Bucknell University
Bucknell Digital Commons

Honors Theses | Student Theses

2011

Equality of P-partition Generating Functions

Ryan Ward
Bucknell University

EQUALITY OF P-PARTITION GENERATING FUNCTIONS

by

Ryan Ward

A Thesis

Presented to the Faculty of

Bucknell University

in Partial Fulfillment of the Requirements for the Degree of

Bachelor of Science with Honors in Mathematics

May 2, 2011

Approved: ______________________

Peter McNamara

Thesis Advisor

____

Karl Voss

Chair, Department of Mathematics

Acknowledgments

First and foremost I would like to thank Professor Peter McNamara for his assistance and guidance over the past two years, without which this thesis would not have been possible. I would also like to thank the faculty, staff, and students of the Mathematics Department for making my studies at Bucknell University both productive and enjoyable.

Contents

List of Figures

Abstract

To every partially ordered set (poset), one can associate a generating function, known as the P-partition generating function. We find necessary conditions and sufficient conditions for two posets to have the same P-partition generating function. We define the notion of a jump sequence for a labeled poset and show that having equal jump sequences is a necessary condition for generating function equality. We also develop multiple ways of modifying posets that preserve generating function equality. Finally, we are able to give a complete classification of equalities among partially ordered sets with exactly two linear extensions.

Chapter 1

Introduction

Combinatorics is concerned with the study of finite sets. Two primary topics in combinatorics are ordered sets and generating functions. The focus of this thesis lies at the intersection of these topics.

Many mathematically important sets have a natural order associated with them. For example, the positive integers have the usual order $1 \leq 2 \leq 3$. A more interesting example is the set of subsets of the positive integers ordered by containment, e.g. $\{\} \subseteq \{1\} \subseteq \{1,2\}$. Note that in this example not every pair of elements can be compared. To illustrate, neither $\{1\}$ is contained in $\{2\}$ nor $\{2\}$ contained in $\{1\}$. We call this order a *partial* order since some pairs of elements are incomparable. The study of general partially ordered sets (posets) is one of the main branches of modern algebraic combinatorics.

Additionally, combinatorialists are often concerned with counting the number of elements in a particular finite set. Sometimes it is possible to derive an exact formula for the number of elements in a set. For example, the number of k-element subsets of

$\{1, 2, \ldots, n\}$ is known to be $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. However, it is often the case that it is more feasible or more useful to construct a polynomial or power series that encodes this counting information. To illustrate, in the polynomial $(1 + x)^n$, the coefficient of x^k equals the number of k-element subsets of $\{1, 2, \ldots, n\}$. A more interesting example is a power series that encodes the number of partitions of n, where a partition of n is a weakly decreasing sequence of positive integers whose sum is n. For example, the partitions of 4 are $(4), (3, 1), (2, 2), (2, 1, 1)$, and $(1, 1, 1, 1)$. We denote the number of unique partitions of n by $p(n)$. While no elementary formula for $p(n)$ exists, the generating function $\sum_{n=0}^{\infty} p(n)x^n$ can be shown to be the expression $\sum_{n=0}^{\infty} p(n)x^n = \prod_{i=1}^{\infty} \frac{1}{1-x^i}$. Polynomials or power series that encode counting information are known as generating functions.

Related to the set of partitions is the set of compositions of n, where a composition of n is a way of writing n as a sum of *unordered* positive integers. For example, $(1, 3, 2)$ is a composition of 6 but not a partition of 6, while $(3, 2, 1)$ is both a composition and partition of 6. In his 1971 Ph.D. thesis, Richard Stanley introduced P-partitions as a way to unify and interpolate between the ideas of partitions and compositions [7]. The "P" in P-partition denotes a partially ordered set. Each poset has an associated P-partition generating function, and these generating functions generalize many different combinatorial objects. Arguably the most important special case of P-partition generating functions are the skew Schur functions; classifying equalities among skew Schur functions is a topic of current research [6; 5; 1; 3]. This thesis is motivated by trying to examine the open question of skew Schur function equality in the more general P-partition generating function setting. The main goal of this thesis is to find necessary conditions and sufficient conditions for two partially ordered sets to have the same P-partition generating function.

The paper is structured as follows. In Chapter 2 we provide the necessary background on partially ordered sets, P-partitions, quasisymmetric functions, and P-partition generating functions. Chapter 3 collects initial results about P-partition generating function equality. In Chapter 4, we give some necessary conditions on the partially ordered sets for their generating functions to be equal. Given a P-partition generating function equality, we can modify the equivalent partially ordered sets in certain ways that preserve generating function equality; these sufficient conditions for equality are the subject of Chapter 5. In Chapter 6, we give a complete classification of equalities among partially ordered sets having exactly two linear extensions. Finally, in Chapter 7, we summarize our work and discuss avenues for further research.

Chapter 2

Mathematical Background

We first introduce the mathematical background and notation necessary to present our results. Further information about these subjects can be found in [8; 9].

2.1 Posets

Given a collection of comparable objects, it is natural to try to associate an order with them. We formalize this idea of ordering a collection of comparable objects with the notion of a partially ordered set (poset). Posets, and orderings in general, play a large role in combinatorics as they generalize many different combinatorial objects, as we will see in the case of P-partitions. We will be using posets to define generating functions whose equalities we examine.

Before defining a poset, let us help build the reader's intuition by introducing some example posets.

(a) The positive integers ordered by $\leq$ form a poset, i.e. $1 \leq 2 \leq 3 \leq \ldots$.

(b) Subsets of $\{a, b\}$ ordered by containment form a poset. For example, $\{\} \subseteq \{a\} \subseteq \{a, b\}$. Note that not every pair of subsets in comparable, since $\{a\}$ is not contained in $\{b\}$ and $\{b\}$ is not contained in $\{a\}$.

(c) A set of integers ordered by divisibility forms a poset. For example, 1 divides 3, 3 divides 6, and 6 divides 30. Note that while $3 \leq 10$ in the usual order on the positive integers, 3 does not divide 10 and is therefore 3 is not "less than" 10 in this poset.

We now give the formal definition of a poset. A *poset* is a set P equipped with a binary relation, usually denoted $\leq$, that has the following properties for all $x, y, z \in P$:

- $x \leq x$, i.e. the relation is reflexive;
- if $x \leq y$ and $y \leq x$, then $x = y$, i.e. the relation is antisymmetric;
- if $x \leq y$ and $y \leq z$, then $x \leq z$, i.e. the relation is transitive.

We will use $\leq_P$ to denote the relation on P to avoid confusion. If no subscript on the relation is given, we mean the usual $\leq$ ordering on the integers. If $x \leq_P y$ and $x \neq y$, we can write $x <_P y$. If there are no incomparable elements in the poset, we call the ordered set a *total order*.

A valuable tool for visualizing posets is the Hasse diagram of a poset. Firstly, we say an element y of the poset *covers* x if $x <_P y$ and there is no z such that $x <_P z <_P y$. For example 2 covers 1 in the positive integers ordered by $\leq$, but 3 does not cover 1. Now, a *Hasse diagram* of a poset is a diagram where each element of the poset is a node and a line is drawn from x up to y if y covers x (we do not consider reflexive relations in the Hasse diagram). The Hasse diagram of the previous example posets can be seen in Figure 2.1.

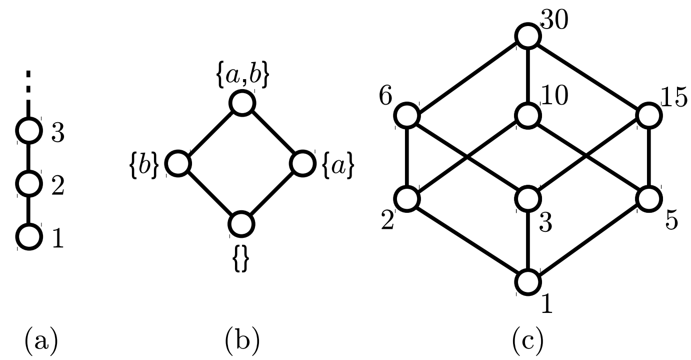


Figure 2.1: Three Hasse diagrams of various posets. The Hasse diagram (a) is the positive integers order by $\leq$, (b) is subsets of $\{a, b\}$ ordered by containment, and (c) is a subset of the integers ordered by divisibility. Note that in both (b) and (c) not every pair of elements is comparable.

We define an *antichain* to be a subset of P where every pair of elements in the subset is incomparable, while we define a *chain* to be a subset of P where every pair of elements in the subset is comparable. In Figure 2.1(c), we can see that $\{2, 3, 5\}$ is an antichain, while $\{1, 10, 30\}$ is a chain.

We now generalize the notion of posets to labeled posets. A *labeling* of a poset P is a bijection $\omega : P \to \{1, \ldots, n\}$. A poset with an associated labeling, denoted (P, ω), is a *labeled poset*. Note that with this definition of labeled poset, we restrict our study to finite posets.

For a labeled poset (P, ω), if y covers x and $\omega(x) < \omega(y)$, then we call the relation $x \leq_P y$ *weak* and will denote such relations in a Hasse diagram with a single line. On the other hand, if y covers x and $\omega(x) > \omega(y)$, then we call the relation $x \leq_P y$ *strong* and will denote such relations in a Hasse diagram with a double line. If a labeled poset consists only of weak relations, then we will say the labeled poset is *naturally labeled*.

An example of a labeled poset can be seen in Figure 2.2. Note that we will often omit the labeling from Hasse diagrams and only note strong and weak relations; the reason for this will be made clear in Section 2.2. We consider two labeled posets (P, ω) and (Q, τ) to be equal if there is a bijection from P to Q that preserves the order relation and the strong and weak relations.

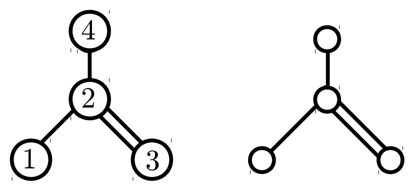


Figure 2.2: The Hasse diagram of a sample labeled poset with and without its associated labeling. This labeled poset has one strong relation and is therefore not naturally labeled.

2.2 P-partition Generating Functions

P-partitions are labelings of the elements of a poset that respect the order of the poset. They were introduced by Stanley as a way to interpolate between partitions and compositions of a number [7]. We will use them to define a generating function on labeled posets.

For a labeled poset (P, ω), a (P, ω)-*partition* is a map σ from P to the positive integers satisfying the following two conditions:

- If $x \leq_P y$, then $\sigma(x) \leq \sigma(y)$, i.e. σ is order-preserving.
- If $x \leq_P y$ and $\omega(x) > \omega(y)$, then $\sigma(x) < \sigma(y)$, i.e. the strong relations must be respected by σ.

Some example P-partitions of the labeled poset in Figure 2.2 can be seen in Figure 2.3. Note that we will denote the labeling of a poset with integers *inside* the nodes of the Hasse diagram (as in Figure 2.2), while a P-partition will be denoted with integers *outside* the nodes of the Hasse diagram (as in Figure 2.3).

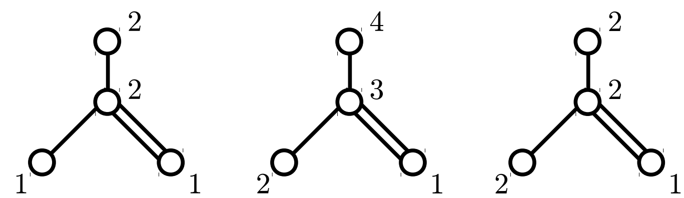


Figure 2.3: Example P-partitions of a labeled poset.

We can now see how P-partitions interpolate between partitions and compositions. P-partitions of a total order with n elements correspond to partitions with n parts, while P-partitions on an antichain with n elements correspond to compositions with n parts. An example of this for three element antichain and the three element total order can be seen in Figure 2.4. Essentially, the total order enforces the requirement that a partition is a weakly decreasing sequence.

(a) (b)

Figure 2.4: A P-partition of the three element antichain (a), which corresponds to the composition $(2, 1, 4)$. The P-partition of the three element total order (b) corresponds to the partition $(4, 2, 1)$.

We now define the main object of our study, P-partition generating functions.

Definition 2.1. *The* P-partition generating function *for a labeled poset* (P, ω) *is denoted* $K_{(P,\omega)}(x_1, x_2, \ldots)$ *and is given by*

$$K_{(P,\omega)}(x_1, x_2, \ldots) = \sum_{(P,\omega)-\text{partition } \sigma} x_1^{|\sigma^{-1}(1)|} x_2^{|\sigma^{-1}(2)|} \ldots, \tag{2.1}$$

where the sum is over all (P, ω)*-partitions* σ*.*

$K_{(P,\omega)}(x_1, x_2, \ldots)$ is a generating function for (P, ω)-partitions, where the number of (P, ω)-partitions σ such that $|\sigma^{-1}(1)| = i_1, |\sigma^{-1}(2)| = i_2, \ldots$ is given by the coefficient of the monomial $x_1^{i_1} x_2^{i_2} \ldots$ in $K_{(P,\omega)}(x_1, x_2, \ldots)$. Even though $K_{(P,\omega)}(x_1, x_2, \ldots)$ is a generating function for (P, ω)-partitions, we will refer to it as the P-partition generating function.

Note that $K_{(P,\omega)}(x_1, x_2, \ldots)$ is a formal power series in an infinite number of variables. We will restrict ourselves to n variables when necessary by denoting the generating function as $K_{(P,\omega)}(x_1, x_2, \ldots, x_n)$. A sample calculation of $K_{(P,\omega)}(x_1, x_2, x_3)$ for a labeled poset can be seen in Figure 2.5. When working in an infinite number of variables, we will abbreviate $K_{(P,\omega)}(x_1, x_2, \ldots)$ by $K_{(P,\omega)}(x)$.

Note that (P, ω)-partitions only rely on the assignment of strong and weak relations given by ω. This allows us to use the notation $K_P(x_1, x_2, \ldots)$ unambiguously when P is naturally labeled, since all labelings of P that are natural give rise to the same generating function. Furthermore, when discussing P-partition generating function equality, we can unambiguously only designate strong and weak relations in a Hasse diagram without specifying the labeling that gives rise to those strong and weak relations.

We can now describe the main goal of this work.

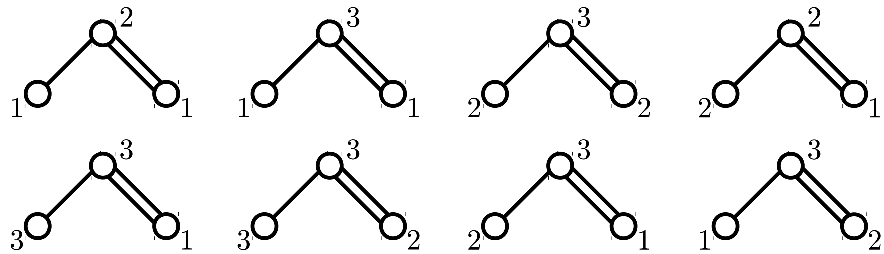


Figure 2.5: All P-partitions of a sample poset, restricting to the codomain $\{1,2,3\}$. Note that nodes on the Hasse diagrams are labeled by the P-partition σ and not the underlying labeling ω. Reading off P-partitions from left to right and top to bottom, we can see that, for the above poset, $K_{(P,\omega)}(x_1,x_2,x_3) = x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + x_1x_2^2 + x_1x_3^2 + x_2x_3^2 + 2x_1x_2x_3$.

Goal: To determine some necessary and some sufficient conditions on the labeled posets (P,ω) and (Q,τ) for $K_{(P,\omega)}(x)$ to equal $K_{(Q,\tau)}(x)$. Our aim is to make these conditions as complete as possible.

A nontrivial example case of when two labeled posets have equal P-partition generating functions can be seen in Figure 2.6.

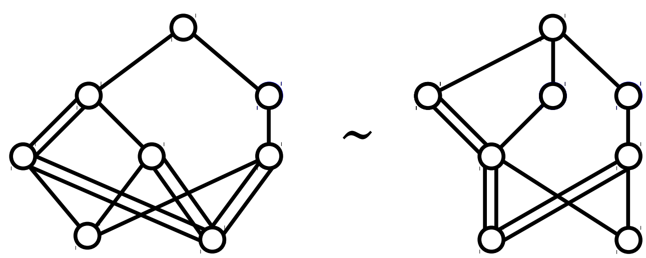


Figure 2.6: Two labeled posets that have the same P-partition generating function.

2.3 Quasisymmetric Functions

As we will see, P-partition generating functions are in a class of functions known as quasisymmetric functions. In this chapter we will develop the theory of quasisymmetric functions, first by starting with symmetric functions. This theory includes a result that will allow us to efficiently calculate $K_{(P,\omega)}(x)$ for any given (P, ω).

The theory of symmetric functions takes the idea of symmetry and applies it to polynomials. Symmetric functions are polynomials that are unchanged under permutation of their variables. For example, consider the polynomial $2x_1x_2 + 3x_1^5 + 3x_2^5$. If we permute the variables of this polynomial, we switch the x_1's and x_2's to get the polynomial $2x_2x_1 + 3x_2^5 + 3x_1^5$, which is in fact equal to the original polynomial. Hence, $2x_1x_2 + 3x_1^5 + 3x_2^5$ is a symmetric function in x_1, x_2. As another example, we can see that $3x_1 + 2x_2$ is not a symmetric function, since $3x_1 + 2x_2 \neq 3x_2 + 2x_1$. Symmetric functions are important objects in algebraic combinatorics and appear in many diverse areas of mathematics.

Our generating functions of interest will be quasisymmetric functions, of which symmetric functions are a subset. To define quasisymmetric functions, we first note that a polynomial is symmetric if, for any sequence of powers $a_1, \ldots, a_n$, the coefficient of $x_{i_1}^{a_1} x_{i_2}^{a_2} \ldots x_{i_n}^{a_n}$ is equal to the coefficient of $x_{j_1}^{a_1} x_{j_2}^{a_2} \ldots x_{j_n}^{a_n}$ for all distinct-element sequences $i_1, i_2, \ldots, i_n$ and $j_1, j_2, \ldots, j_n$. Using our example from before of the symmetric function $2x_2x_1 + 3x_1^5 + 3x_2^5$, we can see that the coefficients of x_1^5 and x_2^5 are both 3. Quasisymmetric functions are polynomials where the coefficient of $x_{i_1}^{a_1} \ldots x_{i_n}^{a_n}$ is equal to the coefficient of $x_{j_1}^{a_1} \ldots x_{j_n}^{a_n}$ whenever $i_1 < \ldots < i_n$ and $j_1 < \ldots < j_n$. This condition is a weakening of the symmetric function condition, hence every symmetric function is quasisymmetric, but not every quasisymmetric function is symmetric.

To illustrate, the polynomial $4x_1^7x_2^5 + 4x_1^7x_3^5 + 4x_2^7x_3^5$ is a quasisymmetric function in x_1, x_2, x_3 since the coefficients of $x_1^7x_2^5$, $x_1^7x_3^5$, $x_2^7x_3^5$ are all equal to 4, but not a symmetric function as switching x_1 and x_2 gives us a new polynomial.

We are now able to show that P-partition generating functions are quasisymmetric functions.

Proposition 2.2. *For any labeled poset (P, ω), we have that $K_{(P,\omega)}(x)$ is quasisymmetric.*

Proof. Let $a_1, \ldots, a_k$, $i_1 < \ldots < i_k$, and $j_1 < \ldots < j_k$ be given. Note that (P, ω)-partitions that give rise to the $x_{i_1}^{a_1} \ldots x_{i_k}^{a_k}$ monomial in Equation (2.1) bijectively correspond to (P, ω)-partitions that give rise to $x_{j_1}^{a_1} \ldots x_{j_k}^{a_k}$; this correspondence is given by replacing corresponding i_r with j_r in the (P, ω)-partition for all $1 \leq r \leq k$, and this correspondence preserves the conditions of a (P, ω)-partition. $\square$

The algebra of quasisymmetric functions has two natural bases. The first basis is the monomial quasisymmetric function basis $\{M_\alpha\}$, indexed by compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ where α is considered to be a composition of $\sum \alpha_i$. A monomial quasisymmetric function is given by

$$M_\alpha = \sum_{1 \leq i_1 < i_2 < \ldots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \ldots x_{i_k}^{\alpha_k}.$$

For example, in three variables we have $M_{(1,2)} = x_1x_2^2 + x_1x_3^2 + x_2x_3^2$.

Before defining the second basis, we first note that a composition β is a refinement of a composition α if α can be obtained from β by adding together adjacent parts of the composition β. If β is a refinement of α, we denote this by $\alpha \succeq \beta$. For example, $(3, 2) \succeq (2, 1, 1, 1)$, but $(1, 4) \not\succeq (2, 1, 1, 1)$.

Another basis for the quasisymmetric functions is the fundamental quasisymmetric function basis $\{L_\alpha\}$. A fundamental quasisymmetric function is given by

$$L_\alpha = \sum_{\beta \preceq \alpha} M_\beta.$$

For example, we have $L_{(3)} = M_{(3)} + M_{(2,1)} + M_{(1,2)} + M_{(1,1,1)}$.

We now make note of an important, non-trivial relation between the labeled poset (P, ω) and the expansion of $K_{(P,\omega)}(x)$ in the fundamental quasisymmetric functions. Before presenting this result, we must first introduce some new ideas. Firstly, the descent set of a permutation $\alpha \in S_n$, denoted $\mathrm{des}(\alpha)$, is $\mathrm{des}(\alpha) = \{i : \alpha(i) > \alpha(i+1)\}$, where $\alpha(i)$ is the i^{th} number in the permutation. For example, for $(3, 1, 2, 5, 4, 6) \in S_6$, we have $\mathrm{des}(3, 1, 2, 5, 4, 6) = \{1, 4\}$. Now, the descent composition of a permutation $\alpha \in S_n$, denoted $\mathrm{co}(\alpha)$, is $\mathrm{co}(\alpha) = (d_1, d_2 - d_1, \ldots, d_k - d_{k-1}, n - d_k)$, where $d_1 < d_2 < \ldots < d_k$ are the elements of $\mathrm{des}(\alpha)$. Continuing with our example, we have $\mathrm{co}(3, 1, 2, 5, 4, 6) = (1, 3, 2)$.

We must now also introduce the idea of linear extensions of a labeled poset. A linear extension of a labeled poset (P, ω) is a permutation of the labels $\omega(P)$ that respects the relations in P. The set of all linear extensions of (P, ω) is denoted $\mathcal{L}(P, \omega)$. The calculation of $\mathcal{L}(P, \omega)$ for a sample labeled poset can be seen in Figure 2.7.

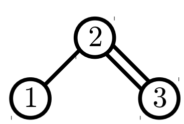


Figure 2.7: The Hasse diagram of a labeled poset (P, ω), where $\omega(P)$ is apparent on the diagram. We can see that $\mathcal{L}(P, \omega) = \{(1, 3, 2), (3, 1, 2)\}$.

With the above definitions, we can now state the following theorem, due to Gessel and Stanley.

Theorem 2.3 ([2; 9])**.** *Let (P, ω) be a labeled poset. Then we have*

$$K_{(P,\omega)}(x) = \sum_{\alpha \in \mathcal{L}(P,\omega)} L_{\mathrm{co}(\alpha)}. \tag{2.2}$$

It is important to note that since the fundamental quasisymmetric functions form a basis for quasisymmetric functions, we can make the crucial observation that P-partition generating functions are equal if and only if the multisets of the descent compositions of the linear extensions of their corresponding labeled posets are equal. A calculation that shows $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x)$ for sample posets (P, ω) and (Q, τ) can be seen in Figure 2.8. Note that we will denote Hasse diagrams that correspond to labeled posets that have the same P-partition generating function by the equivalence relation $\sim$. Note that when computing $K_{(P,\omega)}(x)$, we will often take advantage of Theorem 2.3, as generally a labeled poset will have significantly fewer linear extensions than P-partitions.

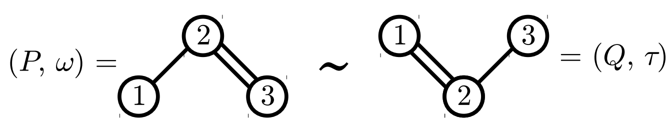


Figure 2.8: The Hasse diagrams of a labeled posets (P, ω) and (Q, τ). We can see that $\mathcal{L}(P, \omega) = \{(1, 3, 2), (3, 1, 2)\}$, so $K_{(P,\omega)}(x) = L_{\mathrm{co}(1,3,2)} + L_{\mathrm{co}(3,1,2)} = L_{(2,1)} + L_{(1,2)}$. We can also see that $\mathcal{L}(Q, \tau) = \{(2, 1, 3), (2, 3, 1)\}$, so $K_{(Q,\tau)}(x) = L_{\mathrm{co}(2,1,3)} + L_{\mathrm{co}(2,3,1)} = L_{(1,2)} + L_{(2,1)}$, which is equal to $K_{(P,\omega)}$.

We will switch between the expansions of $K_{(P,\omega)}(x)$ given by Equation (2.1) and Equation (2.2) as convenient.

An important subclass of P-partition generating functions are the skew Schur functions, which are symmetric functions that arise from a generalization of the Schur basis for symmetric functions. While we don't give the necessary definitions here, we note that classifying equalities among skew Schur functions is a topic of current research [1; 6; 5; 3], and we hope to shed light on this question by studying it in the more general P-partition generating function setting. We refer the interested reader to [9, Ch. 7] for an introduction to skew Schur functions.

Chapter 3

Initial Results

In this chapter we will collect some basic results about P-partition generating functions. These results follow almost immediately from the definition of $K_{(P,\omega)}$, and are already known or are generalizations of basic facts about skew Schur functions.

Proposition 3.1. *If* $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x)$, *then* $|P| = |Q|$.

Proof. Follows immediately from Equation (2.1) since the degrees of monomials in the sum must match. $\square$

Proposition 3.2. *If* $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x)$, *then* $|\mathcal{L}(P,\omega)| = |\mathcal{L}(Q,\tau)|$.

Proof. Follows immediately from Equation (2.2) and the fact that fundamental quasisymmetric functions form a basis for quasisymmetric functions. $\square$

Proposition 3.3. *Suppose that* (P,ω) *is naturally labeled and* $K_{(P,\omega)} = K_{(Q,\tau)}$. *Then* (Q,τ) *is naturally labeled.*

Proof. The proof follows from the observation that $L_{|P|}$ appears with nonzero coefficient in the Equation (2.2) expansion of $K_{(P,\omega)}$ if and only if (P,ω) is naturally labeled, which we now show. We can see that $L_{|P|}$ is in the fundamental quasisymmetric function expansion of $K_{(P,\omega)}$ if and only if $(1, 2, \ldots, |P|) \in \mathcal{L}(P, \omega)$. The previous statement is equivalent to saying there does not exist $x, y \in P$ such that $x \leq_P y$ and $\omega(x) > \omega(y)$, which is the case if and only if (P, ω) is naturally labeled. $\square$

We let the disjoint union of posets (P, ω) and (Q, τ), denoted $(P + Q, \omega + \tau)$, be the set $P \cup Q$ with the relation given by $x \leq_{P+Q} y$ if $x, y \in P$ and $x \leq_P y$, or if $x, y \in Q$ and $x \leq_Q y$, where the labeling $\omega + \tau$ is defined by $(\omega + \tau)|_P(x) = \omega(x)$ and $(\omega + \tau)|_Q(x) = \tau(x) + |P|$.

An example of a disjoint union poset can be seen in Figure 3.1.

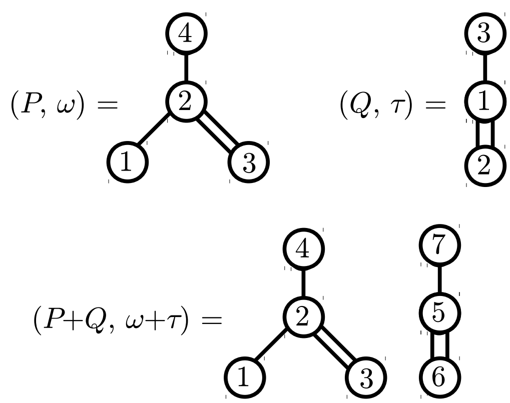


Figure 3.1: $(P + Q, \omega + \tau)$ for sample labeled posets (P, ω) and (Q, τ).

Proposition 3.4. *If (P, ω) and (Q, τ) are labeled posets, then $K_{(P+Q,\omega+\tau)}(x) = K_{(P,\omega)}(x)K_{(Q,\tau)}(x)$.*

Proof. After writing $K_{(P,\omega)}(x)$ and $K_{(Q,\tau)}(x)$ as in Equation (2.1), consider the product $K_{(P,\omega)}(x)K_{(Q,\tau)}(x)$ expanded as a sum of monomials. Each monomial is the product of a monomial coming from a (P, ω)-partition with a monomial coming from a (Q, τ)-partition. Thus each monomial in $K_{(P,\omega)}(x)K_{(Q,\tau)}(x)$ corresponds to a unique

$(P+Q,\omega+\tau)$-partition, and we see that every $(P+Q,\omega+\tau)$-partition arises in this way. Thus, we conclude $K_{(P+Q,\omega+\tau)}(x) = K_{(P,\omega)}(x)K_{(Q,\tau)}(x)$. □

Given a labeled poset (P,ω), we can perform some natural involutions on the poset [4]. We can "reverse" the poset, i.e. reversing the direction of all relations while leaving strong relations strong and weak relations weak. This corresponds to rotating the Hasse diagram 180 degrees. We will denote the rotated poset $(P,\omega)^*$.

Additionally, we can switch the weak and strong relations, i.e. let $\omega(x)$ become $|P|+1-\omega(x)$. We will denote this operation by $\overline{(P,\omega)}$. An example of these operations can be seen in the commutative diagram Figure 3.2.

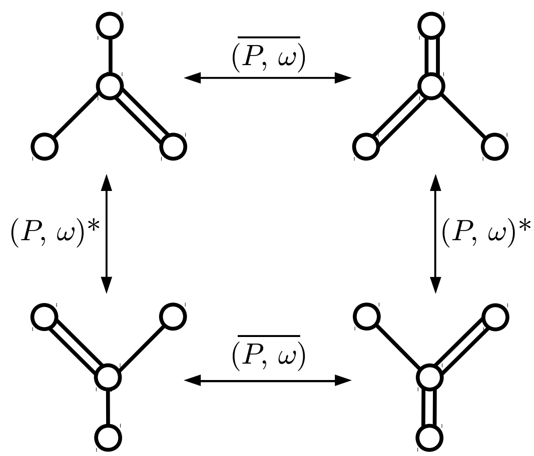


Figure 3.2: (P,ω), $\overline{(P,\omega)}$, $(P,\omega)^*$, and $(\overline{(P,\omega)})^*$ for a sample poset.

The following result is clear.

Proposition 3.5. *The operations $\overline{(P,\omega)}$ and $(P,\omega)^*$ are involutions, that is, $\overline{(\overline{(P,\omega)})} = (P,\omega)$ and $((P,\omega)^*)^* = (P,\omega)$. Furthermore, the operations commute, that is, $(\overline{(P,\omega)})^* = \overline{(P,\omega)^*}$.*

We can see what these involutions do to the fundamental quasisymmetric function expansion given by Equation (2.2) by reasoning what the involutions do to the descent sets of linear extensions.

Proposition 3.6. *The descent sets of the linear extensions of* $\overline{(P,\omega)}$ *are the complements of the descent sets of the linear extensions of* (P,ω).

Proof. By definition the labeling $\tau : P \to \{1,\ldots,|P|\}$ given by $\tau(x) = |P|+1-\omega(x)$ is such that $\overline{(P,\omega)} = (P,\tau)$. We can see that $\omega(x) > \omega(y)$ if and only if $\tau(x) < \tau(y)$. Hence, every linear extension $\alpha \in \mathcal{L}(P,\omega)$ has a corresponding linear extension $\overline{\alpha} \in \mathcal{L}(P,\tau)$, where there is a descent in the i^{th} position in $\overline{\alpha}$ if and only if there is no descent in the i^{th} position in α. □

Similar logic as Proposition 3.6 gives the following result.

Proposition 3.7. *The descent compositions of the linear extensions of* $(P,\omega)^*$ *are the reverse of the descent compositions of the linear extensions of* (P,ω).

Combining Proposition 3.6 and Proposition 3.7, we see the following.

Proposition 3.8. *For labeled posets* (P,ω) *and* (Q,τ), *the following are equivalent:*

- $K_{(P,\omega)} = K_{(Q,\tau)}$
- $K_{(P,\omega)^*} = K_{(Q,\tau)^*}$
- $K_{\overline{(P,\omega)}} = K_{\overline{(Q,\tau)}}$
- $K_{\overline{(P,\omega)^*}} = K_{\overline{(Q,\tau)^*}}$

Proof. This follows from Proposition 3.6, Proposition 3.7, and the observation that $K_{(P,\omega)} = K_{(Q,\tau)}$ if and only if the multisets of the descent compositions of the $\mathcal{L}(P,\omega)$ and $\mathcal{L}(Q,\tau)$ are equal, which is a direct consequence of Equation (2.2). □

Chapter 4

Necessary Conditions for Equality

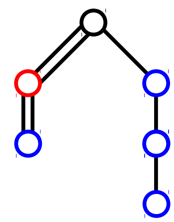
Necessary conditions are conditions on labeled posets that must hold for their corresponding P-partition generating functions to be equal. Practically, they allow us to determine that $K_{(P,\omega)}(x) \neq K_{(Q,\tau)}(x)$ by seeing that a necessary condition on the labeled posets does not hold. Ideally, checking the necessary condition should be much simpler than calculating the labeled posets' generating functions.

We first define the notion of jump sequences, which will give us a necessary condition for P-partition generating function equality.

Definition 4.1. *Let the* jump *of an element be the maximum number of strong relations in a saturated chain from the element down to a minimal element. The* jump sequence *of a labeled poset, denoted* $\mathrm{jump}(P,\omega)$*, is* $\mathrm{jump}(P,\omega) = (j_0, \ldots, j_k)$*, where* j_i *equals the number of elements with jump* i *and* k *is the maximal jump of an element in* P*.*

For an example of the jump sequence of a labeled poset, see Figure 4.1.

Before proving that a necessary condition for P-partition generating function



(4, 1, 1)

Figure 4.1: The jump sequence of a sample labeled poset.

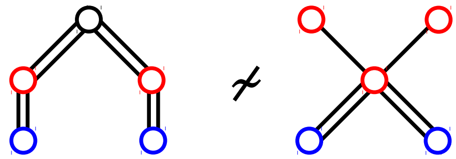
equality is equal jump sequences, we note that the lexicographic order on monomials is given by first comparing the exponents of x_1 in the monomials. The monomial with the lower exponent on x_1 is considered greater. If the exponents on x_1 are equal, then the exponents of x_2 are compared, and so on. For example, in the lexicographic order, $x_1^2x_2x_3^4x_4$ is less than $x_1^2x_2x_3^2x_4^3$.

Proposition 4.2. *If* $K_{(P,\omega)} = K_{(Q,\tau)}$*, then we have* $\mathrm{jump}(P,\omega) = \mathrm{jump}(Q,\tau)$*.*

Proof. Consider a greedy (P,ω)-partition σ_g such that $\sigma_g(x)$ is minimal for all $x \in P$. Then $\mathrm{jump}(P,\omega) = (|\sigma_g^{-1}(1)|,\ldots,|\sigma_g^{-1}(|P|)|)$, which corresponds directly to $x_1^{|\sigma_g^{-1}(1)|}\ldots x_n^{|\sigma_g^{-1}(|P|)|}$, the lexicographically minimal monomial in the Equation (2.1) expansion $K_{(P,\omega)}(x)$. If $\mathrm{jump}(P,\omega) \neq \mathrm{jump}(Q,\tau)$, then $K_{(P,\omega)} \neq K_{(Q,\tau)}$ as their Equation (2.1) expansions will contain different lexicographically minimal monomials. $\square$

See Figure 4.2 for an application of Proposition 4.2. We now note a corollary of Proposition 4.2.

Corollary 4.3. *If P and Q are naturally labeled posets and $K_P(x) = K_Q(x)$, then P and Q have the same number of minimal elements, and P and Q have the same number of maximal elements.*



$(2, 2, 1) \neq (2, 3)$

Figure 4.2: Two labeled posets that cannot have the same P-partition generating function because they have different jump sequences.

Proof. By Proposition 3.8 we have $K_{\overline{P}}(x) = K_{\overline{Q}}(x)$. We can see that the first entry in $\mathrm{jump}(\overline{P}) = \mathrm{jump}(\overline{Q})$ corresponds to the number of minimal elements of P and Q.

Also, by Proposition 3.8 we have $K_{\overline{P^*}}(x) = K_{\overline{Q^*}}(x)$. We can see that the first entry in $\mathrm{jump}(\overline{P^*}) = \mathrm{jump}(\overline{Q^*})$ corresponds to the number of maximal elements of P and Q. □

Note that Corollary 4.3 is not true for labeled posets in general; see Figure 2.8 for an example.

The following is also a direct corollary of Proposition 4.2.

Corollary 4.4. *If P and Q are naturally labeled posets and $K_P(x) = K_Q(x)$, then P and Q have the same maximal chain length.*

Proof. By Proposition 3.8 we have $K_{\overline{P}}(x) = K_{\overline{Q}}(x)$. We can see that the length of the composition $\mathrm{jump}(\overline{P}) = \mathrm{jump}(\overline{Q})$ corresponds to the maximal size chain in P and Q. □

We now define the notion of an antichain sequence to give a conjecture for naturally labeled posets.

Definition 4.5. *Let the antichain sequence of a poset, denoted* $\mathrm{anti}(P)$*, be the finite sequence $(a_1, \ldots, a_n)$, where a_i is the number of antichains in P of size i.*

Conjecture 4.6. *If P and Q are naturally labeled posets and $K_P(x) = K_Q(x)$, then* $\mathrm{anti}(P) = \mathrm{anti}(Q)$.

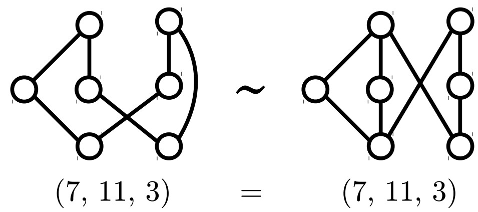


Figure 4.3: Two naturally labeled posets that have the same P-partition generating function and also the same antichain sequence.

Conjecture 4.6 suggests a constructive proof where, given the P-partition generating function, we determine the antichain sequence by utilizing the expansion given by Equation (2.2). In trying to create such a proof, the hardest examples occur when the original poset is actually the disjoint union of two different posets.

Chapter 5

Sufficient Conditions for Equality

Sufficient conditions for P-partition generating function equality give us ways of producing new equalities. Most of the results in this section will take the following form: if $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x)$, then $K_{(\widehat{P},\widehat{\omega})}(x) = K_{(\widehat{Q},\widehat{\tau})}(x)$, where $(\widehat{P},\widehat{\omega})$ and $(\widehat{Q},\widehat{\tau})$ are modifications of (P,ω) and (Q,τ) respectively.

5.1 Removing from Known Equalities

We now describe cases where we can take a P-partition generating equality and make both posets simpler while maintaining equality. Our first result involves removing all jump 0 elements from both labeled posets (see Chapter 4 for the definition of a jump sequence).

Proposition 5.1. *If $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x)$, then, letting $(\widehat{P},\widehat{\omega})$ and $(\widehat{Q},\widehat{\tau})$ be the labeled posets (P,ω) and (Q,τ) with their jump 0 elements removed, we have $K_{(\widehat{P},\widehat{\omega})}(x) = K_{(\widehat{Q},\widehat{\tau})}(x)$.*

Proof. Consider all (P,ω)-partition σ_P such that $|\sigma_P^{-1}(1)|$ is maximal. These (P,ω)-partitions bijectively correspond to monomials in the Equation (2.1) expansion of $K_{(P,\omega)}(x)$, where the exponent of the x_1 term is equal to the number of elements with jump 0 in (P,ω). By Proposition 4.2, the number of elements with jump 0 in (P,ω) is equal to the number of elements with jump 0 in (Q,τ). In turn, this gives a bijection between monomials in $K_{(Q,\tau)}(x)$ where the exponent of the x_1 term is equal to the number of elements with jump 0 and (Q,τ)-partitions σ_Q such that $|\sigma_Q^{-1}(1)|$ is maximal.

(P,ω)-partitions σ_P where $|\sigma_P^{-1}(1)|$ is maximal correspond bijectively to $(\widehat{P},\widehat{\omega})$-partitions $\widehat{\sigma}_P$; the correspondence is described by taking $\widehat{\sigma}_P(x) = \sigma_P(x) - 1$. Furthermore, if σ_P and σ_Q correspond to the same monomial $x_1^{i_1}x_2^{i_2}\dots x_k^{i_k}$ in $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x)$, then $\widehat{\sigma}_P$ and $\widehat{\sigma}_Q$ will correspond to the same monomial $x_1^{i_2}x_2^{i_3}\dots x_{k-1}^{i_k}$ in $K_{(\widehat{P},\widehat{\omega})}(x)$ and $K_{(\widehat{Q},\widehat{\tau})}(x)$. Hence $K_{(\widehat{P},\widehat{\omega})}(x) = K_{(\widehat{Q},\widehat{\tau})}(x)$. $\square$

An example of Proposition 5.1 can be seen in Figure 5.1.

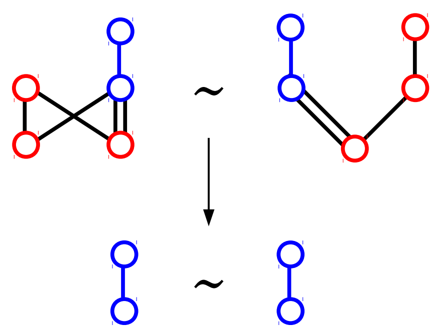


Figure 5.1: Removing jump 0 elements (in red) preserves P-partition generating function equality.

By induction, we get the following corollary of Proposition 5.1.

Corollary 5.2. *If $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x)$, then, letting $(\widehat{P},\widehat{\omega})$ and $(\widehat{Q},\widehat{\tau})$ be the labeled* *posets* (P, ω) *and* (Q, τ) *with their jump* i *elements removed for all* $i < n$ *for some positive integer* n*, we have* $K_{(\widehat{P},\widehat{\omega})}(x) = K_{(\widehat{Q},\widehat{\tau})}(x)$.

We now note a consequence of Proposition 5.1 for naturally labeled posets.

Corollary 5.3. *Let* $\widehat{P}$ *and* $\widehat{Q}$ *be the naturally labeled posets* P *and* Q *with their minimal elements removed. If* $K_P(x) = K_Q(x)$*, then we have* $K_{\widehat{P}}(x) = K_{\widehat{Q}}(x)$.

Proof. By Proposition 3.8 we have $K_{\overline{P}}(x) = K_{\overline{Q}}(x)$. The minimal elements of P and Q correspond to the jump 0 elements of $\overline{P}$ and $\overline{Q}$. By Proposition 5.1, if we let $\widehat{\overline{P}}$ and $\widehat{\overline{P}}$ be the posets $\overline{P}$ and $\overline{Q}$ with their jump 0 elements removed (which corresponds to removing the minimal elements of P and Q), we have $K_{\widehat{\overline{P}}}(x) = K_{\widehat{\overline{P}}}(x)$. Since $\widehat{\overline{P}} = \overline{\widehat{P}}$, by Proposition 3.8 we see that $K_{\widehat{P}}(x) = K_{\widehat{Q}}(x)$. $\square$

An example of Corollary 5.3 can be seen in Figure 5.2.

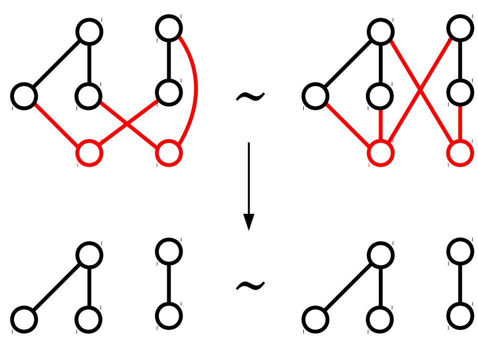


Figure 5.2: Removing the minimal elements (in red) of naturally labeled posets preserves P-partition generating function equality.

Returning to the generally labeled case, if two labeled posets have equal P-partition generating functions and both posets have unique minimal elements, then we can remove those minimal elements and preserve equality. This result will be important in Chapter 6.

Proposition 5.4. *If $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x)$ and both P and Q have unique minimal elements, then, letting $(\widehat{P}, \widehat{\omega})$ and $(\widehat{Q}, \widehat{\tau})$ be the labeled posets (P, ω) and (Q, τ) with their unique minimal elements removed, we have $K_{(\widehat{P},\widehat{\omega})}(x) = K_{(\widehat{Q},\widehat{\tau})}(x)$.*

Proof. This proof is similar to the proof of Proposition 5.1, except we consider the (P, ω)-partitions σ_P where $|\sigma_P^{-1}(1)| = 1$. This is because if $|\sigma_P^{-1}(1)| = 1$, then it must be the case that σ_P of the unique minimal element is 1. By letting $\widehat{\sigma}_P(x) = \sigma_P(x) - 1$, (P, ω)-partitions and (Q, τ)-partitions that correspond to the same monomial in $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x)$ descend to $(\widehat{P}, \widehat{\omega})$-partitions and $(\widehat{Q}, \widehat{\tau})$-partitions that correspond to the same monomial in $K_{(\widehat{P},\widehat{\omega})}(x) = K_{(\widehat{Q},\widehat{\tau})}(x)$.

□

Note that by utilizing Proposition 3.8, Proposition 5.4 allows us to remove unique maximal elements as well.

5.2 Adding to Known Equalities

We now develop ways of adding additional elements to labeled posets that preserves P-partition generating equality. Proposition 5.4 suggests a way of adding minimal elements to posets whose generating functions are known to be equal.

Proposition 5.5. *Suppose that $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x)$. Let m be the number of weak relation minimal elements we wish to add, and let $\overline{m}$ be the number of strong relation minimal elements we wish to add. Let $(\widehat{P}, \widehat{\omega})$ and $(\widehat{Q}, \widehat{\tau})$ be the labeled posets (P, ω) and (Q, τ) with the minimal elements M and $\overline{M}$ added; that is, letting $M = \{1, \ldots, m\}$ and $\overline{M} = \{|P| + m + 1, \ldots, |P| + m + \overline{m}\}$, then $\widehat{P} = P \cup M \cup \overline{M}$ with*

$x \leq_{\widehat{P}} y$ for all $x \in M \cup \overline{M}$ and $y \in P$, where $\widehat{\omega}|_M(x) = x$, $\widehat{\omega}|_P(x) = \omega(x) + m$, and $\widehat{\omega}|_{\overline{M}}(x) = x$ (See Figure 5.3). Then, we have $K_{(\widehat{P},\widehat{\omega})}(x) = K_{(\widehat{Q},\widehat{\tau})}(x)$.

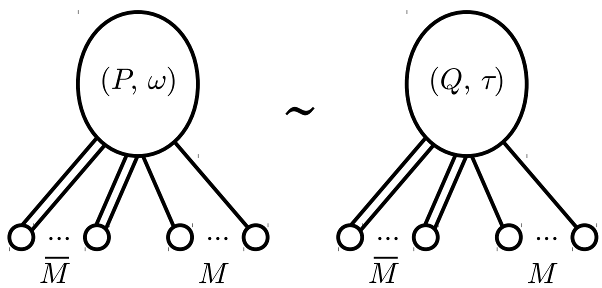


Figure 5.3: The setup for the proof of Proposition 5.5.

Proof. Consider linear extensions of $(\widehat{P}, \widehat{\omega})$. Since the elements in M and $\overline{M}$ come before P in $\widehat{P}$, every linear extension of $(\widehat{P}, \widehat{\omega})$ is of the form $(\mathcal{L}(M + \overline{M}), \mathcal{L}(P))$; i.e. every linear extension of $(\widehat{P}, \widehat{\omega})$ must start with all the elements in $M \cup \overline{M}$. Let $(\alpha_1, \alpha_2) \in \mathcal{L}(\hat{P}, \hat{\omega})$, where $\alpha_1 \in \mathcal{L}(M + \overline{M})$ and $\alpha_2 \in \mathcal{L}(P, \omega)$. Since $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x)$, by Equation (2.2) there exists $\beta_2 \in \mathcal{L}(Q, \tau)$ such that $\mathrm{co}(\alpha_2) = \mathrm{co}(\beta_2)$. Note that $(\alpha_1, \beta_2) \in \mathcal{L}(\widehat{Q}, \widehat{\tau})$.

We want to show that $\mathrm{co}(\alpha_1, \alpha_2) = \mathrm{co}(\alpha_1, \beta_2)$, as this would prove $K_{(\widehat{P},\widehat{\omega})}(x) = K_{(\widehat{Q},\widehat{\tau})}(x)$ by Equation (2.2). We know that $\mathrm{co}(\alpha_1) = \mathrm{co}(\alpha_1)$ and $\mathrm{co}(\alpha_2) = \mathrm{co}(\beta_2)$. Either α_1 ends in an element from M or $\overline{M}$. If α_1 ends in an element from M, then $\mathrm{co}(\alpha_1, \alpha_2) = \mathrm{co}(\alpha_1, \beta_2)$ with no descent between α_1 and both α_2 and β_2. Otherwise α_1 ends in an element from $\overline{M}$ and then $\mathrm{co}(\alpha_1, \alpha_2) = \mathrm{co}(\alpha_1, \beta_2)$ with a descent between α_1 and both α_2 and β_2. $\square$

For an example of Proposition 5.5, see Figure 5.4.

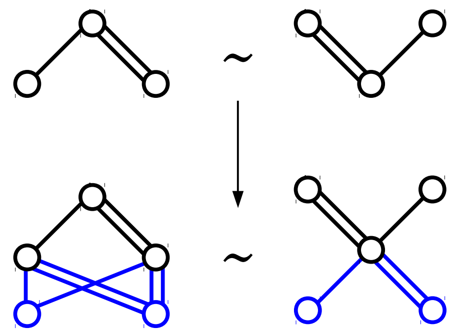


Figure 5.4: Adding one weak relation minimal element and one strong relation minimal element to a known equality.

We can generalize Proposition 5.5 so that, instead of considering adding antichains as minimal elements, we can add on arbitrary posets. The proof of the following result is similar to the proof of Proposition 5.5, although it would require much more meticulous notation.

Proposition 5.6. *Suppose* $K_{(P,\omega)}(x) = K_{(P',\omega')}(x)$, $K_{(Q,\tau)}(x) = K_{(Q',\tau')}(x)$, *and* $K_{(R,\phi)}(x) = K_{(R',\phi')}(x)$. *Then, the labeled posets in Figure 5.5 have the same* P*-partition generating function.*

We now note the following special relation among P-partition generating functions that involves different ways of joining two posets.

Proposition 5.7. *For labeled posets* (P,ω) *and* (Q,τ)*, if* P *has a unique minimal element* p *and* Q *a unique maximal element* q*, then, using the notation from Figure 5.6, we have* $K_A(x) = K_B(x) + K_C(x)$.

Proof. We use the notation in Figure 5.6. Consider an A-partition σ that corresponds to a monomial in the Equation (2.1) expansion of $K_A(x)$. It is the case that either $\sigma(p) \leq \sigma(q)$ or $\sigma(p) > \sigma(q)$, but not both.

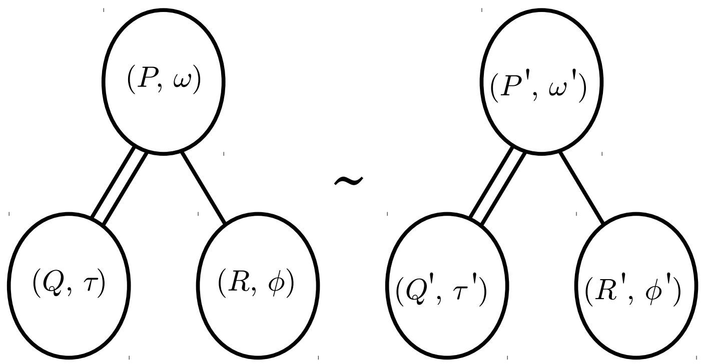


Figure 5.5: Structure of labeled posets with the same P-generating function as mentioned in Proposition 5.6. It is understood that, for example, $q \leq p$ in the left labeled poset for all $q \in Q$ and $p \in P$, and that the poset is labeled is such so that these are strong relations.

If $\sigma(p) \leq \sigma(q)$, then σ is a B-partition and therefore the same corresponding monomial appears in $K_B(x)$. Otherwise, if $\sigma(p) > \sigma(q)$, then σ is a C-partition and therefore the same corresponding monomial appears in $K_C(x)$. We can also see that every B-partition and every C-partition arise in this way from an A-partition. Hence, since every monomial in $K_A(x)$ is in $K_B(x)$ or $K_C(x)$ but not both, we have $K_A(x) = K_B(x) + K_C(x)$, as desired. □

Corollary 5.8. *Suppose $K_{(P,\omega)}(x) = K_{(P',\omega')}(x)$ and $K_{(Q,\tau)}(x) = K_{(Q',\tau')}(x)$. Then, using the notation in Figure 5.6, $K_B(x)$ with (P,ω) and (Q,τ) is equal to $K_B(x)$ with (P',ω') and (Q',τ').*

Proof. From Proposition 5.7, we can conclude that, in Figure 5.6, $K_B(x) = K_A(x) - K_C(x)$. We can see $K_A(x)$ remains equal under switching (P,ω) and (Q,τ) with (P',ω') and (Q',τ') by Proposition 3.4. We can also see that $K_C(x)$ remains equal under switching (P,ω) and (Q,τ) with (P',ω') and (Q',τ') by Proposition 5.6. Hence, since $K_B(x) = K_A(x) - K_C(x)$, $K_B(x)$ also remains equal under switching (P,ω) and

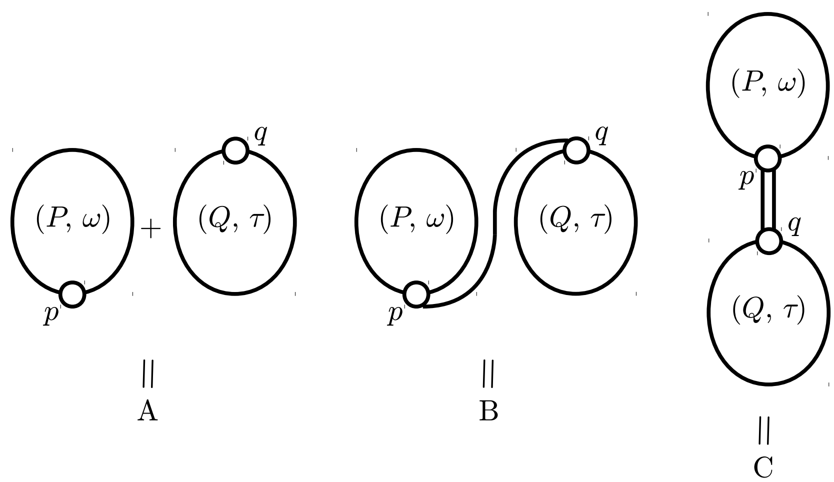


Figure 5.6: Structure of labeled posets A, B, and C as mentioned in Proposition 5.7.

(Q, τ) with (P', ω') and (Q', τ'), as desired. $\square$

Chapter 6

Posets with Two Linear Extensions

We know that from Proposition 3.2 that if two labeled posets satisfy $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x)$, then they have the same number of linear extensions. First, suppose that (P,ω) and (Q,τ) have just one linear extension each. In this case, (P,ω) and (Q,τ) must each be a total order, so $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x) = L_\alpha$, with α determined by the strong and weak relations in the total order. It follows that $(P,\omega) = (Q,\tau)$. Therefore, if we fix the number of linear extensions of $(P,\omega) = (Q,\tau)$, the first interesting case occurs when both labeled posets have two linear extensions.

In this chapter we give a complete classification of P-partition generating function equalities among labeled posets having exactly two linear extensions. It turns out that all equalities among labeled posets with two linear extensions can be derived from the equality of the three element labeled posets in Figure 6.1, which happens to correspond to a skew Schur function equality. The main result from this chapter is the following:

Theorem 6.1. *Suppose that* $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x)$ *and* $|\mathcal{L}(P,\omega)| = |\mathcal{L}(Q,\tau)| = 2.$

Figure 6.1: Two labeled posets with the same P-partition generating function, from which all equalities among labeled posets with two linear extensions can be derived.

Then either $(P, \omega) = (Q, \tau)$ or (P, ω) and (Q, τ) can be constructed from the equality in Figure 6.1 by repeatedly applying the following operations, all which preserve P-partition generating function equality by Proposition 5.5:

- *Add one minimal element to both labeled posets using a weak relation.*
- *Add one minimal element to both labeled posets using a strong relation.*
- *Add one maximal element to both labeled posets using a weak relation.*
- *Add one maximal element to both labeled posets using a strong relation.*

Intuitively, Theorem 6.1 implies all nontrivial equalities are obtained from the equality in Figure 6.1 by adding the same chains to the top and the same chains to the bottom of each poset. An example consequence of this result can be seen in Figure 6.2.

Before proving Theorem 6.1, we will need to describe the structure of posets with two linear extensions.

Lemma 6.2. *If $|\mathcal{L}(P)| = 2$, then P has exactly one antichain of size 2 and no antichains of greater size.*

Proof. If P has an antichain of size greater than 2, then $|\mathcal{L}(P)| \geq 6$, a contradiction. If P has two or more antichains of size 2, then $|\mathcal{L}(P)| \geq 3$, a contradiction. If P

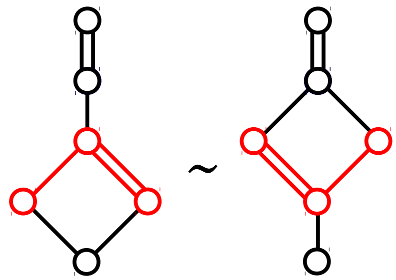


Figure 6.2: Two labeled posets with two linear extensions that have the same P-partition generating function. The underlying equality in Figure 6.1 can be seen in red.

has no antichains of size 2 or greater, then P is a total order and $|\mathcal{L}(P)| = 1$, a contradiction. The result follows. $\square$

Lemma 6.3. *If $|\mathcal{L}(P)| = 2$, then the unlabeled structure of P is as in Figure 6.3, where $\{a, b\}$ is the single antichain of size greater than one and the subsposets Q and R are (possibly empty) total orders.*

Proof. From Lemma 6.2 we know that P has exactly one antichain of size 2, which we shall label $\{a, b\}$. Consider $x \in P \backslash \{a, b\}$. If x and a are incomparable in P or if x and b are incomparable, then either $\{x, a\}$ or $\{x, b\}$ is an antichain, which is a contradiction. If $b \leq_P x \leq_P a$ or $a \leq_P x \leq_P b$, then by transitivity either $b \leq_P a$ or $a \leq_P b$, a contradiction to $\{a, b\}$ being an antichain. Hence, for all $x \in P \backslash \{a, b\}$, we have $x \leq_P a$ and $x \leq_P b$, or we have $a \leq_P x$ and $b \leq_P x$.

Let $Q = \{x \in P : a \leq_P x \text{ and } b \leq_P x\}$. There can be no antichains of length 2 or greater in Q. Hence Q is a total order, as desired. The same holds for $R = \{x \in P : x \leq_P a \text{ and } x \leq_P b\}$. $\square$

Lemma 6.3 describes the complete structure of posets with two linear extensions.

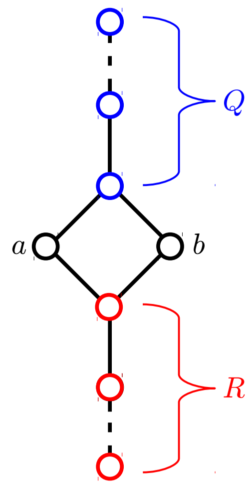


Figure 6.3: The general Hasse diagram for a poset with two linear extensions. In this figure a and b denote poset elements, rather than labels or the images of the elements under a P-partition.

We will need one more result before proving Theorem 6.1.

Lemma 6.4. *Suppose that* $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x)$, $|\mathcal{L}(P,\omega)| = |\mathcal{L}(Q,\tau)| = 2$, *and* $|P| = |Q| \geq 4$. *Then either* P *and* Q *both have unique minimal elements or both have unique maximal elements.*

Proof. Assume not. Then by Lemma 6.3, it is necessarily the case that we have, up to switching P and Q, the unlabeled poset structures shown in Figure 6.4.

Using the notation in Figure 6.4, since $|Q| \geq 4$, we can conclude that $a_Q <_Q b_Q$. Without loss of generality we may assume that $\omega(a_P) < \omega(b_P)$.

Consider the descent compositions of the linear extensions of (P,ω), particularly the first two entries of the descent composition. We have $\mathcal{L}(P,\omega) = \{(\omega(a_P), \omega(b_P), \ldots), (\omega(b_P), \omega(a_P), \ldots)\}$. Hence one linear extension of (P,ω) has a descent in the first position, while the other does not.

We compare this to the first two entries of the descent compositions of the linear

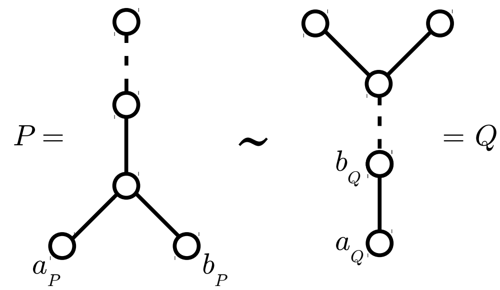


Figure 6.4: Unlabeled poset structure after assuming the negation of Lemma 6.4. In this figure a_P, b_P, a_Q, and b_Q denote poset elements, rather than labels or the images of the elements under a P-partition.

extensions of (Q, τ). We have $\mathcal{L}(Q, \tau) = \{(\tau(a_Q), \tau(b_Q), \ldots), (\tau(a_Q), \tau(b_Q), \ldots)\}$. Either $\tau(b_Q) < \tau(a_Q)$ or $\tau(a_Q) < \tau(b_Q)$, so either both linear extensions of (Q, τ) have a descent in the first position, or both do not. Since one linear extension of (P, ω) has a descent in the first and the other does not, the multisets of the descent compositions of the linear extensions of the labeled posets (P, ω) and (Q, τ) are different. Hence, by a consequence of Equation (2.2), $K_{(P,\omega)}(x) \neq K_{(Q,\tau)}(x)$, a contradiction. $\square$

We now prove the main result of this chapter, Theorem 6.1.

Proof of Theorem 6.1. We proceed by induction on $n = |P| = |Q|$. If $n \leq 3$, then the result holds by enumerating all labeled posets and checking equalities among them. The only nontrivial equality is the one in Figure 6.1.

If $n \geq 4$, then by Lemma 6.4, either P and Q both have unique minimal elements or unique maximal elements. By Proposition 5.4, we can remove both minimal or maximal elements from P and Q to obtain the labeled posets $(\widehat{P}, \widehat{\omega})$ and $(\widehat{Q}, \widehat{\tau})$ that have equal P-partition generating functions. This removal can be reversed while preserving equality by Proposition 5.5. Thus, $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x)$ if and only if $K_{(\widehat{P},\widehat{\omega})}(x) = K_{(\widehat{Q},\widehat{\tau})}(x)$, which gives us the required induction step. $\square$

Chapter 7

Conclusion and Future Work

Our goal was to determine some necessary and some sufficient conditions on the labeled posets (P, ω) and (Q, τ) for $K_{(P,\omega)}(x)$ to equal $K_{(Q,\tau)}(x)$. We developed necessary conditions for equality in Chapter 4, which included creating the notion of jump sequences and a conjecture about antichain sequences for naturally labeled posets.

Chapter 5 was spent developing sufficient conditions, where we were able to modify the posets in known equalities to show new equalities. We were also able to give a complete classification of equalities for posets with two linear extensions in Chapter 6.

Still, these results are far from a complete understanding of the equality of P-partition generating functions. An ambitious goal would be to completely classify all P-partition generating function equalities. However, this goal is currently out of reach. Since equalities among skew Schur functions have yet to be completely classified [6], it is not surprising that the more general problem of classifying equalities among P-partition generating functions remains open.

We are still left with many unanswered questions when it comes to the equality of

P-partition generating functions. Arguably one of the most important is the following:

Question: Can $K_{(P,\omega)}(x) = K_{(Q,\tau)}(x)K_{(R,\phi)}(x)$ for some connected labeled poset (P,ω)?

For skew Schur functions, the answer is no, and allows one to only consider connected skew shapes when trying to describe equalities among skew Schur functions [6]. A resolution of the above question for P-partition generating functions is a critical part of understanding P-partition generating function equalities.

One could build on the result in Chapter 6 by trying to completely classify equalities among labeled posets with three (or more) linear extensions. However, this approach has its limits, since some equalities have no obvious explanation. The example equalities in Figure 7.1 seem to be particularly anomalous.

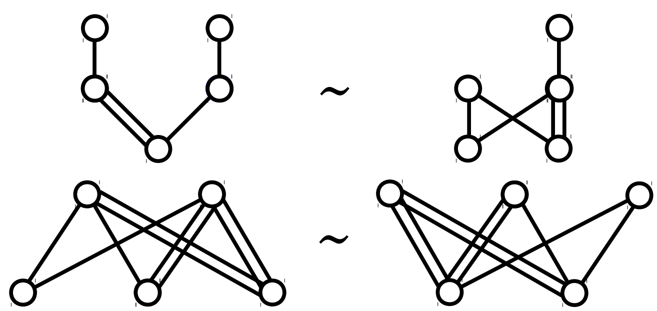


Figure 7.1: Unexplained P-partition generating function equalities.

One of our original motivations was the hope that knowledge developed about the equality of P-partition generating functions might give new insight on skew Schur functions. Unfortunately none of our results about P-partition generating function equality imply new results about skew Schur functions. Hence, completely describing equalities in both the P-partition generating function and skew Schur functions domains is still an open problem.

References

[1] Louis J. Billera, Hugh Thomas, and Stephanie van Willigenburg. Decomposable compositions, symmetric quasisymmetric functions and equality of ribbon Schur functions. *Adv. Math.*, 204(4):204–240, 2006.

[2] Ira M. Gessel. Multipartite P-partitions and inner products of skew Schur functions. In *Combinatorics and algebra (Boulder, Colo., 1983)*, volume 34 of *Contemp. Math.*, pages 289–317. Amer. Math. Soc., Providence, RI, 1984.

[3] Christian Gutschwager. Equality of multiplicity free skew characters. *J. Algebraic Combin.*, 30(2):215–232, 2009.

[4] Claudia Malvenuto and Christophe Reutenauer. Plethysm and conjugation of quasi-symmetric functions. *Discrete Math.*, 193(1-3):225–233, 1998. Selected papers in honor of Adriano Garsia (Taormina, 1994).

[5] Peter R. W. McNamara and Stephanie van Willigenburg. Towards a combinatorial classification of skew Schur functions. *Trans. Amer. Math. Soc.*, 361(8):4437–4470, 2009.

[6] Victor Reiner, Kristin M. Shaw, and Stephanie van Willigenburg. Coincidences among skew Schur functions. *Adv. Math.*, 216(1):118–152, 2007.

[7] Richard P. Stanley. *Ordered structures and partitions.* American Mathematical Society, Providence, R.I., 1972. Memoirs of the American Mathematical Society, No. 119.

[8] Richard P. Stanley. *Enumerative combinatorics. Vol. 1*, volume 49 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1997. With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.

[9] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.