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## A GENERALIZED BIRKHOFF–ROTT EQUATION FOR TWO-DIMENSIONAL ACTIVE SCALAR PROBLEMS\*

HUI SUN<sup>†</sup>, DAVID UMINSKY<sup>†</sup>, AND ANDREA L. BERTOZZI<sup>†</sup>

**Abstract.** In this paper we derive evolution equations for the two-dimensional active scalar problem when the solution is supported on one-dimensional curves. These equations are a generalization of the Birkhoff–Rott equation when vorticity is the active scalar. The formulation is Lagrangian and it is valid for nonlocal kernels  $\mathbf{K}$  that may include both a gradient and an incompressible term. We develop a numerical method for implementing the model which achieves second order convergence in space and fourth order in time. We verify the model by simulating classic active scalar problems such as the vortex sheet problem (in the case of inviscid, incompressible flow) and the collapse of delta ring solutions (in the case of pure aggregation), finding excellent agreement. We then study two examples with kernels of *mixed type*, i.e., kernels that contain both incompressible and gradient flows. The first example is a vortex density model which arises in superfluids. We analyze the effect of the added gradient component on the Kelvin–Helmholtz instability. In the second example, we examine a nonlocal biological swarming model and study the dynamics of density rings which exhibit complicated milling behavior.

**Key words.** Birkhoff–Rott equation, active scalar problems, vortex density functions, Kelvin–Helmholtz instability, biological swarming

**AMS subject classifications.** 76A25, 35B35, 37J25

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**1. Introduction.** The two-dimensional (2D) active scalar problems we study in this paper take the form

$$(1.1) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad \rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}),$$

$$(1.2) \quad \mathbf{v} = \mathbf{K} * \rho = \nabla^\perp N * \rho + \nabla G * \rho,$$

where  $*$  denotes convolution with interaction kernel  $\mathbf{K}$ ,  $(a, b)^\perp = (-b, a)$ , and  $\mathbf{x} \in \mathbb{R}^2$ . Equations (1.1)–(1.2) represent a large class of problems which includes fluids [26, 34, 12], chemotaxis [6], aggregation [3, 2, 19], biological swarming [36], and many others. Our work is focused on two facets of the 2D active scalar problem. First, we consider solutions supported on one-dimensional (1D) curves; second, we look at the case when the velocity field contains both an incompressible and a gradient component. To better characterize the second motivation we assume sufficient smoothness and decay on  $\mathbf{K}$  and apply the Hodge decomposition theorem [26] to the velocity field equation as expressed on the right-hand side of (1.2). As such, the  $\nabla^\perp N * \rho$  term represents the divergence-free or incompressible component of the velocity field and the  $\nabla G * \rho$  is the gradient contribution to the velocity field. We refer to kernels  $\mathbf{K}$  that contain both a divergence-free and a gradient part as kernels of *mixed type*.

Examples of mixed type kernels have been found in vortex models for superfluids, such as superconductors, where the active scalar is a “vortex density” function [8, 14,

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25]. Classical 2D compressible fluids is another example in which  $\mathbf{K}$  will have both an incompressible Newtonian potential and a compressible contribution (the dilatation) [13]. Mixed type kernels also arise in the work of Topaz and Bertozzi [36] in the context of kinematic social interaction models. They considered collective swarming motion kernels that contain both gradient and incompressible terms to better capture biological phenomena such as mill vortices and other rotating structures found in nature [17, 32, 30, 28].

There are many important examples when the kernel  $\mathbf{K}$  creates a purely incompressible velocity field, i.e.,  $G = 0$ . Such examples include quasi-geostrophic models [10, 11, 9] for which  $N = (-\Delta)^{-\alpha}$  and, perhaps more famously, the vorticity formulation of the classical 2D Euler equations for which  $N = \frac{1}{2\pi} \log |r|$  is the Biot–Savart kernel. Vorticity (the curl of the velocity field) is the active scalar in this second example, and the well-studied problem known as the vortex sheet problem [15, 16, 23, 31] is the particular case when vorticity aggregates onto a 1D curve. It was first shown in [5] that the vortex sheet evolves according to the Birkhoff–Rott equation

$$(1.3) \quad \frac{\partial \bar{z}(\Gamma, t)}{\partial t} = \frac{1}{2\pi i} PV \int_S \frac{d\Gamma'}{z(\Gamma, t) - z(\Gamma', t)},$$

where  $\Gamma$  is the local Lagrangian parameterization of the sheet  $S$  and  $z$  is the position of the points on the sheet. The first goal of this paper is to derive an analogous equation to (1.3) for (1.1)–(1.2) where the kernel is general.

Examples of singular “sheetlike” solutions that occur for purely gradient kernels, i.e.,  $N = 0$ , have been observed in chemotaxis models [17, 37] and purely aggregation kernels of the form  $\nabla G(r) = \nabla |r|^\alpha$ . Specifically, the analysis and dynamics of delta ring solutions has been studied in the works of Huang, Bertozzi, Laurent, and others [19, 2, 3]. It has been shown that for  $\alpha > 2$  smooth initial density may collapse to delta ring solutions in infinite time. For dimension two, finite time blow-up “ring solutions” occur if  $1 < \alpha < 2$ . In this regime concentric density rings formally solve (1.1)–(1.2), and differential equations governing the radius of the rings can be computed; see [2]. Moreover, if  $\alpha > 2$ , Huang and Bertozzi [20] have shown numerically that collapsing rings are attractors of radially symmetric solutions to (1.1)–(1.2).

Ring and curve solutions have also been discovered in biologically relevant attraction repulsion kernels such as  $G(r) = \tanh((1-r)a + b)$  and  $G(r) = r^p - r^q$ ; see [21]. There, it was shown that certain parameter regimes for the above kernel have yielded linearly stable rings. In addition, low mode instabilities may develop—depending on parameters—where small perturbations of rings cause the ring to deform into a triangular-like closed curve. In addition, necessary and sufficient conditions were computed when a 1D curve is linearly well-posed.

In section 2 we derive a Lagrangian model using conservation of mass, which is a generalization of the Birkhoff–Rott equation for vortex sheets and fully incorporates both incompressible and gradient velocity fields. In section 3 we define a numerical scheme for implementation and numerically show that it is second order accurate in space and fourth order accurate in time. In this section we also verify our model by simulating two well-studied problems: classic vortex sheet solutions and a purely aggregation model. In both the vortex sheet examples and the evolution of delta rings (in the case of the aggregation model) we obtain excellent agreement using our new model. In section 4 we first study a superfluids example which contains a kernel of mixed type. We observe that by varying the weight of the incompressible versus gradient contribution to  $\mathbf{K}$ , the maximal amount of vortex wind-up occurs when both

pieces of the kernel are present (as opposed to just incompressible contributions). We explore this phenomenon by linearizing our solution and observe the effect the gradient term has on the associated Kelvin–Helmholtz instability. In our second example in section 4, we consider a swarming model and observe that (depending on parameters) the rotational contribution ( $N$ ) to the kernel does not affect the contraction rate of the curve but—somewhat surprisingly—the attractive, gradient contribution ( $G$ ) to the kernel can have a very significant effect on the rotation dynamics of the curve. We conclude in section 5 with a discussion of directions and applications of this model to future research.

**2. Derivation of density sheet equations.** In this section we derive our model for the general active scalar problem for the case when all the mass concentrates on a 1D curve. We start by recalling the elementary derivation that (1.1) is equivalent to conservation of mass. To see this, consider a small box  $\square$  in the spatial domain as shown in Figure 2.1. Between time  $t$  and  $t + \Delta t$ , the amount of mass that flows out through the boundary of the box is approximately  $\int_{\partial\square} \rho \mathbf{v} \cdot \mathbf{n} ds \Delta t$ , while the increase of the mass in the box is  $\int_{\square} \rho d\mathbf{x}(t + \Delta t) - \int_{\square} \rho d\mathbf{x}(t)$ . By conservation of mass, we arrive at the following equality:

$$(2.1) \quad \int_{\square} \rho d\mathbf{x}(t + \Delta t) - \int_{\square} \rho d\mathbf{x}(t) = \int_{\partial\square} \rho \mathbf{v} \cdot \mathbf{n} ds \Delta t.$$

Dividing both sides by  $\Delta t$  and applying the divergence theorem, we get  $\int_{\square} \frac{\partial \rho}{\partial t} d\mathbf{x} = \int_{\square} \nabla \cdot (\rho \mathbf{v}) d\mathbf{x}$ , and (1.1) follows.

For the case when all of the active scalar lies on a 1D curve—hereafter referred to as “the density sheet problem”—conservation of mass is used to derive the equation of motion for the density sheet. To begin, let  $\Gamma^n$  be the sheet at time  $n\Delta t$  and  $\Gamma^{n+1}$  be the sheet at time  $(n + 1)\Delta t$ . We now parameterize our sheet  $\Gamma$  using an arbitrary Lagrangian parameter  $\alpha$ ; thus, we may write the position of the sheet as  $\mathbf{z}(\alpha, t) = (x(\alpha, t), y(\alpha, t)) \in \mathbb{R}^2$ . Now choose  $\alpha_{i-1}$ ,  $\alpha_i$ , and  $\alpha_{i+1}$  to be three reference points on the density sheet; see Figure 2.2.

The mass between  $\alpha_{i-1}$  and  $\alpha_{i+1}$  at time  $n\Delta t$  is exactly  $\int_{\alpha_{i-1}}^{\alpha_{i+1}} \rho |\mathbf{z}_{\alpha}|(n\Delta t) d\alpha$ , while the mass between  $\alpha_{i-1}$  and  $\alpha_{i+1}$  at time  $(n + 1)\Delta t$  is  $\int_{\alpha_{i-1}}^{\alpha_{i+1}} \rho |\mathbf{z}_{\alpha}|((n + 1)\Delta t) d\alpha$ , where  $\mathbf{z}_{\alpha}$  denotes the derivative of  $\mathbf{z}$  with respect to  $\alpha$ . Since the reference points also move with the velocity field  $\mathbf{v}$ , i.e.,

$$(2.2) \quad \frac{\partial \mathbf{z}}{\partial t} = \mathbf{K} * \rho,$$

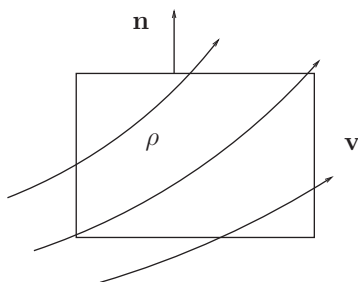


FIG. 2.1. Schematic diagram of a box ( $\square$ ) in a vector field.

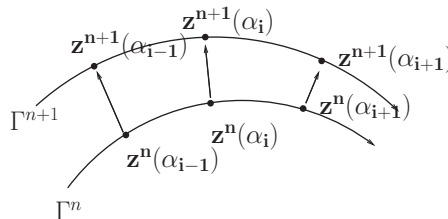


FIG. 2.2. The evolution of the sheet from time  $n\Delta t$  to  $(n + 1)\Delta t$ .

the mass is conserved along trajectories for all time. This implies that

$$(2.3) \quad \int_{\alpha_{i-1}}^{\alpha_{i+1}} \rho|\mathbf{z}_\alpha|(n\Delta t)d\alpha - \int_{\alpha_{i-1}}^{\alpha_{i+1}} \rho|\mathbf{z}_\alpha|((n+1)\Delta t)d\alpha = 0.$$

Dividing (2.3) by  $\Delta t$  and letting  $\Delta t \rightarrow 0$ , we arrive at the equation

$$(2.4) \quad \frac{\partial}{\partial t}(\rho|\mathbf{z}_\alpha|) = 0$$

for the evolution of the density  $\rho$ , which may be written out as

$$(2.5) \quad \frac{\partial \rho}{\partial t} + \rho \frac{\langle \mathbf{z}_\alpha, \mathbf{v}_\alpha \rangle}{\langle \mathbf{z}_\alpha, \mathbf{z}_\alpha \rangle} = 0,$$

where  $\langle \cdot, \cdot \rangle$  represents the usual inner product. Equations (2.5) and (2.2), or equivalently (2.4) and (2.2), yield a closed system of equations for the motion of the density sheet problem.

*Remark 2.1.* We note here that (2.4) and (2.2) represent a generalization of the Birkhoff–Rott equation (1.3). In particular, if one chooses  $N$  to be the Newtonian potential and  $G = 0$ , (2.4) represents a parameterization with respect to circulation, i.e.,  $\rho = -d\Gamma/|dz|$  with  $\alpha = \Gamma$ . It follows then that (2.2) identically reduces to (1.3). Of course (2.4) and (2.2) represent a generalization of the Birkhoff–Rott equation because the combined kernel  $\mathbf{K}$  can take a more general form. With a generic choice of  $\mathbf{K}$ , (2.4) no longer can be interpreted as a parameterization with respect to circulation and either (2.4) or (2.5) may be used. In section 3.2.1, we verify our model by simulating several classical vortex sheet examples. To distinguish our simulations from the direct use of the Birkhoff–Rott equation we use the formulation (2.5) and (2.2), which yields excellent agreement with (1.3) but, in general, either sets of equations can be used.

**3. Numerical method.** We implement (2.5) and (2.2) using a fourth order Runge–Kutta method in time and centered difference discretization in space. In addition, we apply an adaptive mesh method using cubic interpolation for several of the more complicated examples in sections 3.2.1, where more resolution is required. We briefly present this algorithm below.

Let  $\mathbf{z} = (\mathbf{x}, \mathbf{y})$  be the position of the sheet,  $\mathbf{v} = (u, w)$  be the associated velocity field, and  $\rho$  be the density. We denote the associated discretized vectors  $\mathbf{z}_i = (x_i, y_i)$ ,  $\mathbf{v}_i = (u_i, w_i)$  and  $\rho_i$  indexed by  $i$ . Throughout this section,  $N$  denotes the number of space discretization and  $M$  denotes the number of time steps. We then have the following discretized equations for (2.2) and (2.5):

$$(3.1) \quad \mathbf{v}_j = \frac{\partial \mathbf{z}_j}{\partial t} = \sum_i \mathbf{K}(\mathbf{z}_j - \mathbf{z}_i) \cdot \rho(\mathbf{z}_i) |\Delta \mathbf{z}_i|,$$

$$(3.2) \quad \frac{\partial \rho_j}{\partial t} = -\rho_j \frac{(x_{j+1} - x_{j-1})(u_{j+1} - u_{j-1}) + (y_{j+1} - y_{j-1})(w_{j+1} - w_{j-1})}{(x_{j+1} - x_{j-1})^2 + (y_{j+1} - y_{j-1})^2},$$

where

$$(3.3) \quad |\Delta \mathbf{z}_j| = \frac{\sqrt{(x_{j+1} - x_j)^2 + (y_{j+1} - y_j)^2} + \sqrt{(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2}}{2}.$$

Furthermore, let  $\mathbf{F}_1, \mathbf{F}_2$  be vectors defined with the following elements:

$$(3.4) \quad \mathbf{F}_{1,j}(\boldsymbol{\rho}, \mathbf{z}) = \frac{\partial \mathbf{z}_j}{\partial t} = \sum_i \mathbf{K}(\mathbf{z}_j - \mathbf{z}_i) \cdot \rho_i |\Delta \mathbf{z}_i|,$$

$$(3.5) \quad F_{2,j}(\boldsymbol{\rho}, \mathbf{z}, \mathbf{v}) = -\rho_j \frac{(x_{j+1} - x_{j-1})(u_{j+1} - u_{j-1}) + (y_{j+1} - y_{j-1})(w_{j+1} - w_{j-1})}{(x_{j+1} - x_{j-1})^2 + (y_{j+1} - y_{j-1})^2}.$$

Then our Runge–Kutta algorithm for one time step is described below:

- (a)  $\mathbf{V}_1^n = \mathbf{F}_1(\boldsymbol{\rho}^n, \mathbf{z}^n), \mathbf{P}_1^n = \mathbf{F}_2(\boldsymbol{\rho}^n, \mathbf{z}^n, \mathbf{V}_1^n).$
- (b)  $\mathbf{V}_2^n = \mathbf{F}_1(\boldsymbol{\rho}^n + \Delta t \mathbf{P}_1^n / 2, \mathbf{z}^n + \Delta t \mathbf{V}_1^n / 2), \mathbf{P}_2^n = \mathbf{F}_2(\boldsymbol{\rho}^n + \Delta t \mathbf{P}_1^n / 2, \mathbf{z}^n + \Delta t \mathbf{V}_1^n / 2, \mathbf{V}_2^n).$
- (c)  $\mathbf{V}_3^n = \mathbf{F}_1(\boldsymbol{\rho}^n + \Delta t \mathbf{P}_2^n / 2, \mathbf{z}^n + \Delta t \mathbf{V}_2^n / 2), \mathbf{P}_3^n = \mathbf{F}_2(\boldsymbol{\rho}^n + \Delta t \mathbf{P}_2^n / 2, \mathbf{z}^n + \Delta t \mathbf{V}_2^n / 2, \mathbf{V}_3^n).$
- (d)  $\mathbf{V}_4^n = \mathbf{F}_1(\boldsymbol{\rho}^n + \Delta t \mathbf{P}_3^n, \mathbf{z}^n + \Delta t \mathbf{V}_3^n), \quad \mathbf{P}_4^n = \mathbf{F}_2(\boldsymbol{\rho}^n + \Delta t \mathbf{P}_3^n, \mathbf{z}^n + \Delta t \mathbf{V}_3^n, \mathbf{V}_4^n).$
- (e)  $\boldsymbol{\rho}^{n+1} = (\mathbf{P}_1^n + 2\mathbf{P}_2^n + 2\mathbf{P}_3^n + \mathbf{P}_4^n) / 6, \quad \mathbf{z}^{n+1} = (\mathbf{V}_1^n + 2\mathbf{V}_2^n + 2\mathbf{V}_3^n + \mathbf{V}_4^n) / 6.$

After each Runge–Kutta step, we update the tolerance in our adaptive mesh by first setting  $\epsilon = \min(\text{total length of curve}/N, \epsilon)$ . We then consider the distance between consecutive points, and if the distance is greater than  $\epsilon$  we add one point between them using cubic interpolation. In the classical vortex sheet examples (section 3.2.1) and superfluid calculations (section 4.1), the kernel  $\mathbf{K}$  is singular; hence we use Krasny’s desingularization technique, described in detail in section 3.2.1.

**3.1. Convergence study.** To verify the convergence of our method, we use the periodic perturbation example in section 3.2.1. Since the exact solution to this example is unknown, we derive the order of convergence by computing successive differences between numerical solutions. We then double the number of points (in time or in space, respectively) and then apply (3.6) to estimate the convergence rate.

For the convergence in time, let  $(\mathbf{z}_1, \boldsymbol{\rho}_1), (\mathbf{z}_2, \boldsymbol{\rho}_2), (\mathbf{z}_3, \boldsymbol{\rho}_3),$  and  $(\mathbf{z}_4, \boldsymbol{\rho}_4)$  be used to denote the numerical solution for time discretization  $M = 10, 20, 40, 80,$  respectively, at  $T = 0.1,$  with space discretization  $N = 100.$  Then the approximate convergence rate can be calculated as

$$(3.6) \quad \text{Conv. rate} \approx \log(\|\mathbf{e}_i\|_2 / \|\mathbf{e}_{i+1}\|_2) / \log 2,$$

where  $\mathbf{e}_i$  can be taken as vectors  $\mathbf{z}_i - \mathbf{z}_{i+1}$  or  $\boldsymbol{\rho}_i - \boldsymbol{\rho}_{i+1}.$  Notice that subscripts  $i, i + 1$  refer to consecutive refinements in either space or time.

For the convergence in space, we use the same notation with capital letters,  $(\mathbf{Z}_1, \mathbf{P}_1), (\mathbf{Z}_2, \mathbf{P}_2), (\mathbf{Z}_3, \mathbf{P}_3),$  and  $(\mathbf{Z}_4, \mathbf{P}_4),$  to denote the solution for space discretization  $N = 100, 200, 400, 800,$  respectively, at time  $T = 0.1,$  with  $M = 100.$  We use formula (3.6) to compute the approximate convergence rate as before, except that  $\mathbf{e}_i$  is taken to be vectors  $\mathbf{Z}_i - \mathbf{Z}_{i+1}$  or  $\mathbf{P}_i - \mathbf{P}_{i+1}.$  We also compute the convergence in space with the effect of cubic interpolation by starting with the same parameter setting and successively halving the adaptive tolerance  $\epsilon.$  We obtain solutions  $(\mathbf{Z}_1, \mathbf{P}_1), (\mathbf{Z}_2, \mathbf{P}_2), (\mathbf{Z}_3, \mathbf{P}_3),$  and  $(\mathbf{Z}_4, \mathbf{P}_4)$  and then use formula (3.6) to compute the approximate convergence rate. From Table 3.1, we can see that the convergence is approximately fourth order in time and second order in space (as expected). We also see that the convergence for the cubic interpolation adaptive method is approximately second order.

*Remark 3.1.* From the onset we designed a numerical scheme for this general curve evolution problem which is fourth order in time and second order in space. For comparison to the simulation of the classical vortex sheet problem many numerical schemes have been developed which have varying convergence orders [7, 22, 23, 24, 29], including spectrally accurate convergent schemes [33, 18].

TABLE 3.1  
Convergence rate in time and space.

Convergence in time				
M	$\ \mathbf{z}_i - \mathbf{z}_{i+1}\ _2$	conv. rate	$\ \boldsymbol{\rho}_i - \boldsymbol{\rho}_{i+1}\ _2$	conv. rate
10				
20	6.1748e-07		7.7979e-06	
40	4.6074e-08	3.7444	5.7844e-07	3.7528
80	3.1409e-09	3.8747	3.9661e-08	3.8664
Convergence in space				
N	$\ \mathbf{Z}_i - \mathbf{Z}_{i+1}\ _2$	conv. rate	$\ \mathbf{P}_i - \mathbf{P}_{i+1}\ _2$	conv. rate
100				
200	3.6915e-05		1.2083e-03	
400	4.7733e-06	2.9511	2.7878e-04	2.1158
800	1.1654e-06	2.0342	6.9860e-05	1.9966
Convergence rate for cubic interpolation				
$\epsilon$	$\ \mathbf{Z}_i - \mathbf{Z}_{i+1}\ _2$	conv. rate	$\ \mathbf{P}_i - \mathbf{P}_{i+1}\ _2$	conv. rate
0.06				
0.03	4.7311e-06		2.7239e-04	
0.015	1.1345e-06	2.0601	6.5691e-05	2.0519
0.0075	2.8138e-07	2.0115	1.6136e-05	2.0255

**3.2. Verification of method.** Here we test the new algorithm on well-known examples. First, in section 3.2.1 we recompute well-studied solutions of vortex sheets in the literature using the new code and show that the resulting solutions are in excellent agreement to previously published results. Second, in section 3.2.2 we compute concentric collapsing ring solutions in the purely aggregating case and verify good agreement as compared to the known special solutions of the associated ODE theory.

**3.2.1. Case 1: Incompressible vortex sheet examples.** In this section we verify our model by implementing our method to simulate three vortex sheet problems for the 2D Euler equations. This corresponds to setting  $N = \frac{1}{2\pi} \log |r|$  and  $G = 0$  in (1.1)–(1.2). As mentioned, the motion of the vortex sheet is governed by the Birkhoff–Rott equation (1.3), and it is well known that (1.3) is ill-posed due to the Kelvin–Helmholtz instability; see [26, 35]. Thus, in order to implement our model to simulate (1.1)–(1.2) we must desingularize the kernel. Several approaches have been developed to compute the evolution of vortex sheets [1] which address the Kelvin–Helmholtz instability. For our method we use Krasny’s [22] direct desingularization of the kernel  $N$ ,

$$(3.7) \quad \nabla^\perp N_\delta = \frac{(\mathbf{z}(\Gamma, t) - \mathbf{z}(\Gamma', t))^\perp}{|\mathbf{z}(\Gamma, t) - \mathbf{z}(\Gamma', t)|^2 + \delta^2},$$

where  $\delta$  is a regularization parameter, to compute the examples in this section.

Our first verification simulates the classical elliptically loaded wing example [23]. The initial Lagrangian parameterization for the elliptically loaded wing is  $(x, y) = (2\alpha - 1, 0)$ , where  $\alpha \in [0, 1]$ . The initial distribution of vorticity  $\rho$  is set by  $\rho = -d\Gamma/dx$ , where  $\Gamma = \sqrt{1 - x^2}$  is the circulation of the vortex sheet, as depicted in Figure 3.1.

Using our adaptive point method with error tolerance  $\epsilon = 0.075$ , we initialize the sheet using 401 points, and at  $T = 4$  the final number of points is 3171. The results are plotted in Figure 3.2 and the observed roll-up is in excellent agreement with Figure 2 in [23].

For our second example we apply our model to simulate the more complicated fuselage flap configuration which was considered in [23]. The initial conditions are

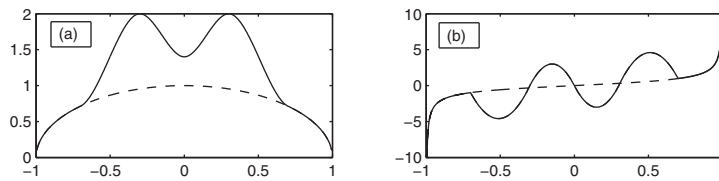


FIG. 3.1. The initial condition for the elliptically loaded example (dashed line) and the simulated fuselage flap configuration example (solid line). Figure (a) is a plot of the initial circulation against  $\alpha$ , and Figure (b) is a plot of the initial density  $\rho$  against  $\alpha$ .

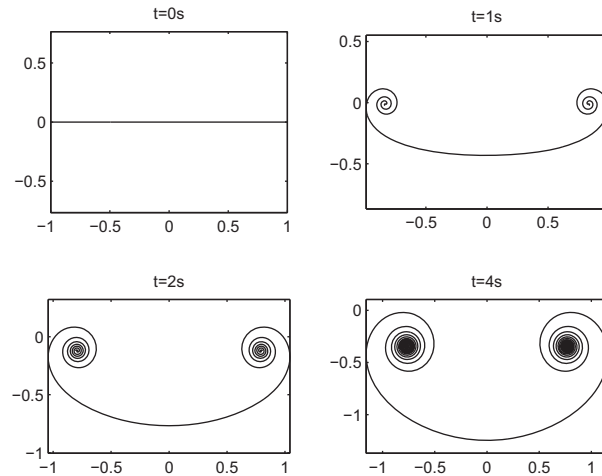


FIG. 3.2. The numerical solution at  $t = 0, 1, 2, 4$  for the elliptically loaded wing example using (2.4) and (2.5). We take  $\delta = 0.05$ ,  $\Delta t = 0.01$ , and we use adaptive mesh refinement.

chosen to simulate the vorticity generated from a fuselage flap and thus our initial  $\rho$  is chosen to be

$$(3.8) \quad \rho(\alpha, 0) = \begin{cases} x/(1-x^2), & x \in [-1, -0.7] \cup [0.7, 1], \\ -3a_3x^2 - 2a_2x - a_1, & x \in [-0.7, -0.3], \\ -3b_3x^2 - 2b_2x - b_1, & x \in [-0.3, 0], \\ 3b_3x^2 - 2b_2x + b_1, & x \in [0, -0.3], \\ 3a_3x^2 - 2a_2x + a_1, & x \in [0.3, 0.7], \end{cases}$$

where  $a_i$  and  $b_i$  are chosen to ensure continuity.

The initial distribution of both  $\rho$  and  $\Gamma$  are plotted in Figure 3.1. We once again initialize our sheet using 401 points and at  $T = 4$  the number of nodes is much higher (10,151) due to the increased stretching and roll-up as compared to the elliptically loaded wing example. The results are plotted in Figure 3.3 and we once again get excellent agreement with Figure 19 in [23].

In our last example we consider periodic perturbations to a uniformly distributed ring of vorticity with  $\rho = 1$ . This example plays an important role in our later studies of both the superfluids and biological examples found in the mixed kernels section (section 4), so we first present simulations in the purely incompressible case. We focus our attention on perturbations of radially symmetric ring distributions in general because they seem to naturally arise as important solutions in several different contexts [21, 2]. The spatially periodic perturbation is chosen to be cosine in the



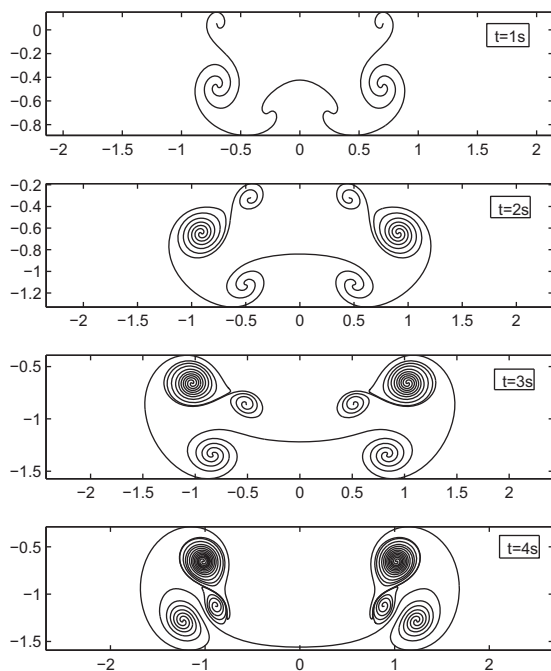


FIG. 3.3. The numerical solution for the simulated fuselage flap configuration example using (2.4) and (2.5). We take  $\delta = 0.1$ ,  $\Delta t = 0.01$ , and we use adaptive mesh refinement.

normal direction with 10 periods and the magnitude being 1% of the radius. In this example we set the radius to 1 and thus our initial conditions are

$$(3.9) \quad r(\alpha) = 1 + 0.01 \cos(20\pi\alpha), \quad \rho(\alpha) = 1,$$

$$(3.10) \quad (x(\alpha), y(\alpha)) = (r(\alpha) \cos(2\pi\alpha), r(\alpha) \sin(2\pi\alpha)).$$

We initialize with 400 points and at  $T = 4$  the number of nodes has grown to 9670. Figure 3.4 demonstrates several stages of periodic roll-up of the ring of vorticity.

**3.2.2. Case 2: Pure aggregation.** We now turn our attention to a verification of our model when the flow is governed by gradient dynamics, i.e.,  $N = 0$ . For this example we focus on a model exhibiting only aggregation, specifically taking the kernel  $\mathbf{K} = \nabla G$ , where

$$(3.11) \quad \nabla G = \nabla \sqrt{x^2 + y^2}.$$

The active scalar equations with this kernel are well studied [4, 2, 19, 20]. It was shown in [2] that because the kernel (3.11) does not satisfy the Osgood condition, finite time blow-up of radially symmetric solutions occurs. In particular, we consider the family of exact solutions of concentric delta rings studied in [2].

To begin, we consider concentric circles (about the origin) with radius  $r_1, r_2, \dots, r_n$ , and positive initial densities  $\rho_1, \rho_2, \dots, \rho_n$  uniformly distributed over each circle. Because kernel (3.11) is purely attractive and the density is all positive, the predicted behavior is that the rings will contract to the origin under the flow of (1.1)–(1.2). In fact, it was shown in [2] that the radii satisfy the simple ODEs

$$(3.12) \quad \frac{dr_i}{dt} = - \sum_{j=0}^n 2\pi r_j \rho_j \psi(r_i, r_j),$$

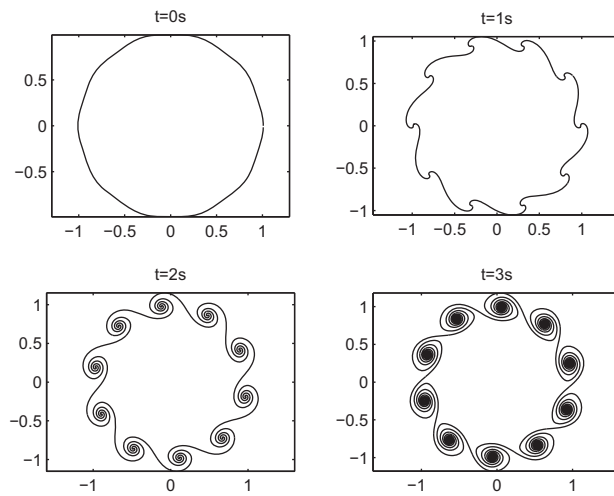


FIG. 3.4. The numerical solution for the periodic perturbed ring example using (1.2) and (2.5). We take  $\delta = 0.05$ ,  $\Delta t = 0.01$ , and we use adaptive mesh refinement.

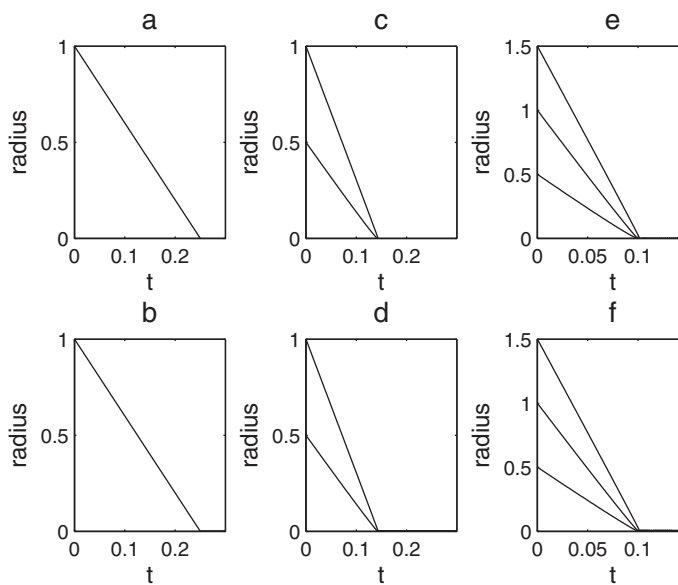


FIG. 3.5. The comparison of the numerical solution of the radius of rings. (a), (c), and (e) are the plot of the radius using (3.12) and (3.13); (b), (d), and (f) are the plot of the radius computed using (1.2) and (2.5). (a) and (b) are the solutions for the one-ring case; (c) and (d) are the solutions for the two-rings case; (e) and (f) are the solutions for the three-rings case.

where

$$(3.13) \quad \psi(r, \tau) = \frac{1}{\pi} \int_0^\pi \frac{r - \tau \cos \theta}{\sqrt{r^2 + \tau^2 - 2r\tau \cos \theta}} d\theta.$$

Thus, to test our method in purely gradient dynamics, we separately simulate our model (2.4) and (2.2) using the kernel (3.11) and then directly solve (3.12)–(3.13). We plot the results in Figure 3.5. Figures 3.5(a), 3.5(c), and 3.5(e) are the plot of

TABLE 3.2  
*Ring collapsing time prediction.*

Initial radius	Ring collapsing time					
	One-ring case		Two-rings case		Three-rings case	
Method	ODE	Ours	ODE	Ours	ODE	Ours
0.5			0.143	0.144	0.100	0.101
1	0.251	0.251	0.145	0.145	0.101	0.101
1.5					0.103	0.103

the radius by directly solving (3.12) and (3.13); Figures 3.5(b), 3.5(d), and 3.5(f) are the plot of the radius computed using (1.2) and (2.5). In each example, all rings have initial density  $\rho = 1$ . Table 3.2 shows the blow-up times for each case, and the agreement between our method and the solutions to the ODEs is excellent.

**4. Kernels of mixed type.**

**4.1. Example 1: Superfluids.** We now turn our attention to examples where the kernels are of mixed type. In this section, we consider a family of equations parameterized by  $\theta$  that arises in the modeling of vortex dynamics for superfluids described in [12]. This family of equations takes the form

$$(4.1) \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\mathbf{u}\rho) = 0, (t, x) \in (0, \infty) \times \mathbb{R}^2,$$

$$(4.2) \quad \mathbf{u} = M\nabla\Delta^{-1}\rho, \quad \rho|_{t=0} = \rho_0,$$

where  $\rho$  is known as a vortex density function of the superfluid and  $M(\theta)$  is a constant orthogonal matrix of the form

$$M(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

This model is derived from the hydrodynamic equations for Ginzburg–Landau vortices [14]. In [12, 27] the authors found that when  $\cos \theta = 0$ , smooth solutions to (4.1) and (4.2) may blow up in finite time. In addition, if  $\rho_0$  changes sign, it was shown that concentration phenomena exist in the approximate solutions sequence of (4.1) and (4.2) regardless of the initial data’s degree of regularity. Thus it is interesting to study the vortex sheet problem for (4.1) and (4.2), which is simply a generalization of the classic vortex sheet problem studied in section 3.2.1.

To match our notation, we may write  $\Delta^{-1}\rho = G * \rho$ , where  $G(r) = -\frac{1}{2\pi} \ln r$  is the fundamental solution of the Laplace equation in  $\mathbb{R}^2$ . Notice that the parameter  $\theta$  controls the contribution to the kernel given by both the incompressible and gradient parts, which can be explicitly written out as

$$(4.3) \quad \mathbf{K}(x, y) = \mathbf{K}_1(x, y) \cos \theta - \mathbf{K}_2(x, y) \sin \theta,$$

where  $\mathbf{K}_1(x, y) = -(x, y)/(2\pi r^2)$  and  $\mathbf{K}_2(x, y) = (-y, x)/(2\pi r^2)$ . Thus if we take  $\theta = 0$ ,  $M(\theta)\nabla\Delta^{-1}\rho = \mathbf{K}_1 * \rho$  is purely a gradient flow of the Newtonian potential. However, if we take  $\theta = -\pi/2$ ,  $M(\theta)\nabla\Delta^{-1}\rho = \mathbf{K}_2(x, y) = (-y, x)/(2\pi r^2)$ , we recover the Biot–Savart kernel exactly. For the purpose of our study we consider values of  $0 \leq \theta \leq -\pi/2$  which yield kernels of mixed type (except, of course, the end points). This regime of parameter values ensures the correct sign of the gradient part of the kernel  $\mathbf{K}_1(x, y)$  is attractive.

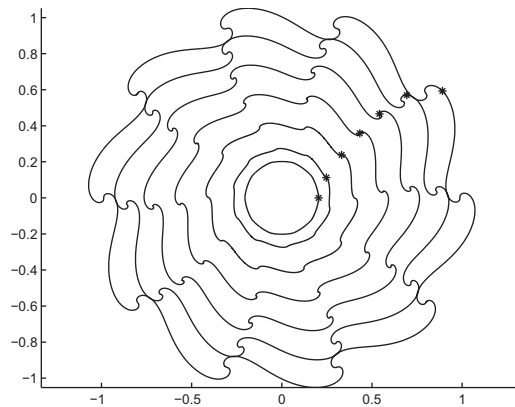


FIG. 4.1. Plot of the vortex density sheet for several values of  $\theta$  with initial conditions at  $t = 1$ . From outside to inside  $\theta = -\pi/2, -5\pi/12, -\pi/3, -\pi/4, -\pi/6, -\pi/12$ , and  $0$ . The asterisks represent the point that was initially positioned at  $(1, 0)$ .

We are specifically interested in using our model to better understand the dynamics of vortex density sheets as we vary the parameter  $\theta$ . From our discussion above it is clear that as  $\theta$  increases from  $\theta = -\pi/2$  to  $\theta = 0$  the amount of contribution to our kernel  $\mathbf{K}$  from the gradient component (attraction) increases while simultaneously the amount of incompressible component (rotation) decreases. What is surprising, though, is that linearly increasing  $\theta$  has several nonlinear effects on the curve dynamics.

To begin, we use our model to solve for the curve solutions by simply replacing (4.1) with (2.5). Since  $\mathbf{K}_1$  and  $\mathbf{K}_2$  are singular, we use Krasny's desingularization method for both kernels  $\mathbf{K}_{1\epsilon} = -(x, y)/(2\pi(r^2 + \epsilon^2))$  and  $\mathbf{K}_{2\epsilon} = (-y, x)/(2\pi(r^2 + \epsilon^2))$  with  $\epsilon = 0.1$ . We take perturbations of a ring of vorticity as our first example with the following initial conditions:

$$(4.4) \quad (x(\alpha), y(\alpha)) = (r(\alpha) \cos(2\pi\alpha), r(\alpha) \sin(2\pi\alpha)), \quad \rho(\alpha) = 1,$$

where  $r(\alpha) = (1 + 0.01 \cos(20\pi\alpha))$ . We solve (2.5) and (4.2) with initial conditions (4.4) for  $\theta = -\pi/2, -5\pi/12, -\pi/3, -\pi/4, -\pi/6, -\pi/12$ , and  $0$ , plotting in Figure 4.1 the position of the sheet at  $t = 1$ .

If we record the angular coordinates of the asterisks in Figure 4.1 to the horizontal axis we can use this to measure the amount of angular rotation of the ring. The innermost curve corresponds to  $\theta = 0$ , which is the pure gradient case for the kernel, and the curve clearly exhibits no rotation. The outermost curve corresponds to  $\theta = -\pi/2$ , which is the purely incompressible case for the kernel; we measure the rotation angle to be approximately  $0.187\pi$ . One may expect that as we move from the outermost to the innermost curve (increasing  $\theta$  by  $\pi/12$  between any of the two consecutive curves) we should observe a monotonic decrease in rotation angle. Instead, Figure 4.1 shows that the amount of rotation actually increases initially (and peaks near  $\theta = -\pi/3$ ), before eventually decreasing to zero.

We separately plot this rotation angle at  $t = 1$  as a function of  $\theta$  for both the perturbed ring (4.4) and an unperturbed ring in Figure 4.2, seeing that in both cases a maximum occurs on the interior of this range of  $\theta$ . The maximum angle for the perturbed case is  $0.7123$ , attained at  $\theta \approx -14/36\pi$ ; while the maximum angle for the unperturbed case is  $0.5835$ , attained at  $\theta \approx -11/36\pi$ . In general, the

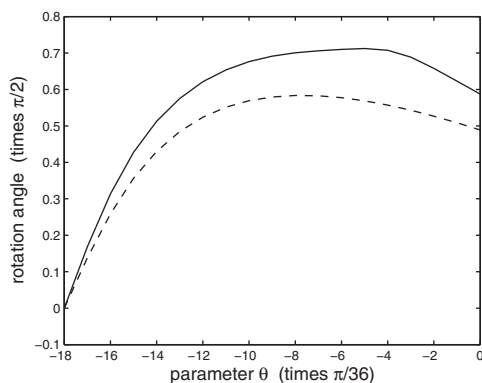


FIG. 4.2. Plot of the rotation angles at  $t = 1$  with respect to parameter  $\theta$ . The solid curve corresponds to the initial condition of a perturbed ring. The dashed curve corresponds to an initial condition of an unperturbed ring.

value of  $\theta$  for which the maximum angle of rotation occurs is time-dependent but for  $t \gg 0$  we observe that a maximum is always found in the interior of  $(-\pi/2, 0)$ . For  $t$  sufficiently small, the maximum angle occurs at the parameter  $\theta = -\pi/2$ , corresponding to a purely incompressible kernel. Hence, the incompressible kernel dominates the initial rotation dynamics but for slightly longer times the aggregation term plays an important role.

The second aspect of the curve dynamics we would like to study as we vary  $\theta$  is the amount of roll-up that occurs as a result of the perturbation to the ring. We are also interested in the amplification in time of the perturbation as measured from the unperturbed ring as we vary  $\theta$ . To study these aspects we selected  $\theta = -\pi/2, -5\pi/12, -\pi/3$ , and  $-\pi/4$  and plotted the position of the curve at the later time  $t = 1.5$  in Figure 4.3. Noting the initial position (marked by an asterisk), it becomes clear that the solutions with  $\theta = -\pi/3$  and  $-\pi/4$  rotate more than  $\theta = -\pi/2$ . In addition, we can see in Figure 4.3 that the amplitude of the perturbation also decreases as  $\theta$  decreases from  $\theta = -\pi/2$  to  $\theta = -\pi/4$ . The amount of roll-up appears to decrease, but unfortunately it is difficult to see in Figure 4.3 due to the smaller amplitude. To better investigate this phenomenon we focus in Figure 4.4 on one of the roll-ups shown in Figure 4.3(d). Figure 4.4 shows that there are many roll-ups seen by zooming in on the wind up structure. We remark as well that this roll-up structure is robust and does not change when we halve either the error tolerance or the time step. In order to calculate the wind-up numbers precisely, we calculate the tangential angle  $\phi$  at each point numerically using the following formula:

$$(4.5) \quad \phi_i = \arctan \left( \frac{y_{i+1} - y_{i-1}}{x_{i+1} - x_{i-1}} \right).$$

Based on this, we calculate the absolute value of the increase of  $\phi$  by

$$(4.6) \quad d\phi_i = |\phi_{i+1} - \phi_i|.$$

In one period, the roll-up rotates first counterclockwise and then clockwise an identical amount. Thus, since the perturbation has 10 periods, we define the wind-up number as  $\sum_i d\phi_i/20$ . As seen in Table 4.1, the amount of roll-up actually *increases* with  $\theta$ , eventually peaking at around  $\theta = -\pi/3$ , where there are approximately 2.92 rounds

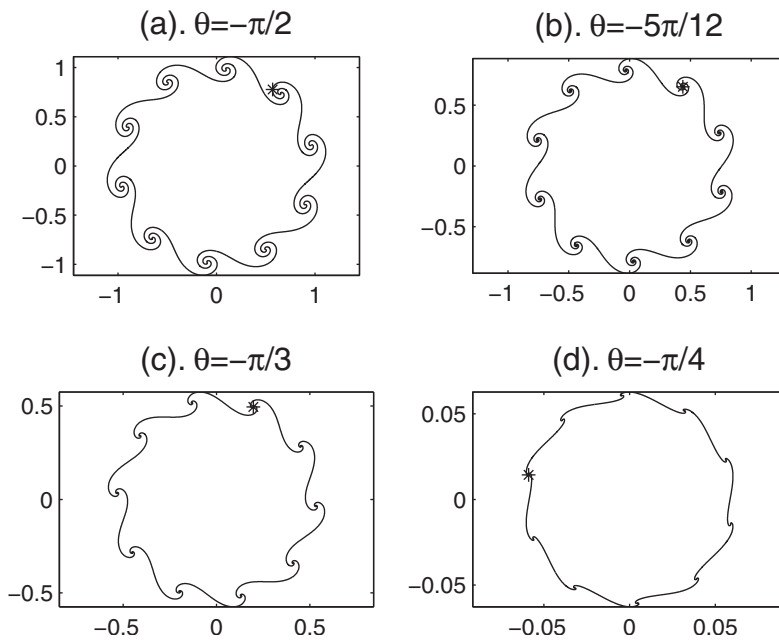


FIG. 4.3. The solution at time  $t = 1.5$  for four different values of  $\theta$ . The asterisk indicates the position of the point initialized at  $(1, 0)$ .

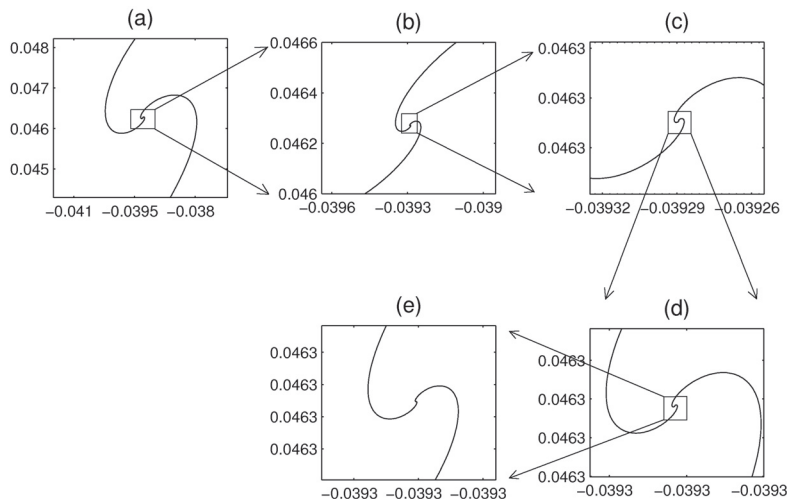


FIG. 4.4. Subsequent enlargements of a particular roll-up in picture in Figure 4.3(d) using 12,530 grid points.

TABLE 4.1  
Table of wind-up numbers.

Parameter $\theta$	$-\pi/2$	$-5\pi/12$	$-\pi/3$	$-\pi/4$
Wind-up number	1.5	2.45	2.92	2.47

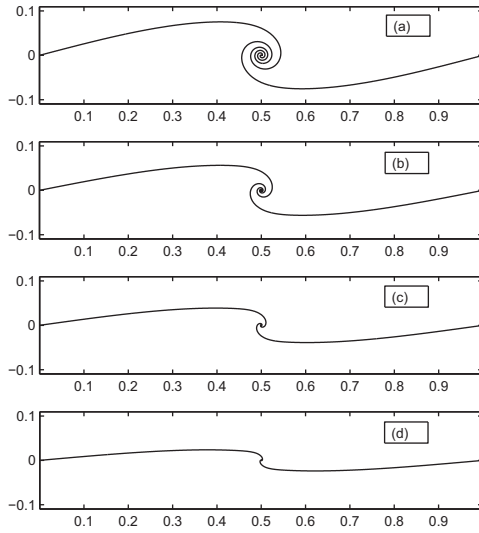


FIG. 4.5. The solution to the periodic line problem at time  $t = 1$  with initial condition  $\epsilon \sin(2\pi\alpha)$ . (a)  $\theta = -\pi/2$ , wind-up number = 2.64; (b)  $\theta = -5\pi/12$ , wind-up number = 5.04; (c)  $\theta = -\pi/3$ , wind-up number = 4.12; (d)  $\theta = -\pi/4$ , wind-up number = 1.60.

of roll-up. The amount of roll-up then begins to decrease. At  $\theta = -\pi/4$ , which represents an equal amount of incompressible part and gradient part for the kernel, there are only 2.47 rounds of roll-up in the picture.

Thus, we find that both the maximum amount of rotation of the vortex density ring and the amount of roll-up are not monotone functions of  $\theta$ . For a fixed time  $t > 0$  these maxima occur when there is a fully mixed kernel, i.e., a contribution from both the gradient part and the incompressible part. The amplitude of the perturbation monotonically decreases as  $\theta$  increases from  $\theta = -\pi/2$  to  $\theta = 0$ . Ultimately, as  $\theta$  increases and the gradient flow (the attraction) becomes the dominant contributor to the velocity field, both the roll-up and the rotation are damped out.

To explain this behavior physically and mathematically, we consider the linear stability analysis associated with the Kelvin–Helmholtz instability for this more general problem of a fully mixed kernel. Specifically, we study the linear stability theory of perturbations of a flat constant solution on a periodic domain. Recall that the linear stability analysis of the classic vortex sheet problem [26] demonstrates that the  $k$ th Fourier mode grows like  $e^{|k|t/2}$ , which implies that the linear evolution problem is linearly ill-posed. This ill-posedness explains the rapid development of the complicated roll-up behavior seen in section 3.2.1, classically known as the Kelvin–Helmholtz instability. Following the calculations in [26] we choose the flat vortex density solution to perform this calculation.

Our initial conditions for the flat density sheet problem can be expressed as  $z(\alpha, 0) = \alpha + \eta(\alpha, 0)$  with  $\alpha \in [-\infty, \infty]$ , where  $\eta = \eta_2 + \eta_1 i$  is a small perturbation to the position of the sheet. By choosing  $\rho|z_\alpha| = 1$  over a fixed period, it is clear that  $\eta$  also represents a perturbation of the density which takes the form  $\rho = 1 - \eta_2' + O(\eta_1'^2) + O(\eta_2'^2)$ .  $\eta_1$  represents a perturbation which is perpendicular to the flat sheet.  $\eta_2$  is a parallel perturbation and is the leading order contribution to the density perturbation. Figure 4.5 shows the evolution of the curve at  $t = 1$  for several different values of  $\theta$  where  $\eta_1$  is a small Fourier mode 1 perturbation and  $\eta_2 = 0$ .

We observe all the same phenomena that we saw in the ring perturbation calculation. As  $\theta$  increases from  $-\pi/2$  to  $-\pi/4$ , the number of roll-ups first increases and then decreases. Second, the roll-ups become smaller and smaller in structure as the amplitude of the perturbation (measured from the flat line) lowers as  $\theta$  increases.

For our stability calculation we use  $\mathbf{K}(x, y) = \lambda_1 \mathbf{K}_1(x, y) + \lambda_2 \mathbf{K}_2(x, y)$ , where  $\lambda_1 = \cos \theta$  and  $\lambda_2 = -\sin \theta$ . By (2.5) it is sufficient to understand the linearized evolution equation for  $z(\alpha, t)$ , which has the form

$$(4.7) \quad \frac{\partial \bar{z}(\alpha, t)}{\partial t} = \frac{\lambda_2 - \lambda_1 i}{2\pi i} PV \int \frac{d\alpha'}{z(\alpha, t) - z(\alpha', t)}.$$

By linearizing around our flat sheet  $z(\alpha, t) = \alpha + \eta(\alpha, t)$ , we get the equation

$$(4.8) \quad \frac{\partial \bar{\eta}}{\partial t} = \frac{\lambda_2 - \lambda_1 i}{2} \mathcal{H}\eta',$$

where  $\mathcal{H}\eta'$  is the Hilbert transform of  $\eta'$ , where  $\eta'$  is the derivative of  $\eta$  with respect to the parameterization and  $\bar{\eta}$  is the complex conjugate of  $\eta$ .

*Remark 4.1.* Note that if  $\lambda_1 = 0$  and  $\lambda_2 = 1$ , then (4.8) recovers the classical Kelvin–Helmholtz instability calculation. Thus, the inclusion of a gradient term can be seen directly in the  $-\lambda_1 i$  term in (4.8).

Letting  $\eta(\alpha, t) = A_k(t)e^{i2\pi k\alpha} + B_k(t)e^{-i2\pi k\alpha}$ , we get the relations

$$(4.9) \quad A'_k = (\lambda_1 - \lambda_2 i)\pi k \bar{B}_k, \quad B'_k = (\lambda_1 - \lambda_2 i)\pi k \bar{A}_k,$$

which yield solutions of the form

$$(4.10) \quad A_k(t) = A_k^+ e^{\pi kt} + A_k^- e^{-\pi kt}, \quad B_k(t) = B_k^+ e^{\pi kt} + B_k^- e^{-\pi kt}.$$

We now select an initial condition for our perturbation that contains both a spatial displacement perturbation (perpendicular to the flat sheet in the  $y$  direction) and a density perturbation (parallel to the flat sheet in the  $x$  direction). If we choose  $\eta(\alpha, 0) = \epsilon_1 i \sin 2\pi m_1 \alpha + \epsilon_2 \sin 2\pi m_2 \alpha$ , then for  $k \neq m_1$  or  $m_2$  we get  $A_k(t) = B_k(t) = 0$ . Otherwise,

$$(4.11) \quad A_{m_1}^+ = \frac{\epsilon_1}{4}(1 - \lambda_1 + \lambda_2 i), \quad A_{m_1}^- = \frac{\epsilon_1}{4}(1 + \lambda_1 - \lambda_2 i),$$

$$(4.12) \quad B_{m_1}^+ = \frac{\epsilon_1}{4}(-1 + \lambda_1 - \lambda_2 i), \quad B_{m_1}^- = \frac{\epsilon_1}{4}(-1 - \lambda_1 + \lambda_2 i),$$

$$(4.13) \quad A_{m_2}^+ = -\frac{i\epsilon_2}{4}(1 - \lambda_1 + \lambda_2 i), \quad A_{m_2}^- = -\frac{i\epsilon_2}{4}(1 + \lambda_1 - \lambda_2 i),$$

$$(4.14) \quad B_{m_2}^+ = -\frac{i\epsilon_2}{4}(-1 + \lambda_1 - \lambda_2 i), \quad B_{m_2}^- = -\frac{i\epsilon_2}{4}(-1 - \lambda_1 + \lambda_2 i).$$

The solution to the linearized problem is then

$$(4.15) \quad \eta(\alpha, t) = i[\epsilon_1(\sin 2\pi m_1 \alpha \cosh(\pi m_1 t) - \lambda_1 \sin 2\pi m_1 \alpha \sinh(\pi m_1 t)) + \epsilon_2 \lambda_2 \sin 2\pi m_2 \alpha \sinh(\pi m_2 t)] + \epsilon_2(\sin 2\pi m_2 \alpha \cosh(\pi m_2 t) - \lambda_1 \sin 2\pi m_2 \alpha \sinh(\pi m_2 t)) - \epsilon_1 \lambda_2 \sin 2\pi m_1 \alpha \sinh(\pi m_1 t).$$

From (4.15), we can now explain the effect of including a gradient term on the dynamics of the flat vortex density sheet and the Kelvin–Helmholtz instability. If we first consider purely perpendicular perturbations to the vortex density sheet (corre-



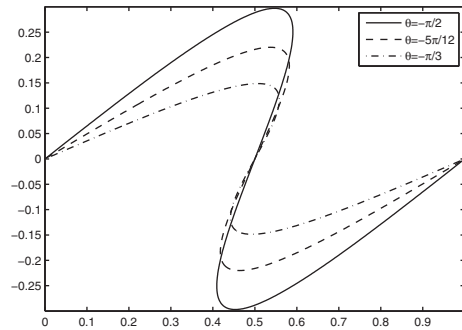


FIG. 4.6. The solution to the linearized problem at time  $t = 1.3$  with initial condition  $\epsilon_1 \sin(2\pi\alpha)$ . The solid curve is for  $\theta = -\pi/2$ ; the dashed curve is for  $\theta = -5\pi/12$ ; the dotted-dashed curve is for  $\theta = -\pi/3$ .

sponding to  $\epsilon_2 = 0$ ), our calculation above yields that the  $k$ th Fourier mode grows like  $e^{|k|t/2}$ . This implies that the linear evolution problem is still linearly ill-posed in the fully mixed case. Hence, just as in the classical Kelvin–Helmholtz instability, we expect a singularity in the curvature of our solution in finite time. The linearization calculation provides the mechanism for the dampened amplitude that we see in the nonlinear calculations in Figure 4.5.

When  $\theta$  is a bit greater than  $-\pi/2$ ,  $\lambda_1$  is a small positive number. We can see from (4.15) that this is the direct cause of the dampening out of the growth in the  $y$  direction. This is observed in Figure 4.5 and is explicitly exhibited in the linearized solutions plotted in Figure 4.6 for various  $\theta$  values. We can now also argue why we observe more roll-up in fully mixed kernels as opposed to just incompressible motion. At the point of a roll-up, the dampened amplitude along with the added attractive behavior of the gradient kernel forces the vorticity to remain closer together and aggregate at the roll-up point. Thus, by having more “mass” in a closer proximity, the rotational rate of  $r^{-1}$  causes this aggregated mass to rotate more quickly than if no gradient dynamics were included.

We can also understand from (4.15) the linearized dynamics of a pure density perturbation to the curve which corresponds to  $\epsilon_1 = 0$ . The linearized solution also predicts that the  $k$ th Fourier mode in the density grows like  $e^{|k|t/2}$ , implying that the linear evolution problem is also linearly ill-posed. Thus another effect of including a gradient term is the growth of singularities *in the density* in addition to the singularities in the curvature. In general, an arbitrary small perturbation to the vortex density sheet will generate singularities in both the curvature and the density; an example of this phenomenon is plotted in Figure 4.7. In this example, it appears that the curvature and density singularities occur at the same spatial point. Whether curvature singularities and density singularities must occur at the same place and time is unknown and is an interesting open question.

**4.2. Biological swarming.** We conclude this section by turning our attention from vortex density sheets to a biological model for swarming. In [36], Topaz and Bertozzi study the continuum model (1.1) and (1.2) with the Gaussian kernel  $G_d(x, y) = \frac{1}{d^2} e^{-(x^2+y^2)/d^2}$ . The parameter  $d$  is the relevant length scale and  $G_d$  is used as a biological kernel to model swarming and milling behavior for both incompressible motion  $N$  and gradient motion  $G$ . They considered localized continuous distributions of the density but ultimately study the dynamics of the incompressible

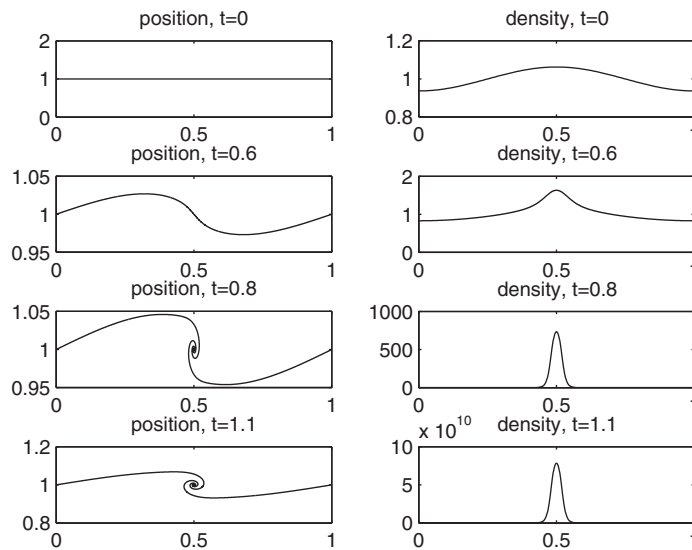


FIG. 4.7. Time evolution of both the curve and density with  $\eta(\alpha, 0) = 0.01 \sin(2\pi\alpha)$  with  $\theta = -5\pi/12$ . This pure density perturbation leads to both a curvature and density singularity formation.

motion and the gradient motion separately. Using our model, we study the dynamics of curve solutions with a fully mixed kernel of the form  $\mathbf{K} = \lambda_1 \nabla G_d + \lambda_2 \nabla^\perp G_d$ , where  $\lambda_1$  is a weight for the gradient contribution to the kernel and  $\lambda_2$  is a weight for the incompressible contribution to the kernel. Using the same approach as the superfluids example, we would like to understand how incompressible motion and gradient motion affect each other by controlling the weights  $\lambda_1$  and  $\lambda_2$  for each.

We study the initial value problem (1.1) and (1.2) using a perturbed density ring with initial condition of the form (4.4), where

$$(4.16) \quad r(\alpha) = \bar{r} + \tilde{r} \cos(12\pi\alpha) \text{ with } \alpha \in [0, 1].$$

For our first two experiments we take  $d = 3$  for both  $\nabla G_d$  and  $\nabla^\perp G_d$  in the kernel and choose  $\bar{r} = 1$  with the very large perturbation of  $\tilde{r} = 0.2$ . We fix the weight of the gradient part in our first simulation to be  $\lambda_1 = 1$  and vary the amount of the incompressible part from  $\lambda_2 = 0$  to  $\lambda_2 = 9$ , plotting the solution curves at  $t = 50$  in Figure 4.8. By keeping  $\lambda_1 = 1$  fixed we can observe how changing the value of  $\lambda_2$  (the incompressible motion) affects the dynamics with a fixed rate of contraction. From Figure 4.8 it is clear that the amount of rotational shear that occurs on the “spiral arms” increases as  $\lambda_2$  increases, as one would expect. It is also easy to see that the rate of contraction (using the magnitude of the scale of the curves  $5 \times 10^{-5}$ ) is identical regardless of how much incompressible part is added to the kernel. This is also consistent with the superfluids example.

Next, we fix the incompressibility coefficient  $\lambda_2 = 1$  and vary the gradient coefficient  $\lambda_1$  to see how the increase of the gradient affects the rotation and shear of the curve solutions. Figure 4.9 gives the solutions for different  $\lambda_1$ 's at time  $t = 25$ . There are several important features to observe in Figure 4.9. First, it is clear from the axis that as  $\lambda_1$  increases the rate of contraction increases as expected. Second, we note that the rotational shear of the arms decreases and the amount of rotation of the shape increases as  $\lambda_1$  increases. We calculate the degree of rotation by measuring

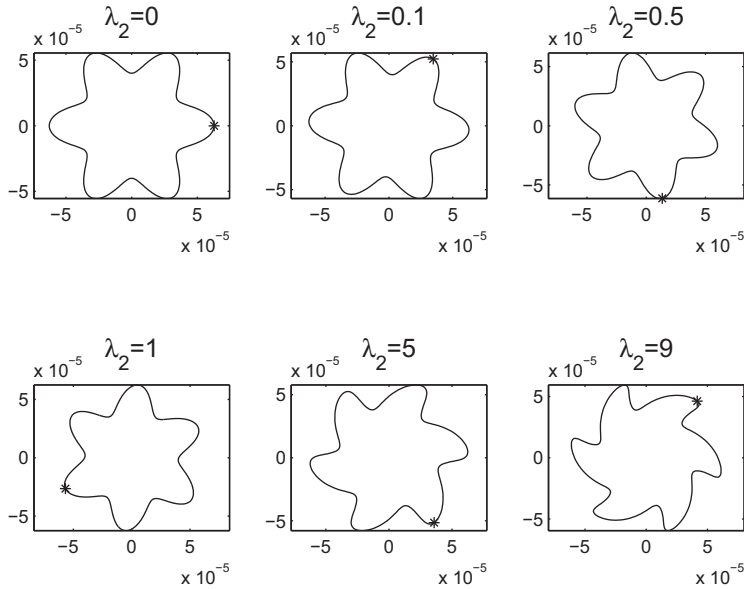


FIG. 4.8. The solution at time  $t = 50$  for  $\lambda_1 = 1$  and varying values of  $\lambda_2$ .

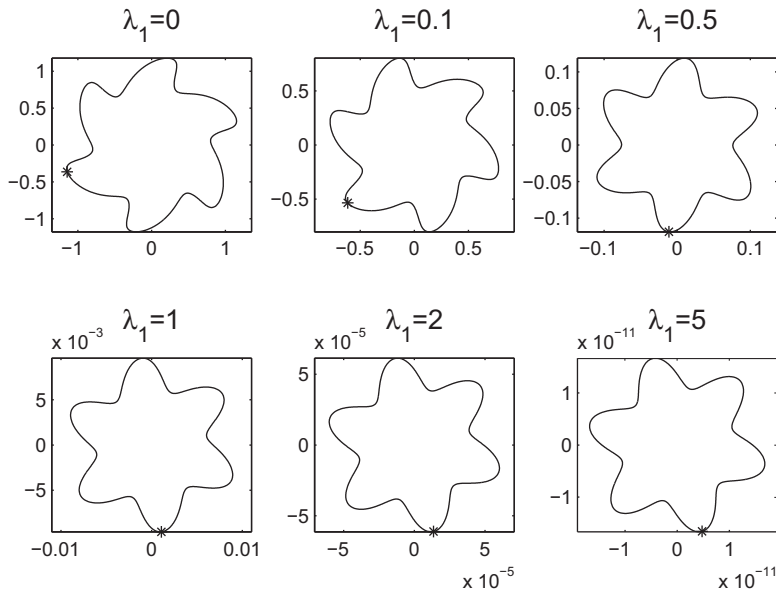


FIG. 4.9. The solution at time  $t = 25$  for  $\lambda_2 = 1$  and varying values of  $\lambda_1$ .

the angle from the asterisks to the point  $(1, 0)$ ; the values are recorded in Table 4.2. One noticeable change in the angle occurs between  $\lambda_1 = 0.1$  to  $\lambda_1 = 0.5$ , where the angle of rotation changes from  $1.23\pi$  to  $1.47\pi$ . Increasing values of  $\lambda_1$  beyond  $\lambda_1 > 1$  yields only small increases in the angle of rotation. Thus, even with a much smoother Gaussian kernel the same theme from the superfluids example persists: the gradient contribution can have a strong effect on the rotational dynamics but the reverse does not occur.

TABLE 4.2  
Table of wind-up numbers.

Parameter $\lambda_1$	0	0.1	0.5	1	2	5
Rotation angle	1.0976	1.2276	1.4700	1.5358	1.5693	1.5894

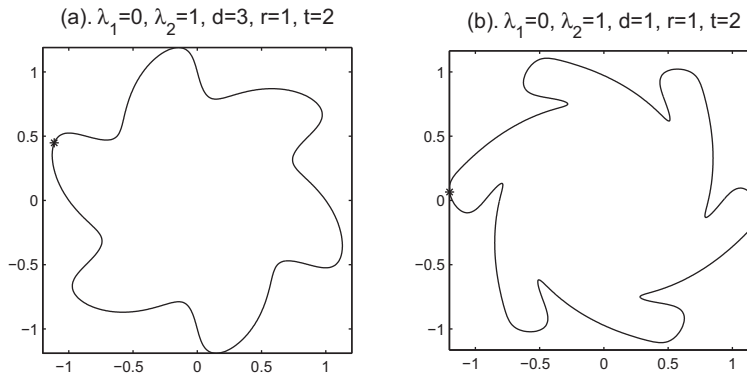


FIG. 4.10. By choosing parameters  $d$  and  $r$ , the spin directions of the outer arms are different.

Perhaps the most interesting behavior we observe in this example is that different spin directions of the perturbation arms occur depending on the relationship between the size of the ring  $r$  and the length scale of the kernel  $d$ . In our examples  $\rho > 0$ ; the curve thus rotates counterclockwise by the right-hand rule. In Figure 4.10(a), which corresponds to  $d = 3$  and  $r = 1$ , the outer arms spin slower in the clockwise direction relative to the curve's speed of rotation; hence the arms appear to be "falling behind." In contrast, Figure 4.10(b) uses the parameters  $d = 1$  and  $r = 1$ , producing a counterclockwise spin of the arms, which is faster than the curve's speed of rotation. This forces the arms to "get ahead" of the curve. We can suppose, then, that there must be a critical ratio  $\gamma_0 = d/r$  in the behavior of the spiral arms as we increase the parameter  $d$  from 1 to 3 where the speeds match.

To estimate  $\gamma_0$  we first consider the simpler problem of an unperturbed ring and the velocity  $\epsilon$  away from the ring depicted in Figure 4.11. Let us assume our initial condition is a circle with radius  $r$  and density normalized to  $\rho = 1$ . For this estimate we also set  $\lambda_1 = 0$  and  $\lambda_2 = 1$  in our model to isolate the effect of the incompressible velocity field (which is the cause of the rotation rates). This results in a constant radius  $r$  (as opposed to a contracting one), allowing us to pinpoint  $\gamma_0$  more precisely. We have seen that the amount of gradient in the kernel has an effect on the rotational shearing, but we will observe below that the predicted  $\gamma_0$  seems to be independent of  $\lambda_2$ .

To find the value of  $\gamma_0$  we need to compute the angular velocity  $\omega_p$  of a point  $p = (1 + \epsilon, 0)$  just outside the ring, i.e., where  $\epsilon \ll 1$ ; see Figure 4.11. This point represents a small radial perturbation of the circle. If this point is moving faster than on the ring, then perturbations of the ring will result in spiral arms that shear in the counterclockwise direction relative to the ring, as in Figure 4.10(b). If the point is moving slower than on the ring the spiral arms will fall behind the ring, as in Figure 4.10(a). To calculate the angular velocity  $\omega_p$  of the point  $p$  which is distance  $\epsilon$  from the circle we compute the integral

$$(4.17) \quad \omega_p = \frac{1}{r + \epsilon} \int_0^{2\pi} \partial_x G_d(r + \epsilon - r \cos \theta, -r \sin \theta) r d\theta.$$

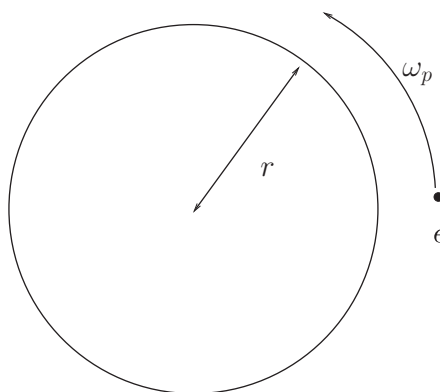


FIG. 4.11. The initial condition as a circle, with the angular velocity it generates to a point with distance  $\epsilon$  on the right of the circle.

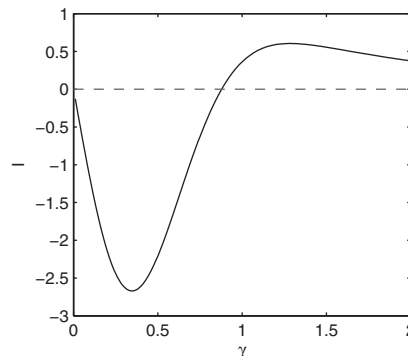


FIG. 4.12. Integral  $I$  as a function of  $\gamma$ .  $I(0.879) = 0.000171$  and  $I(0.878) = -0.003721$ , indicating that the zero lies between 0.878 and 0.879.

We then differentiate (4.17) with respect to  $\epsilon$  and we get to leading order

$$(4.18) \quad \frac{d\omega_p}{d\epsilon} = \frac{2}{d^5} \cdot I, \quad \text{where } I = \int_0^{2\pi} \left[ -\frac{r}{d} 2(1 - \sin \theta)^2 + \frac{d}{r} \sin \theta \right] e^{-2(1 - \sin \theta) \frac{r^2}{d^2}} d\theta.$$

We see that the sign of  $\frac{d\omega}{d\epsilon}$  depends solely on  $\gamma = r/d$ . When  $I < 0$ , i.e.,  $\frac{d\omega}{d\epsilon} < 0$ , the points on the arm which are closer to the circle have a faster angular velocity. Then the arms appear to wind up in the opposite direction of the spin. When  $I > 0$ , i.e.,  $\frac{d\omega}{d\epsilon} > 0$ , the points on the arm which are outside the circle have a faster angular velocity, which makes the arms appear to wind up in the same direction as the spin. Thus, our critical value  $\gamma_0$  is precisely when  $I(\gamma_0) = 0$ , which is the critical ratio of radius-to-kernel length scale. Figure 4.12 is a numerical calculation of  $I$  as a function of  $\gamma$ . From this we see that  $\gamma_0 \approx 0.88$  for our example.

The existence of a critical  $\gamma_0$  provides the explanation of why we see qualitatively different dynamics in the spiral arms between Figures 4.10(a) and 4.10(b). Since the ratio  $r/d$  is what determines the shearing behavior in our simpler problem, we can measure the accuracy of  $\gamma_0 = 0.88$  once we include both a nonzero  $\lambda_1$  and  $\lambda_2$  in our fully nonlinear perturbation problem. By including a positive value for  $\lambda_1$  the curve solution will attract toward the origin. Thus, if we start with a ring whose large perturbations initially start outside of the critical radius, we should initially see the arms shear faster than the ring. This faster rotation will cause the arms to move ahead of the ring. However, as the entire curve shrinks and crosses our critical estimate of  $\gamma_0 = 0.88$ , we would expect the spiral arms to reverse directions. The initial conditions we use for this experiment are described in (4.4) and (4.16), with  $d = 1$ ,  $\bar{r} = 1$ , and  $\tilde{r} = 0.2$ . In addition, we take  $\lambda_1 = 0.01$  and  $\lambda_2 = 0.5$ . The plot of the initial condition in Figure 4.13 shows that the large perturbations do in fact lie outside the critical radius.

As predicted from our calculation, the  $t = 3$  plot shows the arms located outside the critical radius moving faster in the counterclockwise direction. However, by  $t = 11$  most of the spiral arm has contracted inside the critical radius and begins to reverse course. By  $t = 15$  the entire curve and spiral arms are inside the critical radius and the rotational shear becomes pronounced in the reverse direction—shown in Figure 4.13.

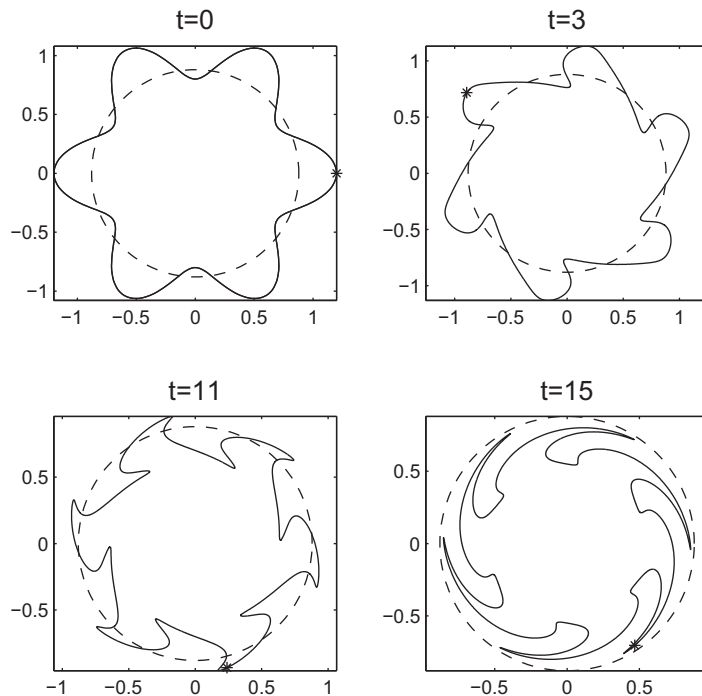


FIG. 4.13. The solution at time  $t = 0, 3, 11, 15$  with initial conditions (4.4) and (4.16) with  $d = 1$ ,  $\bar{r} = 1$ ,  $\tilde{r} = 0.2$ ,  $\lambda_1 = 0.01$ , and  $\lambda_2 = 0.5$ .

Thus, our idealized calculation for the critical radius based on the assumption of an unperturbed ring approximates the reversal quite well, though it appears that the reversing of the spiral arm direction in the  $t = 11$  picture of Figure 4.13 does occur just outside the ring.

**5. Discussion.** In this paper we have derived the evolution equations for the solutions to the general 2D active scalar problem in the case when the active scalar lies on 1D curves. We implement an adaptive Lagrangian scheme which is second order in space and fourth order in time. The model is then shown to reproduce classical vortex sheet dynamics as compared to the desingularized Birkhoff–Rott equation. When the velocity is purely a gradient, the model accurately predicts the finite time collapse of power law aggregation equations. When the velocity field contains both an incompressible and a gradient component, our model exhibits new dynamics, which include an increased roll-up associated with the Kelvin–Helmholtz instability for vortex density functions of superfluids. In the second example of mixed-type velocity fields we consider a model for biological swarming and aggregation and show that the length scale of the kernel plays an important role in the milling and rotational shear behavior of the active scalar curve.

The active scalar curve equation has made an immediate impact in the study of the active scalar problem in the context of attraction-repulsion kernels found in swarming models. It has been successfully used to understand and predict complex patterns that arise (see [21]) when the interaction kernel is purely gradient and has a long-range attraction and short-range repulsion structure. Moreover, this model has been generalized to  $n$  dimensions (see [38]) and used to perform the same stability

analysis for predicting complex patterns that aggregate on the  $n$  sphere; see also [39]. In the context of studying attraction-repulsion kernels the model developed in this paper has been of great use and we expect it to find uses in many other contexts.

Finally, another interesting direction of future research is to better understand the interaction between gradient and incompressible components, which we have seen provides very rich and nonlinear dynamics. In the superfluids example it would be an interesting physical and mathematical question to better understand the different types of singularity formation (curvature versus density) and their relation.

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