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SO(2,1) Conformal Anomaly: Beyond Contact Interactions

Horacio E. Camblong
University of San Francisco, camblong@usfca.edu

SO(2,1) conformal anomaly: Beyond contact interactions

Gino N. J. Añaños,[1,2] Horacio E. Camblong,[3] and Carlos R. Ordóñez[1,2]

[1]*Department of Physics, University of Houston, Houston, Texas 77204-5506, USA*

[2]*World Laboratory Center for Pan-American Collaboration in Science and Technology, University of Houston Center, Houston, Texas 77204-5506, USA*

[3]*Department of Physics, University of San Francisco, San Francisco, California 94117-1080, USA*

The existence of anomalous symmetry-breaking solutions of the SO(2,1) commutator algebra is explicitly extended beyond the case of scale-invariant contact interactions. In particular, the failure of the conservation laws of the dilation and special conformal charges is displayed for the two-dimensional inverse square potential. As a consequence, this anomaly appears to be a generic feature of conformal quantum mechanics and not merely an artifact of contact interactions. Moreover, a renormalization procedure traces the emergence of this conformal anomaly to the ultraviolet sector of the theory, within which lies the apparent singularity.

I. INTRODUCTION

The relevance of conformal quantum mechanics has been recognized for decades [1] in the context of the scale-invariant Hamiltonian dynamics [2] of the inverse square potential, which is characterized by an SO(2,1) commutator algebra. A formally identical symmetry algebra was discovered for the magnetic monopole [3], the magnetic vortex [4], and the two-dimensional contact interaction [5]. Remarkably, this algebra has also been identified within the maximal "Schrödinger group" of symmetries of nonrelativistic field theories [6] and related applications [7,8].

Most importantly, the central role played by conformal quantum mechanics in theoretical physics has been highlighted in recent years in a wide variety of problems. First, insights into the physics of black holes have been directly gleaned from the concept of near-horizon SO(2,1) conformal invariance [9–11], as well as from its supersymmetric extensions [12–16]. This is in large part due to the remarkable connections provided by the AdS/CFT correspondence [17]. In addition, the ubiquity of the Calogero model [18], from black holes [19] to applications in condensed-matter physics [20,21], has led to alternative applications of a formally identical algebra of conformal generators. Finally, the use of field-theory renormalization techniques has promoted novel methods for the treatment of singular interactions, including those within the conformal quantum mechanics class, by means of Hamiltonian [5,22–26] as well as path-integral techniques [27–29].

The underlying property common to all the problems mentioned above is the presence of a particular conformal symmetry after an appropriate *reduction* framework is applied. As such, this is a particular realization of conformal invariance for an effective (0+1)-dimensional field. It is the corresponding reduced problem that is described within the conformal quantum mechanics class, typically with an *effective Hamiltonian* $H \equiv p^2/2M + V(\mathbf{r})$, or with many-body generalizations thereof. In particular, in its reduced form, a conformally invariant interaction is characterized by an interaction potential $V(\mathbf{r})$ that is a homogeneous function of degree -2. This property alone implies that these interac-

tions satisfy a set of classical symmetries under time reparametrizations [22,30]. The associated quantum-mechanical generators are the Hamiltonian H, the dilation operator $D \equiv tH - \Lambda/2$, in which $\Lambda = (\mathbf{p} \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{p})/2$, and the special conformal operator $K \equiv 2tD - t^2H + Mr^2/2$; these generators yield an SO(2,1) Lie algebra [31] at the "classical" level

$$[D,H]_{\text{regular}} = -i\hbar H,$$

$$[K,H]_{\text{regular}} = -2i\hbar D,$$

$$[D,K]_{\text{regular}} = i\hbar K. \tag{1}$$

Appropriate anomalous modifications of this "regular" algebra will be discussed below, when the theory is quantized.

The main purpose of this paper is to explore the quantum symmetry breaking of the algebra (1) for the inverse square potential. The existence of this *conformal anomaly* was first recognized for the two-dimensional contact interaction by indirect methods in the seminal work of Ref. [5] and was recently confirmed by a direct calculation at the level of the commutator algebra in Refs. [30,32]. Even though a draft of the more general theory was developed in [30], the proof of its actual realization for the all-important inverse square potential is still lacking. This is the problem to which we now turn our attention, for the particular case of spatial dimensionality $d = 2$.

II. ULTRAVIOLET ORIGIN OF THE ANOMALY FOR THE TWO-DIMENSIONAL INVERSE SQUARE POTENTIAL

The inverse square potential is of fundamental importance because of its applications to black holes [9–16], nuclear physics [26,33], and molecular physics [34]. Even though the existence of this conformal anomaly had been anticipated by other indirect arguments [34], in this work we present the first conclusive direct computation at the level of the commutator algebra (1). More precisely, as the next step towards establishing a more general framework, we show that the two-dimensional case of the inverse square potential confirms the conclusions drawn in Ref. [30]. The advantage of this particular dimensionality lies in the remarkable similari-

ties that the inverse square potential and the δ-function interaction exhibit for $d=2$. Not only is the dimensionality the same, but both interactions are characterized by a vanishing critical coupling, and the corresponding expressions for the anomalous terms can be considerably simplified.

The fundamental quantity encoding the nature of the anomaly is [30]

$$\mathcal{A}(\mathbf{r}) \equiv \frac{1}{i\hbar}[D,H] + H = \left[1 + \frac{1}{2}\mathcal{E}_{\mathbf{r}}\right]V(\mathbf{r}), \qquad (2)$$

where 1 is the identity operator and $\mathcal{E}_{\mathbf{r}} = \mathbf{r} \cdot \boldsymbol{\nabla}$ stands for the Eulerian derivative. In particular, the two-dimensional form of Eq. (2) simplifies to

$$\mathcal{A}(\mathbf{r}) = \frac{1}{2}\boldsymbol{\nabla} \cdot \{\mathbf{r}V(\mathbf{r})\}. \qquad (3)$$

For the case of the two-dimensional inverse square potential, the Hamiltonian

$$H = \frac{p^2}{2M} - \frac{g}{r^2} \qquad (4)$$

is conformally invariant, with $\lambda = 2Mg/\hbar^2$ being the dimensionless form of the coupling constant. Then, the formal two-dimensional identity

$$\boldsymbol{\nabla} \cdot \left[\frac{\hat{\mathbf{r}}}{r}\right] = 2\pi\delta^{(2)}(\mathbf{r}) \qquad (5)$$

implies that

$$\mathcal{A}(\mathbf{r}) = -g\pi\delta^{(2)}(\mathbf{r}), \qquad (6)$$

whose expectation value for a normalized state $|\Psi\rangle$ becomes

$$\frac{d}{dt}\langle D\rangle_\Psi = \langle\mathcal{A}(\mathbf{r})\rangle_\Psi = -g\pi\int d^2\mathbf{r}\,\delta^{(2)}(\mathbf{r})|\Psi(\mathbf{r})|^2. \qquad (7)$$

Equation (7) can be used to shed light on the nature of the possible conformal symmetry breaking. Specifically, two important features are immediately apparent:

(i) The correct evaluation of Eq. (7) requires an appropriate regularization procedure, because of the well-known vanishing or asymptotically free value of g. This behavior competes against the logarithmic singularity of the *renormalized* ground-state wave function $\Psi(\mathbf{r})$ at the origin [22,23,25],

$$\Psi_{(gs)}(\mathbf{r}) = \frac{\kappa}{\sqrt{\pi}}K_0(\kappa r), \qquad (8)$$

where $\kappa = \sqrt{2M|E_{(gs)}|}/\hbar$. Consequently, Eq. (7) has to be regularized concurrently with other observables in the theory.

(ii) The existence of an anomaly [nonvanishing value of Eq. (7)] arises from the "singularity" at the origin, which is encoded in the δ function. The presence of this generalized function can be physically interpreted as representing the "core" of the interaction near the singular point, according

to Eq. (5). In short, the origin of this conformal anomaly can be conclusively traced to the apparent singularity at the origin, which lies within the ultraviolet sector of the theory.

In the following sections, we will regularize the theory using an ultraviolet real-space regulator.

III. REAL-SPACE REGULARIZATION OF THE INVERSE SQUARE POTENTIAL

Real-space regularization of the ultraviolet physics is implemented by an appropriate modification of the interaction for $r \lesssim a$. This procedure amounts to the introduction of a regular potential $V^{(<)}(\mathbf{r})$ for $r \lesssim a$, where it succinctly describes the short-distance physics. Moreover, in order to maintain the intrinsic physics of the inverse square potential, the core interaction $V^{(<)}(\mathbf{r})$ should implement a continuous transition from the long- to the short-distance physics [35]; i.e., it should satisfy the continuity requirement $V^{(<)}(r=a) = -g/a^2$. The simplest and most convenient choice is afforded by a finite square well $V^{(<)}(\mathbf{r}) = -g\,\theta(a-r)/a^2$, so that the unregularized Hamiltonian (4) undergoes the replacement

$$H \rightarrow H_a = \frac{p^2}{2M} - \frac{g}{r^2}\theta(r-a) - \frac{g}{a^2}\theta(a-r), \qquad (9)$$

in which $\theta(z)$ stands for the Heaviside function. Then, for a wave function $\Psi(\mathbf{r}) = e^{im\phi}u_{|m|}(r)/\sqrt{r}$, the corresponding reduced radial Schrödinger equation is given by

$$\left\{\frac{d^2}{dr^2} + \left[\frac{2M}{\hbar^2}E + \lambda\frac{\theta(a-r)}{a^2}\right]\right.$$
$$\left. - \frac{l^2 - \lambda\theta(r-a) - 1/4}{r^2}\right\}u_l(r) = 0, \qquad (10)$$

in which $l = |m|$, with m being the usual quantum number. A bound-state solution ($E<0$) to Eq. (10) can be written in terms of Bessel functions [36],

$$R_l(r) \equiv \frac{u_l(r)}{\sqrt{r}} = \begin{cases} \{J_l(\tilde{k}r), N_l(\tilde{k}r)\} & \text{for } r<a, \\ \{I_{i\Theta}(\kappa r), K_{i\Theta}(\kappa r)\} & \text{for } r>a, \end{cases} \qquad (11)$$

where the effective coupling becomes

$$\Theta \equiv \Theta_l = \sqrt{\lambda - l^2}, \qquad (12)$$

the energy parameters are

$$\tilde{k}^2 = \frac{2M}{\hbar^2}E + \frac{\lambda}{a^2} \qquad (13)$$

and

$$\kappa^2 = -\frac{2M}{\hbar^2}E, \qquad (14)$$

and the symbol $\{,\}$ stands for linear combination. In Eq. (11) the regular boundary conditions at the origin and at infinity lead to the particular selection

$$\Psi(\mathbf{r}) = e^{im\phi} \times \begin{cases} B_l J_l(\tilde{k}r) & \text{for } r < a, \\ A_l K_{i\Theta}(\kappa r) & \text{for } r > a, \end{cases} \quad (15)$$

where the relative values of A_l and B_l can be determined from the continuity condition

$$B_l J_l(\tilde{k}a) = A_l K_{i\Theta}(\kappa a). \quad (16)$$

In addition, the continuity of the logarithmic derivative at $r = a$ provides the equation for the energy eigenvalues

$$\tilde{k} \frac{J_l'(\tilde{k}a)}{J_l(\tilde{k}a)} = \kappa \frac{K_{i\Theta}'(\kappa a)}{K_{i\Theta}(\kappa a)}. \quad (17)$$

Furthermore, the values of A_l and B_l can be fixed from the normalization condition

$$1 = \int d^2\mathbf{r} |\Psi(\mathbf{r})|^2$$

$$= A_l^2 2\pi\kappa^{-2} \left\{ \mathcal{K}_{i\Theta}(\kappa a) + \left(\frac{\kappa}{\tilde{k}} \right)^2 \left[\frac{K_{i\Theta}(\kappa a)}{J_l(\tilde{k}a)} \right]^2 \mathcal{J}_l(\tilde{k}a) \right\}, \quad (18)$$

where the functions

$$\mathcal{K}_{i\Theta}(\kappa a) = \int_{\kappa a}^{\infty} dz\, z[K_{i\Theta}(z)]^2 \quad (19)$$

and

$$\mathcal{J}_l(\tilde{k}a) = \int_0^{\tilde{k}a} dz\, z[J_l(z)]^2 \quad (20)$$

are conveniently defined. Equations (18), (19), and (20) will be further simplified when the theory is renormalized in Sec. V.

IV. CALCULATION OF THE CONFORMAL ANOMALY

We are now ready to start the computation of the regularized anomaly. First, from Eqs. (3), (5), and (9), the conformal anomaly manifests as the failure of the dilation operator to yield a zero time derivative; explicitly, the regularized counterpart of Eq. (7) is obtained with the replacement

$$\mathcal{A}(\mathbf{r}) \to \mathcal{A}_a(\mathbf{r}) = -g\pi\delta^{(2)}(\mathbf{r})\theta(r-a) - \frac{g}{a^2}\theta(a-r). \quad (21)$$

Therefore, the corresponding expectation value for a renormalized and normalized state $|\Psi\rangle$ becomes

$$\frac{d}{dt}\langle D\rangle_{\Psi} = \lim_{a\to 0}[\langle \mathcal{A}_a(\mathbf{r})\rangle_{\Psi_a}^{(<)} + \langle \mathcal{A}_a(\mathbf{r})\rangle_{\Psi_a}^{(>)}]. \quad (22)$$

In Eq. (22), $|\Psi_a\rangle$ is the regularized counterpart of $|\Psi\rangle$, as given in Eq. (15). Furthermore, $\langle \mathcal{A}_a(\mathbf{r})\rangle_{\Psi_a}^{(j)}$ stands for the contribution to the expectation value from the ultraviolet region $(r < a)$, for $j = <$; and from the region $r > a$, for $j = >$. Remarkably, Eq. (21) shows that

$$\langle \mathcal{A}_a(\mathbf{r})\rangle_{\Psi_a}^{(>)} = 0, \quad (23)$$

which confirms that *only* the singularity at the origin can be the source of the conformal anomaly. As a consequence,

$$\frac{d}{dt}\langle D\rangle_{\Psi} = \lim_{a\to 0}\langle \mathcal{A}_a(\mathbf{r})\rangle_{\Psi_a}^{(<)}$$

$$= -2\pi \lim_{a\to 0}\frac{g}{a^2} B_l^2 \int_0^a dr\, r[J_l(\tilde{k}r)]^2. \quad (24)$$

This expression for the anomaly can be most easily interpreted by rewriting it in the form

$$\frac{d}{dt}\langle D\rangle_{\Psi}$$

$$= E\lim_{a\to 0}\left(\left\{ \frac{\lambda(a)}{(\tilde{k}a)^2} \right\} \left\{ \frac{\pi A_l^2}{\kappa^2} \right\} \left\{ \frac{2\mathcal{J}_l(\tilde{k}a)}{[J_l(\tilde{k}a)]^2}[K_{i\Theta}(\kappa a)]^2 \right\} \right), \quad (25)$$

as follows from Eqs. (14), (16), and (24). In Eq. (25), E is the finite renormalized value of the energy associated with $|\Psi\rangle$, and $\lambda(a)$ is the running coupling constant. Correspondingly, the anomalous time derivative of Eq. (25) is scaled with the bound-state energy E of the state $|\Psi\rangle$. Moreover, as we will show below, upon renormalization, each one of the three additional factors enclosed in braces is asymptotically equal to one (with respect to the limit $a \to 0$). As a result,

$$\frac{d}{dt}\langle D\rangle_{\Psi} = E, \quad (26)$$

which agrees with the expected answer: the right-hand side of Eq. (26) becomes the energy of the stationary normalized state [30].

Finally, once the value of the anomalous commutator $[D,H]$ has been identified, the corresponding value of the commutator $[K,H]$ is determined [30], with

$$\frac{d}{dt}\langle K\rangle_{\Psi} = 2t\frac{d}{dt}\langle D\rangle_{\Psi} = 2tE. \quad (27)$$

V. RENORMALIZATION

The final required step is the renormalization of the system. This is implemented by finding the behavior of the running coupling constant from the consistency requirement that Eq. (17) admit a *finite* bound-state energy, when $a \rightarrow 0$. From the small-argument expansion of the Macdonald function

$$K_{i\Theta}(z) \stackrel{(z\rightarrow 0)}{=} -\sqrt{\frac{\pi}{\Theta \sinh(\pi\Theta)}}$$
$$\times \sin\left(\Theta\left[\ln\left(\frac{z}{2}\right)+\gamma\right]\right)[1+O(z^2)], \quad (28)$$

Eq. (17) becomes

$$\Theta \cot\left(\Theta\left[\ln\left(\frac{z}{2}\right)+\gamma\right]\right) \stackrel{(a\rightarrow 0)}{=} \tilde{k}a\frac{J'_l(\tilde{k}a)}{J_l(\tilde{k}a)}[1+O([\kappa a]^2)]. \quad (29)$$

The renormalization condition consists in taking the limit $a \rightarrow 0$, with a running coupling $\Theta(a)$ to be determined self-consistently so that κ remains fixed, thus guaranteeing a finite energy. As argued in Refs. [24] and [25], Eq. (29) is ill defined unless $\Theta(a)$ has the appropriate logarithmic running

$$\Theta(a) \propto -[\ln(\kappa a)]^{-1} \stackrel{(a\rightarrow 0)}{\rightarrow} 0. \quad (30)$$

In other words, this behavior drives the coupling λ towards its critical value, which is exactly zero for $d=2$; in particular, when $l=0$, $\lambda(a)=\Theta^2(a) \stackrel{(a\rightarrow 0)}{\rightarrow} 0$. Once this running behavior sets in, the only bound state that survives the renormalization process will occur for $l=0$, because the other channels ($l \neq 0$) will be automatically placed in the weak-coupling regime, for which binding is suppressed [23,25]. In addition, this analysis shows that binding will always occur for $d=2$, when the critical coupling is zero; this fact alone places the two-dimensional case in a unique position. Moreover, the condition (29) for the energy eigenvalues becomes

$$\cot\left(\Theta\left[\ln\left(\frac{z}{2}\right)+\gamma\right]\right) \stackrel{(a\rightarrow 0)}{=} -\frac{\Theta}{2}[1+O(\Theta^2)], \quad (31)$$

which logically enforces the limits $\cos\alpha \stackrel{(a\rightarrow 0)}{\rightarrow} 0$ and $|\sin\alpha| \stackrel{(a\rightarrow 0)}{\rightarrow} 1$, where $\alpha = \Theta[\ln(z/2)+\gamma]$.

We are now ready to prove the fact that the three additional factors in Eq. (25) are asymptotically equal to one. First, Eq. (13) implies that $(\tilde{k}a)^2 = \lambda + O([\kappa a]^2)$, in which $\lambda = \Theta^2$ is the leading logarithmic term with respect to a, according to Eq. (30); thus,

$$\frac{\lambda(a)}{(\tilde{k}a)^2} \stackrel{(a\rightarrow 0)}{\rightarrow} 1, \quad (32)$$

for a finite energy level E. The second additional factor in Eq. (25) has a limiting value of one because

$$A_0 \stackrel{(a\rightarrow 0)}{=} \frac{\kappa}{\sqrt{\pi}}\{1+o(\kappa a)\}. \quad (33)$$

This can be deduced from Eqs. (18), (19), and (20), for $l=0$, $\kappa a \ll 1$, and $\Theta \sim \tilde{k}a \ll 1$, which collectively imply that $2\mathcal{K}_{i\Theta}(\kappa a)=1+O(\Theta^2)$ and $2\mathcal{J}_{l=0}(\tilde{k}a)=\Theta^2[1+O(\Theta^2)]$. Finally, the third additional factor in Eq. (25) becomes

$$2\frac{\mathcal{J}_l(\tilde{k}a)}{[J_l(\tilde{k}a)]^2}[K_{i\Theta}(\kappa a)]^2$$
$$\stackrel{(a\rightarrow 0)}{=} 2\left(\frac{\Theta^2}{2}\right)\left[-\frac{\sin\alpha}{\Theta}\right]^2[1+O(\Theta^2)]$$
$$\stackrel{(a\rightarrow 0)}{=} 1+O(\Theta^2). \quad (34)$$

In closing, this renormalization procedure, based on the modification of the ultraviolet behavior, shows that the ground state wave function reduces to Eq. (8) in the limit $a \rightarrow 0$. However, as discussed in Sec. II, for the computation of the anomaly, this limit can only be taken as the last step, after all expressions have been properly regularized. In this paper we have shown that the ensuing procedure is implemented at the level of Eq. (25) and yields the anticipated answer, Eq. (26).

VI. CONCLUSIONS

In conclusion, we have shown the existence of a conformal anomaly of the SO(2,1) algebra associated with the dynamics of the two-dimensional inverse square potential. The corresponding violations of the conservation laws of the dilation and special conformal charges follow patterns very similar to those encountered earlier for contact interactions. In particular, this work is closely related to the conformal interactions of maximal physical relevance, involved in applications from molecular physics to the physics of black holes. Consequently, this analysis leads to new insights into the emergence of anomalies within the framework of conformal quantum mechanics. Finally, these ideas can be generalized beyond the two-dimensional case and for a more general modification of the ultraviolet physics; additional details are in progress and will be reported elsewhere.

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