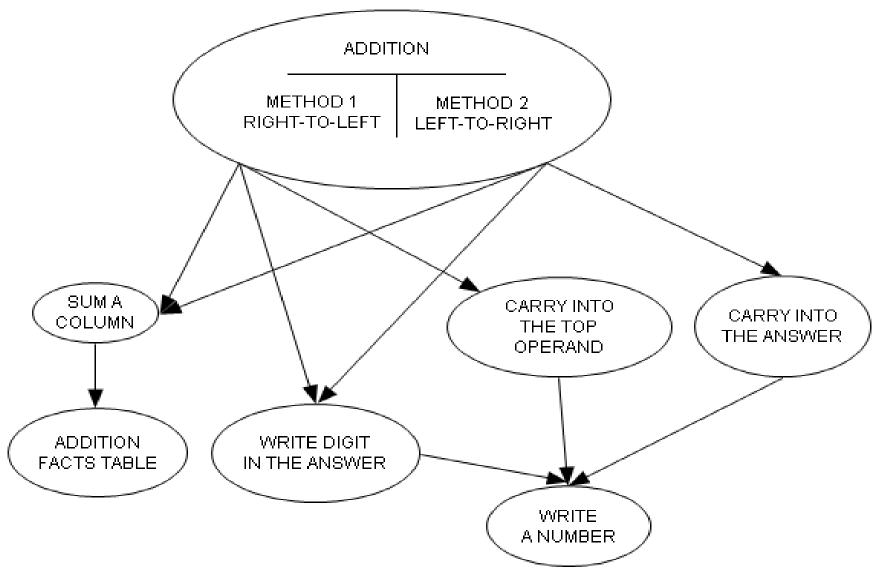

Figure 2. Partial network of operations for the task of addition. Excerpted from "Diagnostic Models for Procedural Bugs in Basic Mathematical Skills," by J. Brown and R. Burton, 1978, Cognitive Science, 2, p.160.

The game provided the teachers with an awareness of the span of errors that students may have and with clinical practice in diagnosing the errors. The game also highlighted strategy errors that the teachers could make. One strategic error was to jump too quickly to a hypothesis before incorporating all the symptoms. A more subtle strategy error sometimes made by the teachers was to focus on only one type of symptom, which would steer the teachers to incorrectly characterize the problem. As well as a tool for training teachers, the Buggy game has been used with middle school students and was influential in training the students to visualize their own errors as clues to misconceptions, not as signs of incompetence.

Over the last 30 years, the science of computerized diagnostic tools has been carried on by many others (e.g. Corbett, Koedinger, & Anderson, 1997; Graesser, vanLehn, Rosé, Jordan, & Harter, 2001; Sleeman, 1984) under the category of Intelligent Tutoring Systems. During this time, Intelligent tutoring Systems has invoked the field of Artificial Intelligence to expand the role of diagnostic systems for remedial mathematics to constructive learning environments and has extended the scope to support physics, computer science, and electronics technical training.

For example, another computer-based diagnostic program for algebra students, the Leeds Modelling System (LMS), was developed by Sleeman (1984), and was based on correct and incorrect rules (mal-rules) observed in Buggy and other programs, leading to a more efficient model. Examples of algebraic mal-rules are (a) M + N*X = implies M*N + X = and (b) M*X = N implies X = N. Results of early tests with a group of 15-year-old students showed excellent correlation with the results of individual interviews and uncovered a basic flaw in the program: the program incorrectly assumed that a rule correctly applied once would always be correctly applied in similar problems. Subsequent tests uncovered new mal-rules, which were integrated into a revised LMS program.

The revised LMS program (Sleeman, 1984) categorized the types of errors that students make: manipulative, parsing, clerical, and random. Manipulative errors are errors of omission caused by cognitive overload or inattention. Parsing errors are genuine errors caused by a misunderstanding of algebraic notation. Manipulation and parsing errors are not always easy to discriminate, but a quite different remediation approach is required by each. Clerical errors are errors in reading the mathematical notation or numbers. Random errors are those errors in arithmetic computation that do not occur consistently.

With the advances in understanding misconceptions and in computer technology, efforts to computerize formative diagnostic tests in the mathematics domain continue today. Russell et al. (2009) have developed the online Diagnostic Algebra Assessment System (DASS), which consists of sets of 10 to 12 multiple choice questions, each set focusing on one specified misconception. To be comprehensive, the DAAS would need to contain an enormous library of misconceptions and a more enormous bank of questions. The menu of four responses for each item of the test contains the correct answer, the answer obtained by the target misconception, and two distractors obtainable by other errors or misconceptions. A sample test item is

> m is a positive number. How many possible values can 10m have?
> a) 5 b) 10 c) 20 d) infinitely many (p. 416)

The correct response is d), infinitely many, because m may be assigned any real value from 0 and up. Some students ignore the literal (the misconception) and respond b) 10, the number associated with m in the question. A student's test may be graded either for an assessment of his ability or for an assessment of his misconception status. His ability is scored as the number of correct responses; his misconception status is scored as the number of items in which the misconception response was selected. If more than 35% of the set was answered with the misconception choice, the student is deemed to hold the misconception. Both test scores are immediately available to the test administrator. A class-wide summary report is part of a kit provided to the class teacher that also includes explanatory material, a lesson plan for an instructional intervention, a list of materials, and suggested activities.

In the pilot test of the DASS diagnostic program, a pretest-intervention-posttest study, Russell et al. (2009) studied the effects of the complete DASS program on students' ability scores and on their misconception scores. They also compared the individual effects of the misconception scores and of the supporting kit provided by DASS. In all, there were four treatment groups in the pilot test that varied in the misconception analysis and supportive instructional materials provided, but all treatment groups, including a control group, participated in the online DASS tests and received the ability report. Group 2 also was given the misconception report, but not the instructional materials. On the other hand, Group 3 was given the instructional materials, but not the misconception report. Table 1 lists descriptions of each of the four treatment configurations. Russell et al. limited the scope of misconceptions in the pilot test of the DASS to just one misconception from each of just three categories of (a) literal symbol: assigning a concrete numerical value to the variable; (b) equality: the operational definition; and (c) graphing: treating a graph as a picture representing a given scenario instead of treating it as showing the relationship of two variables. The reliability ratings of the 22 test items for both ability and misconception were moderate to high, for both areas of literal symbols and equivalence.

Forty-four algebra teachers with their 905 students voluntarily participated in the pilot test and were randomly assigned to a treatment group. The number of teachers and the number of students in each group is shown in Table 1. The students ranged from grade 6 to grade 12, but 90% of them were in grades 8 and 9.

The three week intervention was not strictly structured. Each teacher based an intervention on the pretest analysis of his or her class, using the resources provided by DASS, when supplied.

The pretest analysis showed that 14% of the students held the targeted misconception for the literal symbol and 11% of the students held the targeted misconception for equality. The improvements in the students' ability and misconception scores, averaged over all participating students, were small, which is not surprising in light of the small proportion of students with misconceptions to begin with.

Table 1

Configurations of Treatment Groups for the DASS Pilot Test

Configuration	Group 1	Group 2	Group 3	Group 4
Intervention	Control	Partial	Partial	Full
Number of teachers	11	17	7	9
Number of students	227	278	153	247
Ability report	Yes	Yes	Yes	Yes
Misconception report	No	Yes	No	Yes
Instructional materials	No	No	Yes	Yes

However, the comparison of achievement in both ability scores and misconception scores between the treatment groups all showed that full intervention was more effective than partial intervention or no intervention (effect sizes moderate to high). Comparing the results of the configurations with instructional materials (Groups 3 and 4) to those without instructional materials, Russell et al. (2009) found improvements in both ability scores and in misconception scores, but the effect sizes were small in both cases.

Studies of computer-aided diagnostic programs have made it abundantly clear that the diagnoses of students' errors, either by computer or by one-on-one interviewing, are not a simple task. In any case, knowing procedural errors that are common among students is valuable information.

Not all misconceptions are procedural in nature, however. Many misconceptions are conceptual in nature. Macgregor and Stacey (1997) studied how students cope with the concept of literal symbols when first confronted with them in the classroom. They tested two classes of seventh grade students twice, once before they had received any instruction in algebra and again after eight weeks of their introductory algebra course. Two loaded algebra problems were asked:

> 1) David is 10 cm taller than Con. Con is h cm. tall. What can you write for David's height? Answer: h + 10 cm.
> 2) Sue weighs 1 kg less than Chris. Chris weighs y kg. What can you write for Sue's weight? Answer: y – 1 kg. (p. 5)

Both questions required skills in interpreting word problems mathematically, a beginning knowledge of algebraic notation, and an acceptance of an answer that is not numerical. MacGregor and Stacey expected the students at the pretest to be unable to respond or to provide responses at the level of Küchemann's (1981a) lower division (letter ignored, letter assigned a value, or letter as a label for an object). In fact, for the first problem, two-thirds of the 42 students were unable to respond, but the other 14 students provided reasoned answers. For example, one student responded with the correct answer (10 + h) and two others responded with a different letter (t, or g). The responses were similar for the second problem.

For the post-test following eight weeks of algebra instruction, two equivalent questions were asked:

> 1) Con is 8 cm taller than Kim. Kim is y cm tall. What can you write for Con's height? Answer: y + 8 cm.
> 2) Sam is x cm shorter than Eva. Eva is 95 cm tall. What can you write for Sam's height? Answer: 95 – x cm. (p. 5)

In the post-test, 14 of 34 students responded with the correct answer to the first problem, 16 students for the second problem. The errors were not of Küchemann's (1981a) lower division of rationales, but showed some improvement in understanding the concept of a literal. For example, 10 students responded to problem one with 8y, y8, 8 – y or y – 8, all four of which showed a recognition of the basic definition of a literal as a quantity and an attempt to combine the quantities y and 8, but they erred in the manner of joining them. Seven students made similar errors in problem two. The recognition of the literal as a quantity, the basic definition of a literal symbol, increased from 4% pretest to 63% post-test. The proportion of no-responses decreased from 67% pretest to 11% post-test.

The students' responses to the pretest corroborate the theory that students, even at the elementary level, often begin their formal study of a concept with prior conceptions, sometimes correct, but often not (Booth, 1981; Clement et al., 1989). The progress shown by these students over eight weeks of introductory instruction indicates that the preconceptions can be readily correctable, but not always. Success is highly dependent on the curriculum and the teacher's expertise, at least.

Clement et al. (1989) pointed out that not all misconceptions are incorrect, but may be incomplete or limited in scope. They defined "misconception" as "… students' ideas that are incompatible with currently accepted scientific knowledge" (p. 555). Reasonably correct conceptions that can be used as a basis for correcting misconceptions of a related phenomenon they called "anchoring conceptions." They tested 137 high school students who had not taken a physics course to detect their misconceptions and their anchoring conceptions. They found, for instance, that in the area of Newton's Second Law, 80% of the students correctly believed with some confidence that holding down a bed spring would cause the spring to apply an upward force against the hand. Meeting the criteria for an anchoring conception, this example could be used to correct a common misconception among the students who did not believe that a wall exerts a force on fist when the fist strikes the wall. Clement et al. focused on science misconceptions, but their notion of anchoring conceptions is equally applicable to mathematics. A student may have an anchoring conception of the additive property of equations (adding equal amounts to both sides of an equation produces another valid equation), but may harbor a misconception that the multiplicative property of equations (multiplying both sides of an equation by equal amounts produces a valid equation) may only be applied if there is a single term on each side of the equation. The additive property can be used as a springboard to fully explain the multiplicative property.

Christou et al. (2007) collected from Freudenthal (1983) and Küchemann (1981a) four popular preconceptions and misconceptions of literal symbols that students exhibit, which they described as insidious carry-overs from arithmetic. The first misconception was that the literal symbol is a label for an object. For example, when solving word problems in arithmetic, students and teachers often use abbreviations or initials for the units involved. This practice keeps the emphasis on the arithmetic, but breeds a habit of thinking 3m represents 3 meters. In an algebra problem, the student may automatically interpret 3m the same way, ignoring the m because it is an object (a misconception) and focusing on the quantity 3. However, in algebra, m represents a quantity and 3m means 3 times the quantity m.

The second area of misconceptions related to the literal symbol is the group of misconceptions arising from the difference in concatenations between arithmetic and algebra. In arithmetic, two adjoined digits, such as 27, imply addition (20 + 7), and 8½ inches implies 8 inches + ½ inch. In algebra, on the other hand, the concatenation of a number and a literal, such as 3a, implies multiplication (3 * a). Unless this difference is made clear to students, they can develop misconceptions.

The third misconception related to the literal symbol is that a literal symbol represents a specific number, which it occasionally does, but not always. The fourth misconception is that an answer cannot contain a literal, but in algebra, an answer is not always numeric. For some students, arithmetic, and presumably algebra, is an empirical subject, requiring numerical answers. Answers in algebraic form (containing one or more literals) are disconcerting, leaving the student with a feeling that the problem is not yet solved, that is, a lack of closure. For Christou et al. (2007), these misconceptions related to the literal symbol were the result of inappropriate transfer of prior knowledge from arithmetic.

The typical algebra curriculum also engenders misconceptions in other ways. Harel et al. (2008) observed high school teachers ($n < 30$) for their attention to meanings of terms, symbols, and operations while teaching. They chose to report on two teachers who exemplified the gamut of faults related to inattention to the meanings of terms and symbols in the classroom, and they reported the consequences for the students. The four classes of faults the teachers committed under the pressure of classroom instruction were: (a) the introduction of topics and terms without a stated purpose; (b) failure to make distinctions between different terms, or between an object and an operation; (c) inattention to the meanings of terms, inconsistency in use, incorrect definitions, and emphasizing the importance of procedures over meaning; and (d) use of symbols without attention to their meaning, incorrect use of symbols, emphasizing form of expressions over meaning, and emphasizing symbols as inputs to a procedure rather than representations of quantities. The four consequences for student leaning were reported as (a) mathematics involves executing procedures more than meaning and reasoning; (b) mathematical definitions are coincidental or arbitrary; (c) symbolic reasoning is performed without attention to the meaning of the symbols; (d) the form of an expression is more important than its meaning. While these reported observations and conclusions are important for classroom practices, what Harel et al. did not report is also important. They did not specifically report that the confusion resulting from not having a guiding framework of well-defined meanings creates an environment conducive to the development of misconceptions.

Research Related to Equivalence

The concepts of equivalence and the literal symbol originated independently, but their applications are not entirely disconnected, as shown in Figure 1. The applications of relational equivalence only rarely require some understanding of a literal symbol, but applications of the literal symbol almost always require an understanding of relational equivalence. This lack of symmetrical dependence between the concepts and the applications has two consequences. The first consequence is that the concept of relational equivalence and its applications are learned first. Current standards (National Council of Teachers of Mathematics, 2000; National Governors Association NGA Center for Best Practices, 2010) call for the relational concept of equivalence to be learned by the second grade and for the concept of the literal symbol to be learned between the sixth and the eighth grades. The second consequence of the lack of symmetrical dependence between the concepts and their applications is that assessment of students' understanding of the literal symbol assumes their understanding of equivalence; the assessment will be confounded unless the student's understanding of equivalence is known in advance or assessed simultaneously.

In the typical course of mathematics education, the twin concepts of addition and subtraction of two quantities and their symbols are often introduced before the first grade. The concepts of addition and subtraction, and later those of multiplication and division, are bona fide operations. The concept of addition, as indicated by the + symbol, is always placed in a context with two numbers that are to be combined. Similarly, the other three operations of subtraction, multiplication, and division are operations performed on two numbers. At about the same time, the concept of equality and its symbol are introduced. The equal sign, however, is not an operator, and equality is not an operation. Equality is an expression of a relationship between two mathematical quantities, specifically the relationship of equivalence. Other relationship symbols are *greater than* and *less than*. Hattikudur and Alibali (2010) suggested that these two relational signs along with the equal sign should be introduced as a group to reinforce the difference between relations and operators. If the significance of the difference between operator and relationship becomes lost, the young student may be left with the misconception that the equal sign is also an operator, a direction to perform an indicated operation and supply the result in an indicated place. This misconception has been dubbed *the operational concept of equality* (or *of the equal sign*). This misconception suffices for problems in the canonical form of a + b = __, where a and b may be any two numbers and the addition operation may be replaced with a subtraction, a multiplication, or a division operation (Alibali et al., 2007). For example, the misconception would produce the expected result (30) for the problem 5 * 6 = __.

The use of mathematical problems in the canonical format, and its many variations, is not a recent invention. Such open mathematical statements were used extensively by Davis (1964) in the Madison Project, which was a supplementary mathematics discovery program covering mathematics from the elementary arithmetic operations through beginning algebra.

An arguable assumption pervading the literature related to the misconception related to equivalence and its related symbol is that similar misconceptions manifest themselves in similar symptoms. The assumption has led to a wide search for common interpretations, correct and incorrect, of the equivalence concept. Behr et al. (1976) were among the first to publicize the tendency of elementary students to interpret the equality sign as an operator.

Falkner et al. (1999) discovered that the simple act of changing the format of the problem to a + b = __ + c had devastating results. They posed the problem 8 + 4 = __ + 5 to classes in the elementary schools. Students with the operational misconception of equivalence were prompted by the equal sign to perform the addition and to put the result

(12) in the blank, but they were unsure what to do with the “+5” in the problem. Some ignored the extra term, putting 12 in the blank; others included the extra term in the result and put 17 in the blank. On the other hand, students who understood that the equality sign indicated that the two quantities must be the same recognized that 7 must be entered in the blank position, making both quantities denote the same quantity of 12. Other variations of the canonical problem produced similar confusion among Falkner et al.’s students.

Molina Gonzales, Ambrose and Martinez (2004) studied third grade students as they responded to four-term open sentences in which the blank might be in any of the four positions. They observed that all 15 of the third grade students were dependent on the operational definition of the equal sign; none showed any inclination to use the equal sign, or equality, as a relationship between two quantities. As a result, the students were pressured to deduce a solution in real time. Error counts for each question type ranged from 12 to 15. Among the errors, two symptoms were discovered: (a) a tendency to use a mirror image of the opposite number when the blank was adjacent to the equal sign (14 + <u>13</u> = 13 + 4) and (b) a tendency to insert the other side’s operation when the blank was in the first or fourth position (<u>12</u> + 4 = 5 + 7). The same test administered to 26 fifth and sixth grade students resulted in random errors and in dramatically lower error counts, ranging from 0 to 5 except for one question type. The outlier question type had a blank in the third position, and when the operations were subtractions rather than additions, the error count changed from 5 to 10. Molina Gonzales et al. attributed the improvement in error counts and the error types to an improved knowledge of the relational definition of equivalence.

A similar study of kindergarten, grade three, and grade six students was conducted by Freiman and Lee (2004). These students were enrolled in a mathematics program that was attempting to smooth the abrupt transition between arithmetic and algebra by eliminating barriers developed and misconceptions encouraged in the elementary curriculum. Freiman and Lee tested the students with a set of three-term open statements and four-term open statements. They observed the same errors as did Molina Gonzales et al. (2004) plus a tendency to insert in the blank the difference of two terms instead of the indicated sum of two terms.

Molina and Ambrose (2006) assessed the understanding of equivalence of third grade students (n=13) using three- and four-term open equivalence statements, in which the blank might appear in any of the four positions, and closed statements which were either *True* or *False*. Following a pretest of the open equivalence statements, the students participated in a class-wide discussion of the proposed answers and the relational concept of equivalence. Two months later, the students were given seven closed statements to evaluate as *True* or *False*, which could be answered based on the relational nature of equivalence. The results showed that two-thirds of the class failed to retain or transfer their introduction to the relational nature of equivalence. More discussion of equivalence followed, as well as activities and assessment, and several more students acquired the relational view of equivalence. The experiment demonstrated that the relational concept of equivalence cannot be remediated with a simple show and tell approach, but must be fostered with the students over time, at least with third-grade students, although the combination of open equivalence statements and closed equivalence statements appear to be an effective medium of learning.

Other studies of the understanding of equivalence among elementary students have also attempted intervention methods with varying degrees of success, especially at the higher elementary grades (e.g., Alexandrou-Leonidou & Philippou, 2007; Molina & Ambrose, 2008; Jacobs et al. 2007; Warren & Cooper, 2009). Because it is not clear yet if college freshmen hold persistent misconceptions related to equivalence or literal symbols that are similar to those of younger students, or if they hold misconceptions at all, it is premature to design an intervention to correct the misconceptions. Interventions used in the elementary grades would have to be adapted prior to using them with college students, and the results at the college level might be different. It is not clear whether college students hold more entrenched misconceptions or their broader background would facilitate remediation. Because intervention is not part of this study, research related to remediation methods for lower grade students are not pertinent to this study.

Linguistically analyzing the language of algebra, Freudenthal (1981) described five ways in which the equality sign and the concept of equivalence are correctly used. He did not give the usages names, but they corresponded exactly to the basic applications of equivalence: the applications of arithmetical and algebraic identities, the application of arithmetic or algebraic assignments, and the application of an alias as used in this study. Warren (2007) and Küchemann (1981a) cataloged the same five ways that students interpreted the equal sign. By defining subcategories of the five interpretations and by adding other popular, but improper, usages of the equal sign, Molina et al. (2009) expanded the list to 11 interpretations. They included, for example, the common habit of students to separate steps of a calculation with an equal sign rather than placing the separate steps on separate lines. The result of using equal signs as “splitters” of sequential steps is an undecipherable statement of sequential equalities.

In an ideal academic environment, students would learn the relational view of equivalence in the first two grades (e.g., Warren, 2007) and begin a study of algebra in middle school without ever being handicapped by the operational view of equality. However, not all students enter middle school so well prepared. The team of Alibali et al. (2007) conducted a longitudinal investigation of the algebraic knowledge of middle school students that included a study of their knowledge of equivalence and their knowledge of operations related to equations, which is thought to be related to their knowledge of equivalence (Kieran, 1981).

The research questions posed by Alibali et al. (2007) were:

> What meanings do middle school students ascribe to the equal sign, and how do these change over time? What is the relationship between the meanings ascribed to the equal sign and performance on problem solving equations? How is performance at solving equations related to when students acquire a more sophisticated understanding of the equal sign? (p. 1)

They found that some students enter middle school with a handicapping operational view of equality and that some of those students exited middle school with the same handicap. Most of the students showed a small gradual achievement throughout middle school, both in their knowledge of equivalence and in their skills related to the equation-solving strategy of recognizing and forming equivalent equations.

The participating students in the study by Alibali et al. (2007) were 81 students from a middle school in the Midwest. The sample was 62% White, 25% African American, 7% Asian, and 5% Hispanic. The standard curriculum for the school was *Connected Mathematics*, which included some development of algebraic concepts in the sixth grade and algebraic topics in the seventh and eighth grades, but no explicit instruction on the topic of equivalence.

Data were collected four times over a three-year period. The first data were collected at the beginning of the sixth grade to establish a reference point for the remaining data and to estimate the proportion of students exiting elementary school with the relational view of equivalence and the ability to recognize equivalent equations. Longitudinal data were also collected at the beginnings of the seventh and eighth grades and at the end of the eighth grade. The instrument for collecting the data was a written assessment, consisting of two items, developed by Alibali et al. (2007) based on items from prior research related to equality and equation skills. The first item asked the students for their written interpretation(s) of the equal sign. Originally, this question was asked orally of first and second grade students by Behr et al. (1976) but in the research since that time the question has been developed further. Item 10 on the instrument for this study (see Appendix B.) is a simplified version of item 1 on the instrument by Alibali et alia. Two items in Alibali et al.'s instrument were related to equivalent equations. One of them is similar to items 5, 8, and 17 through 21 on the instrument for this study, the other is similar to item 9 on the instrument for this study (see Appendix B.).

The assessment administered by Alibali et al. (2007) came in three forms randomly assigned to students. All three forms contained the equality sign question and one of the two equivalent equation problems. Fifty-five students received one of the two equivalent equation items; 26 students received the other. The data for the students as a whole clearly indicated that students gradually acquired an understanding of the relational view of equivalence during the three years of middle school. As the students entered middle school, approximately 20% of them understood the relational view; by the end of the eighth grade, nearly 60% of them understood the relational view of equivalence.

The statistical significances of these increases were not reported, possibly because the results were very curriculum- and classroom-dependent. The trend, however, may be typical. As Alibali et al. (2007) pointed out, the curriculum in use at their study site did not contain explicit instruction on equivalence; the students who acquired the relational view of equivalence did so by external influences or spontaneously in the class, which would be a function of time and exposure. Not discussed by Alibali et al. was that the proportion of students who entered middle school with the operational understanding of equivalence dropped from about 70% as they entered middle school to 30% at the beginning and at the end of grade eight, while those holding *other* views remained constant at 10%, except for an unexplained spike to 25% at the beginning of the eighth grade. Without understanding the spike in the *other* understandings of equivalence at the beginning of the eighth grade, it is unclear if the plateau in the operational understanding of equality is significant.

The data for the 81 students as a whole also indicated a gradual increase of students who can recognize equivalent equations, starting at nearly 50% as they enter middle school, to over 75% as they exit middle school, consistently 20% to 30% higher than the proportion holding the relational view of equivalence across the three-year period. The same data showed that the proportion of students able to recognize and apply the concept of equivalent equations similarly improved across the grades, from 20% to 40%. The association between the relational view of equivalence and the recognition of equivalent equations was only significant at the beginning of the seventh grade. The association between the relational view of equivalence and recognizing and applying equivalent equations was only significant at the seventh and eighth grade levels as had been found by Knuth et al. (2006). However, as noted by Alibali et al. (2007), almost half the students exited middle school without a relational view of equivalence and fewer still with an understanding of equivalent equations. It is, therefore, reasonable to suspect that a significant proportion may also enter college without these skills.

On the individual level, over half of the students who acquired the relational view continued to use the relational view steadfastly; only 12% of the students showed some vacillation. Fifteen percent of the students demonstrated a relational understanding of equivalence from the beginning of middle school, but 25% of the students entered middle school without a relational knowledge of equivalence and exited middle school without it. Without explicit instruction on the topic of equivalence, it seems that the concept is not easy to acquire.

Of the 64 students who showed an ability to recognize and apply equivalent equations, 13 did so at the same time as they demonstrated the relational view of equivalence. Of the remaining 51 students who demonstrated an ability to recognize and use equivalent equations separately from demonstrating the relational view of equivalence, 35 students demonstrated the relational view of equivalence first. Because the binomial probability of such a biased proportion is .005, the relational view of equivalence generally precedes the ability to recognize and apply equivalent equations.

Although the student sample in the study by Alibali et al. (2007) consisted entirely of middle school students, the data indicated that students pass through elementary school without acquiring a relational view of equivalence and without developing the ability to recognize equivalent equations, both of which are crucial to learning algebra. (See also Knuth et al., 2006.) Without explicit attention to the subject of equivalence in the classroom and in the textbooks (McNeil et al., 2006), many students will not acquire the relational view of equivalence in middle school. If the trend of no explicit instruction on equivalence continues through high school, many students may arrive at college without the necessary understanding of equivalence and, therefore, an understanding of equivalent equations, which dooms them to failure in an algebra class.

As acknowledged by Alibali et al. (2007), the assessment was constrained to one item for each measurement, but more items might have provided more opportunities for students to demonstrate understanding and might have produced different results. The same reasoning has led this study to use a broader menu of test items. Alibali et al. also suggested that middle school students' low usage of the relational definition of equivalence can be related to their lack of algebraic experience; more experience in algebra will produce more opportunities to invoke the relational definition, which will lead to greater familiarity and usage of the concept. For the college freshman students in this study, who have been more exposed to algebra, the use of the more sophisticated relational concept was more common.

No studies of the status of understanding of equivalence by college freshmen exists, but a study by McNeil and Alibali (2005) investigated the status of understanding equivalence over the span of students from elementary school through graduate school. The students were shown an equal sign in one of three contexts: (a) a bare equal sign (=), or (b) an equal sign used in an open addition statement ($4 + 5 + 4 =$ __), or (c) an equal sign in an equivalence relation (4 + 8 + 5 = 4 + __). In all contexts, the students were asked to write their own meanings of the equal sign.

At the lower end of the grade range, the elementary students (n = 55) predominately offered the operational view of equality: 60% of the students who were shown the bare equal sign and over 80% of the cases of each of the other two contexts. The seventh grade students (n = 25) offered the operational view in approximately 60% of the bare sign and the addition contexts, but in less than 10% of the cases in the equivalence context, very similar to the college students in remedial classes of this study.

McNeil and Alibali (2005) administered the same test to 35 undergraduate students, all of whom had one semester of calculus experience. The undergraduate students offered the operational view in less than 20% of the bare sign and addition contexts, and not at all in the equivalence context. The physics graduate students (n = 12) offered only the relational view in all three contexts. The results of the study by McNeil and Alibali (2005) showed a tendency toward the relational view of equivalence as a function of mathematical training.

The 12th International Commission on Mathematical Instruction (ICMI) was a four-year study of the Future of Teaching and Learning of Algebra, which concluded in 2004. The study consisted of nine working groups, including one on Symbols and Language. Each group published a final synopsis of their conclusions. For the Symbols and Language working group, Drouhard and Teppo (2004) explained that written mathematical communication takes place on three levels: natural language, symbolic writings, and compound representations. Natural language is the text written in the vernacular language, which may contain technical terms that must be defined and clearly understood. Symbolic writings are non-literal figures that represent concrete concepts, such as the numerals that represent quantities. Compound representations contain elements of both natural language and symbols.

Because of its reliance on signs and symbols to convey meaning, the language of algebra is inherently ambiguous. Symbols, or signs, include words, numerals and operational symbols.

> …collections of symbols can be seen as either representations of procedures or can be taken structurally to stand for mathematical objects. There are other sources of ambiguity associated with the language's use of the minimal symbols set. …As an additional example, consider the different kinds of algebraic sentences in which = appears. This symbol can be used to indicate equality of numbers (5 + 3 = 8), equivalence of expressions (a –(-b) = a + b), or to define a function (f(x) = 2x + 7). (Drouhard & Teppo, 2004, p. 241)

An expert's behavior "…relies on the capability to reach the meaning of the symbols on demand" (p. 251). A learner's behavior, on the other hand, is characterized by a lack of fluency with words and symbols. Where these two meet, as in a classroom, confusion reigns if the expert is not meticulous and timely when explaining the meanings of the words and symbols he or she is using (Harel et al., 2008).

Drouhard and Teppo (2004) adopted the definition of *meaning* developed by Frege (1848-1925) as having two components: denotation and sense. Meaning was then applied to four cases: the arithmetic expression, the arithmetic statement, the algebraic expression, and the algebraic statement. The denotation of an arithmetic expression is the single numerical value equivalent to the expression. For instance, the denotation of 8 + 5 is 13. Drouhard and Teppo expanded the denotation of the arithmetic expression to include *undefined,* necessary for expressions such as 1 / 0, which have no numerical denotation. Similarly, algebraic expressions like 2x + 3 denote a number, but one that is indeterminate, a number that is only specified when the denotation of x is known. For the algebraic expression, the denotation might be in the form of a table of literal values with associated values of the expression. Frege, as well as Drouhard and Teppo, regarded arithmetic and algebraic statements not as factual statements of a mathematical relation, but as statements that have truth values or undefined as denotations. For instance, the arithmetic expression $2*4+3=11$ denotes True. An algebraic statement is also denoted as True, False, or undefined, together with a range of values for which the statement is True or False. For instance, the algebraic statement 2x + 3 = 11 denotes True only for x = 4. Liebenberg, Sasman, and Olivier (1999) attempted to teach this concept to ninth grade students, but had limited success.

The sense of an expression or statement is the format in which it is given. Two arithmetic or algebraic expressions or statements mean the same and are interchangeable if the formats in which they appear, though they may look different, have the same denotation. For instance, 2 * 4 + 3 has the same sense as 8 + 3 or a 5 * 2 + 1. Similarly, in algebra, 2*(x + 3) has the same denotation as 2x + 6 because the same table of possible values result for each expression in spite of having a different sense.

Often, however, the denotation component of meaning is emphasized much more than the sense component, but both are necessary to succeed in mathematics (Drouhard & Teppo, 2004). By substituting expressions with the same denotation (equivalent), one can greatly simplify the solving of arithmetic and algebraic problems. For instance, to add 48 + 33, one could replace 48 with 50 – 2 because they have the same denotation. Then the problem becomes 50 – 2 + 33. Rearranging the terms gives 50 + 31, which is 81. The exercise serves no purpose for a pencil and paper operation, but is easier if attempting to solve the problem mentally. The use of transformations of the same denotation is mandatory in algebra, where the standard method of solving linear equations is a sequence of same-denotation transformations that systematically reduce the complexity of the equation (the sense) without changing the solution set.

Liebenberg et al. (1999) described a student's first structural interpretation of the meaning of arithmetic and algebraic expressions, or snippets of an expression, as the surface interpretation. The surface interpretation takes an expression at face value and treats it as directions to perform a series of basic operations. For instance, students may interpret the surface structure of the expression 5(x -2) as an instruction to "first subtract two from the value of x and then multiply that result by five." Libenberg et al. described the second interpretation of algebraic expressions as an interpretation of the systemic structure, i.e., its sense, which provides an ability to exchange one expression for another apparently different but equivalent expression, which has the same sense. For instance, students need to be able to recognize that the structural equivalence of the expression snippet 5(x-2) is structurally equivalent to the snippet 5x - 10 because one structure may be more calculable than the other in a given problem. Libenberg et al. obtained mixed results when they tested 40 sixth grade students for recognition (without calculation) of the equivalence of two pairs of arithmetic expressions: (a) (208 + 59) * 61 * 48 and 208 + 59 * 61 *48 (not =) and (b) (415 * 58) * (232 / 29) and 415 * 58 * 232 /29 (=). The students were also asked for the rationale for their choice of equal or not equal. For question (a), 31 of the 40 students answered correctly, but two students gave incorrect rationales. The incorrect rationales both showed some misunderstanding of the order of operations convention. For question (b), 22 students answered incorrectly, citing differences in the structure that they mistakenly thought would lead to different results. Of the 18 students who answered correctly, 8 did so with a correct rationale. Eight others were correct but thought it was necessary to multiply before dividing. At the sixth grade level, over 50% of the students had a correct understanding of the order of operations convention that permitted them to correctly assess the truthfulness of simple arithmetic expressions, but not consistently.

Given an opportunity, some students, even fourth grade students, will recognize and take advantage of structurally equivalent exchanges to simplify problems. McNeil and Alibali (2004) investigated the hypothesis that prior learning can create impediments to learning. They studied 70 fourth-grade students attempting to solve noncanonical problems in the formats of a + b + c = d + __ and a + b + c = __ + d. The emphasis of the study was on errors and incorrect strategies, but the authors did not discuss one noteworthy result from the list of correct strategies. Their hypothesis was that the first format, with a blank at the end, resembled the canonical format (a + b = __) and would be more influenced by prior learning to add up all the numbers, obtaining an incorrect answer. Problems in the second format, with the blank not at the end, would be novel to the students, and if solved, must have been encoded by the student unencumbered by any prior learning.

The students were administered three blocks of four problems each, two of which were problems in the format of a + b + c + d = __ (very similar to the canonical format), one problem of the blank-at-the-end format, and one problem in the blank-not-at-the-end format. The following is a sample block of questions (underlines added):

3 + 6 + 7 + 3 = __	4 + 8 + 5 + 7 = __
4 + $\underline{3 + 5}$ = 4 + __	$\underline{6 + 8}$ + 4 = __ + 4 (p. 455)

Success rates and strategies used were different for each of the two noncanonical problems, as expected. Incorrect strategies, such as add-all-numbers, or add-all-numbers up to =, were used consistently by 40 students in the blank-final problem and by 46 of the students in the non-blank-final problem. Correct strategies were used by 29 students in the blank-final problem and by 24 students in the non-blank-final problem.

Among the correct strategies described by McNeil and Alibali (2004) for one of the non-blank-final problems ($\underline{3 + 4} + 5 = \_\_ + 5$) (underline added) was a strategy of grouping: by judicious grouping of the numbers (as indicated by the underline) and one simple addition, the problem was quickly and efficiently solvable. McNeil and Alibali did not report on the number of students who observed that the underlined group in the above problem could be added to obtain 7, an equivalent expression. The left side of the equation becomes 7 + 5, clearly indicating that 7 is the correct number to place in the blank on the right side of the equation. Similarly, combining the underlined terms in the third and fourth problems of the sample block of problems above, clearly and efficiently indicate that 8 and 14 are the numbers to place in the blanks. This strategy is an example of using same-denotation exchanges or using systemic structure changes to solve or simplify problems in the arithmetic domain, even though the subject of structural relations was not a component of the curriculum.

Most of the equivalence-elated items in the questionnaire for this study were adopted from the study of elementary students by Rittle-Johnson, Mathews, Taylor,and McEldoon (2010). Items used by Rittle-Johnson et al. were themselves excerpted from prior research (e.g. Jacobs, et al., 2007; McNeil & Alibali, 2004; Rittle-Johnson & Alibali, 1999). The items in the Rittle-Johnson questionnaire constituted a matrix of items of four levels of difficulty and three problem classes: equation-solving, equation structure, and equal-sign. In this study, the items have been reclassified, based on the format of the items, as related to definition, properties, and applications of the equivalence and are itemized in Table 2. The four levels of difficulty have been retained.

In order to develop a scale to mark students' progress on the path to fluency with the relational definition of equality, Rittle-Johnson et al., (2010) developed a construct map of the typical progression that students follow as they develop that fluency by solving increasingly difficult problems. The continuous progression was graduated by four signposts: level 1 through level 4. Level 1, *rigid operational*, is the lowest grade: a hardcore dependence on the operational definition of equality. By level 2, *flexible operational*, students are beginning to handle some atypical problem formats, but are still relying on the operational definition of equality. At level 3, *basic relational*, students are using the relational definition to solve equations with operations on both sides of the equal sign. Fluency with the relational definition of equality comes at level 4, *comparative relational*, where the students recognize that performing the same operation on both sides of an equation maintains the equality relation, and can use other sophisticated strategies.

Research Related to the Literal Symbol

Older literature defines variables in mathematics as changeable quantities, not constants (Philipp,1992), as if all literal symbols were variables. Modern literature does not contain a specific definition of the general term *literal symbol*, but the conventional understanding of the term, expressed by Philipp, is any letter, upper or lower case, that refers to a quantity. In algebra, typically Roman letters are used, but other branches

Table 2

Matrix of research Questions, Variables, and Questionnaire Items

Research Question	Variable	Source[a]	Level	Item
1	Equivalence Definition	Rittle-Johnson	1	1
1	Equivalence Definition	Rittle-Johnson	1	6
1	Equivalence Definition	Rittle-Johnson	1	11
1	Equivalence Definition	Rittle-Johnson	1	13
1	Equivalence Definition	Rittle-Johnson	2	12
1	Equivalence Definition	Rittle-Johnson	2	14
1	Equivalence Definition	Rittle-Johnson	3	2
1	Equivalence Definition	Rittle-Johnson	3	3
1	Equivalence Definition	Rittle-Johnson	3	10
1	Equivalence Definition	Rittle-Johnson	3	15
1	Equivalence Definition	Rittle-Johnson	3	16
1	Equivalence Definition	Rittle-Johnson	4	7
2	Equivalence Property: Reflection	Rittle-Johnson	2	25
2	Equivalence Property: Reflection	Rittle-Johnson	3	24
2	Equivalence Property: Symmetry	Rittle-Johnson	3	4
3	Equivalence Application: Assign	Küchemann	1	31
3	Equivalence Application: Assign	Küchemann	2	32
3	Equivalence Application: Assign	Küchemann	3	33
3	Equivalence Application: Eq Expression	Rittle-Johnson	3	17
3	Equivalence Application: Eq Expression	Rittle-Johnson	3	18

(continued)

Research Question	Variable	Source[a]	Level	Item
3	Equivalence Application: Eq Expression	Rittle-Johnson	3	19
3	Equivalence Application: Eq Expression	Rittle-Johnson	4	5
3	Equivalence Application: Eq Expression	Rittle-Johnson	4	8
3	Equivalence Application: Eq Expression	Rittle-Johnson	4	9
3	Equivalence Application: Eq Expression	Rittle-Johnson	4	20
3	Equivalence Application: Eq Expression	Rittle-Johnson	4	21
3	Equivalence Application: Identity	Rittle-Johnson	2	22
3	Equivalence Application: Identity	Rittle-Johnson	2	23
4	Literal Definition	Knuth	na[b]	26
5	Literal Property	Küchemann	1	42
5	Literal Property	Küchemann	2	43
5	Literal Property	Küchemann	4	34
6	Literal Application: Generalized Number	Küchemann	2	40
6	Literal Application: Generalized Number	Küchemann	2	41
6	Literal Application: unknown	Küchemann	1	35
6	Literal Application: unknown	Küchemann	3	37
6	Literal Application: unknown	Rittle-Johnson	4	36
6	Literal Application: variable	Küchemann	1	27
6	Literal Application: variable	Küchemann	1	28
6	Literal Application: variable	Küchemann	2	29
6	Literal Application: variable	Küchemann	3	30
6	Literal Application: variable	Küchemann	3	38
6	Literal Application: variable	Küchemann	4	39

[a] Knuth et a. (2005), Küchemann (1981), Rittle-Johnson et al. (2010). [b] Level not assigned by author

of mathematics and the natural sciences also use Greek letters. The term *variable*, which is a common application of literals, became the popular term for literals in the literature beginning in the 1960s, but since 2006, the terms *literal symbol*, *literal*, and *letter* have completely replaced the term variable. The two basic properties of a literal symbol are (a) same literals refer to the same quantity and (b) different literals refer to quantities that may be different or the same (Davis, 1964).

Unless students understand well the basic definition of the literal symbol and its properties, they will have difficulties with algebra (Malisani & Spagnolo, 2005; Rosnick,1981). Küchemann (1981a) and Weinberg et al. (2004), for example, observed that some students have misconceptions related to the basic definition. Küchemann found that a common misconception among 14-year old students was interpretation of literal symbols as the objects themselves rather than some numeric property or count of the objects. The literature also contains examples of misunderstanding of the two properties of the literal symbol. Steinle et al. (2006) found that some elementary students would accept different values for the three like literals in an equation, and they found that some elementary students would not accept the same value for literals that were different.

To probe the depth of students' conception of a literal symbol, many researchers have taken the direct approach of asking the students to explain in writing their conception (Asquith et al., 2007; Knuth et al., 2005; Weinberg et al. 2004). The question has been typically presented as:

> In the expression 2n + 3, the arrow points to a symbol.
> ↑
> What does the symbol stand for? (Knuth, et al., 2005, p. 70)

Middle school teachers were asked how their students would respond to the written variable question and how a district-wide group would respond (Asquith et al., 2007). The responses were predictions of the proportions of students answering correctly (multiple values) or incorrectly (specific number, object, other, no response). The predictions compared relatively well with the results of a district-wide assessment. Yet, in the eighth grade, 76% of the students answered correctly, leaving 24% entering high school with misconceptions about the meaning of a literal symbol.

Other questions may follow that use a literal in other applications, to probe the breadth of a student's understanding of a literal symbol. For example, a frequent question is:

Which is larger? 3n or n + 6? (Knuth et al., 2005, p,70)

This question employs n as a variable. As n is assigned different values, the two expressions take on different values. For some values of n, 3n will be greater than n + 6 (for n greater than 3); for other values of n, n + 6 will be greater (n less than three). Students were correct if they realized that the answer depended on the value of n. Asquith et al. (2007) also asked the teachers to predict the students' responses, but they had less success in this case. Only 64% of the 8th grade students who took the district-wide test answered this question correctly, leaving 36% to enter high school without a firm understanding of a variable.

Literal symbols are used in many applications. The four applications of the literal symbols that are critical for beginning algebra are generalized number, variable value, unknown value, and alias.(e.g., Malisani & Spagnolo, 2005; McNeil, Weinberg, Hattikidur, Stephens, Asquith, Knuth, & Alibali, 2010; Philipp, 1992; Usiskin,1999). To this list, Philipp would add *parameter* (a special type of variable) as an application and Usiskin would add *formula* (a special usage of variables). For all but one application of the literal symbol, a knowledge of equivalence as a relation is a prerequisite, because each application takes the form of an equation. The one exception is the application of *generalized number,* in which the literal refers to no one number in particular but is unspecified to preserve generality. A generalized number is not used by itself, but is used to form generalized algebraic identities. For example, the literal n could be used to refer to a general number, which could then be used to make the algebraic identity $n * 0 = 0$, as Jacobs et al. (2007) suggested. The identity would be read as "any number multiplied by zero equals zero." Kieran (2007) suggested that elementary students internalize these verbal forms of identities. Generalized numbers also appear in generalized definitions, such as $n^2 = n * n$, the definition of the superscript 2.

The application of variable is the use of a literal to refer to a quantity that is under the control of the user. Variables are never used alone, but appear in equations relating two or more variables. After assigning values to all but one of the variables, the remaining variable is dictated by the equation that relates them all. An example is the equation relating total price (T) to unit price (U) and quantity (Q): $T = U * Q$. Varying either or both U and Q, the user may determine T for any number of purchasing scenarios. In this example, the variables were given mnemonic letters, which helps to keep track of the referents of the literals. However, for some students, this practice can encourage the misconception of treating the literal as the label of an object (McNeil et al., 2010). Several researchers (e.g. Chick & Harris, 2007; English & Warren, 1998; Kieran, 2007) have recommended that students explore the application of variable by building a growing pattern of triangles or squares with toothpicks, keeping track of the number of toothpicks and the number of triangles, as the figure grows. If they notice a relation between the pairs of numbers, they have discovered a variable relation they can at least verbalize, if not put into symbolic form.

A literal may represent an unknown quantity in an equation. Using the equation, the user can determine the value(s) of the unknown quantity. For example, in the equation $5x + 3 = 15 + x$, the x is some unknown quantity. By systematic algebraic procedures, the value of x (answer = 3) in both sites can be determined.

A literal used as convenient representation of an awkward number is an alias. Examples of alias are π which represents the number 3.14159…, which is an irrational number and awkward to write, and c, the speed of light, which is also awkward because it is so big ($2.998 * 10^{10}$ cm/sec). There are two kinds of aliases, those without units, like π, and those with units, like c. The difference is that the numeric values of aliases for unitless quantities are constant, but the numeric values of aliases for quantities that bear units depend on the units used. For instance, it does not matter if the radius of a circle is measured in meters or in inches, π is still 3.14159…. However, the speed of light measured in ft/sec is $9.836*10^{8}$. The literature is silent regarding misconceptions of aliases and this study also ignores the alias application.

Misconceptions regarding literal symbols are often developed in the elementary grades. MacGregor and Stacey (1997) found that students entering middle school exhibited some of the same misconceptions as Küchemann's (1981a) beginning high school students, plus two new misconceptions of assigning values to literals based on their position in the alphabet, and creating new variables rather than operating on the given variables. MacGregor and Stacey assessed students across four grade levels (6 through 9) using a set of four progressively difficult word problems. Two new misconceptions appeared in the higher grade levels. Some students interpreted a literal as 1 unless told otherwise; others interpreted a literal as a general referent (h meant height, either Con's height or David's height). MacGregor and Stacey recorded a gradual increase in the percentage of correct responses except for a small drop in performance for two geometry-based problems in the fourth year, but not exceeding 73% for any of the problems. MacGregor and Stacey attributed some of the newer misconceptions to interference from new learning that was itself flawed, or to inappropriate teaching materials.

Christou et al. (2007) categorized the common misconceptions of students new to the literal symbol as outgrowths of the arithmetic curriculum and they are prime candidates for the conceptual change approach. These misconceptions included Küchemann's (1981a) misconceptions plus some caused by the general shift in conventions between arithmetic and algebra. For instance, in arithmetic, letters are sometimes used to indicate units of measurement: 8 m for eight meters, for example. (The space between 8 and m is not always present.) The expression 8m in algebra, however, represents 8 * m. In addition, concatenation of two numbers in arithmetic signifies addition (8½ indicates 8 + ½ and 35 indicates 30 + 5); in algebra, the concatenation of a number and a literal, or two literals, indicates multiplication (2xy = 2 * x * y). Christou et al. studied the responses of beginning high school students to questions of the form: Are there some numbers among the following alternatives that you think cannot be assigned to 4g?, followed by 12 alternatives of whole numbers, fractions, and decimal numbers, some of which were negative numbers. The last alternative was "No, all numbers can be assigned to it," which was the correct response. Of the 34 participants, 18.6% answered correctly, 30.3% accepted only the positive whole numbers, and 25.4% did not discriminate between the negative sign in the menu of choices and a negative sign in the given expression, if there was one. Christou et al. attributed these errors to an influence from the elementary arithmetic curriculum in which numbers are usually whole numbers and unsigned numbers are assumed to be positive.

Jacobs et al. (2007) conducted a year-long mathematical development program for the teachers of 19 schools of one of the lowest performing elementary school districts in California. As part of that program, the students were administered a final assessment of equivalence topics and literal symbol applications. The portion of the test dedicated to equivalence consisted of open equivalence statements to measure knowledge of the relational nature of equivalence and a set of numerical items to assess the students' ability to use numerical relations to simplify calculations. The next portion of the assessment was dedicated to solving equations in which the unknown quantity was represented by a literal. This portion of the assessment was not administered to first grade students, but was administered to second grade through fifth grade students, using equations of grade-appropriate difficulty. For example, second and third grade students were given equations similar to $c + c + c + 4 = 16$; fourth and fifth grade students were given equations similar to $3 * c + 5 = 23$. Equations of this caliber are solvable using relational thinking. Elementary students not familiar with relational thinking would be unable to solve the equation because they would also not have learned algebraic techniques. Another portion of the final assessment was dedicated to the use of algebraic identities, such as $c + b = b + c$, which the students were to indicate as *always true* or *not always true.*

Students' achievement in understanding equivalence was significantly better for the students of participating teachers, but was not significantly better in the areas of equation solving and algebraic identities.

Summary

The segment of the literature reviewed here has been related to the development of concepts and of misconceptions, primarily in the domain of mathematics. Misconceptions in science were never a surprise, possibly because describing the inner workings of observed natural phenomena is at first an interpretation of one's observations, which later is adapted by many people, but it is always an interpretation. Mathematics was thought to be immune from misconceptions, possibly because mathematics is not observation-based, but is a rigorously derived, defended, and peer-approved explanation of mathematical relations and properties.

However, the research shows that misconceptions in mathematics are quite common, notably in the grade levels up to mid-high school, but in adults as well. Piaget predicted it, and research has confirmed it. Because mathematics is inherently modular and sequential by nature, misconceptions at any level of mathematics can be a severe impediment to learning higher levels of mathematics. Students attempting to learn algebra are no exception.

In problem-solving situations, two approaches are possible. One could adopt the "scientific" approach, which is a thorough analysis of the problem that identifies the root causes of the problem. The scientific approach can be time-consuming and laborious, but the result is certain. Brown and Burton (1978) took the scientific approach when they attempted to detail the steps of every mathematical operation. Although the entire program was computerized, it was clearly going to be enormous on a full scale and the enormity might render the program impractical. Since then, other programs, such as DASS, by Russell et al. (2009) have been written in an attempt to develop a more efficient computer-based diagnostic program, but they are still limited in scope.

The other approach, in problem-solving terms, is the "shotgun" approach. Used in situations where there is an urgency to the problem, or where there are many possible basic causes, all possible causes are remedied at once, with the hope of catching all the real problems. This approach is not cheap, but it is fast. The research to date has been taking a modified shotgun approach by identifying as many of the most popular causes as possible and then treating them in parallel. This study continues that approach, focusing on students at the college freshman level.

CHAPTER III

RESEARCH METHODOLOGY

Students in the K-12 system perennially demonstrate difficulties in learning algebra. There has been much promising research into the genesis of students' difficulties, and into the nature of the difficulties, but high school exit exams and college placement tests clearly show that many students continue to exhibit problems when they exit high school. Only 26% of the nation's high school graduates in 2009 were in the top two proficiency levels, on a scale of three levels (National Center for Education Statistics, 2010). Many of those students come to college, or later return to college, underprepared for college mathematics, but little is known about the extent or nature of their difficulties at that point.

The research of student's difficulties is abundant at the elementary grades and middle school grades, but diminishes at the high school level, vanishing completely after the 10th grade. The research at the middle school level and at the entry high school levels have identified the most popular sources of difficulties as misconceptions of fundamental mathematics concepts, particularly the concepts of equivalence and the literal symbol, which for many students are not yet resolved by the 10th grade.

This chapter describes the methodology for this study, addressing the research design, the sample, the instrument, the pilot test, the data collection process, and the data analysis. There were six research questions guiding this study:

1. What proportion of students in remedial mathematics classes recognizes the relational definition of equivalence?

2. What proportion of students in remedial mathematics classes recognizes the reflective and symmetrical properties of equivalence?
3. What proportion of students in remedial mathematics classes recognizes and can apply the three applications of equivalence: arithmetic identity, arithmetic equivalent expressions, and assignment?
4. What proportion of students in remedial mathematics classes recognizes the quantitative definition of a literal symbol?
5. What proportion of students in remedial mathematics classes recognizes the two properties of a literal symbol that same literals represent the same quantities and that different literals represent same or different quantities?
6. What proportion of students in remedial mathematics classes recognizes and can apply the three applications of a literal symbol: an unknown quantity, a variable quantity and a generalized quantity?

Research Design

This study was a descriptive study of college students who had been placed into remedial mathematics classes. The purpose of this research was to estimate the proportion of college students in remedial mathematics classes who held no misconceptions (were completely knowledgeable) and those who held limiting misconceptions (were less knowledgeable) of the definition, the properties, and the applications for each concept of equivalence and the literal symbol. Students may hold nonfatal misconceptions related to equivalence and the literal symbol that limit their abilities to some degree either by not being aware of the many applications of the concept (moderate in breadth) or by being only able to answer less complex problems (moderate in depth), as observed by Küchemann (1981a).

The six dependent variables measured in this study were the numbers of students who could demonstrate: (a) the recognition and application of the relational definition of equivalence, (b) knowledge related to two properties of equivalence, (c) knowledge related to three basic applications of equivalence, (d) knowledge of the definition of the literal symbol, (e) knowledge related to two properties of the literal symbol, and (f) knowledge of three basic applications of the literal symbol. In each case, the method of measurement was the average score of a set of questions related to the variable.

There were five control variables of interest to this study: age group (four levels), gender, course level (two levels), and college seniority (five levels), and the time of day the classes were held (two levels).

Description of the Sample

The site for the research was an urban community college in Northern California and was chosen by the researcher because of his access to the study sample and his acquaintance with the participating professors. The senior administration of the host college approved the research project (See Appendix A.).

The community college offered two levels of remedial, non-credit mathematics courses as support for mathematically underprepared students. The lower level course, Basic Mathematics, covers the four basic operations of addition, subtraction, multiplication and division with numbers in the formats of whole numbers, fractions, and decimal numbers. In the second half of the semester, the emphasis turns to basic problem-solving situations involving percentage, proportion, and unit conversions that introduce the students to one equation type involving one literal symbol representing an unknown quantity.

The higher level course, Prealgebra, introduces the students to signed numbers and the four basic operations in terms of signed numbers. In the second half of the semester, the students are exposed to solving problems with general linear equations involving one literal symbol representing an unknown quantity, but the literal symbol may appear more than once in the equation.

In all, 191 students participated in the study, in four sections of Basic Mathematics (n = 105) and three sections of Prealgebra (n = 86). Four of the classrooms were daytime classes (n = 118) and three were evening classes (n = 73). Class sizes were typically 30 to 40 students, but typically 25 to 30 students in each class were present and agreed to participate in this study. All sections for each course followed a standardized curriculum and used the same textbook. Students in remedial mathematics classes included not only recent graduates from high school, but also, many reentrant adult students, which resulted in a wide range of ages and prior experiences in each classroom. The students ranged in age from less than 20 (n = 100), 21 to 25 (n = 51), 26 to 30 (n = 13), and over 30 (n = 21). The students were predominantly female (n = 121). The majority of the students were freshmen (n = 77 in first semester, n = 41 in second semester). Forty-seven students were in their second year (n = 38 in third semester, n = 9 in their fourth semester) and 21 students claimed five or more semesters of college.

Prealgebra classes, for which Basic Mathematics or the equivalent is a prerequisite, and Basic Mathematics itself were appropriate for this study because both courses are remedial level courses, emphasizing different but complementary ranges of basic arithmetic skills. In both courses, the students had a history of serious difficulties with mathematics, but their prior experience permitted some of the students to be placed in the higher Prealgebra course. The results of the survey showed that the Prealgebra students, as a group, did perform better than the Basic Mathematics students in general, and both groups showed a distribution of skill levels.

Both evening classes and daytime classes were appropriate for this study because students in either time period exhibited the same symptoms of mathematical difficulties, but they differed, in general, by age, by motivation, and by obligations external to education (Long, 2004; Lundberg, 2003). Therefore, evening students may have had different root causes for their difficulties and could generate different results from daytime students. Similarly, students of differing college seniority (1^{st} semester, 2^{nd} semester, for example) could be expected to have different experiences and different maturity as college students and could have provided different results. However, the results of the survey showed no significant differences between day and evening students, nor were there significant differences between students of different college seniority, or different genders.

Protection of Human Subjects

An application for approval to conduct an offsite student-involved research project was presented to the University of San Francisco Institutional Review Board for the Protection of Human Subjects (IRBPHS). After the initial approval by the Board (See Appendix A), the questionnaire was modified once to provide a better arrangement of items without lengthening the questionnaire and the forms, and once more the procedure was modified to facilitate administering the questionnaire. In both cases, the IRBPHS approved the modifications (See Appendix A).

Instrument

The instrument for this study was a 43-item questionnaire (See Appendix B), the purpose of which was to explore the level of participating students' knowledge in each of the two areas of equality and the literal symbol. Just one form of the questionnaire was prepared. Individually completed questionnaires gathered the students' data, which were analyzed to calculate the proportions of college freshmen who could recognize and apply the relational definition of equivalence (Research Question 1), two properties of equivalence (Research Question 2), three distinct basic applications of the relational definition (Research Question 3), the basic definition of a literal symbol (Research Question 4), two properties of the literal symbol (Research Question 5), and three distinct basic applications of the literal symbol (Research Question 6).

The questionnaire items selected for this study have all been reported in prior research. Because most of the prior research articles focused primarily on just one concept, three earlier research articles were required to provide items that encompassed the two concepts to be investigated in this study: Rittle-Johnson et al. (2010); Knuth et al. (2005); and Küchemann (1981a) and were used in this study with permissions of the authors and the publishers (See Appendix A.). Not all the items in the source studies were selected for this study; some items the original authors found uninformative and some items thought to be redundant were excluded. Detailed descriptions and the object of each selected item are found in Appendix C. Among the items were multiple-choice questions, True-False questions, fill-in-the-blank questions, open ended questions, and mathematical exercises; all formats familiar to college students.

Each of the 43 items selected for this questionnaire was associated with one research question, providing a measure of a student's knowledge of one of six variables: each concept's definition, its properties, or its applications. The matrix of questionnaire items and their associations with the research questions and variables is shown in

. However, the items were not always uniquely and obviously relatable to one variable. According to the basic diagram shown in Figure 1, the definition of each concept underlies its properties, which, in turn, underlie its applications. Furthermore, the definition of equivalence underlies the applications of the literal symbol. An item that attempts to assess an application, for example, must presume an understanding of the definition and its properties. For this study, items were assigned to the variable that matched the deepest level of knowledge required to answer the question.

Multiple items for the same variable were necessary to assess the extent of a student's knowledge. Fundamental concepts, like the Properties of Equivalence concept, can be very subtle in different contexts, and associated misconceptions can be different and show different symptoms. Several items probing different aspects are necessary to effectively assess a student's knowledge of a concept. Furthermore, a student's usage of an application can be limited by the complexity of the item (Küchemann, 1981a). To determine a student's flexibility with any application, several test items of progressive difficulty were necessary; more items expanded the measured range and increased the precision of the assessment.

Of the 28 items on the questionnaire related to the equivalence concept, three items were excerpted from Küchemann (1981a) and 25 items were excerpted from Rittle-Johnson et al. (2010), which were edited only to standardize the numbering system and the format of the questions. One item from Rittle-Johnson et al., which assessed the ability of the student to find missing numbers if the missing number was indicated by a literal symbol rather than a blank, was better defined for this study as an unknown application of the literal symbol. The item was grouped, therefore, with like items from Küchemann as item 36.

The common theme of the first 12 items in Table 2 was specifically to identify students who were knowledgeable of the relational definition of equivalence). These 12 items formed the basis for the Equivalence Definition variable. Three items were included to assess the student's knowledge of two of the three properties of equivalence; no items were available in the literature to assess the transitive property of equivalence. These three items formed the basis for the Equivalence Properties variable. Three items taken from Küchemann (1981a) were included to assess the student's knowledge of the Assignment application of equivalence. Eight items were included for the more subtle and more varied application of Equivalent Structure and two items were included for assessment of the Identity Application. These last 13 items formed the basis for the Equivalence Applications variable.

Of the 15 items on the questionnaire related to the concept of the literal symbol, 13 items were excerpted from Küchemann (1981a), one from Knuth et al. (2005), and one from Rittle-Johnson et al. (2010) (See Table 2.). The definition of a literal symbol (a literal represents a quantity) was not addressed directly by Küchemann. The one item in the questionnaire addressing the definition of a literal symbol was adapted from Knuth et al. because it was analogous to one of the items addressing the definition of equivalence taken from Rittle-Johnson et al., 2010. That one item defined the Literal Definition variable. Three items addressed the two properties of a literal symbol. These three items defined the Literal Properties variable. Two items addressed the Generalized Number application, three items addressed the Unknown application of a literal symbol, and six items addressed the Variable application. Together, these 11 items defined the Literal Applications variable.

The Küchemann items were excerpted from the Concepts in Secondary Mathematics and Science (CSMS) test as reported by Küchemann (1981a). Küchemann calculated difficulty levels for each question based on students' success rates, which he further grouped into four levels of difficulty. Although the success rates for college students differed from those of Küchemann's high school students, the relative ranking and grouping of the items were similar for the college students of this study.

Validity and Reliability

Rittle-Johnson et al. (2010) established validity and reliability of all items in their study of elementary students. Küchemann (1981a) established the validity of the items used in his Algebra test of late middle-school to early high school students. Rittle-Johnson et al. (2010) established the reliability ratings of their items in several ways. Internal consistency of the two alternate forms of the Rittle-Johnson et al. questionnaires was high. Cronbach's alpha was .94 for form 1 of the questionnaire and was .95 for form 2. Comparing the initial form of the questionnaire used for a pilot study with the revised format, high test-retest correlations of .94 and .95 were obtained for the two forms. An independent rater for five of the items on each test form showed interrater reliabilities of .99 and .97. These excellent reliability ratings attest to the reliability of both forms of Rittle-Johnson et al.'s questionnaire. However, because the items in Rittle-Johnson et al.'s study assessed students for knowledge of equivalence, unlike this study that assessed students on three subdivisions of equivalence: definition, properties, and applications; and because the students in this study (college) were different from those in Rittle-Johnson's study (elementary school), the reliability of the data was different for this study. For the Definition of Equivalence, Cronbach's alpha was .65 (marginally acceptable) for this study. Alpha for Properties of Equivalence was .36 (unacceptable) and alpha for Applications of Equivalence was .72 (marginally acceptable).

Validity of the questionnaire items were also established in several ways. Face validity was evaluated by four independent experts, resulting in a validity rating of 4.1, on a scale of 0 to 5, where 3 was *important* to the goal of tapping equality and 5 was *essential* to tapping equality. In addition, Rittle-Johnson et al. (2010) analyzed the data to confirm that the variance was related to a single variable: equality understanding. They used a Rasch model by Winsteps that explained 57% of the variance; a second component would have contributed an increase of only 2.2%. A confirmatory factor analysis for up to three factors (three classes of test items) corroborated the unidimensionality of the data.

In designing their questionnaire, Rittle-Johnson et al. (2010) selected items from prior literature on the basis of their class and their level of difficulty as described by the construct map developed at the outset of the research. A product of the Rasch model was a Wright map that compared, on a common scale, student abilities with test item success rates. With few exceptions, the most difficult items were the test items presumed to be most difficult and the least difficult items were those presumed to be least difficult, which confirmed the basic structure of the construct map, but some adjustments were necessary based on the few discrepancies. In summary, the items from Rittle-Johnson et al. appeared valid, and the construct map appeared valid.

Küchemann (1981a) demonstrated the validity of the literal symbol related items by comparing items within the Algebra test to each other using a PHI coefficient, which is a special case of the product-moment correlation for dichotomous data (Hart, 1981). For the Algebra test, the mean PHI coefficient was .44 for the 14-year-old test group, which is among the highest coefficients for all tests, in all age groups. In addition, the number of students responding correctly to 2/3 or more of the items at a given difficulty level were compared across all the mathematics tests using Pearson's correlation coefficient. The coefficients for the Algebra test ranged from .60 to .73 (Küchemann, 1981b). Küchemann was not sure if these correlation figures were good or not. Küchemann (1981a) also reported that the algebra test correlated highly with the Calvert DH test of non-verbal reasoning ($r = .7$). It appeared that there was some consistency in the items for measuring students' skills, and therefore some validity.

No attempts were made to establish the reliability of test items in Küchemann's Algebra test. For this study, the reliability coefficient for Literal Definition was not applicable because there was only one item in the questionnaire for this variable. Cronbach's alpha for Properties of the Literal Symbol was .48 (unacceptable) and for the Application of the Literal Symbol was .64 (marginally acceptable). These low values may be due to items not best suited to assessing the subdivisions of a literal symbol as defined in this study.

Item 29 from Knuth et al. (2005), and its extension, asking for a verbal description of a literal symbol and other meanings, has not been validated. The item bears a strong resemblance to item 10, which asks for the same information about the equal sign, and which was validated by Rittle-Johnson et al. (2010).

Pilot Study

Although all the questionnaire items were taken from previous research, the composite questionnaire for this study was novel and the instructions were unique. A pilot study of the questionnaire was conducted with one of the Basic Mathematics classes to determine the amount of time required to administer and to complete the questionnaire. The pilot test also identified clarity problems in the instructions and identified procedural difficulties, which were resolved and approved prior to administering the questionnaire to other classes.

Findings for the pilot study were that the introduction took 25 minutes to complete and that the handling of the consent forms, the demographic data, and the questionnaire was awkward. During the pilot test, the students were able to complete the questionnaire in 35 minutes. A revised checklist was prepared to more efficiently guide the researcher through the introduction. The package of materials for each student consisted of two copies of the consent form, one copy of the form requesting demographic data, and a paper copy of the questionnaire. Improvements to the demographic form and using colored paper for the copies to be returned to the researcher significantly expedited the distribution and collection of the paper forms. After these adjustments, the total time for the administration of the survey was reduced to 50 minutes, which met the most stringent request of the cooperating professors. Four items in the questionnaire were replaced with four alternate items to provide additional items for one of the variables. However, four items that were initially included as questions to identify students whose knowledge exceeded the expected range of knowledge were converted in the scoring scheme to items assigned to assess existing variables.

Data Collection

All data were collected in the Fall semester, 2011. That semester the college scheduled four sections of Basic Mathematics and four sections of Prealgebra. Electronic mail was sent to six professors in the week preceding the first week of classes, explaining this study and requesting their cooperation. The request was followed up with individual meetings just prior to the beginning of a class. All but one professor agreed to cooperate. The one exception was a section of Prealgebra for deaf students, taught by a deaf professor. That one section was excused because it was agreed that the survey would be difficult for all to administer properly. After negotiating schedules with the remaining professors, the survey was administered to the seven participating sections over 16 days, spanning weeks five and six of the 13-week semester.

For each section, the questionnaire was administered at the beginning of a regular class period. The students were given a package of two consent forms (one to keep and one to sign and return), one copy of a demographic survey to sign and return, and a copy of the questionnaire to fill in and return. To preserve the confidentiality of the students' private data, the items in each package were given a unique serial number that encoded the student, the class level, the section, and the date. Only the consent forms and the demographic data sheet contained both students name and serial number. For analysis purposes, the students' demographic data and serial number were entered into a computer database, but their names were not. The only connection between a student's name and serial number was on the paper forms which were kept under lock and key by the researcher.

In accordance with the requirements of the IRBPHS, the researcher explained the purpose of the study, the role of the student, and described the confidential handling of the data obtained. The students received no remuneration for their participation, but potential broad benefits were described. Those students who accepted the invitation to participate completed, signed a consent form, and responded to a short written demographic survey (Appendix D). These preliminaries consumed up to 15 minutes. Immediately after collecting the consent forms, the participating students began the questionnaire, which took an additional 35 minutes to complete.

The first three items on the questionnaire formed a memory test: for each item, the students were shown an algebraic expression for 10 seconds on a large screen using the class video projector system. Then the students had 15 seconds to record on the questionnaire form the expression as they remembered it. This set of three progressively complex items was completed by the whole class at one time, after which the students were on their own schedule to answer the remaining items of the questionnaire directly on the questionnaire form.

There were little missing data. The only case of missing data within the questionnaire data was one student started the survey late and missed the group presentation of the first three items. All three items were related to the Definition of

Equivalence variable and that student's data were excluded from analyses of the Definition of Equivalence. Within the demographic data, six students chose not to respond to the age group question, five students did not respond to the gender question, and five students did not respond to the seniority question. The missing personal data were confined to just six individual students, who were excluded from the analyses that were based on the missing data.

After scoring individual students questionnaires, the data were entered into a spreadsheet along with student serial numbers and demographic data. Initial summary analyses at each classroom level were prepared and shared with the classroom professor. The data were then exported to a statistical analysis program for further analysis.

Limitations

There were several limitations associated with this study. The first was that the students participating in the survey were not randomly selected. The students who participated in this survey were a convenience sample consisting of all but one section of the two developmental mathematics courses at the community college. They approximated demographically the students in remedial classes across the state of California. Of the 50,000 students in community college remedial mathematics classes in California in 2002, 55% were female (Perry, Bahr, & Woodward, 2010). In this study, the proportion of female students was 65%. The students in this study were also generally older than those students across California. In the state, 79% of the students were younger than 20, compared to 54% for this study; across the state, 9% of the students were older than 25, compared to 18% in this study. The results of this study are, therefore, only weakly generalizable to the state-wide population of remedial mathematics students.

The questionnaire that was the instrument for the survey exhibited a general shortcoming. It consisted of 43 items that were previously validated and used in different environments, but not used together. For students who are weak in mathematical skills and mathematics self-confidence, eight pages of 43 mathematics-related questions in 35 minutes was a daunting and exhausting task. The reliability and validity of the students' responses in the latter half of the questionnaire, therefore, may be questionable.

At the same time, from an analysis point of view, this study suffered from an inadequate number of items for some of the variables. The variables of Equivalence Properties and Literal Symbol Properties were assessed on just three items each; the Literal Symbol Definition variable was assessed on only one item. Three items were just enough to produce four possible scores, but were not enough to provide resolution on a multifaceted variable.

A serious limitation of this study is the low reliability coefficients for all the variables. The source items were related to either equivalence or to literal symbols in general, but, in this study the items were associated with one of three subdivisions of a concept. The items were not originally designed to assess one subdivision (definition, properties, or applications) and were dependent on more than one in many cases. Specifically designed items would have higher reliability coefficients.

There is one challenge to conducting a survey on the subject of mathematics. No matter how it is described to the students, the survey looks like a mathematics test to the students and they adopt an attitude of striving for as high a score as possible. They may use deduction, induction, and association to arrive at a correct answer, but that may not be the intent of the survey, as in the case of this study. The intent of this study was to obtain a snapshot of each student's knowledge related to the two concepts of equivalence and literal symbols early in their college career. The questionnaire was not meant to be a learning experience for the students. In most classrooms in this study, the last students to complete the questionnaire were observed to be working very diligently to arrive at the correct answers, but the intent was to obtain the status of their knowledge background.

Data Analysis

At the first level of analysis, scores were recorded for each item of the questionnaire for each student. Each item was scored as *knowledgeable* (1) or *not knowledgeable* (0). The students were instructed to leave blank those items that they could not answer. Unanswered and incorrectly answered questionnaire items were therefore assumed to be not knowledgeable and were scored as 0.

At the second level of analysis, item scores were aggregated by variable for each student and a mean score for each student was computed for each of the six variables: a) Equivalence Definition, b) Equivalence Properties, c) Equivalence Applications, d) Literal Symbol Definition, e) Literal Symbol Properties, and f) Literal Symbol Applications. Each student's mean score for each variable ranged from zero to one. For example, the 12 items corresponding to the Equivalence Definition variable are 1, 2, 3, 6, 7, 10, 11, 12, 13, 14, 15, and 16. If a student's scores on these items were 1, 1, 1, 1, 1, 0, 0, 1, 0, 1, 0, and 0, his score for Equivalence Definition variable would be 7 / 12, or .58.

The students' mean scores for each variable were tested for differences at the .05 significance level between male and female students, day and evening students, class levels of students, students in different age groups, and students of different college seniority. Prior to the testing of the students' variable scores, the data needed to be scrutinized for independence, normality, and equal variances to verify conformance to the assumptions of the analyses.

For data to be independent, the data produced by one student must not be affected by the data produced by another student. All of the data that were used for this research were gathered during the administration of the survey in seven separate classes. The classes were distributed in time and space, which inhibited diffusion effects. The classes spanned two campuses and were split between day and evening hours, all over a period of 16 calendar days. Each administration of the questionnaire was proctored by the researcher and usually also by the class professor to ensure no local collaboration between students. Independence of the data can therefore be assumed.

There were departures from normality in the data in this study. For example, the students' scores for the Equivalence Definition variable were biased toward high scores (M = .81), which produced a ceiling effect and a pronounced negative skew. To assess the degree of normality of the aggregated data, histograms, boxplots, Fisher skewness factors, and kurtosis factors were calculated for each variable. Skewness and kurtosis factors were considered significant if their absolute values were greater than two times their standard error (Miles & Shevlin, 2001). The statistics of the distributions for each variable are shown in Table 3. The numbers of survey items are shown in the first row of Table 3 because the numbers of survey items associated with each variable drastically affected the resolution and the statistics of the data. Fewer items provided a coarser assessment and less information. Row two of the table lists the number of students contributing to each variable. All 191 students were scored on all variables except for one student who was not scored for Equivalence Definition because he was missing responses to three survey items associated with that variable.

Table 3

Statistics of the Overall Mean Scores for Each Variable

	Equivalence			Literal symbols		
	Definition	Properties	Applications	Definition	Properties	Applications
Survey item frequency	12	3	13	1	3	11
Number of students	190	191	191	191	191	191
Mean	.81	.93	.61	.33	.42	.35
Median	.83	1.00	.62	.00	.33	.36
St. dev.	.16	.17	.18	.47	.29	.19
Skewness	-1.36*	-2.76*	-.31	.73*	-.12	.31
SE of skewness	.18	.18	.18	.18	.18	.18
Kurtosis	3.32*	8.29*	.09	-1.48*	-.99*	-.29
SE of kurtosis	.35	.35	.35	.35	.35	.35

* Significant per Miles and Shevlin (2001)

The data for the Equivalence Properties variable, for a worst case example, were based on just three items, which is the theoretical minimum of items necessary to subdivide a range of data into four categories. The median score was 1.00, which is also the upper limit of the range of possible scores. As a result, the distribution of the data was most unlike a normal distribution and was the most severely skewed. Better models would be exponential ($r^2 = .9998$) or cubic ($r^2 = 1$). Similarly, the data for the Literals

Definition were based on just one item of the questionnaire, making the data dichotomous. The median score was .00, which was also the lower range of possible scores. These data were also quite unlike a normal distribution.

However, the comparison of means tests is robust to nonnormal distributions if the sample sizes are large ($n > 30$). The sample sizes for the six variables in this study were all large, as shown in Table 3, and testing of the mean scores between variables was not constrained.

The assumptions for t-tests include the assumption of equal variances as well as the assumptions of independence and normality previously addressed. Levine's test for equality of variances is conducted with every t-test and results were obtained for cases of equal variances, as well as the results for a version of the t-test that does not require equal variances. For this study, Levine's tests were significant only for the data for Equal Applications and for Literal Properties, for which the t-test results for unequal variances were used.

T-tests were used to compare the six variable scores between male and female students, between day and evening students' scores, and between students in Basic Mathematics and Prealgebra classes. ANOVAs were used to compare students in different age groups and at different levels of college seniority. Because there were five tests being made for each variable, a Bonferroni adjustment was made to the significance level (.05 / 5 = .01). No significant differences were found between male students and female students, nor between daytime and evening students. Consequently, all subsequent analysis did not distinguish between male and females or between day and evening classes.

One-way ANOVA tests were used to compare student mean scores across college seniority (five levels) and across ages (four levels). No significant differences were found between the students' mean scores on all the variables across the four age groups (Refer to Table 4). Nor were significant differences found between students' mean scores at different levels of college seniority (Refer to Table 5).

The third level of scoring was to categorize students' variable scores into the four levels, following the examples of Rittle-Johnson et al. (2010) and Küchemann (1981a). The lowest breakpoint for every research question in this study was 0.25. Students with research question scores =< .25 on a variable were deemed to be *unskilled* on that variable. Students with variable scores above 0.25 to 0.55 were deemed to be *weak* for the variable. Students with variable scores above 0.55 to 0.85 were deemed to be *moderate* for that variable. Students with variable scores greater than 0.85 were deemed to be *expert* for that variable.

The fourth level of analysis for this study was to calculate the proportion of students in each skill category for each variable. The hypothetical student used as an example in the first level of analysis would be deemed to be moderate for the Equivalence Definition variable. The direct answer to each research question was the proportion of students deemed to be expert relative to that variable; the two-part answer to each research question was the proportion of students deemed to be at the expert skill supplemented by the proportion of students at the moderate skill level.

Table 4

One-way ANOVA F-test Results between Student Age Groups

	df	SS	MS	F	p
			Equivalence		
Definition					
Between groups	3	.01	.00	.13	.94
Within groups	180	4.46	.03		
Properties					
Between groups	3	.05	.02	.56	.65
Within groups	181	5.09	.03		
Applications					
Between groups	3	.19	.06	1.85	.14
Within groups	181	6.10	.03		
			Literal symbols		
Definition					
Between groups	3	.70	.23	1.05	.37
Within groups	181	40.19	.22		
Properties					
Between groups	3	.13	.04	.50	.68
Within groups	181	15.38	.09		
Applications					
Between groups	3	.01	.00	.07	.98
Within groups	181	6.54	.04		

Table 5

One-way ANOVA F-test Results between Student College Seniority Levels

	df	SS	MS	F	p
		Equivalence			
Definition					
Between groups	4	.12	.03	1.20	.31
Within groups	180	4.36	.02		
Properties					
Between groups	4	.14	.04	1.28	.28
Within groups	181	5.00	.03		
Applications					
Between groups	4	.24	.06	1.76	.14
Within groups	181	6.06	.03		
		Literal symbols			
Definition					
Between groups	4	.66	.17	.74	.57
Within groups	181	40.67	.23		
Properties					
Between groups	4	.95	.24	2.97	.02
Within groups	181	14.56	.08		
Applications					
Between groups	4	.10	.02	.66	.62
Within groups	181	6.53	.04		

CHAPTER IV

RESULTS

The purpose of this study was to estimate the proportions of college students in remedial mathematics courses who were able to recognize and apply the basic mathematical concepts of a) equivalence and b) the use of literal symbols in arithmetic and in beginning algebra, both of which are essential to a student's ability to progress in his or her studies of algebra. For each of the two concepts, the students' levels of ability was assessed on the three variables of their recognition of (a) the basic definition of the concept,)(b) the general properties of the definition, and (c) the three most basic applications of each concept. The primary misconception for the definition of the equivalence concept is the concept of treating an equality statement as an instruction to perform the indicated operation, which is adequate for arithmetic studies in the elementary grades, but is inappropriate for more general problems found in higher arithmetic and in algebra. The primary misconception for the definition of the literal symbol is the interpretation of a literal symbol as a representation of an object, as opposed to a quantity related to an object (frequency, cost, weight, etc.).

The research questions that guided this study were:

1. What proportion of students in remedial mathematics classes recognizes the relational definition of equivalence?
2. What proportion of students in remedial mathematics classes recognizes the reflective and symmetrical properties of equivalence?

3. What proportion of students in remedial mathematics classes recognizes and can apply the three applications of equivalence: arithmetic identity, arithmetic equivalent expressions, and assignment?
4. What proportion of students in remedial mathematics classes recognizes the quantitative definition of a literal symbol?
5. What proportion of students in remedial mathematics classes recognizes the two properties of a literal symbol that same literals represent the same quantities and that different literals represent same or different quantities?
6. What proportion of students in remedial mathematics classes recognizes and can apply the three application of a literal symbol: an unknown, a variable and a generalized number?

The instrument for this study was 43-item questionnaire that was designed to survey the students' knowledge of the two fundamental concepts: equivalence and the literal symbol. The survey, including the pilot test, was conducted early in the Fall semester of 2011 in seven classrooms usurping some normal class time to be as unobtrusive as possible.

Findings

In general, students in the Prealgebra course outperformed students in the Basic Mathematics course, except for the variable of Literal Symbol Definition, but the difference of the means was just .03. The data and the results of the t-tests for all variables are shown in Table 6. The difference, however, was significant only for the three variables: Literal Symbol Properties, $t(189) = -5.70$, $p= .00$, where the difference of the means was moderate (.2 to .4). The table also show significant results for Literal

Table 6

Data and Results of t-tests Between the Remedial Mathematics Courses for All Variables

Variable	Course	N	Mean	Mean difference	s.d.	Levene,s significance	t	df	p
Equivalence									
Definition	Basic Math	104	.78	.07	.16	.01*	2.98	185.84	.00*
	Prealgebra	86	.85		.14				
Properties	Basic Math	104	.91	.04	.18	.00*	1.85	188.87	.07
	Prealgebra	86	.96		.01				
Applications	Basic Math	104	.60	.03	.20	.00*	1.13	188.01	.26
	Prealgebra	86	.63		.15				
Literal symbols									
Definition	Basic Math	104	.34	-.03	.48	.40	-.42	189	.67
	Prealgebra	86	.31		.47				
Properties	Basic Math	104	.32	.22	.28	.62	5.70	189	.00*
	Prealgebra	86	.55		.26				
Applications	Basic Math	104	.30	.12	.18	.68	4.85	189	.00*
	Prealgebra	86	.42		.17				

Note: * < .01

Symbol Applications, $t(189) = -4.58$, $p = .00$ and for Equivalence Definition, $t(185.84) = -2.98$, $p = .003$ where the differences of the means were small (<.2), as were all the nonsignificant variables. This result could be indicative of greater exposure to and experience with algebra and the use of literal symbols accumulated by students in the Prealgebra class.

One-way ANOVA tests were used to compare student mean scores across college seniority (5 levels) and across ages (four levels). Because the variables were being compared on five measurements, a Bonferroni adjustment was made to the significance level. (.05/5 = .01) No significant differences were found between the students' mean scores on all the variables across the four age groups. Nor were significant differences found between students' mean scores at different levels of college seniority. (Refer to Tables 4 and 5.) The most senior groups by age and by college seniority both demonstrated better performance for Equivalence Applications and Literal Symbol Applications. In addition, the students in their 5th or higher semester outperformed the other students in the Properties of Equivalence. (Refer to Tables 7 and 8.) This improved performance by more senior students could be accounted for greater exposure over time to algebraic applications.

Answers to Research Questions

Each student earned a score between zero and one for each variable, based on the proportion of correct responses to the items associated with that variable. Students whose scores were greater than .85 for each variable were deemed to be at the *expert* level for that variable. Students with scores greater than .55 were deemed to be at the *moderate* knowledge level for that variable. Students with scores greater than .25 were deemed to be at the *weak* knowledge level and the remaining students were considered to be *unskilled* for that variable. The proportions of the total number of students at each knowledge level for each variable (and Research Question) are listed in Table 9.

Table 7

ANOVA Mean Score Results for Each Variable, by Age Group

	Equivalence					Literal symbol				
Variable by age groups	N	Mean score	sd	Minimum	Maximum	N	Mean score	sd	Minimum	Maximum
Definition										
< 20	99	.81	.17	.00	1.00	100	.37	.49	.00	1.00
21-25	51	.82	.14	.42	1.00	51	.33	.48	.00	1.00
26-30	13	.81	.13	.58	1.00	13	.23	.44	.00	1.00
> 30	21	.79	.14	.50	1.00	21	.19	.40	.00	1.00
Properties										
< 20	100	.92	.17	.33	1.00	100	.40	.31	.00	1.00
21-25	51	.93	.19	.00	1.00	51	.45	.26	.00	.67
26-30	13	.97	.09	.67	1.00	13	.46	.32	.00	1.00
> 30	21	.95	.12	.67	1.00	21	.43	.28	.00	1.00
Applications										
< 20	100	.58	.18	.08	1.00	100	.34	.19	.00	.82
21-25	51	.64	.21	.15	1.00	51	.35	.19	.00	.91
26-30	13	.60	.19	.23	.92	13	.35	.23	.00	.64
> 30	21	.66	.14	.46	1.00	21	.36	.17	.00	.73

Table 8

ANOVA Mean Score Results for Each Variable, by College Semester

	Equivalence					Literal symbol				
Variable by semester	N	Mean score	sd	Minimum	Maximum	N	Mean score	sd	Minimum	Maximum
Definition										
1st semester	76	.79	.18	.00	1.00	77	.35	.48	.00	1.00
2nd semester	41	.79	.15.	.42	1.00	41	.24	.43	.00	1.00
3rd semester	38	.83	.13	.50	1.00	38	.32	.47	.00	1.00
4th semester	9	.90	.10	.67	1.00	9	.44	.53	.00	1.00
5th or higher semester	21	.82	.14	.50	1.00	21	.43	.51	.00	1.00
Properties										
1st semester	77	.91	.18	.33	1.00	77	.36	.28	.00	1.00
2nd semester	41	.92	.21	.00	1.00	41	.40	.27	.00	.67
3rd semester	38	.94	.13	.67	1.00	38	.50	.28	.00	1.00
4th semester	9	.96	.11	.67	1.00	9	.63	.35	.00	1.00
5th or higher semester	21	1.00	.00	1.00	1.00	21	.44	.30	.00	1.00
Applications										
1st semester	77	.60	.20	.08	1.00	77	.34	.19	.00	.82
2nd semester	41	.60	.19	.15	1.00	41	.32	.19	.00	.73
3rd semester	38	.59	.16	.23	1.00	38	.37	.21	.00	.91
4th semester	9	.62	.17	.38	.92	9	.36	.20	.00	.55
5th or higher semester	21	.71	.13	.46	1.00	21	.39	.17	.09	.73

Table 9

Proportion of Students (in Percentages) at Each Knowledge Level for Each Research Question

Research question	Variable	Number of survey items	Overall mean score	Knowledge level: Expert (>.85)	Moderate (>.55)	Weak (>.25)	Unskilled (0-.25)
			Equivalence				
1	Definition	12	.81	40.8	52.9	5.2	.5
2	Properties	3	.93	83.8	13.1	2.6	.5
3	Application	13	.61	5.2	53.9	36.1	4.7
			Literal symbols				
4	Definition	1	.33	33	0	0	67
5	Properties	3	.42	4.7	40.8	31.4	23
6	Application	11	.35	.5	11	61.3	27.2

Note: The number of students contributing to each variable is 191, except for the Equivalence Definition variable (n= 190).

Research question 1.

> What proportion of students in remedial mathematics classes recognizes the relational definition of equivalence?

The direct answer to the research question is that, of the students who participated in the survey, 40.8% (n= 78) were expert at Equivalence Definition, demonstrating excellent ability to recognize and apply the relational definition of equivalence by succeeding in at least 85% of the cases.

The majority (n = 102) of the remaining students (53.4%) were at the level of moderate knowledge, demonstrating an ability to recognize the relational definition in at least 55% of the cases. They appeared to have some comprehension of the relational definition of equivalence, but could not always recognize when to use it or could not always apply the definition to more complex questions. Only 5.7% of the students demonstrated knowledge levels of *weak* or *unskilled* related to Research Question 1.

Research question 2.

> What proportion of students in remedial mathematics classes recognizes the reflective and symmetrical properties of equivalence?

The proportion of the students who were expert at recognizing the properties of equivalence was 83.8% (see Table 9). More students were expert at recognizing and applying the properties of equivalence (correctly answering the assigned items in the Questionnaire in more than 85% of the cases) than were adept at the definition of equivalence. This may be a result of the properties of equivalence not being completely dependent on the relational definition; the operational definition of equivalence is sometimes correct, but only in some applications. Even using the operational definition, some students can recognize and apply the properties of equivalence. Additional students (13.1%) exhibited a moderate ability to recognize the properties of equivalence by correctly answering more than 55% of the items associated with the Equivalence Properties variable. Only 3.1% of the students were able to answer correctly the items in less than 55% of the cases, putting them in the weak category (2.6%) or the unskilled category (.5%).

Research question 3.

> What proportion of students in remedial mathematics classes recognizes and can apply the three applications of equivalence: arithmetic identity, arithmetic equivalent expressions, and assignment?

The direct answer to the research question is that 5.2% of the students were expert at recognizing and applying the basic applications of equivalence, being able to respond correctly in more than 85% of the cases. This proportion was a marked difference from the proportions for the definition and the properties of equivalence. Although the students seemed to have the underlying concept knowledge, they were not able to broadly apply the knowledge to problem situations.

However, the majority of students (53.9%) fell into the moderate category, able to answer the questionnaire items associated with the Equivalence Applications variable in more than 55% of the cases and up to 85% of the cases. This indicates that the students may have found some of the questions too difficult to answer correctly, reflecting their lack of experience Some students (40.8%) were weak, lacking the knowledge to answer the questions correctly 55% of the time, of whom 4.7% could not answer correctly 25% of the time. . It may be that they lacked reading knowledge or problem-solving strategies.

To summarize the equivalence related research questions, over 90% of the students in remedial mathematics classes appeared to be either expert or to have a moderate ability to recognize the Definition and Properties of Equivalence. With some help and additional experience, they could all be expert in these two variables. However, nearly half of the students were lacking severely in their ability to recognize and implement the fundamental Applications of Equivalence, which are basic to solving word problems and real world problems. Perhaps these data explain, in part, the common difficulty that students in remedial mathematics classes have with solving word problems and real world problems.

Research question 4.

> What proportion of students in remedial mathematics classes recognizes the quantitative definition of a literal symbol?

The answer to Research Question 4 is that 33% of the students were at the expert level, succeeding in answering the only definition-related item on the questionnaire. This result indicated that many students who were in arithmetic and prealgebra level mathematics courses had accumulated some knowledge related to literal symbols in high school or in early college.

No students were categorized as moderate or as weak, in relation to Research Question 4. The remaining 67% of the students appeared here as unskilled. The reason for this dichotomous distribution is that there was only one item in the questionnaire associated directly with the definition of a literal symbol and students' scores for this variable could only be dichotomous, not spread over a scale of four levels. With more items, the students may have been able to demonstrate other degrees of knowledge.

Research question 5.

> What proportion of students in remedial mathematics classes recognizes the two properties of a literal symbol that same literals represent the same quantities and that different literals represent same or different quantities?

The direct answer to Research Question 5 is that 4.7% of the participating students were expert concerning the properties of a literal symbol, answering correctly the questions related to the properties at least 85% of the time. The remaining students demonstrated a range of skill levels. At the moderate knowledge level, successful in at least 55% of the cases, there were 40.8 % of the students. At the weak level, there were another 31.4% of the students. At the unskilled level, there were 23% of the students. These low scores reflect the less exposure and experience that students in the remedial level courses have with algebra.

Research question 6.

> What proportion of students in remedial mathematics classes recognizes and can apply the three application of a literal symbol: an unknown, a variable and a generalized number?

The data showed that a mere 0.5% of the students were expert in this knowledge. An additional 11% of the students demonstrated moderate knowledge, but the majority (61.3%) demonstrated weak knowledge related to applications of a literal symbol. A large number (27.2%) were unskilled related to Research Question 6.

Most learning of applications of a literal symbol occurs in early algebra, with which these students in remedial courses have little experience. It is not a surprise, therefore, that the students' knowledge related to applications of a literal symbol were generally poorer than other knowledge.

Summary

The proportion of students who were fluent with the relational definition of equivalence (Research Question 1) was 40.8%. That is not to say that the other 60% were ignorant of the relational definition of equivalence, because over one half of the students (52.9%) in this study clearly demonstrated moderate, but significant knowledge of the relational definition. Less than 1% of the students in this study appeared to be completely unaware of the relational definition of equivalence. Students with moderate knowledge, given carefully selected support and instruction, need only to improve and expand their knowledge to become expert at the relational definition of equivalence.

In addition to being knowledgeable of the definition of the relational equivalence, students need to be fluent with the properties of equivalence before they can skillfully apply their knowledge to real-world problems. The proportion of students in this study who were expert with the properties of equivalence (Research Question 2) was 83.8%. An additional 13.1% of the students showed moderate knowledge. It would appear that the properties of equivalence were well understood by students in the remedial mathematics classes and little support or structure would be required to improve students' knowledge.

The results of this study clearly indicate, however, that college students in remedial mathematics classes are deficient in the basic applications of equivalence (Research Question 3). A low proportion (5.2%) of the students was expert in the area of basic applications of equivalence and fewer than 5% of the students were unskilled in Equivalence Applications. The majority of the students (53.9%) demonstrated some knowledge, but moderate, and 36.1% of the students demonstrated weak knowledge. In the area of equivalence education, it is the applications that require the most attention. Most of the students had an understanding of the concept, but they were as yet unskilled in recognizing opportunities to put them to use.

Students' knowledge related to the definition of a literal symbol was not as high as the knowledge related to equivalence. Only 33% of the students in this study demonstrated an expert understanding of the definition of a literal symbol (Research Question 4). Because literal symbols have little place in mathematics classes prior to algebra, it is not surprising that fewer students exhibited expert level knowledge related to literal symbols. The students who have acquired some knowledge of the definition of a literal symbol may have gathered this knowledge from high school algebra classes.

Fewer than 5% of the students in this study were expert in the properties of a literal symbol (Research Question 5), demonstrating the hierarchal nature of the knowledge. A large proportion (40.8%) of students demonstrated moderate skills and these students, with proper instruction and support, could readily improve their knowledge to the expert level after becoming fluent with the definition of a literal symbol. The majority of students (54.4%), however, demonstrated weak or unskilled knowledge levels. The properties of a literal symbol, therefore, need considerable attention before students can be expert.

Less than 1% of the students in this study were expert with the applications of a literal symbol and 88.5% were weak or unskilled, again demonstrating the hierarchal nature of the knowledge. The entire topic of literal symbols needs to be emphasized for the students in remedial mathematics classes, beginning with the definition and continuing with properties and applications.

The results of this study have demonstrated clearly that many students have a moderate understanding of equality; they are not ignorant of equivalence. In contrast, this study showed that the majority of the students are not knowledgeable of the concept of literal symbols. Because college students in remedial mathematics classes are generally weak in the concept of literal symbols, they require much help with this topic.

CHAPTER V

SUMMARY, IMPLICATIONS, CONCLUSIONS, AND RECOMMENDATIONS

This chapter presents a summary and conclusion to the study in five parts. The first part is a summary of the study, describing in brief the fundamental problem, the purpose of this study, the research questions, and the methodology. A summary of the findings of the study are included in the second part of this chapter and the third part discusses the limitations of this study. The fourth part of this chapter discusses the conclusions of the study and the fifth part discusses the implications and recommendations for research, methodology, and practice based on the results of this study.

Summary of the Study

Some students at all grade levels appear to struggle with the basic concepts of mathematics. In his Constructivist theory, which underlies this study, Piaget observed children, through adolescence, develop and resolve misconceptions. He predicted that they eventually would enter adulthood retaining some incomplete or incorrect conceptions (Piaget & Inhelder, 2000). Research of misconceptions and interventions over the past few decades has been concentrated at the elementary and middle school grades (Alibali, et al., 2007; Knuth et al., 2005; McNeil et al., 2006) where arithmetic instruction and introductory algebra instruction are concentrated. Some research has been focused on middle school students that indicate over 50% of eighth grade students did not understand the relational definition of equivalence (e.g. Knuth et al., 2005), but no research has been conducted at the senior high school levels. Research at the college level (e.g. McNeil & Alibali, 2005; Stigler et al., 2010) has been typically at higher levels of mathematics or at misunderstandings of higher-level concepts, creating a gap in the research. It is not known, for example, what proportion of high school students continue to have difficulties with fundamental concepts, such as equivalence and literal symbols, similar to those found at earlier grade levels. Misconceptions at these basic levels, or others, will be a serious impediment to the study of algebra (Baroudi, 2006; Russell et al., 2009; Warren, 2003).

Purpose of the study.

The purpose of this study was to investigate the proportion of college students in remedial mathematics classes who exhibit no misconceptions related to the two basic concepts of equivalence and literal symbols, thereby partially filling the gap in the research. For this study, each concept was described by three parameters: definition, properties, and basic applications.

Equivalence is one aspect of relational thinking, which also includes other relations of quantities, such as greater than, less than, and congruency. Associated with the definition are particular properties, which underpin many applications, of which only three were considered in this study.

The definition of equivalence, which is the contemporary term for equality and its properties, is a statement of the relation of sameness between two quantities. Equivalence has three properties and a corollary. The properties are reflection (any number equals itself), symmetry (if quantity A equals quantity B, then quantity B equals quantity A), and transitivity (if quantity A equals quantity B and if quantity B equals quantity C, then quantity A equals quantity C). The corollary of the definition is the concept that equivalent quantities are interchangeable, that is, one quantity may be substituted for the other in other equations or expressions (McNeil, 2008).

There are many applications of equivalence (Freudenthal, 1983), but only the three applications likely encountered first by students were included in this study. The three applications of equivalence were statements of equivalent expressions (eg., 4 + 7 = 9 + 2), statements of arithmetic identity (e.g., 7 + 3 = 10), and statements of assignment (e.g., X = 4). The applications are similar in appearance (as equations), but they differ greatly in purpose and usage and students need to be very fluent in order to discriminate them on sight.

Jacobs et al. (2007) described relational thinking as a general process of noticing the relations between distinct items by comparing items in terms of their size, color, weight, et cetera. In mathematics, relational thinking involves comparisons of items, or groups of items, in quantifiable terms, such as counts, cash value, cost, et cetera. The relations between relational thinking, the definition and properties, and the applications of equivalence were described by Jacobs et al. and are diagrammed in the Equivalence side of Figure 3 to highlight the parameters and their interrelations. Literal thinking spawns Equivalence and a parent-child relation exists between the two, which is indicated by an arrow between the Literal Thinking and the Equivalence concept ellipse in Figure 3. Similarly, arrows indicate the parent-child relations between the Equivalence concept ellipse and each of the three Application ellipses for the concept of Equivalence, as well as two other arrows to application ellipses of Literal Symbols concept. Figure 3 also shows that the applications of Arithmetic Identity and Arithmetically Equivalent Expressions underpin the usage of Simplifying Arithmetic Expressions and Equations.

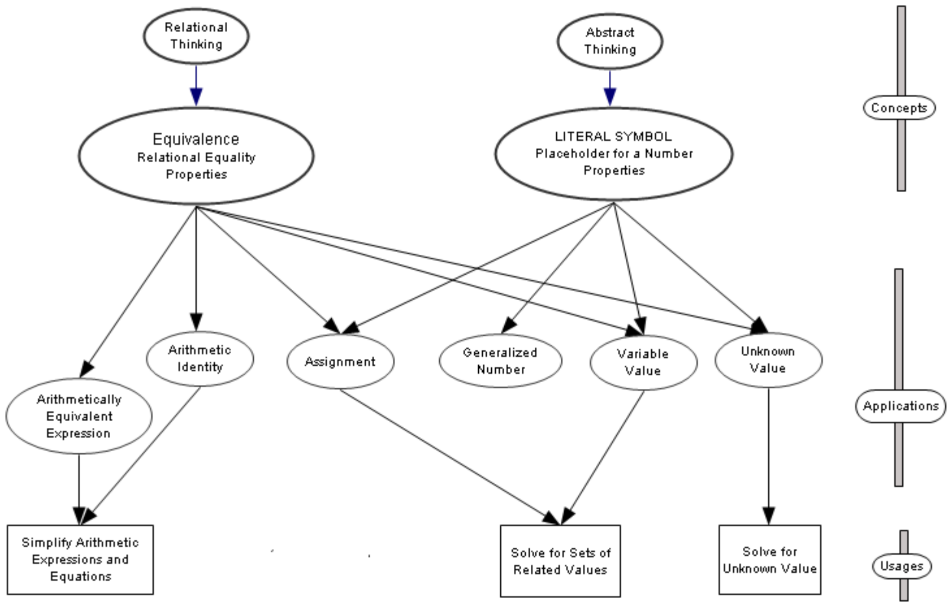

Figure 3. Simplified diagram of equivalence and literal symbols concepts: definitions, properties, and applications.

Similarly, literal symbols are an aspect of abstract thinking in which quantities are referred to abstractly as literal symbols, as opposed to numerals. Analogously to Equivalence, Literal Symbols also have a definition, properties, and many applications. A literal symbol in any expression or equation is defined in algebra as a letter that represents an unspecified numerical quantity in any numerical format (Knuth et al., 2005; Steinle et al., 2006; Usiskin, 1999; Warren, 2003). The two properties are: (a) in any one expression, like letters represent the same quantities, and (b) unlike letters in an expression or equation represent separate quantities that may be the same or not (Steinle, et al, 2006; Weinberg et al., 2004). The three applications of a literal symbol most likely first encountered by students are a generalized number, a variable value, and an unknown value (Kieran, 1991). The applications are similar in appearance (as equations), but differ greatly in purpose and usage and, here also, students need to be very fluent in order to discriminate them on sight.

Students' fluencies with the definition, the properties and the applications of each concept formed the six dependent variables for this study. Because the concepts of equivalence and literal symbols are fundamental to the study of algebra and higher mathematics, misconceptions and ignorance on the part of the students within the six variables constitutes a serious impediment to his or her studies of algebra (Baroudi, 2006; English & Warren, 1998; Knuth et al., 2005; Mathews et al., 2012; Warren, 2003). The purpose of this study was, therefore, to investigate the proportion of remedial mathematics students in college who could exhibit knowledge of the accepted meanings for each parameter of the equivalence and the literal symbol concepts. These results helped fill the gap that exists in the research.

Research questions.

The six research questions guiding this study were:

1. What proportion of students in remedial mathematics classes recognizes the relational definition of equivalence?
2. What proportion of students in remedial mathematics classes recognizes the reflective and symmetrical properties of equivalence?
3. What proportion of students in remedial mathematics classes recognizes and can apply the three applications of equivalence: arithmetic identity arithmetic equivalent expressions, and assignment?
4. What proportion of students in remedial mathematics classes recognizes the quantitative definition of a literal symbol?

5. What proportion of students in remedial mathematics classes recognizes the properties of a literal symbol that same literal symbols represent the same quantities and different literal symbols represent same or different quantities?
6. What proportion of students in remedial mathematics classes recognizes and can apply the three applications of literal symbol: an unknown quantity, a variable quantity, and a generalized number?

Methodology.

The six dependent variables for this study are the proportions of students who demonstrate fluencies with the definition, the properties, and the applications for each of two concepts: equivalence and literal symbol. Students were surveyed to assess their fluencies with each of the six variables using a questionnaire of 41 multiple choice or True-False items and two open items: 29 equivalence items taken from Rittle-Johnson et al. (2010), 13 literal symbols items taken from Küchemann (1981a), and one literal symbol item taken from Knuth et al. (2005). The items, except for Knuth's item, had been previously validated as indicative of particular misconceptions related to either equivalence or to literal symbols. For this study, the items were segregated further as indicative of the definition, the properties, or an application of each concept.

Two levels of remedial mathematics courses were available to the students: Basic Mathematics (arithmetic) and Prealgebra (introduction to signed numbers, literal symbols and equations). Placement testing dictated which level of remedial class was initially appropriate for each student. The questionnaire was administered to seven sections of both remedial mathematics courses (n = 191) over a two-week period early in the Fall 2011 semester. In each classroom, following a short orientation, the students completed the questionnaire in 35 minutes or less. Scoring of individual questionnaires provided individual student scores for each variable, from which sample-wide statistics were derived and reported as the responses to the research questions.

Sample Description

The students who agreed to participate in this study were enrolled in one or the other of Basic Mathematics (n = 105) or Prealgebra (n = 86), both remedial, non-credit mathematics courses. Four of the sections were daytime sections (n = 86) and three were evening sections (n = 73). The students ranged in age from less than 20 (54%), but some were over 30 (11%). The female population was 65%. Freshmen accounted for 63% of the total participants.

Summary of the Findings

There were five general findings in this study. The first finding was that the 43 items selected for this study from prior studies of elementary school and high school students were appropriately difficult for college students in remedial mathematics classes. Students' batch scores for all 43 items ranged from .12 to 1.00, with a mean of .60 and a standard deviation of .14. Because only 1% of the student's scores exceeded .90, the range and mean indicated that the set of 43 items adequately spanned the students' abilities.

The second general finding in this study was that students in the Prealgebra classes did consistently outperform those in the Basic Mathematics classes, as expected, but the differences were significant only for the variables of Equivalence Definition, Literal Symbol Properties, and Literal Symbol Applications, and the differences of the means were small to moderate. This finding suggests that Prealgebra students had a better knowledge of the meaning of equivalence and more experience using literal symbols, which is consistent with the curricula of the two remedial mathematics courses. A third general finding was that there were no significant differences in performance between groups of students based on gender, age, class time (day vs. evening), or college seniority.

The fourth general finding from the results of this study was that the dependencies depicted in the diagram of concept relations (Figure 3) were corroborated. The proportions of students at the expert level for each variable generally decrease, reflecting the hierarchal dependency within each concept. Similarly, the proportions of students showing expert knowledge with the variables related to Literal Symbols are generally less than those for Equivalence, consistent with the general dependency of Literal Symbol variables on the concept of Equivalence.

The fifth general finding in this study was that students exhibited a range of knowledge on each variable, as Küchemann (1981a) and Rittle-Johnson et al. (2011) observed, not simply an knowledge or lack of knowledge (e.g. Alibali, et al., 2007; McNeil et al., 2010). In this study, students were grouped according to knowledge level on each research question as indicated by their scores on each variable. The four levels were labeled as: (a) expert (score >.85, demonstrating near perfect knowledge of the concept), (b) moderate (score >.55, demonstrating basic knowledge of the concept, but not always able to recognize and use it, (c) weak (score >.25, demonstrating knowledge of the concept only in the more basic usages), and (d) unskilled (score ≤.25, demonstrating near complete lack of knowledge of the concept. The proportions of students in each category for the equivalence-related variables are shown in Figures 4 and 5, which re-present the numerical data in Table 9.

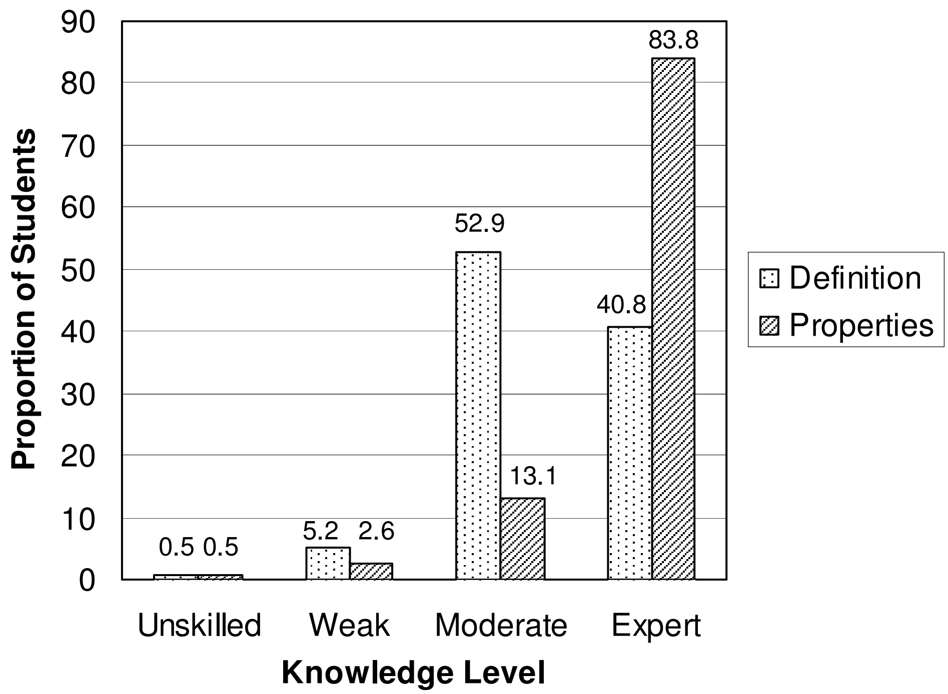

Figure 4. Proportions of students at each knowledge level for Equivalence Definition and Equivalence Properties.

Equivalence knowledge.

A large proportion of the students demonstrated expert knowledge of the relational definition of equivalence (40.8%), which answered Research Question 1. A larger proportion showed expert knowledge related to the properties of equivalence (83.8%), which answered research Question 2. The apparent contradiction of dependency in these data may be explained by the fact that the dominant misconception for Equivalence Definition is the operational definition (equality is a direction to perform an indicated numerical operation) that does not contradict the relational definition, but is not flexible enough for higher arithmetic and algebra. It is possible to know the properties of equivalence while holding the operational definition in some instances.

The results for Equivalence Definition and Equivalence Properties show similar profiles. Very low proportions of students demonstrated unskilled or weak knowledge, less than 6% for both variables. This minority may have a firm belief in the operational definition or some different misconception, perhaps difficult to identify and remedy (Vosniadou & Verschaffel, 2004). Expert students demonstrated fluency with the relational definition, with no misconceptions. Students in the moderate category also demonstrated knowledge of the relational definition, but only on the simpler items (Kuchemann, 1981a; Rittle-Johnson et al., 2010). In this case, the moderate and expert categories share the understanding of the basic concept and can be described together. Over 90% of the students demonstrated at least a moderate knowledge of the relational definition of equivalence. Similarly, over 90% of the students demonstrated at least a moderate knowledge of the properties of Equivalence. Some guidance on transferring the knowledge to additional types of problems may accelerate most of these students to the expert ranks for equivalence definition and properties.

Beginning with Research Question 3, related to applications of equivalence, the picture changes drastically: students' performances were considerably lower. Only 5.2% of the students exhibited expert knowledge related to the three applications of equivalence, which is the answer to Research Question 3. The profile for the proportions of students at different knowledge levels for Equivalence Application knowledge is shown in Figure 5, which re-presents the numerical data in Table 9.

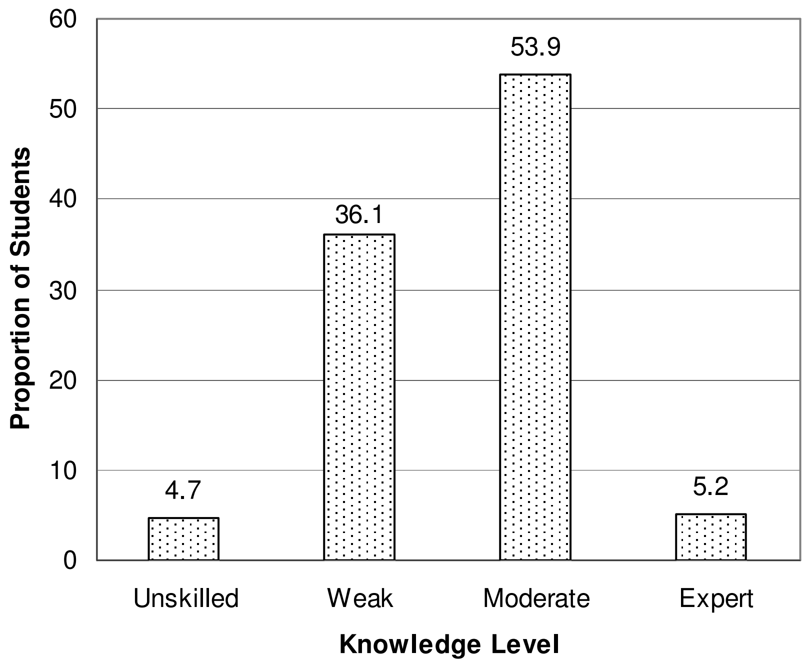

Figure 5. Proportions of students at each knowledge level for Equivalence Applications.

The majority (53.9%) of the students exhibited a moderate knowledge level and many (36.1%) demonstrated weak knowledge. This drastic difference between the proportions for definition and properties knowledge, and applications knowledge may reflect the relative emphases given these topics in textbooks and curricula. The lower proportions suggest that the students have received less exposure and experience related to applications.

Summarizing the students' knowledge of Equivalence, most students have at least a fair knowledge of the relational definition of Equivalence, but many need to transfer their knowledge to include a broader recognition of where and when the definition and properties can be used to advantage. For instance, a student may recognize an opportunity to conveniently replace 13 in a calculation with the equivalent 10 + 3, but cannot recognize an opportunity in the more complicated case of replacing 347 in a calculation with the equivalent 300 + 50 - 3.

The most serious deficiency the students showed was the knowledge of when or how to apply even the most basic applications of equivalence, although 40% of the students were expert in definition of Equivalence. Less than 5% of the students appeared to have no knowledge of Equivalence Applications and less than 6% demonstrated expert knowledge. The majority of the students (90%) fell between these extremes. The large disparity between the applications and the other variables may be related to the curricular and assessment emphases on application algorithms, rather than concepts that rationalize the algorithms (Givven et al., 2011). The number of algorithms grows tremendously in algebra and, if dependent on rote memory, a student's mental processing capacity is soon overloaded. More emphasis on the underlying concepts may reduce the memory load.

Literal symbols knowledge.

Students in remedial mathematics classes performed very poorly overall on items related to literal symbols. These students may continue to have the same difficulties discovered among middle school and high school students. Some K-12 students are intimidated by the increased use of symbology and the increased abstractness in algebra (Hadjidemetriou, Pampala, Petridou, Williams, & Wo, 2007; Stephens, 2005). In addition, the definition of equivalence underpins the use of literal symbols; an incomplete understanding of equivalence and the operations related to equations would be detrimental to learning the properties and applications of literal symbols (Williams & Cooper, 2002). MacGregor and Stacey (1997) found that teaching styles, pedagogical approaches, teaching materials, and the learning environment all affect the learning of algebra. For these reasons, in part, college students in the remedial mathematics classes have acquired little successful experience in the meaning and applications of literal symbols before arriving at college.

Within the two most remedial mathematics classes at the college where this study was conducted, students are introduced to one or two applications of literal symbols in the later half of the Basic Mathematics class but are immersed in their use in the Prealgebra class. The proportions of students at each knowledge level for both Literal Symbol Definition and Literal Symbol Properties are sown in Figure 6, which re-presents the numerical data in Table 9.

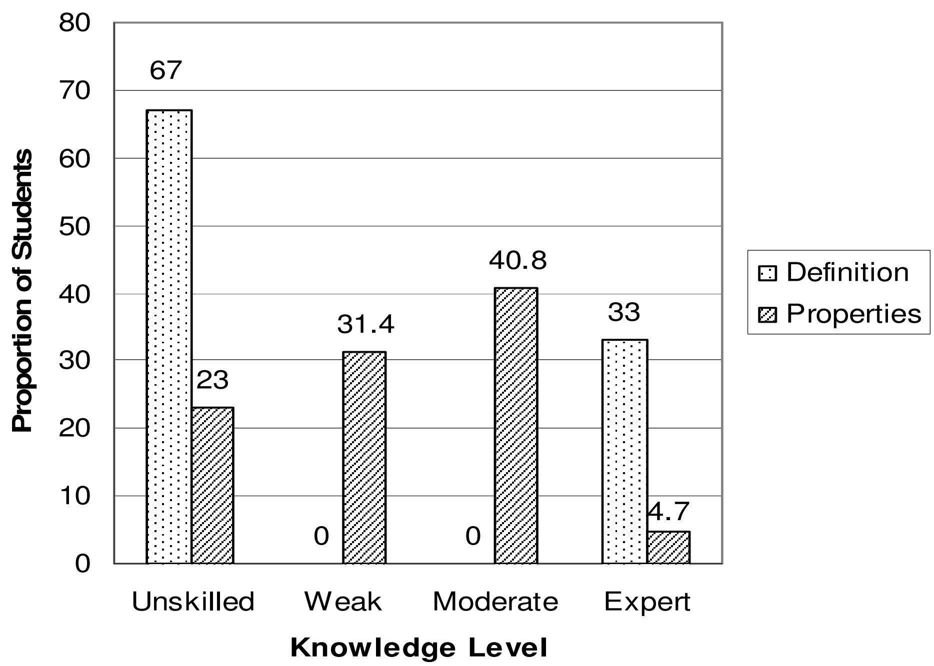

Figure 6. Proportions of students at each knowledge level for Literal Symbols Definition and Properties.

Because the questionnaire was administered early in the semester, it was not surprising that few students (33%) were expert related to the definition of a literal symbol, answering Research Question 4. All of the remaining students (67%) were unskilled. Because there was only one item on the questionnaire that related to the definition of a literal symbol, students cold only demonstrate expert or unskilled knowledge levels. More items related specifically to the definition of a literal symbol may have provided a more descriptive distribution of knowledge levels.

A smaller proportion of students (4.7%) were expert with the properties of literal symbols, answering Research Question 5. Within the structure of the two remedial mathematical courses, students are shown and work with problems in which there is only one literal symbol, for which the properties are simplified. The properties of literal symbols are more complicated when there are two or more symbols involved. Students in remedial mathematics classes, therefore, have little exposure in college to the complete properties of literal symbols, but they have collected, over time, some knowledge of the properties of literal symbols: 40.8% of them demonstrated moderate knowledge and 31.4 % exhibited weak knowledge. Some students (23%) were categorized as unskilled, able to answer questions related to literal properties in less than 25% of the cases. The results for this study may be biased toward lower proportions because two of the three items on the questionnaire related to Properties of Literal Symbols involved more than one literal symbol.

As in the case of Applications of Equivalence, students' performances in applications of Literal Symbols also showed a precipitous drop. The results of the study related to Applications of Literal Symbols are shown in Figure 7, which re-presents the numerical data in Table 9. Less than 1% of the students demonstrated an expert capability in applications, answering Research Question 6, and only 11% of the students were considered as having moderate knowledge. The majority of the students (61.3%)

demonstrated weak knowledge in the area of applications of literal symbols and 27.2% of the students were unskilled.

The low performance of students may be a result of low performance on the definition and properties of a literal symbol. Low performance on three variables related to Equivalence may also be a contributing factor to the low performance on Literal Symbols.

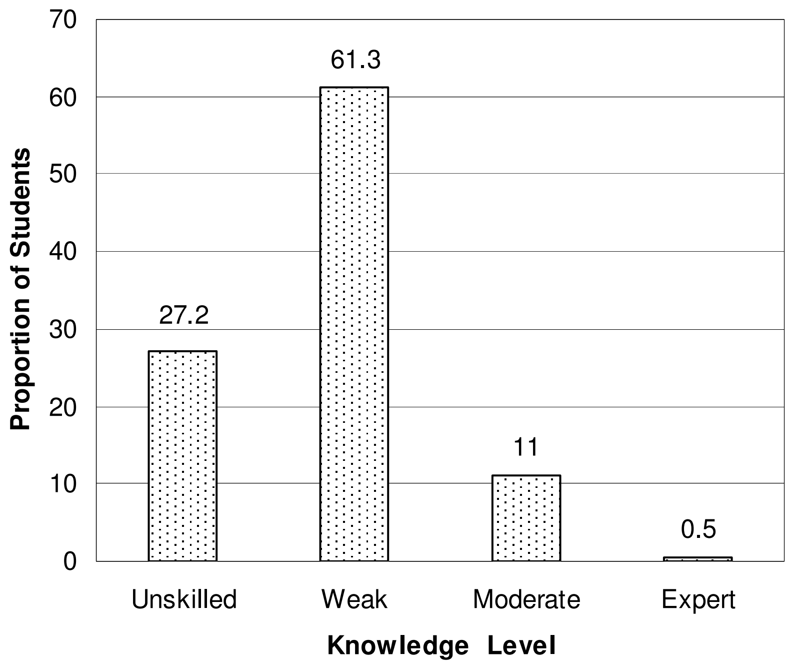

Figure 7. Proportions of students at each knowledge level for Applications of Literal Symbols.

The poor results for Literal Symbol variables were not surprising, in light of the scarce prior experience with Literal Symbols by the students in this study. The restricted exposure to literal symbols the students obtain in the Basic Mathematics course could be used to bolster students' knowledge of the definition and properties literal symbols before they are exposed to literal symbols in force in the Prealgebra course.

This study reflected prior studies of general K-8 students by assessing students on the same concepts of equivalence and literal symbols. Most prior studies focused on either the concept of equivalence (e.g. Molina et al., 2009; McNeil & Alibali, 2004; Rittle-Johnson et al., 2010) or the concept of literal symbols (e.g., Christou et al, 2007; Küchemann, 1981; Weinberg et al., 2004). Very few studies attempted assessment of both concepts (e.g. Asquith et al., 2007; Knuth, et al. 2005) and they were constrained to a short list of items. This study differed from prior research by using a biased sample of students assigned to remedial mathematics courses as opposed to a general unbiased sample. This study also differed from prior research by assessing students on many assorted items simultaneously on both concepts and by looking at the spectrum of students' scores instead of just the proportion of successful students.

This study has provided a unique insight into a mathematical preparedness of beginning college students, something that has been lacking in the research to date. In addition to finding the proportion of students who were expert in the six areas of concern, this study has found that students' knowledge of each variable can be divided into four levels of knowledge and that the proportions of students were not uniformly distributed across the four levels.

Summary and Discussion of Limitations

There were several limitations associated with this study. The first was that the students participating in the survey were not randomly selected. The students who participated in this survey were a convenience sample consisting of all but one section of the two developmental mathematics courses at the community college. Furthermore, by limiting the sample to students screened for mathematics underpreparedness, the sample was biased toward the less skilled students. In addition, this sample of students in remedial mathematics classes only approximated the demography of the students in remedial classes across the state of California. Of the 50,000 students in community college remedial mathematics classes in California in 2002, 55% were female (Perry et al., 2010). In this study, the proportion of female students was 65%. The students in this study were also generally older than those students across California. In the state, 79% of the students were younger than 20, compared to 54% for this study; across the state, 9% of the students were older than 25, compared to 18% in this study. The quantitative results of this study are, therefore, not generalizable to freshmen or first-semester students, but, at best, the relative results may be indicative of community college students in remedial mathematics classes.

The questionnaire that served as the instrument for this study exhibited a general shortcoming. It consisted of 43 items that were previously validated and used in different environments, but not used together. For students who are weak in mathematical skills and mathematics self-confidence, 43 mathematics-related questions on eight pages in 35 minutes was a daunting and exhausting task. The reliability and validity of the students' responses in the latter half of the questionnaire, therefore, may be questionable.

On the other hand, from an analysis point of view, this study suffered from an inadequate number of items for some of the variables. The variables of Equivalence Properties and Literal Symbol Properties were assessed on just three items each; the Literal Symbol Definition variable was assessed on only one item. Three items were just enough to produce four possible scores, but were not enough to provide resolution on a multifaceted variable. The limitations of inadequate items and the overall length of the questionnaire are opposing concerns.

There is one challenge to conducting a survey on the subject of mathematics. No matter how it is described to the students, the survey looks like a mathematics test to the students and some students set out to obtain as high a score as possible. They will use deduction, induction, and association to arrive at a correct answer, but that may not be the intent of the survey. The intent of this study, for example, was to obtain a snapshot of each student's entering knowledge related to the two concepts of equivalence and literal symbols. In most classrooms in this study, the last students to complete the questionnaire were observed to be working very diligently to arrive at the correct answers, but the intent was to obtain the instantaneous status of their knowledge background.

Discussion of the Findings

The finding that there were no significant differences in knowledge related to the equivalence concept and the literal symbols concept between most groupings of students (male versus female, day versus evening, college seniority levels, and age groups) was not surprising. Mathematics instruction in high school is aimed at the students on track to college mathematics, and little, if any, further instructional emphasis is placed on the fundamental concepts of equivalence and literal symbols (Knuth et al., 2005; McNeil et al., 2006). Any learning on the part of the students related to the basic concepts past middle school appears to be a result of self-study or osmosis. At the community colleges, the groups of students are composed of a wide range of students and the groups tend to be similarly heterogeneous. Other research articles (e.g. Stigler, et al., 2010) at college level remedial mathematics classes made no distinction between these groupings.

The one grouping that did show some differences in knowledge levels was the Prealgebra class and the Basics Mathematics class, which were populated according to incoming knowledge levels through placement testing. The Prealgebra class did in fact show superior knowledge levels in all variables, as expected, but the differences were significant only for Research Question 1 (Equivalence Definition), $t\ (185.84) = 2.98$, $p < .01$; for Research Question 5 (Literal Symbol Properties), $t(189) = 5.70$, $p < .01$; and for Research Question 6 (Literal Symbol Applications), $t(189) = 4.58$, $p < .01$. At the point in the semester that this survey was administered, the courses would not have had a polarizing effect on the results. Within the two remedial mathematics classes in this study, literal symbols were introduced in conjunction with the unknown application in the latter half of the Basic Mathematics course. At the time of this study, the students in the Basic Mathematics course had not yet begun to work with literal symbols. At the same time, students in the Prealgebra course were just beginning to study literal symbols and algebraic equations. But a student's learning was not restricted to these two courses; both groups of students may have been exposed to literal symbols and equations in high school, which would tend to homogenize the results

The finding that the diagram developed for this study to help explain the concepts and their parts was corroborated by the results suggests that the partitioning of the concepts into definition, properties, and applications is a valid approach, although the reliability results were low. A questionnaire of items designed to address these particular parameters is needed to establish the approach.

The choice of subdividing the results into four categories of expert, moderate, weak, and unskilled was appropriate. The two higher categories captured all the students who held at least a fair understanding of the concept, differing by degree; the two lower categories captured the students who held little or no understanding of the concept, differing by degree. The remediation needs are very different for the two groups.

Conclusions

This study found that a large proportion of the students was considered at the expert level on the variable of Equivalence Properties, but fewer students were expert on all the remaining five variables (see Figures 4 through 7). For the Equivalence variables, this study found that low proportions of students fell into the unskilled category. The majority of the remaining students fell into the category of moderate knowledge. In general, over 90% of the students were expert or moderate, indicating a fair, but not complete, understanding of the equivalence relation and its properties. Over 50% of the students were expert or moderate in the applications of equivalence, indicating that more misconceptions may be active in the area of applications. These results are skewed higher than the results shown by the general 6^{th} grade students in Rittle-Johnson et al.'s (2010) study. In their study the 6^{th} grade students were near-normally distributed centered on the middle of the ability range.

The proportions of students at the different knowledge levels for the literal symbol-related variables are drastically different from the proportions for equivalence-related variables. Only 33% of the students were expert in the definition of a literal symbol. The Prealgebra course, which emphasizes the literal symbol and its applications, may remediate some of the problems students have with literal symbols, as it is designed to do. The results of this study of related to literal symbols were lower than the results of Küchemann's (1981a) Algebra test of general 14 year old students across all algebra tasks. Küchemann's categories of students were based on their success with problems of different difficulties. Six percent of the students were in the highest category, 29% in the next lower category, 24% in the next lower category, and 31% in the lowest category.

Lubliner (2004) wrote that students in elementary school appeared to suddenly obtain lower reading scores in the fourth to sixth grades. Lubliner explained that it was not that the students' knowledge deteriorated, but that the expectations and norms were suddenly raised from vocalization of the written word to interpretation and understanding of the written words. Perhaps the change from numerical operations to abstract thinking of numbers is a similarly sudden change in mathematics expectations that is a difficult hurdle for some students.

In a recent study of remedial mathematics students at a community college, Stigler et al. (2010) assessed students' knowledge of arithmetic operations with fractions and decimals, operations under a radical sign, and percentage problems. Although these are basic mathematical operations, they are not as fundamental as equivalence and literal symbols, which the authors seem to have presumed had been mastered. Stigler et al. concluded that some students did not know the basic arithmetic operations of their study, but could have missed observing more fundamental misconceptions. In a separate report on the same study, Givvin, Stigler, and Thompson (2011) developed a model of the learning trail from kindergarten to college followed by typical developmental students. The model clearly showed that their instruction had emphasized procedures and had been lacking instruction in concepts. This study adds that, for some students, the missing concepts may start with equivalence and literal symbols.

In general, this study found that students in remedial mathematics classes were generally moderate to expert regarding equivalence, and they could come up to par if the curricula and the resources were to apply appropriate emphasis on the three parameters of the fundamental concepts. However, they were weak to fair regarding literal symbols. It may be that those few who did appear to be unskilled had more deep-rooted impediments to their learning of mathematics, perhaps they are candidates for a concept change approach (Vosniadou & Verschaffel, 2004).

This study supplemented the existing research by extending the research to students at the college level, which had been rarely done before, and by looking a little deeper at the concepts of equivalence and literal symbols in terms of their definition, properties, and applications. The additional insight into the details of students' conceptual understanding informs future research, methodology, and practice.

Implications for Research

The results of the study strongly suggest that the fundamental concepts of equivalence and literal symbols cannot be assumed to be understood in research related to college students in remedial mathematics classes. In this study, most students exhibited a good but incomplete understanding of equivalence and a much weaker knowledge of literal symbols, which shortcomings could confound the results of other research.

Based on the large proportion of students showing moderate knowledge of equivalence in this study, further research regarding the precise limits of their knowledge is indicated. Knowing exactly the common misunderstandings of equivalence would underscore the topics in which freshmen students need most instruction and remediation.

Further research is also warranted to develop effective interventions tuned to students' needs.

Similarly, the results of this study regarding literal symbols indicate that a thorough study of current misconceptions held by students is necessary. Many of these have been identified for K-8 students in prior research (e.g., Kücheman, 1981a; Wagner, 1983). Not all of those misconceptions may be found among college students, but, once the misconceptions are exposed, broadly effective interventions for literal symbol misconceptions need to be researched also. A systematic approach to interventions may be warranted, focusing first on the definition, then on the properties, and, finally, on primary applications of literal symbols.

A particular area of such research is a broader study of a student's ability to recognize and use equivalent arithmetic statements (an application of equivalence) as substitutions to more easily solve other problems, an application of equivalence only touched in Rittle-Johnson et al.'s (2010) study and in this study (four items). For example, Jones and Pratt (2012) studied middle school students' abilities to use substitutions using computerized exercises. A recommendation for future research is to adapt this technique to college level students.

Implications for Methodology

The questionnaire items developed by Riddle-Johnson et al. (2010) and Küchemann (1981a) served as a good initial study, but items that are more accurately tuned to the knowledge levels of college remedial mathematics students need to be developed. In particular, questionnaire items that are directly related to definition, properties, or applications are needed to create and validate scales. They would contribute to reliability of the items and to a more valid and more efficient questionnaire.

Having more than three items per variable would be desirable to maintain a four level scale, but more items imply a longer test, which is problematic by itself. It is not necessary, and in fact may be self-defeating, to develop a questionnaire spanning too many variables. Future research might be limited in scope to perhaps three variables (one concept) on the questionnaire, limiting the time involved for the student to no more than 20 minutes.

Another disadvantage of lengthy surveys is the opportunity for students to learn by association: the response to one question provides clues to the correct response to other questions, violating the purpose of the survey to acquire an assessment of the student's prior knowledge. As an example, during this study, one student remarked to the researcher after turning in her survey that she was reminded that equality also meant sameness and that she had returned to earlier questions to improve her answers. A method of administering the questionnaire needs to be developed that prohibits a student's ability to change previous answers.

For example, a change in the presentation of the questionnaire may be in order. The questionnaire could be segmented and administered at different times, which was the approach of Rittle-Johnson et al. (2010), or the questionnaire could be shortened by randomly assigning a shorter list of items to students. If enough computers are available, the questionnaire could be administered simultaneously on individual terminals and the response time controlled for each item. Computerized questionnaires could be designed that prohibit a student from returning to previous questions.

Implications for Practice

Attention to the details of equivalence and literal symbols is necessary for students in remedial mathematics because these details are what many of of the students lack. Textbooks do not take up the slack. Professors of remedial mathematics need to continually point out instances as they occur where different properties and applications of equivalence, literal symbols, and other fundamental concepts are being used either in the textbook or in the classroom (see Harel et al., 2008). The repetition and reminders might help instill in the students well-grounded concepts, which they may be able to transfer to other applications. This would constitute a change of attitude for most professors who are in the habit of taking fundamental concepts for granted and glossing over them while explaining deeper topics, in the interest of efficiency.

Recommendations

It could prove helpful and informative for college students to take a short, in-class, reduced scope, formative survey early in the semester. It would not only inform the students of their weaknesses but would also inform the professor of common weaknesses that need addressing. A more effective intervention for the Basic Mathematics course may be a short in-class workshop could be held very early in the semester covering the definition, properties, and applications of equivalence.

To more efficiently administer the survey, the questionnaire could be computerized with a number of items related to the each application, but which span a range of difficulty. If the items were presented in reverse order (most difficult first), the next item for a student who answers an item unsatisfactorily would be the next easier item in the group. If a student answers correctly, the remaining easier items of the group could be assumed correct also. In this way a larger group of items can be included without sacrificing the quality of the results. More sophisticatedly, the questionnaire could be designed to be interactive, like some aptitude tests, where the response to one item in a set of related items dictates the choice of the next item: if correct, the next item is a more difficult item of the same set, or is an item from a set of different types of items; an unsatisfactory response may dictate an easier item from the same set.

Closing Remarks

This study has successfully contributed to the literature associated with the difficulties students have with learning algebra by investigating college students in remedial mathematics classes. The study established that some students continue to have misconceptions related to equivalence but many have misconceptions related to the literal symbol. The students in remedial mathematics classes at college exhibit a distribution of knowledge related to Equivalence that is biased, in general, toward the moderate and expert levels. The trend for student's knowledge regarding the Literal Symbol is biased toward the middle range of scores. Overall, the students show weakness on the parameter of applications, both for equivalence and for literal symbols, which is a serious impediment the ultimate goal of algebra: the solution of real world problems.

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Permission from Rittle-Johnson for Questionnaire Items

e: Permission to use the test instrument published February 2011

Subject: Re: Permission to use the test instrument published February 2011
From: "Rittle-Johnson, Bethany" <bethany.rittle-johnson@Vanderbilt.Edu>
Date: 5/28/2011 3:17 PM
To: Terrence J Maguire <tjmaguire@dons.usfca.edu>

Dear Terrence,

Of course! My assessments are available online at my project website – search for "earlyalgebra" on the vandy webpage. We have a newer version you may want to use. I don't think you need my permission, or the journals, as a note – it's federally funded work and should be available to all.

Best wishes,
Bethany

PS Be sure to check out Martha Alibali and Mitch Nathan's work with middle school kids.

Permission from Publisher for Questionnaire Items from CSMS

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Terrence J Maguire <tjmaguire@dons.usfca.edu>

permissions to use assessment items for research

Permissions <permissions@gl-assessment.co.uk> **Thu, May 26, 2011 at 7:08 AM**
To: Terrence J Maguire <tjmaguire@dons.usfca.edu>

Dear Terence

Once the Assessments go out of print the copy right permission reverts to the Authors and Permission needs to be sought from them.

Kind Regards

Permissions

From: Terrence J Maguire [mailto:tjmaguire@dons.usfca.edu]
Sent: 26 May 2011 01:00

Permission for Questionnaire Items from Küchemann (1981a)

Re: Permission to use test items from CSMS

Subject: Re: Permission to use test items from CSMS
From: "Kuchemann, Dietmar" <dietmar.kuchemann@kcl.ac.uk>
Date: 5/28/2011 4:59 PM
To: Terrence J Maguire <tjmaguire@dons.usfca.edu>

Terence
I attach the full CSMS Algebra test and the teacher's guide for using it. I am pleased to give you permission to use it for the study you've outlined, below, and hope you find it useful

Best wishes

Dietmar K

Permission for Item from Knuth et al. (2005)

Re: Permission to use test item

Subject: Re: Permission to use test item
From: Eric Knuth <knuth@education.wisc.edu>
Date: 6/24/2012 8:31 AM
To: Terrence J Maguire <tjmaguire@dons.usfca.edu>

Hi Terry,

Sure, you are welcome to use the item. I have also attached a relevant JRME paper my colleagues and I wrote that you might also find helpful (you may have already read it).

Good luck with your dissertation study.

Eric Knuth

On Jun 23, 2012, at 10:05 PM, Terrence J Maguire wrote:

> Dr. Knuth,
> My name is Terry Maguire and I am a doctoral candidate at the University of San Francisco, School of Education.My dissertation is a survey of students in remedial math classes at a community college. They appear to have the same difficulties with math as students in elementary school and middle school: they don't understand equivalence and the literal symbol.
> In my survey, I placed an item I adopted from your article Middle School students' understanding of core algebraic concepts: Equivalence & Variable (2005). ZDM 37(1), which you co-authored with Aliballi, McNeil, Weinberg, and Stephens. I would like your permission to include your item in Figure 1: "The following question is about this expression: 2n + 3 The arrow points to a symbol. What does the symbol stand for?" (Arrow not shown here) I have appended an additional question:"What else might the symbol stand for?" to elicit additional response.
> Other questions in my survey are from Rittle-Johnson, Mathews, Taylor, and McEldoon (2010) for equivalence and Kuchemann (1981) for literal symbols.
> Thank you for your support.
> Terry Maguire

****
Eric J. Knuth, Ph.D.
Professor, Mathematics Education
University of Wisconsin
Teacher Education Building 476C
225 N. Mills Street
Madison, WI 53706
(608) 263-3209 office
(608) 263-5141 sec'y
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Permission from Community College to Conduct Research

June 20, 2011

Institutional Review Board for the Protection of Human Subjects
University of San Francisco
2130 Fulton Street
San Francisco, CA 94117

Dear Members of the Committee:

On behalf of Ohlone College, I am writing to formally indicate our awareness of the research proposed by Terry Maguire, a student at USF. We are aware that Terry intends to conduct research by administering a written survey to our students. I am responsible for overseeing outside research at the college and am an executive officer here. I give Terry Maguire permission to conduct his research at Ohlone College.

If you have any questions or concerns, please feel free to contact my office at (510) 659-6202.

Sincerely,

James E. Wright, Ph.D.
Vice President of Academic Affairs and Deputy Superintendent

Initial Permission from IRBPHS (USF) to Conduct Research

Subject: IRB Application #11-073-Approved
From: irbphs <irbphs@usfca.edu>
Date: 7/19/2011 7:00 PM
To: <tjmaguire@usfca.edu>
CC: Mathew Mitchell <mitchellm@usfca.edu>

July 19, 2011

Dear Mr. McGuire:

The Institutional Review Board for the Protection of Human Subjects (IRBPHS) at the University of San Francisco (USF) has reviewed your renewal request for human subjects approval regarding your study.

Your renewal application has been approved by the committee (IRBPHS #11-073). Please note the following:

1. Approval expires twelve (12) months from the dated noted above. At that time, if you are still in collecting data from human subjects, you must file a renewal application.

2. Any modifications to the research protocol or changes in instrumentation (including wording of items) must be communicated to the IRBPHS. Re-submission of an application may be required at that time.

3. Any adverse reactions or complications on the part of participants must be reported (in writing) to the IRBPHS within ten (10) working days.

If you have any questions, please contact the IRBPHS at (415) 422-6091.

On behalf of the IRBPHS committee, I wish you much success in your research.

Sincerely,

Terence Patterson, EdD, ABPP
Chair, Institutional Review Board for the Protection of Human Subjects
----
IRBPHS - University of San Francisco
Counseling Psychology Department
Education Building - Room 017
2130 Fulton Street
San Francisco, CA 94117-1080
(415) 422-6091 (Message)
(415) 422-5528 (Fax)
irbphs@usfca.edu
----
http://www.usfca.edu/soe/students/irbphs/

Permission from IRBPHS (USF) to Conduct Research

IRB Modification Application #11-073-Approved

Subject: IRB Modification Application #11-073-Approved
From: IRBPHS <pattersont@usfca.edu>
Date: 9/8/2011 2:10 PM
To: <tjmaguire@usfca.edu>
CC: Mathew Mitchell <mitchellm@usfca.edu>

September 9, 2011

Dear Mr. McGuire:

The Institutional Review Board for the Protection of Human Subjects (IRBPHS) at the University of San Francisco (USF) has reviewed your renewal request for human subjects approval regarding your study.

Your modification application has been approved by the committee (IRBPHS #11-073). Please note the following:

1. Approval expires twelve (12) months from the dated noted above. At that time, if you are still in collecting data from human subjects, you must file a renewal application.

2. Any modifications to the research protocol or changes in instrumentation (including wording of items) must be communicated to the IRBPHS. Re-submission of an application may be required at that time.

3. Any adverse reactions or complications on the part of participants must be reported (in writing) to the IRBPHS within ten (10) working days.

If you have any questions, please contact the IRBPHS at (415) 422-6091.

On behalf of the IRBPHS committee, I wish you much success in your research.

Sincerely,

Terence Patterson, EdD, ABPP
Chair, Institutional Review Board for the Protection of Human Subjects
----
IRBPHS ^ University of San Francisco
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Education Building ^ Room 017
2130 Fulton Street
San Francisco, CA 94117-1080
(415) 422-6091 (Message)
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irbphs@usfca.edu
----
http://www.usfca.edu/soe/students/irbphs/

Permission from IRBPHS (USF) to Conduct Research

RB Modification Application #11-073 - Modification Approved

Subject: IRB Modification Application #11-073 - Modification Approved
From: "USF IRBPHS" <irbphs@usfca.edu>
Date: 10/6/2011 8:27 AM
To: <tjmaguire@usfca.edu>
CC: <mitchellm@usfca.edu>

October 6, 2011

Dear Mr. Maguire:

The Institutional Review Board for the Protection of Human Subjects (IRBPHS) at the University of San Francisco (USF) has reviewed your request for modification of your human subjects approval regarding your study.

Your modification application has been approved by the committee (IRBPHS #11-073).

1. Approval expires twelve (12) months from the dated noted above. At that time, if you are still in collecting data from human subjects, you must file a renewal application.

2. Any modifications to the research protocol or changes in instrumentation (including wording of items) must be communicated to the IRBPHS. Re-submission of an application may be required at that time.

3. Any adverse reactions or complications on the part of participants must be reported (in writing) to the IRBPHS within ten (10) working days.

If you have any questions, please contact the IRBPHS at (415) 422-6091.

On behalf of the IRBPHS committee, I wish you much success in your research.

Sincerely,

Terence Patterson, EdD, ABPP
Chair, Institutional Review Board for the Protection of Human Subjects
\-
IRBPHS - University of San Francisco
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Education Building - Room 017
2130 Fulton Street
San Francisco, CA 94117-1080
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Appendix B

Research Instrument

Questionnaire

[“I’ll show you a problem for a few seconds. After I take the problem away, I want you to write the problem exactly as you saw it.”]

1) [__ + 2 = 5]
2) [5 + 2 = __ + 4]
3) [__ + 5 = 8 + 7 + 5]

For items 4 through 6, decide if the number sentence is True. In other words, does it make any sense? After each problem, mark True, False, or I don’t know.

4) 31 + 16 = 16 + 31 ☐ T ☐ F ☐ I don’t know

5) 7 + 6 = 6 + 6 + 1 ☐ T ☐ F ☐ I don’t know

6) 5 + 5 = 5 + 6 ☐ T ☐ F ☐ I don’t know

7) The following problem has two sides. Circle the choice that correctly breaks the problem into problem into its two sides.

8 + 2 + 3 = 4 + __

a)

Side A	Side B
8 + 2 +	3 = 4 + __

b)

Side A	Side B
8 + 2 + 3 + 4	= __

c)

Side A	Side B

I don’t know

d)

Side A	Side B
8 + 2 + 3	4 + __

e)

Side A	Side B
8 + 2 + 3 = 4 + __	__ + 4 = 3 + 2 + 8

8) Without adding 67 + 86, can you tell if the statement below is True or False?

67 + 86 = 68 + 85 ☐ T ☐ F ☐ I don't know

How do you know?

9) It is true that 56 + 85 = 141.

Without subtracting the 7, can you tell if the statement below is True or False?

56 + 85 - 7 = 141 – 7 ☐ T ☐ F ☐ I don't know

How do you know?

10) What does the equal sign (=) mean?

Can it mean anything else?

11) Which of these pairs of numbers is equal to 6 + 4 ? Circle your answer.

a) 5 + 5

b) 4 + 10

c) 1 + 2

d) None of the above

12) Which answer choice would you put in the empty box to show that five pennies are the same amount of money as one nickel? Circle your answer.

a) 5¢

b) =

c) +

d) I don't know.

For items 13 through 19, find the number that goes in each blank.

13) 4 + __ = 8 ☐ I don't know.

14) 8 = 6 + __ ☐ I don't know

15) 3 + 4 = __ + 5 ☐ I don't know

16) __ + 2 = 6 + 4 ☐ I don't know

17) 7 + 6 + 4 = 7 + __ ☐ I don't know

18) 8 + __ = 8 + 6 + 4 ☐ I don't know

19) 6 – 4 + 3 = __ + 3 ☐ I don't know

For items 20 and 21, find the number that goes in each blank. You can try to find a shortcut so that you don't have to do all the adding. Show your work and write your answer in the blank.

20) 898 + 13 = 896 + __

21) 43 + __ = 48 + 76

For items 22 through 25, mark the statements as True, False, or I don't know. Then, explain how you know.

22) 5 + 3 = 8 ☐ T ☐ F ☐ I don't know

23) 7 = 3 + 4 ☐ T ☐ F ☐ I don't know

24) 6 + 4 = 5 + 5 ☐ T ☐ F ☐ I don't know

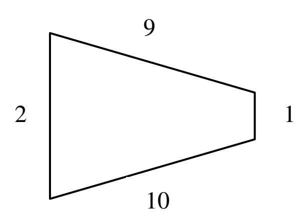
25) 3 = 3 ☐ T ☐ F ☐ I don't know

For item 26, consider the following expression 2n + 3
↑

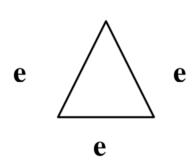
26) The arrow points to a symbol. What does the symbol stand for?

What else might the symbol stand for?

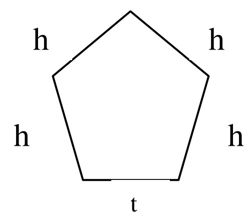
27) What is the perimeter of the figure?



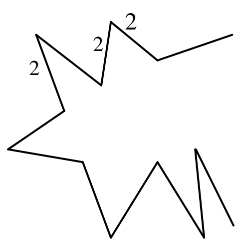
28) What is the perimeter of the figure?



29) What is the perimeter of the figure?



30) What is the perimeter of the figure?



Part of this figure is not drawn. There are n sides altogether all of length 2.

31) If a + b = 43

a + b + 2 = _______

32) What can you say about u if

u = v + 3 and

v = 1

33) If e + f = 8

e + f + g = ________

34) $L + M + N = L + P + N$

a) Always True

b) Sometimes True (when?)

c) Never True

35) What can you say about a if

$a + 5 = 8$

36) Find the value of n. Explain your answer. $n + n + n + 2 = 17$

☐ I don't know

37) What can you say about r if

$r = s + t$, and

$r + s + t = 30$?

38) What can you say about c

if $c + d = 10$ and c is less than d?

39) Which is larger: $2n$ or $n + 2$?

Explain.

40) Write down the smallest and the largest of these:

$n + 1$, $n + 4$, $n – 3$, n, $n – 7$

Smallest _______________ Largest ________________

41) 4 added to n can be written as $n + 4$.

Add 4 to $n + 5$ _________________

For items 42 and 43, fill in the blanks

42) $2a + 5a =$ ___

43) $2a + 5b + a =$ ___

Appendix C

Detailed Description of Questionnaire Items

Items Related to Equivalence

This appendix consists of a detailed explanation of each questionnaire item, including its specific objective and scoring procedure. The objective of items 1 through 25 and items 31 through 33 of the questionnaire is to identify those students who are knowledgeable of the relational definition of equivalence, its properties and some of its applications. Students with only the operational view of equality will have much difficulty with most of these items; students with a relational view of equality will have little difficulty with the items, depending on their previous exposure to the different applications. Students with other misconceptions of equality may answer correctly some of the items. The proportion of items answered correctly will indicate a student's level of knowledge of the concept of equivalence, its properties, and its applications.

Items 1, 2, and 3 of the questionnaire are three memory tests of a mathematical equation. McNeil and Alibali (2004) found that students holding the operational view of equality tend to reformulate nonstandard equations in their memory to conform to the canonical format. On the other hand, students with the relational view of equality have less difficulty in recalling nonstandard equations, depending on their exposure to nonstandard equations.

["I'll show you a problem for a few seconds. After I take the problem away, I want you to write the problem exactly as you saw it."]

1) [__ + 2 = 5]
2) [5 + 2 = __ + 4]
3) [__ + 5 = 8 + 7 + 5]

For each of these three items, the test administrator will read the directions to the students and then show them the equation for five seconds. The students then have ten seconds to record on the questionnaire the equation as they recall it. The sequence is repeated for items 2 and 3. Scoring will be *correct* for each item if all numbers, operators, equal sign and blanks are correctly arranged. Numerals can be misquoted and the scoring will be considered correct.

Items 4, 5, and 6 of the questionnaire form a set of three items assessing whether a student can apply the relational definition and its properties and applications to a closed numerical equation.

> For items 4 through 6, decide if the number sentence is True. In other words, does it make any sense? After each problem, mark True, False, or I don't know.
>
> 4) 31 + 16 = 16 + 31 □ T □ F □ I don't know
> 5) 7 + 6 = 6 + 6 + 1 □ T □ F □ I don't know
> 6) 5 + 5 = 5 + 6 □ T □ F □ I don't know

Item 4 is an instance of the symmetry property of equivalent statements, a component of the relational definition of equivalence and is True. Item 5 is an assessment of applying an equivalent expression (6 + 1 = 7) to part of an equality statement and is True. Item 6 is an instance of equality meaning *the same as*, which is the basic part of the relational definition of equality and is False. These three items will be scored *correct* if correct answers are given.

To evaluate a statement of equality for its truthfulness such as 4 + 5 = 9, two quantities, 4 + 5 and 9, are compared for *same value*. The two quantities are referred to in structural terms as the two *sides of the equation*.

7) The following problem has two sides. Circle the choice that correctly breaks the problem into problem into its two sides.

8 + 2 + 3 = 4 + __

a)

Side A	Side B
8 + 2 +	3 = 4 + __

b)

Side A	Side B
8 + 2 + 3 + 4	= __

c)

Side A	Side B
I don't know	

d)

Side A	Side B
8 + 2 + 3	4 + __

e)

Side A	Side B
8 + 2 + 3 = 4 + __	__ + 4 = 3 + 2 + 8

Item 7 of the questionnaire assesses the student's familiarity with the basic significance of the equal sign (=). Item 7 will be scored *correct* for a choice of d).

Item 8 of the questionnaire continues the assessment of the student's understanding of the relational definition of equivalence; item 8 provides a nonstandard, closed statement further complicated by using two-digit terms.

8) Without adding 67 + 86, can you tell if the statement below is True or False?

67 + 86 = 68 + 85 □ T □ F □ I don't know

How do you know?

In item 8, the student is directed to determine the truthfulness of the statement, not by evaluating each side of the equation independently, but by noticing two subtle embedded equivalences (68 = 67 + 1 and 85 = 86 – 1) that can be applied to the terms in the right side of the equation, simplifying the decision process The student is asked to explain his method for his or her decision. Item 8 will be scored *correct* if it is answered True and if an equivalent expression is used.

Item 9 of the questionnaire is another assessment of the Equivalent Expression application, as in item 8.

9) It is true that 56 + 85 = 141.

Without subtracting the 7, can you tell if the statement below is True or False?

56 + 85 - 7 = 141 – 7 □ T □ F □ I don't know

How do you know?

In item 9, the student is directed to determine the truthfulness of a statement of equality, not by independently evaluating each side of the equation, but by applying embedded equivalent expressions to simplify the decision process. The student is also asked in item 9 to provide a rationale for his decision. Item 9 will be scored *correct* if choice (True) is made, and if the student uses an equivalent expression in his rationale.

Items 10, 11, and 12 of the questionnaire are related to a student's interpretation of the equal sign (=). Students who hold the operational definition of equality are said to interpret the equal sign as a directive to perform the indicated operation on the given numbers.

10) What does the equal sign (=) mean?
Can it mean anything else?

In item 10 of the questionnaire, the student is asked to write a definition of a naked equal sign; no context for the symbol is provided. The item provides the student with an opportunity to describe in his own words a definition of the equal symbol. The purpose of this question is to obtain uninfluenced information about the student's personal conception of the equal sign and, perhaps, to elicit alternative definitions of the equal sign, which may or may not be correct. Item 10 will be scored as *correct* if the student provides at least one relational definition of the symbol.

Item 11 of the questionnaire is an assessment of equality meaning *same value as*.

11) Which of these pairs of numbers is equal to 6 + 4 ? Circle your answer.

a) 5 + 5
b) 4 + 10
c) 1 + 2
d) None of the above

The student is asked in item 11 to recognize a two-term numerical expression that has the same value as a given two-term expression. Item 11 will be scored *correct* if choice a) is selected.

Item 12 of the questionnaire continues the investigation of the meaning of the equal sign, or equality. The student is asked to recognize a correct application of the equal sign.

12) Which answer choice would you put in the empty box to show that five pennies are the same amount of money as one nickel? Circle your answer.

(a) 5¢
(b) =
(c) +
(d) I don't know.

Item 12 of the questionnaire provides a graphic depiction of a group of five pennies and a separate image of one nickel, separated by a box. The student is asked to place in the box a symbol from a menu of 4 choices that signifies the two groups have the same value. Scoring for this item will be *correct* if choice b, the equal sign (=) is selected.

Items 13 through 21 are items related to solving open equations, that is, finding a number that goes into the blank in a statement of equality that makes the equality statement true. Items 13 through 19 of the questionnaire form a seven-item set assessing the student's ability to apply the relational definition of equivalence to find a missing value in progressively complex nonstandard formats.

For items 13 through 19, find the number that goes in each blank.

13) 4 + __ = 8	□ I don't know.
14) 8 = 6 + __	□ I don't know

Item 13 is a variation of the canonical three-term problem format, where the blank is in the second position, rather than in the final position and the answer is 4. Item 14 is a symmetrical variation of the format used in item 13 and the answer is 2. Items 13 and 14 can be solved directly from number facts and could be solved by students who adhere to the operational definition of equality.

However, the operational definition will lead a student to an incorrect answer in item 15.

15) 3 + 4 = __ + 5	□ I don't know
16) __ + 2 = 6 + 4	□ I don't know

Solving item 15 through 19 requires a relational definition of equality. Item 15 progresses to a four-term problem, in the format a + b = c + d, where the blank is in the third position and the answer is 2. Item 16 is similar to item 15, but the blank is in the first position and the answer is 8.

Items 17, 18, and 19 may be solved by independently evaluating each side of the equation and comparing the results, or by noticing embedded equivalent expressions in the equation which can be used to simplify the equation and the decision process. No distinction is made in the scoring for the method used.

17) 7 + 6 + 4 = 7 + __	□ I don't know
18) 8 + __ = 8 + 6 + 4	□ I don't know
19) 6 – 4 + 3 = __ + 3	□ I don't know

Item 17 is a five-term problem, in the format a + b + c = d + e, and the blank is in the fifth position. The correct answer is 10. Item 18 is similar to item 17, but the blank is in the second position and the answer is 10. Item 19 is also similar to item 17 but the blank is in the fourth position, and the left side of the equation involves both addition and subtraction. The correct answer for item 19 is 2. Items 13 through 19 each will be scored *correct* if the correct answers are given.

Items 20 and 21 of the questionnaire continue the equation-solving tasks of items 13 through 19 with the added complication of multidigit terms in place of the single-digit terms of previous items

> For items 20 and 21, find the number that goes in each blank. You can try to find a shortcut so that you don't have to do all the adding. Show your work and write your answer in the blank.
>
> 20) 898 + 13 = 896 + __
>
> 21) 43 + __ = 48 + 76

Students are asked in these 2 items to show their calculations as well as their answers because nonrelational methods may be used to solve this problem. Item 20 is a four-term problem similar to problem item 15, but with multidigit numbers and the blank in the fourth position. The answer to item 20 is 15. Item 21 is similar to item 16 but with the blank in the second position. The answer for item 21 is 81. Because multidigit numbers may require side calculations instead of mental calculations and may contain minor errors, Items 20 and 21 each will be scored as *correct* if the answer is within one of the correct answer and if equivalent expressions are used.

Items 22 and 23 of the questionnaire form a set of progressively difficult items related to the identity application of equality. All the identities are numerical (no literal symbols) in this set

> 22) 5 + 3 = 8 □ T □ F □ I don't know
>
> 23) 7 = 3 + 4 □ T □ F □ I don't know

Item 22 is a closed equality in canonical problem; no difficulty for students with a relational view of equality, but a possible dilemma for those with the operational view because it contains no blank. Item 23 is a symmetrical variation of the format of item 22 and will more likely confuse a student with the operational view of equality. Items 22 and 23 will each be scored *correct* if answered True.

Items 24 and 25 are assessments of a student's knowledge of the reflection property of equivalence. Item 24 is a four-term closed statement, which is more confusing to a student with an operational view of equality. Item 25 is a two-term equation, which would be clearly True for a student with a relational view, but incomprehensible to a student with an operational view. Items 24 and 25 will be individually scored *correct* if the correct answer is given.

24) $6 + 4 = 5 + 5$ □ T □ F □ I don't know

25) $3 = 3$ □ T □ F □ I don't know

Items 31 through 33 were excerpted from Küchemann (1981a) and are related in this study to the assignment application of equality to assign a value to a literal symbol

31) If $a + b = 43$

$a + b + 2 =$ ______

32) What can you say about u if

$u = v + 3$ and

$v = 1$

33) If $e + f = 8$

$e + f + g =$ _______

The group of a + b in item 31 is assigned a value of 43, making the correct answer 45. Item 32 is more difficult than item 31 because the presentation is not in progressive order, requiring the student to read the first equation, then to read the second equation, and finally to return to the first equation. The correct answer for item 32 is $u = 4$. Item 33 is more difficult than item 32 because the answer is algebraic $(8 + g)$ rather than numeric, which some students consider unfinished (a lack of closure). Küchemann (1981a) assigned item 31 to level one, item 32 to level two, and item 33 to level three. Items 31 through 33 will be scored *correct* if the correct answers are given.

Items Related to Literal Symbol

Items 26 through 30 progressively probe the student's understanding of the basic meaning of the literal symbol

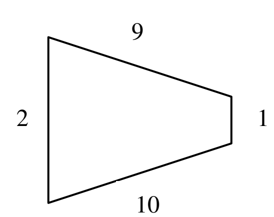
For item 26, consider the following expression 2n + 3
↑

26) The arrow points to a symbol. What does the symbol stand for?
What else might the symbol stand for?

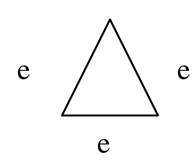
Item 26 is excerpted from Knuth, et al. (2005). Item 26 provides an opportunity for the student to describe in his or her own words the meaning of a literal as shown in a sample algebraic expression. Knuth's item is extended in this study by adding a prompt to the student for other, perhaps incorrect, but telling meanings of the literal. The item is analogous to item 10, which are the same open questions related to the equal sign. Item 26 will be scored as *correct* if any response indicates that a literal symbol represents a quantity, not an object. The item compares to Küchemann's level one (easiest) difficulty.

Items 27 through 30 continue the exploration of the student's understanding of the basic definition of a literal symbol.

27) What is the perimeter of the figure?



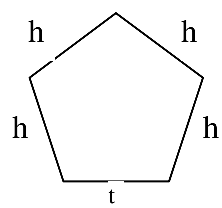
28) What is the perimeter of the figure?



Item 27 asks for the perimeter of an irregular four-sided polygon where the given lengths of the sides of the polygon are all numeric. The correct response is 22. The item is not related to the literal symbol and will not be scoreed, but can be used to filter out the students who do not understand the meaning of *perimeter,* which is needed for the following three items. Item 28 is a simple figure, but the lengths of the sides are given as literals. The correct response for item 28 is 3e, or some algebraic equivalent. Küchemann assigned items 27 and 28 to level one.

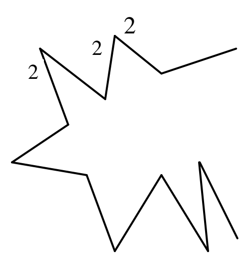
Items 29 and 30 are a set of items about the perimeter of progressively difficult polygonal figures where some or all of the lengths of the sides are given as literals.

29) What is the perimeter of the figure?



The figure in item 29 is more complex because there are two literal symbols involved. The correct answer for item 29 is 4h + t, or some algebraic equivalent. Küchemann assigned item 29 to level 2.

30) What is the perimeter of the figure?



Part of this figure is not drawn. There are n sides altogether all of length 2.

The incomplete diagram, with only a note to describe the complete diagram, contributes to the difficulty of item 30. The correct response for item 30 is 2n and item 30 was assigned a difficulty level of three. Item 27 will not be scored; items 28 through 30 will each be scored *correct* if the answer is correct.

Item 34 assesses the student's knowledge of the properties of a literal symbol. The item requires a solid conception of equality as a relation.

34) L + M + N = L + P + N
   a) Always True
   b) Sometimes True (when?)
   c) Never True

The statement is sometimes True (choice b): when M = P both sides of the equation are L + P + N. The student needs to understand that like literals have like values, but different literals may also have like values. The item is assigned to level four by Küchemann (1981a). With only one test item for this property, and a relatively difficult one at that, it is only possible to obtain a dichotomous assessment of a student's knowledge of the properties of a literal symbol. The item will be scored as *correct* if choice (b) is selected and the condition is explained correctly.

Items 35 through 37 probe the depth of a student's understanding of the literal as an unknown quantity

35) What can you say about a if

$$a + 5 = 8$$

36) Find the value of n. Explain your answer.

n + n + n + 2 = 17 ☐ I don't know

37) What can you say about r if

r = s + t, and

r + s + t = 30 ?

Item 35 is a simple example of the *unknown* application and can be solved from number facts alone. The item was assigned to level 1 by Küchemann (1981a). Item 36 is an exception to the Kuchemann set of questions; the item is excerpted from Rittle-Johnson et al. (2010). It is more complex than item 35 because of the repeated use of the variable n, which implies the same value is repeated. For scoring purposes, this item is assumed to be at level two. Item 37 is a more difficult example of the *unknown* application of a literal because three literals are used; solving the item also requires a firm relational understanding of equality. The increase of complexity comes from the reversed assignment of an algebraic value (r) to the group s + t, which can be applied to the second equation, and finally solved (r = 15). Other strategies are also possible, but are not less difficult. The item is assigned to level three by Küchemann. Scoring of items 35 through 37 will be *correct* if the correct answers are given.

Items 38 and 39 assess the student's familiarity with the application of a literal as a variable quantity. The items involve inequalities, rather than equalities, which some students may not notice. Students typically receive less instruction on inequalities and are therefore less comfortable with inequality problems

38) What can you say about c
if c + d = 10
and c is less than d?

Several strategies will lead to the correct result: c is less than 5, or some equivalent statement. Küchemann assigned item 38 to level 3. Item 38 will be coded *correct* if the correct answer is given.

39) Which is larger: 2n or n + 2 ?
Explain.

Neither 2n nor n + 2 is universally larger; which is larger depends on the value of n. Several strategies will lead to the correct answer: 2n is the larger for n greater than 2; n + 2 is the larger when n is less than 2. Item 39 was assigned a difficulty of level 4 by Küchemann because it is an item about an inequality, requiring a strong understanding of literal symbols and an understanding of equality flexible enough to accommodate inequalities. Item 39 will be scored *correct* if the range of values is correctly identified and a valid explanation is provided.

Items 40 and 41 probe the student's knowledge of the Generalized Number application of a literal symbol.

40) Write down the smallest and the largest of these:
n + 1, n + 4, n – 3, n, n – 7
Smallest _______________ Largest ________________

41) 4 added to n can be written as n + 4.
Add 4 to n + 5 _________________

In item 40, the student is asked to recognize five expressions, involving the literal n, as generalized numbers and to select the largest and smallest numbers from the set of expressions. In item 41, the student is asked again to recognize a generalized number and to perform an addition operation on it. An example of a similar operation is provided.

Items 42 and 43 form a pair of items that probes a student's proficiency with algebraic operations. The problems require a firm understanding of the meanings and applications of both literal symbols and equality. Students who are able to answer these questions may be better prepared than the others and may not belong in the sample.

For items 42 and 43, fill in the blanks
42) 2a + 5a = ___

43) 2a + 5b + a = ___

Item 42 is a relatively simple algebraic expression that requires the strategy of collecting like terms, resulting in an algebraic (nonnumeric) answer. The correct response for item 40 is 7a, which Küchemann assigned to level one. Item 43 is more sophisticated because two literal symbols are involved. The correct response to item 41 is 3a + 5b, which is also nonnumeric, and which Küchemann assigned to level two.

Appendix D

Instructions for Students and Consent Form

Instructions for Student and Consent Form

NAME ____________________

Instructions

This questionnaire is a survey of the current knowledge of specific mathematical concepts by this class. It is not a quiz graded for correctness. Your spontaneous, honest answers, whether mathematically correct or not, are the information sought. If you believe you know the mathematically correct answer, respond accordingly. However, if you do not understand the question, or if you think the question can not be answered, or if you do not know how to answer the question, answer “I don’t know”. Do not guess an answer or make any attempt to deduce an answer.

Examples

1. Fill in the blank:

8 + 5 = ____ ☐ I don’t know.

For this question, you might likely enter 13, or maybe 11, in the blank.

2. True or False

Sin 30° = 1 T F ☐ I don’t know.

For this question, you would likely check the box for “I don’t know”. Whatever answers you provide are the correct answers for this survey.

The questions will ask for a written response, or will ask you to “Fill in the blank”, or will ask you to evaluate a statement as “True or False”. All questions will also have an option to respond “I don’t know”. Because your immediate responses are the information sought, you will be allowed 30 minutes to answer all the questions. Please be sure to answer all the questions. Thank you for your cooperation in this survey.

Your individual responses will be held in strict confidence, analyzed only after the information is collected to the class, or higher level. A summary of the class results may be shared with your professor, who may use the information to supplement the course material. Ultimately, this may be to your benefit. The class and higher level summary analyses will also be reported in a doctoral dissertation.

Part of the analysis will be to compare answers between groups within the class, e.g., males vs females, older vs younger students. For this reason, some demographic information is needed. Please provide the following supportive information about yourself.

Gender: ____ Male ____ Female

Age group: ____ under 20 ____ 21 – 25 ____ 26 – 30 ____ over 30

Ethnicity: ____ Caucasian ____ Hispanic ____ Afro-American ____ Other

College seniority: ____ 1st semester ____ 2nd semester ___3rd semester

___ 4th semester ___ 5th or more semester

Consent to be a Research Subject

Purpose and Background

Mr. Terrence Maguire, a graduate student in the School of Education at the University of San Francisco is doing a study on the mathematical backgrounds of college freshmen and students recently returning to college. In particular, the researcher is interested in the depth of understanding that students entering college have of the concepts of equality and literal symbols (letters) and their applications, which is information not well known at the college level.

I am being asked to participate because I am enrolled in a freshman level class in mathematics.

Procedures

If I agree to be a participant in this study, the following will happen:

1. I will complete a short questionnaire giving basic information about me, including gender, age group, ethnicity, college seniority, and most recent studies in mathematics
2. I will complete a survey of short mathematics questions related to the meanings and applications of equality and literal symbols.

Risks and Discomforts

1. It is possible that I will not be able to answer many of the questions in the survey. This may make me feel uncomfortable, but I am free and encouraged to respond "I don't know" or to leave any question blank. I am also free to stop participating at any time.
2. Participation in research may mean a loss of confidentiality. For this study, records of individual survey results will be kept confidential as far as possible. Individual results will be entered into a database for analysis under an encoded identity. Individual identities and results will be kept only as hard copy in a locked file, accessible only to the researcher and which will be shredded upon completion of the study.
3. The time to complete the survey may be an hour and I may become bored or exhausted.

Benefits

There will be no direct benefit for me from participating in this study, nor any penalty for not participating. After the results are summarized for the class, my professor may find in the report some benefits for the entire class. The general benefit from this study may be a better understanding of the mathematical needs of college freshmen.

Cost / Financial Considerations

There will be no financial costs to me from participating in this study. The responses to the survey will be answered on the copy of the survey provided by the researcher.

Payment/Reimbursement

There will be no financial reimbursement to me for my participation in this study.

Questions

I have had an opportunity to ask Mr. Maguire questions about the survey. If I have further questions, I may email Mr. Maguire at tmaguire@Ohlone.edu for a prompt reply.

If I have any questions or comments about this research, I will first attempt to communicate with Mr. Maguire. If I am unable to make contact with him, or if I feel uncomfortable in doing so, I may contact the Institutional Review Board for the Protection of Human Subjects (IRBPHS) at the university of San Francisco, which concerned with the protection of volunteers in research projects. I may reach the IRBPHS at (415) 422-6091 t leave a voicemail message, or by emailing IRBPHS at IRBPHS@usfca.edu. I may also write to IRBPHS, Department of Psychology, University of San Francisco, 2130 Fulton Street, San Francisco, CA 94117-1080.

Consent

I have been given a signed copy of this consent form to keep.

PARTICIPATION IN RESEARCH IS VOLUNTARY. I am free to decline to be in this study, or to withdraw from it at any point. My decision as to whether or not to participate in this study will have no influence on my future status as a student or employee at USF.

My signature below indicates that I agree to participate in this study.

Subject's signature Date of signature

Signature of person obtaining consent Date of signature