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Generalized multiplicities of edge ideals

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Abstract We explore connections between the generalized multiplicities of square-free monomial ideals and the combinatorial structure of the underlying hypergraphs using methods of commutative algebra and polyhedral geometry. For instance, we show that the j-multiplicity is multiplicative over the connected components of a hypergraph, and we explicitly relate the j-multiplicity of the edge ideal of a properly connected uniform hypergraph to the Hilbert–Samuel multiplicity of its special fiber ring. In addition, we provide general bounds for the generalized multiplicities of the edge ideals and compute these invariants for classes of uniform hypergraphs.

1 Introduction

The theory of multiplicities is centuries old, and it involves a rich interplay of ideas from various fields, including algebraic geometry, commutative algebra, convex geometry,

and combinatorics. The first rigorous general algebraic treatment of multiplicities was given by Chevalley and Samuel for zero-dimensional ideals [7,8,33,34], and soon they became ubiquitous in commutative algebra. For instance, the Hilbert–Samuel multiplicity plays a prominent role in the theory of integral dependence of ideals due to the influential work of Rees [30]. Multiplicity theory has also close ties with polyhedral geometry via Ehrhart theory. In addition, the Hilbert–Samuel multiplicity of zero-dimensional monomial ideals has an elegant interpretation in convex geometry and combinatorics. Indeed, the multiplicity of a zero-dimensional monomial ideal is equal to the normalized full-dimensional volume of the complement of its Newton polyhedron in the positive orthant [38]. More recently, Achilles and Manaresi introduced the concept of j-multiplicity [1], and Ulrich and Validashti proposed the notion of ε-multiplicity [40], extending the classical Hilbert–Samuel multiplicity to arbitrary ideals in a general algebraic setting. These invariants have been proven useful in commutative algebra and algebraic geometry for their connections to the theory of integral closures and Rees valuations, the study of the associated graded algebras, intersection theory, equisingularity and local volumes of divisors [10,22,23,29,40]. Recently, Jeffries and Montaño showed that these numbers measure certain volumes defined for arbitrary monomial ideals, similar to the zero-dimensional case [20]. Currently, there is a rising interest in finding formulas for the j-multiplicity of classes of ideals [21,25]. The main objective of this paper is to further understand how the j-multiplicity and the ε-multiplicity manifest in various combinatorial structures and invariants. In particular, we consider square-free monomial ideals associated with hypergraphs, called the edge ideals, which are not zero-dimensional, and we explore connections between the generalized multiplicities of such ideals and the combinatorial properties of the underlying hypergraphs. It is notable that [20, Theorem 3.2] implies that the j-multiplicity of the edge ideal of a uniform hypergraph and the normalized volume of the associated edge polytope are the same up to a constant factor. Thus, the theory of j-multiplicity in particular provides a new perspective on the edge polytopes which may contribute to the currently limited information about these objects, and vice versa. Geometric features of edge polytopes as well as algebraic properties and invariants of the edge ideals such as regularity, Cohen–Macaulayness, their symbolic Rees algebras and core have been studied extensively in commutative algebra and combinatorics [24,27,35,39,41–43]. Our main results concerning the generalized multiplicities of the edge ideals are the following.

Let G be a hypergraph on n nodes with edge ideal $I(G)$ and Newton polyhedron $P(G)$. We show that the normalized volume is multiplicative with respect to free sums of co-convex sets (Proposition 4.5) which produces a multiplicativity formula for the j-multiplicity for monomial ideals (Theorem 4.6). In particular, if $G_1, \ldots, G_c$ are the connected components of G, then we obtain $j(I(G)) = j(I(G_1)) \cdots j(I(G_c))$ (Proposition 5.3), but this relation is not true for the ε-multiplicity (Remark 10.8). Assume each connected component of G is properly connected. Then we observe the analytic spread of $I(G)$ equals $n - p + c$, where p is the number of the node pivot equivalence classes of G (Proposition 6.1). In particular, this implies the j-multiplicity and the ε-multiplicity of the edge ideal of G are not zero if and only if the nodes in each connected component of G are pivot equivalent (Proposition 6.2). In this case, we prove that $j(I(G)) = m^c e(k[G])$, where $e(k[G])$ is the Hilbert–Samuel multiplicity

of the edge subring $k[G]$ (Theorem 7.5). As an application, we obtain a formula relating the Hilbert–Samuel multiplicity of the edge subring of G to the volume of its edge polytope (Corollary 7.7). Moreover, we note that the height of the toric edge ideal of G is $e - n + p - c$, where e is the number of edges in G (Proposition 8.1). As an application we obtain the following when $j(I(G))$ is not zero: If $e = n$ then $j(I(G)) = m^c$ (Proposition 8.2), and if $e = n + 1$ then $j(I(G)) = m^c l$, where l is half the length of the unique nontrivial minimal monomial walk in G up to equivalence (Proposition 8.4). We also prove $j(I(G))$ is greater than or equal to $j(I(H))$ for any subhypergraph H of G, provided $j(I(G))$ is not zero (Theorem 9.2), and equality holds when H is obtained from G by removing a free node (Proposition 9.6). These statements fail to be true for the ε-multiplicity (Remark 10.8). As a corollary, we conclude $j(I(G))$ is bounded above the j-multiplicity of the complete m-uniform hypergraph on n nodes as in Example 3.3. In particular, if G is a simple graph on n nodes such that $j(I(G))$ is not zero, then $j(I(G))$ is between 2^{τ_0} and $2^n - 2n$, where τ_0 is the odd tulgeity of G (Corollary 9.5). In addition, we show that if G is an odd cycle of length n, then $\varepsilon(I(G)) = \frac{2}{n+1}$ (Proposition 10.4) and we compute the ε-multiplicity of the edge ideals of complete m-uniform hypergraphs (Proposition 10.3). Throughout the paper, we develop results from the perspective of both commutative algebra and polyhedral geometry which reveals a beautiful interaction of ideas between the two approaches.

The paper is organized as follows. In Sect. 2, we review the notion of j-multiplicity in a general algebraic setting. In Sect. 3, we recall the connection between the j-multiplicity of monomial ideals and the associated polytopes. In Sect. 4, we describe a connection between the j-multiplicity and the free sum of co-convex sets and prove the multiplicativity of the j-multiplicity of edge ideals over the connected components. In Sect. 5, we further explore the j-multiplicity of edge ideals via volumes. In Sect. 6, we give a formula for the analytic spread of edge ideals and we obtain a combinatorial characterization of the vanishing of their j-multiplicity and ε-multiplicity using pivot equivalence relation. In Sect. 7, we study the relation between the j-multiplicity of the edge ideal of a hypergraph and the associated edge subring. In Sect. 8, we use toric edge ideals to obtain a formula for the j-multiplicity of the edge ideal of classes of hypergraphs. In Sect. 9, we provide general bounds for the j-multiplicity of edge ideals. In Sect. 10, we compute the ε-multiplicity of the edge ideals of cycles and complete hypergraphs.

2 The j-multiplicity

Let R be a Noetherian local ring with maximal ideal $\mathfrak{m}$ and Krull dimension n. We recall the notion of j-multiplicity $j(I)$ of an ideal I in R as introduced and developed in [11, 6.1] and [1]. Let S be a standard graded Noetherian R-algebra, that is, a graded R-algebra with $S_0 = R$ and generated by finitely many homogeneous elements of degree one. Then $\Gamma_{\mathfrak{m}}(S) \subset S$ is a graded ideal in S, where $\Gamma_{\mathfrak{m}}$ denotes the zeroth local cohomology with respect to the ideal $\mathfrak{m}$ of R. In particular, $\Gamma_{\mathfrak{m}}(S)$ is finitely generated over S. Thus, there exists a fixed power $\mathfrak{m}^t$ of $\mathfrak{m}$ that annihilates $\Gamma_{\mathfrak{m}}(S)$. Therefore, $\Gamma_{\mathfrak{m}}(S)$ is a finitely generated graded module over $S/\mathfrak{m}^t S$, which is a standard graded

Noetherian algebra over the Artinian local ring $R/\mathfrak{m}^t$. Hence, $\Gamma_{\mathfrak{m}}(S)$ has a Hilbert function that is eventually polynomial of degree at most $\dim S - 1$, whose normalized leading coefficient is the Hilbert–Samuel multiplicity $e(\Gamma_{\mathfrak{m}}(S))$. We define the j-multiplicity $j(S)$ to be $e(\Gamma_{\mathfrak{m}}(S))$ when $\dim \Gamma_{\mathfrak{m}}(S) = \dim S$ and zero otherwise. If S_k is the graded component of S of degree k and λ denotes the length, we may write

$$j(S) = (\dim S - 1)! \lim_{k \to \infty} \frac{\lambda_R(\Gamma_{\mathfrak{m}}(S_k))}{k^{\dim S - 1}}.$$

If the graded components of S have finite length, then $j(S)$ is the same as the Hilbert–Samuel multiplicity $e(S)$. In addition, one can see that the condition $\dim \Gamma_{\mathfrak{m}}(S) < \dim S$ is equivalent to $\dim S/\mathfrak{m}S < \dim S$. Therefore, one has the following statement.

Remark 2.1 $j(S) = 0$ if and only if $\dim S/\mathfrak{m}S < \dim S$.

Recall that the associated graded ring of R with respect to an ideal I, which we denote by $\mathcal{G}$, is a standard graded Noetherian R/I-algebra of dimension n. Then, the j-multiplicity $j(I)$ is defined as the j-multiplicity of the graded ring $\mathcal{G}$. In terms of the length of the graded components of $\Gamma_{\mathfrak{m}}(\mathcal{G})$, we may write

$$j(I) = (n - 1)! \lim_{k \to \infty} \frac{\lambda_R(\Gamma_{\mathfrak{m}}(I^k/I^{k+1}))}{k^{n-1}}.$$

If I is $\mathfrak{m}$-primary, then the graded components of the associated graded ring of R with respect to I have finite length, and $j(I)$ is indeed the Hilbert–Samuel multiplicity $e(I)$. Moreover, $j(I) = 0$ if and only if $\dim \mathcal{G}/\mathfrak{m}\mathcal{G} < \dim \mathcal{G} = n$ by Remark 2.1. The dimension of the special fiber ring $\mathcal{G}/\mathfrak{m}\mathcal{G}$ is denoted by $\ell(I)$ and is called the analytic spread of I. Thus, we have the following statement.

Remark 2.2 $j(I) = 0$ if and only if $\ell(I) < n$.

We refer the reader to [11] for further properties of j-multiplicities and to [6] for unexplained terminology.

3 The j-multiplicity of monomial ideals and volumes

We begin with recalling some definitions and notation from convex geometry related to monomial ideals. Consider the integer lattice $\mathbb{Z}^n$ in $\mathbb{R}^n$. A *lattice polytope* F in $\mathbb{R}^n$ is the convex hull of finitely many lattice points. A *unimodular n-simplex* is the convex hull of $n + 1$ lattice points $\{v_0, v_1, \ldots, v_n\}$ such that $\{v_1 - v_0, \ldots, v_n - v_0\}$ is a basis for the lattice. We use Vol_n to denote the *normalized n-dimensional volume* in $\mathbb{R}^n$ defined such that $\mathrm{Vol}_n(\Delta) = 1$ for any unimodular n-simplex Δ. Then for any lattice polytope F we have $\mathrm{Vol}_n(F) = n!\mathrm{vol}_n(F)$, where vol_n is the usual Euclidean volume in $\mathbb{R}^n$. Similarly, we can define the normalized k-dimensional volume with respect to any sublattice in $\mathbb{Z}^n$ of rank k. We will be concerned with the

following particular situation. Suppose F is a lattice polytope lying in a rational affine hyperplane

$$L = \{z \in \mathbb{R}^n \mid \langle u, z \rangle = b\},$$

where $b \in \mathbb{Z}, b \geq 0$, and $u = (u_1, \ldots, u_n)$ is a primitive integer vector, that is $\gcd(u_1, \ldots, u_n) = 1$. We use $\langle u, z \rangle$ to denote the inner product of u and z in $\mathbb{R}^n$. Then we write $\mathrm{Vol}_{n-1}(F)$ to denote the normalized $(n-1)$-dimensional volume with respect to the sublattice $L \cap \mathbb{Z}^n \subset \mathbb{Z}^n$. Note that the integer b is the *lattice distance* from L to the origin. For a lattice polytope $F \subset \mathbb{R}^n$ of dimension at most $n-1$, we write $\mathrm{pyr}(F)$ for the convex hull of F and the origin, which we call the pyramid over F. Clearly, $\mathrm{Vol}_n(\mathrm{pyr}(F)) = 0$ if $\dim F$ is less than $n-1$. When $\dim F = n-1$, we have the following formula which is standard in lattice geometry:

$$\mathrm{Vol}_n(\mathrm{pyr}(F)) = h(F)\mathrm{Vol}_{n-1}(F), \tag{1}$$

where $h(F)$ is the lattice distance from the affine span of F to the origin. More generally, let $a \in \mathbb{Q}^n$ be such that $\langle u, a \rangle \leq b$. Then the convex hull $\mathrm{pyr}_a(F)$ of a and F is the pyramid over F with apex a and lattice height $h(F) - \langle u, a \rangle$. Therefore, we obtain

$$\mathrm{Vol}_n(\mathrm{pyr}_a(F)) = (h(F) - \langle u, a \rangle)\mathrm{Vol}_{n-1}(F). \tag{2}$$

Here $h(F) - \langle u, a \rangle$ is the *lattice distance* from the affine span of F to a.

Now let I be a monomial ideal in $R = k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$. The *Newton polytope* $F(I)$ is the convex hull in $\mathbb{R}^n$ of the exponent vectors of the minimal generators of I, and the *Newton polyhedron* $P(I)$ is the convex hull in $\mathbb{R}^n$ of the exponent vectors of all monomials in I. The following result due to Jeffries and Montaño [20, Theorem 3.2] relates the j-multiplicity of a monomial ideal to the underlying Newton polyhedron.

Theorem 3.1 *Let I be a monomial ideal and $F_1, \ldots, F_k$ be the compact facets of $P(I)$. Then*

$$j(I) = \sum_{j=1}^{k} \mathrm{Vol}_n(\mathrm{pyr}(F_j)) = \sum_{j=1}^{k} h(F_j)\mathrm{Vol}_{n-1}(F_j),$$

where $h(F_j)$ is the lattice distance from the affine span of F_j to the origin.

Recall that by Remark 2.2, $j(I) = 0$ if and only if $\ell(I)$ is less than n. On the other hand, by a result of Bivià-Ausina [4], the analytic spread of I is the maximum of the dimensions of the compact faces of $P(I)$ plus one. Therefore, we obtain the following statement.

Remark 3.2 $j(I) = 0$ if and only if all compact faces of $P(I)$ have dimension less than $n-1$, that is $P(I)$ has no compact facets.

Example 3.3 Let I be the ideal generated by all square-free monomials of degree m in R. Then, the Newton polytope of I is the convex hull of all vectors in $\mathbb{R}^n$ with exactly

m entries being 1 and the rest 0. Therefore, I corresponds to a hypersimplex of type (m, n) lying in the hyperplane $z_1 + \cdots + z_n = m$. It is classical that $\mathrm{Vol}_{n-1}(F(I))$ equals the Eulerian number $A(n - 1, m)$. Therefore, by Theorem 3.1 we obtain a closed formula

$$j(I) = m \cdot A(n - 1, m) = m \cdot \left(\sum_{k=0}^{m} (-1)^k \binom{n}{k} (m - k)^{n-1} \right).$$

For instance, if $m = 2$ then $j(I) = 2^n - 2n$, and if $m = n - 1$ then $j(I) = n - 1$. Note that $j(I) = 0$ if and only if $m = n$.

Below we provide a simple proof of Theorem 3.1 when I is a monomial ideal of the form wJ, where w is a monomial and J is a zero-dimensional monomial ideal in R, using the volume interpretation of the Hilbert–Samuel multiplicity of zero-dimensional monomial ideals due to Teissier [38]. Note that all monomial ideals of a polynomial ring in two variables are of form wJ as above.

Proof First note that by Theorem [22, 3.12], $j(I) = j(wJ) = e(J) + e(J\bar{R})$, where $\bar{R} = R/(w)$. Write w as $x_1^{a_1} \cdots x_n^{a_n}$. By the associativity formula for the Hilbert–Samuel multiplicity,

$$e(J\bar{R}) = \sum_{i=1}^{n} \lambda((\bar{R})_{(x_i)}) \cdot e(J(\bar{R}/x_i \bar{R})) = \sum_{i=1}^{n} a_i \cdot e(J R_i)$$

where $R_i = R/(x_i)$. Hence, we obtain

$$j(I) = e(J) + \sum_{i=1}^{n} a_i \cdot e(J R_i). \tag{3}$$

For a polyhedron P denote by $c(P)$ the union of the pyramids over the compact faces of P. Using Teissier's result for the zero-dimensional ideal J, we have $e(J) = \mathrm{Vol}_n(c(P(J)))$. For $i = 1, \ldots, n$, let P_i be the facet of $P(J)$ with the inner normal vector e_i. Then P_i is the Newton polyhedron of the zero-dimensional ideal $J R_i$ and, hence, $e(J R_i) = \mathrm{Vol}_{n-1}(c(P_i))$, again by Teissier's result. Therefore,

$$j(I) = \mathrm{Vol}_n(c(P(J))) + \sum_{i=1}^{n} a_i \mathrm{Vol}_{n-1}(c(P_i)).$$

We claim that the latter equals $\mathrm{Vol}_n(c(P(I)))$. Note that $P(I) = P(J) + a$, where $a = (a_1, \ldots, a_n)$ as above. Let F_j be the compact facets of $P(J)$ with primitive inner normals $\eta_j \in \mathbb{Z}^n$, for $1 \leq j \leq k$. As the compact facets of $P(I)$ are translates of the F_j, we have

$$\mathrm{Vol}_n(c(P(I))) = \sum_{j=1}^{k} \min_{u \in P(I)} \langle u, \eta_j \rangle \mathrm{Vol}_{n-1}(F_j) = \sum_{j=1}^{k} \min_{u \in P(J)} \langle u, \eta_j \rangle \mathrm{Vol}_{n-1}(F_j)$$

$$+ \sum_{j=1}^{k} \langle a, \eta_j \rangle \mathrm{Vol}_{n-1}(F_j). \tag{4}$$

The first summand in the right-hand side of (4) equals $\mathrm{Vol}_n(c(P(J)))$. For the second summand, we have

$$\sum_{j=1}^{k} \langle a, \eta_j \rangle \mathrm{Vol}_{n-1}(F_j) = \sum_{i=1}^{n} a_i \sum_{j=1}^{k} \langle e_i, \eta_j \rangle \mathrm{Vol}_{n-1}(F_j). \tag{5}$$

Lemma 3.4 below implies that the projection of the union of the F_j onto L_i gives a polyhedral subdivision of $c(P_i)$. As the projection of F_j onto L_i has volume $\langle e_i, \eta_j \rangle \mathrm{Vol}_{n-1}(F_j)$, we get

$$\mathrm{Vol}_{n-1}(c(P_i)) = \sum_{j=1}^{k} \langle e_i, \eta_j \rangle \mathrm{Vol}_{n-1}(F_j).$$

Combining this with (5) and (4), we obtain

$$\mathrm{Vol}_n(c(P(I))) = \mathrm{Vol}_n(c(P(J))) + \sum_{i=1}^{n} a_i \mathrm{Vol}_{n-1}(c(P_i)),$$

as claimed. □

Lemma 3.4 *Let P be a polyhedron in the n-orthant $\mathbb{R}^n_{\geq 0}$ whose complement $\mathbb{R}^n_{\geq 0} \setminus P$ is bounded. Let $L_i = \{z \in \mathbb{R}^n \mid z_i = 0\}$ be a coordinate hyperplane. Then the projection $\pi_i : \mathbb{R}^n \to L_i$ gives a bijection between the union of the compact facets of P and the closure of the complement of $P \cap L_i$ in the $(n-1)$-orthant $\mathbb{R}^n_{\geq 0} \cap L_i$.*

Proof First note that the non-compact facets of P are precisely the intersections $P \cap L_i$ for $1 \leq i \leq n$. This implies that the union of the compact facets $\mathcal{F}$ of P equals the closure of $\partial P \cap \mathbb{R}^n_{>0}$. In addition, the inner normals of the compact facets of P have all their coordinates positive. To simplify notation, we assume $i = n$ and let $P' = P \cap L_n$ and $c(P')$ be the closure of the complement of P' in $\mathbb{R}^n_{\geq 0} \cap L_n$.

First, we check that π_n restricted to $\mathcal{F}$ is one-to-one. Indeed, suppose $a_1 = (a', t_1)$ and $a_2 = (a', t_2)$ lie in $\mathcal{F}$ for some $(a', 0) \in L_n$ and $t_1, t_2 \geq 0$ and assume $t_1 \leq t_2$. Let η be an inner normal to a facet containing a_2. Then $\langle \eta, z \rangle$ attains its minimum on P at $z = a_2$, but since $a_1 \in P$ and $\eta_n > 0$ we must have $t_2 \leq t_1$. Therefore, $t_1 = t_2$ and so $a_1 = a_2$.

Now we show that $\pi_n(\mathcal{F}) = c(P')$. Let $a_0 = (a', 0)$ be an interior point of $c(P')$ (relative to L_n) and thus $a_0 \notin P$. Since $\mathbb{R}^n_{\geq 0} \setminus P$ is bounded, $(a', t) \in P$ for $t \gg 0$. Since P is closed, there exists the smallest value of $t > 0$ such that $a = (a', t)$ lies

in P and, hence, in the boundary of P. Thus, a lies in a compact facet of P, as all coordinates of a are positive. Therefore, the interior of $c(P')$ is contained in $\pi_n(\mathcal{F})$. Since $\mathcal{F}$ is closed, by continuity, $c(P') \subseteq \pi_n(\mathcal{F})$. Finally, if $\pi_n(a) = (a', 0) \in P'$ for some $a = (a', t_1) \in \mathcal{F}$ then the entire ray $\{(a', t) \mid t \geq 0\}$ lies in P. By the same argument as in the previous paragraph we must have $t_1 \leq 0$, thus $t_1 = 0$. In other words, $\pi_n(a) = a$ lies in the boundary of P'. Therefore, $\pi_n(\mathcal{F}) \subseteq c(P')$. $\qquad\square$

4 The j-multiplicity of monomial ideals and free sums

In this section, we observe that if I is a sum of monomial ideals whose sets of minimal monomial generators involve pairwise disjoint collections of variables, then the j-multiplicity of I is the product of the j-multiplicities of the summands, see Theorem 4.6. The combinatorial counterpart here is the free sum of co-convex bodies.

Recall the notion of a co-convex body. Let $C \subset \mathbb{R}^n$ be a closed convex cone with non-empty interior which does not contain nontrivial linear subspaces. Let $P \subset C$ be a convex set such that $C \setminus P$ is bounded. Then the closure of $C \setminus P$, denoted by $c(P)$, is called a *co-convex* body. Furthermore, let $F(P) = c(P) \cap P$ which is the union of the bounded faces of P. For example, let $F(I)$ be the Newton polytope and $P(I)$ be the Newton polyhedron of a monomial ideal I in $R = k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$. Let C be the cone over $F(I)$ and $P = P(I) \cap C$. Then the co-convex body $c(P)$ is the union of pyramids over the bounded faces of $P(I)$. Its normalized volume equals the j-multiplicity of the ideal I

$$j(I) = \mathrm{Vol}_n(c(P)), \tag{6}$$

according to Theorem 3.1.

Definition 4.1 Let $P_i \subset C_i \subset \mathbb{R}^{n_i}$, for $i = 1, 2$, be convex sets contained in convex cones as above and $K_i = c(P_i)$ the corresponding co-convex bodies. Define the *free sum* $P_1 \oplus P_2$ to be the convex hull of the union $(P_1 \times \{0\}) \cup (\{0\} \times P_2)$ in $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. The closure of the complement of $P_1 \oplus P_2$ in $C_1 \times C_2$ is called the *free sum* of the co-convex bodies K_1 and K_2, and is denoted by $K_1 \oplus K_2$.

Example 4.2 Let Δ_1 be an n_1-simplex generated by integer vectors $v_1, \ldots, v_{n_1}$ in $\mathbb{R}^{n_1}$ and Δ_2 be an n_2-simplex generated by integer vectors $w_1, \ldots, w_{n_2}$ in $\mathbb{R}^{n_2}$ and let $n = n_1 + n_2$. Then $\Delta_1 \oplus \Delta_2$ is the n-simplex generated by $(v_1, 0), \ldots, (v_{n_1}, 0), (0, w_1), \ldots, (0, w_{n_2})$. Moreover, the normalized volumes of Δ_1, Δ_2, and $\Delta_1 \oplus \Delta_2$ satisfy

$$\mathrm{Vol}_n(\Delta_1 \oplus \Delta_2) = \mathrm{Vol}_k(\Delta_1)\mathrm{Vol}_l(\Delta_2).$$

Indeed, the volume on the left equals the absolute value of the determinant of the block matrix with blocks corresponding to the two sets of vectors.

The above property about normalized volumes extends to free sums of arbitrary convex sets containing the origin, as well as to co-convex bodies. For convex centrally symmetric bodies, this follows from [32, p. 15] but the argument can be adapted to

the case of co-convex bodies as sketched below. A different proof for convex sets containing the origin was found by T. McAllister (private communication).

Let $K = c(P) \subset C$ be a co-convex body. The *Minkowski functional* of K is defined on C by

$$|x|_K = \inf\{r \geq 0 \mid x \in rK\}.$$

Note that K is the set of those $x \in C$ with $|x|_K \leq 1$ and $F(P)$ is the set of $x \in C$ with $|x|_K = 1$. Furthermore, for any $r \geq 0$, the dilation $rF(P)$ is the set of $x \in C$ with $|x|_K = r$.

Lemma 4.3 *Let $K_1 \oplus K_2$ be a free sum of co-convex sets $K_i = c(P_i) \subset C_i \subset \mathbb{R}^{n_i}$, for $i = 1, 2$. Then*

(a) $F(P_1 \oplus P_2) = \{((1-t)p_1, tp_2) \in C_1 \times C_2 \mid p_i \in F(P_i), 0 \leq t \leq 1\}$,
(b) $|x|_{K_1 \oplus K_2} = |x_1|_{K_1} + |x_2|_{K_2}$ *for any* $x = (x_1, x_2) \in C_1 \times C_2$.

Proof (a) First, by convexity of the P_i we have

$$P_1 \oplus P_2 = \{((1-s)v_1, sv_2) \in C_1 \times C_2 \mid v_i \in P_i, 0 \leq s \leq 1\}. \tag{7}$$

Pick $p_i \in F(P_i)$, for $i = 1, 2$, and consider $p = ((1-t)p_1, tp_2)$ for some $0 \leq t \leq 1$. Let Γ_i be a bounded face of P_i containing p_i with inner normal u_i, and let $b_i = \min_{v_i \in P_i} \langle u_i, v_i \rangle = \langle u_i, p_i \rangle$. Note that $b_i > 0$ since $0 \notin \Gamma_i$, so by rescaling the u_i we may assume that $b_i = 1$. Put $u = (u_1, u_2)$. Then $\langle u, p \rangle = 1$. On the other hand, for any $v = ((1-s)v_1, sv_2) \in P_1 \oplus P_2$ we have

$$\langle u, v \rangle = (1-s)\langle u_1, v_1 \rangle + s\langle u_2, v_2 \rangle \geq 1.$$

This shows that p belongs to a bounded face of $P_1 \oplus P_2$.

Conversely, if $p \in F(P_1 \oplus P_2)$ then $\langle u, p \rangle = \min_{v \in P_1 \oplus P_2} \langle u, v \rangle$ for some $u = (u_1, u_2)$. As above, by (7), we have

$$\langle u, p \rangle = (1-t)\langle u_1, p_1 \rangle + t\langle u_2, p_2 \rangle \quad \text{for some } 0 \leq t \leq 1.$$

Therefore, $\langle u_i, p_i \rangle = \min_{v_i \in P_i} \langle u_i, v_i \rangle$ for $i = 1, 2$, i.e., $p_i \in F(P_i)$.

(b) Let $r = |x|_{K_1 \oplus K_2}$. Then $x \in rF(P_1 \oplus P_2)$, hence, by *(a)* $x = (x_1, x_2) = (r(1-t)p_1, rtp_2)$ for some $p_i \in F(P_i)$ and $0 \leq t \leq 1$. This implies that $|x_1|_{K_1} = r(1-t)$ and $|x_2|_{K_2} = rt$ and so

$$|x_1|_{K_1} + |x_2|_{K_2} = r = |x|_{K_1 \oplus K_2}.$$

$\square$

The following lemma is an easy adaptation of the calculation given in the proof of Lemma 3.2 in [32, p. 15].

Lemma 4.4 *Let $K \subset C$ be a co-convex body. Then*

$$\int_C e^{-|x|_K}\, dx = n!\mathrm{vol}_n(K) = \mathrm{Vol}_n(K).$$

Now the above-mentioned property of the free sum follows from the two lemmas and the Fubini theorem.

Proposition 4.5 *Let $K_1 \oplus K_2$ be a free sum of co-convex sets $K_i = c(P_i) \subset C_i \subset \mathbb{R}^{n_i}$, for $i = 1, 2$. Then*

$$\mathrm{Vol}_{n_1+n_2}(K_1 \oplus K_2) = \mathrm{Vol}_{n_1}(K_1)\mathrm{Vol}_{n_2}(K_2).$$

Proof Indeed, by Lemma 4.4 and Lemma 4.3, part (b)

$$\mathrm{Vol}_{n_1+n_2}(K_1 \oplus K_2) = \int_{C_1 \times C_2} e^{-|x|_{K_1 \oplus K_2}}\, dx = \int_{C_1} e^{-|x_1|_{K_1}}\, dx_1 \int_{C_2} e^{-|x_2|_{K_2}}\, dx_2$$
$$= \mathrm{Vol}_{n_1}(K_1)\mathrm{Vol}_{n_2}(K_2).$$

□

Now let an ideal $I \subset R = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$ be the sum of monomial ideals whose sets of generators involve pairwise disjoint collections of variables. Then Proposition 4.5 provides us with the following multiplicativity property of the j-multiplicity.

Theorem 4.6 *Assume that the set of the variables $\{x_1, \ldots, x_n\}$ is partitioned into subsets $X_1, \ldots, X_s$ and consider the ideal $I = I_1 R + \cdots + I_s R$ for some monomial ideals $I_k \subset R_k = k[X_k]_{(X_k)}$ for $k = 1, \ldots, s$. Then*

$$j(I) = j(I_1) \cdots j(I_s).$$

Proof Let $C \subset \mathbb{R}^n$ be the cone over $F(I)$ and $P = P(I) \cap C$ as above. Then the j-multiplicity $j(I)$ equals the normalized volume of the co-convex body $c(P)$, as in (6). Similarly, let $C_k \subset \mathbb{R}^{n_k}$, where $n_k = |X_k|$, be the cone over $F(I_k)$ and $P_k = P(I_k) \cap C_k$. Then $j(I_k)$ equals the normalized volume of $c(P_k)$. On the other hand, $c(P)$ equals the free sum $c(P_1) \oplus \cdots \oplus c(P_s)$. Therefore, by Proposition 4.5 we have

$$j(I) = \mathrm{Vol}_n(c(P)) = \mathrm{Vol}_{n_1}(c(P_1)) \cdots \mathrm{Vol}_{n_s}(c(P_s)) = j(I_1) \cdots j(I_s).$$

□

Remark 4.7 It would be interesting to give an algebraic proof of Theorem 4.6. For instance, using Theorem 7.2 and Theorem 7.5 one may give an algebraic proof for the case of edge ideals of m-uniform hypergraphs with properly connected components. Moreover, using methods of commutative algebra we can show Theorem 4.6 holds for arbitrary zero-dimensional ideals, or for arbitrary homogenous ideals generated in the same degree. This leads us to believe that Theorem 4.6 holds true even if the ideals involved are not monomial. These results will be addressed in a subsequent paper.

5 The j-multiplicity of edge ideals and volumes

Consider a hypergraph G with the node set $V(G) = \{x_1, \ldots, x_n\}$ and the edge set $E(G)$. By definition, $E(G)$ consists of finitely many subsets of $V(G)$, called edges of G. We say G is m-*uniform* if each edge of G has size m. Note that a *simple graph* is a 2-uniform hypergraph. By abuse of notation, we let $k[x_1, \ldots, x_n]$ be a polynomial ring generated by the x_i as indeterminates over a field k. To every edge $\{x_{i_1}, \ldots, x_{i_k}\}$ in G we associate a square-free monomial $x_{i_1} \cdots x_{i_k}$ in the local ring $R = k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$. Then the *edge ideal of* G is

$$I(G) = (x_{i_1} \cdots x_{i_k} \mid \{x_{i_1}, \ldots, x_{i_k}\} \in E(G)) \subset R.$$

We denote the Newton polyhedron and the Newton polytope of $I(G)$ simply by $P(G)$ and $F(G)$, respectively. Following [27,43], we call $F(G)$ the *edge polytope* of G.

Assume G is m-uniform. Then it can be readily seen that the monomials in R associated with the edges of G are the minimal generators of $I(G)$. Note that $F(G)$ is the convex hull of some lattice points in $\mathbb{Z}^n$ in which all entries are zero except for m entries which are 1. Thus, $F(G)$ lies in the hyperplane

$$L = \{(z_1, \ldots, z_n) \in \mathbb{R}^n \mid z_1 + \cdots + z_n = m\},$$

and so the dimension of $F(G)$ is at most $n-1$. Therefore, the edge polytope $F(G)$ is the unique maximal compact face of $P(G)$, and if the dimension of $F(G)$ is exactly $n-1$, then $F(G)$ is the unique compact facet of $P(G)$. Recall the formula in Theorem 3.1 on the j-multiplicity of a monomial ideal and the volume. For the edge ideal $I(G)$, there is only one term in the sum corresponding to $F(G)$ as the unique compact facet when the j-multiplicity is not zero. In this case, the volume of the pyramid $\mathrm{pyr}(F(G))$ is computed by (1) where the lattice distance $h(F(G)) = m$. Therefore, we obtain the following result connecting the j-multiplicity to the volume of the edge polytope.

Corollary 5.1 *Let G be an m-uniform hypergraph on n nodes. Then*

$$j(I(G)) = m \cdot \mathrm{Vol}_{n-1}(F(G)).$$

Let G be a hypergraph on n nodes. If G has an isolated node, then every generator of $I(G)$ will be missing at least one of the variables which makes $F(G)$ of dimension less than $n - 1$. Therefore, $j(I(G))$ is zero. Similarly, if the number of edges of G is less than the number of nodes, then $j(I(G))$ is zero. We conclude the following.

Remark 5.2 If G is a hypergraph with an isolated node, or if the number of edges of G is less than the number of nodes, then $j(I(G)) = 0$. Thus, for the rest of this paper we will assume that the hypergraphs in question do not have isolated nodes, and the number of edges of each connected component is at least the number of its nodes.

A hypergraph G is called *connected* if for any two nodes $x_i, x_j \in V(G)$, there is a sequence of edges in $E(G)$ such that x_i and x_j belong to the first and the last edges of the sequence, respectively, and consecutive edges in the sequence have a common

node. Let $G_1, \ldots, G_c$ be the connected components of G. Then the edge ideal $I(G)$ is the sum of the extensions of the edge ideals $I(G_k)$ for $k = 1, \ldots, c$ whose generators depend on pairwise disjoint collections of variables. Therefore, by Theorem 4.6 we obtain the following result.

Proposition 5.3 *Let $G_1, \ldots, G_c$ be the connected components of a hypergraph G. Then*

$$j(I(G)) = j(I(G_1)) \cdots j(I(G_c)).$$

Recall that by a result of Bivià-Ausina [4], for a monomial ideal the analytic spread equals one plus the maximum of the dimensions of the compact faces of the Newton polyhedron. If $I(G)$ is the edge ideal of an m-uniform hypergraph G on n nodes and e edges, then $F(G)$ is the unique maximal compact face of the Newton polyhedron $P(G)$. Therefore,

$$\ell(I(G)) = 1 + \dim F(G) = \operatorname{rank} M(G),$$

where $M(G)$ denotes the $e \times n$ *incidence matrix* of G. If G is a simple graph, then rank $M(G)$ is equal to $n - c_0$, where c_0 is the number of connected components of G that contain no odd cycles, i.e., the number of bipartite components of G [16]. We conclude the following.

Remark 5.4 If $I(G)$ is the edge ideal of an m-uniform hypergraph G, then $\ell(I(G))$ is the rank of the incidence matrix of G. In particular, if G is a simple graph on n nodes, then $\ell(I(G)) = n - c_0$.

Using Remark 2.2 and Remark 5.4, we obtain the following characterization for positivity of the j-multiplicity of edge ideals of simple graphs.

Proposition 5.5 *If G is a simple graph, then $j(I(G)) \neq 0$ if and only if all connected components of G contain an odd cycle, that is they are non-bipartite.*

In Sect. 6, we generalize Proposition 5.5 to m-uniform hypergraphs. If a simple connected graph has the same number of nodes as the number of edges, then it contains exactly one cycle; hence, it is called *unicyclic*. Therefore, in a simple graph the number of nodes is equal to the number of edges if and only if the connected components are unicyclic. The following result computes the j-multiplicity of the edge ideals of simple graphs with unicyclic components. In the following proof, τ_0 stands for the maximum number of node-disjoint odd cycles in G, called *odd tulgeity* of G.

Proposition 5.6 *Let G be a simple graph with c connected components and $e = n$. If $j(I(G)) \neq 0$, then $j(I(G)) = 2^c$. In particular, if G unicyclic, then $j(I(G)) = 2$ when G has an odd cycle, and it is zero otherwise.*

Proof Since $e = n$, by Proposition 5.5 we obtain $j(I(G)) \neq 0$ if and only if each connected component has exactly one odd cycle. Thus in this case, $\tau_0 = c$. By [16, Theorem 2.6], the maximal minor of the incidence matrix $M(G)$ with maximum

absolute value is $\pm 2^{\tau_0}$. But $M(G)$ is a square matrix in our case. Therefore, the absolute value of $\det(M(G))$ is 2^c. Note that $\mathrm{pyr}(F(G))$ is an n-simplex and the vertices of $F(G)$ are exactly the rows of the incidence matrix $M(G)$. Thus, the normalized volume of $\mathrm{pyr}(F(G))$ equals the absolute value of $\det(M(G))$. Now the result follows from Theorem 3.1. □

In Proposition 8.2, we prove an extension of Proposition 5.6 for m-uniform hypergraphs.

Remark 5.7 If G is the *complete m*-uniform hypergraph on n nodes, then Example 3.3 provides a closed formula for the j-multiplicity of $I(G)$ in terms of m and n.

6 The pivot equivalence relation and analytic spread

Let G be an m-uniform hypergraph. By Remark 5.2, we will always assume that G has no isolated nodes. Then G is called *properly connected* if for any two edges u, v in $E(G)$, there is a sequence of edges of G starting with u and ending with v, such that the intersection of consecutive edges has size $m - 1$. Note that simple connected graphs are properly connected. As in [5], we define a relation $\approx$ on the set of nodes of G by letting $x_i \approx x_j$ if there is a subset $A \subset \{x_1, \ldots, x_n\} \setminus \{x_i, x_j\}$, such that $\{x_i\} \cup A$ and $\{x_j\} \cup A$ are edges of G. Then we define an equivalence relation $\sim$ on the set of nodes of G by declaring $x_i \sim x_j$ for two nodes x_i, x_j if there is a sequence of nodes $x_{i_1}, \ldots, x_{i_r}$ such that

$$x_i = x_{i_1} \approx x_{i_2} \approx \cdots \approx x_{i_r} = x_j.$$

Note that $x_i \sim x_i$ for $i = 1, \ldots n$ as we assume G has no isolated nodes. This equivalence relation is called *pivot equivalence*, and it gives a partition of the nodes of G into *pivot equivalence classes*.

Proposition 6.1 *Let G be an m-uniform hypergraph on n nodes in which the connected components are properly connected. Let c be the number of connected components and p be the number of pivot equivalence classes of G. Then*

$$\ell(I(G)) = n - p + c.$$

Proof Let $G_1, \ldots, G_c$ be the connected components of G. Since the G_i are properly connected, then by the main theorem of [5] the rank of the incidence matrix of G_i is $n_i - p_i + 1$, where n_i is the number of nodes and p_i is the number of pivot equivalence classes in G_i. Recall from Remark 5.4 that the analytic spread of the edge ideal of G can be computed as the rank of its incidence matrix, which is the sum of the ranks of the incidence matrices of the G_i. Hence, the analytic spread of the edge ideal $I(G)$ is given by $\sum_{i=1}^c (n_i - p_i + 1)$. Therefore, we may write $\ell(I(G)) = n - p + c$. □

Using Remark 2.2 and Proposition 6.1, we obtain the following characterization for positivity of the j-multiplicity of edge ideals of m-uniform hypergraphs.

Proposition 6.2 *Let G be an m-uniform hypergraph in which the connected components are properly connected. Then $j(I(G)) \neq 0$ if and only if the nodes in each connected component of G are pivot equivalent.*

If G is a properly connected m-uniform hypergraph admitting pivot equivalence classes $V_1, \ldots, V_p$, then by the first proposition of [5] there are fixed positive integers $b_1, \ldots, b_p$ such that each edge of G contains exactly b_i nodes from V_i for $i = 1, \ldots, p$. Hence, $m = b_1 + \cdots + b_p \geq p$. We conclude the following.

Remark 6.3 If G is a properly connected m-uniform hypergraph, then G has at most m pivot equivalence classes.

For instance, if G is a simple connected graph, then G admits at most two pivot equivalence classes since two nodes are pivot equivalent if by definition they are connected by a *walk* of even length (see the definition of a walk in Sect. 8). Indeed, one may observe that G admits only one pivot equivalence class if and only if G contains an odd cycle. It follows that if G is not connected, then $p = c + c_0$, where c_0 is the number of connected components of G that contain no odd cycles. Hence, $\ell(I(G)) = n - p + c = n - c_0$ as in Remark 5.4.

7 The j-multiplicity of edge ideals and edge subrings

As in the previous section, let $I(G) \subset R = k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$ be the edge ideal of an m-uniform hypergraph G on n nodes. Then the *edge subring* of G, denoted by $k[G]$, is the subalgebra of R generated by the edges of G. In other words,

$$k[G] := k[x_{i_1} \cdots x_{i_m} \mid \{x_{i_1}, \ldots, x_{i_m}\} \in E(G)] \subset R.$$

Note that the edge subring of G is a graded algebra generated in degree m, thus it can be regarded as a standard graded algebra by assigning degree 1 to its generators. The Hilbert–Samuel multiplicity of the edge subring with respect to this grading is denoted by $e(k[G])$. Let G be an m-uniform hypergraph on n nodes with properly connected components. Then there is a natural homogeneous isomorphism between the edge subring $k[G]$ and the special fiber ring of the edge ideal of G. Therefore, the Krull dimension of $k[G]$ is the analytic spread of $I(G)$. Hence, by Proposition 6.1 we obtain the following statement.

Remark 7.1 If G is an m-uniform hypergraph with properly connected components, then

$$\dim k[G] = n - p + c,$$

where n is the number of nodes, p is the number pivot equivalence classes and c is the number of connected components of G.

If G is a simple graph on n nodes in which all connected components contain an odd cycle, then $\mathrm{Vol}_{n-1}(F(G))$ is equal to $2^{c-1}e(k[G])$ by [13, Theorem 4.9]. Therefore,

$j(I(G)) = 2^c e(k[G])$ by Corollary 5.1. The following result is an extension of this statement to m-uniform hypergraphs. Our proof is an algebraic argument that does not rely on the relation between multiplicities and volumes. We begin with the case that G is properly connected.

Theorem 7.2 *Let G be a properly connected m-uniform hypergraph. If $j(I(G)) \neq 0$, then*

$$j(I(G)) = m \cdot e(k[G]).$$

Proof Let I denote the edge ideal of G and assume $j(I) \neq 0$. Then $j(I) = e(\Gamma_{\mathfrak{m}}(\mathcal{G}))$ by definition, where $\mathcal{G}$ is the associated graded ring of R with respect to I, and $\mathfrak{m}$ is the maximal ideal $(x_1, \ldots, x_n)R$. By the associativity formula for multiplicities of graded modules over graded algebras,

$$e(\Gamma_{\mathfrak{m}}(\mathcal{G})) = \sum \lambda((\Gamma_{\mathfrak{m}}(\mathcal{G}))_P) \cdot e(\mathcal{G}/P),$$

where λ denotes the length, and the sum runs over all minimal primes P in the support of $\Gamma_{\mathfrak{m}}(\mathcal{G})$ of dimension n. Recall the special fiber ring $\mathcal{G}/\mathfrak{m}\mathcal{G}$ is isomorphic to $k[G]$, which is a domain. Therefore, $\mathfrak{m}\mathcal{G}$ is a prime ideal of $\mathcal{G}$ of dimension n, since $\dim \mathcal{G}/\mathfrak{m}\mathcal{G} = \ell(I) = n$ by Remark 2.2. Moreover, $\mathfrak{m}\mathcal{G}$ is in the support of $\Gamma_{\mathfrak{m}}(\mathcal{G})$ and any prime ideal in the support of $\Gamma_{\mathfrak{m}}(\mathcal{G})$ contains $\mathfrak{m}\mathcal{G}$ as some power of $\mathfrak{m}\mathcal{G}$ annihilates $\Gamma_{\mathfrak{m}}(\mathcal{G})$. Thus, $\mathfrak{m}\mathcal{G}$ is the only minimal prime in the support of $\Gamma_{\mathfrak{m}}(\mathcal{G})$ of dimension n. Therefore,

$$j(I) = e(\Gamma_{\mathfrak{m}}(\mathcal{G})) = \lambda((\Gamma_{\mathfrak{m}}(\mathcal{G}))_{\mathfrak{m}\mathcal{G}}) \cdot e(\mathcal{G}/\mathfrak{m}\mathcal{G}) = \lambda(\mathcal{G}_{\mathfrak{m}\mathcal{G}}) \cdot e(k[G]).$$

It remains to show that $\mathcal{G}_{\mathfrak{m}\mathcal{G}}$ has length m. Let $\mathcal{R}$ denote the Rees algebra of I, which is defined as

$$\mathcal{R} = R[It] = R[x_{i_1} \cdots x_{i_m} t \mid \{x_{i_1}, \ldots, x_{i_m}\} \in E(G)].$$

Then $\mathcal{G} = \mathcal{R}/I\mathcal{R}$ and so $\mathcal{G}_{\mathfrak{m}\mathcal{G}} \simeq \mathcal{R}_{\mathfrak{m}\mathcal{R}}/I\mathcal{R}_{\mathfrak{m}\mathcal{R}}$. We claim that the ideal $\mathfrak{m}\mathcal{G}_{\mathfrak{m}\mathcal{G}} = \mathfrak{m}\mathcal{R}_{\mathfrak{m}\mathcal{R}}/I\mathcal{R}_{\mathfrak{m}\mathcal{R}}$ is principal. Since G is properly connected and $j(I)$ is not zero, any two nodes x_i and x_j in G are pivot equivalent by Proposition 6.2. Then by Lemma 7.3 below, we have $(x_i)\mathcal{R}_{\mathfrak{m}\mathcal{R}} = (x_j)\mathcal{R}_{\mathfrak{m}\mathcal{R}}$. Thus, $\mathfrak{m}\mathcal{R}_{\mathfrak{m}\mathcal{R}} = (x_i)\mathcal{R}_{\mathfrak{m}\mathcal{R}}$ for any node x_i in G, which proves the claim. Let $\{x_{i_1}, \ldots, x_{i_m}\}$ be an edge in G. Then

$$\mathfrak{m}^m \mathcal{R}_{\mathfrak{m}\mathcal{R}} = (x_{i_1})\mathcal{R}_{\mathfrak{m}\mathcal{R}} \cdots (x_{i_m})\mathcal{R}_{\mathfrak{m}\mathcal{R}} = (x_{i_1} \cdots x_{i_m})\mathcal{R}_{\mathfrak{m}\mathcal{R}} \subset I\mathcal{R}_{\mathfrak{m}\mathcal{R}} \subset \mathfrak{m}^m \mathcal{R}_{\mathfrak{m}\mathcal{R}}.$$

Thus, $I\mathcal{R}_{\mathfrak{m}\mathcal{R}} = \mathfrak{m}^m \mathcal{R}_{\mathfrak{m}\mathcal{R}}$. Hence, the principal ideal

$$\mathfrak{m}^k \mathcal{G}_{\mathfrak{m}\mathcal{G}} = (\mathfrak{m}^k + I)\mathcal{R}_{\mathfrak{m}\mathcal{R}}/I\mathcal{R}_{\mathfrak{m}\mathcal{R}}$$

is zero if and only if $k \geq m$. Therefore,

$$\lambda(\mathcal{G}_{\mathrm{m}\mathcal{G}}) = \sum_{k=1}^{m} \lambda\left(\mathrm{m}^{k-1}\mathcal{G}_{\mathrm{m}\mathcal{G}}/\mathrm{m}^{k}\mathcal{G}_{\mathrm{m}\mathcal{G}}\right) = m.$$

$\square$

Lemma 7.3 *Let G be an m-uniform hypergraph. Let $\mathcal{R}$ denote the Rees algebra of the edge ideal of G. If x_i and x_j are two nodes in G that are pivot equivalent, then $(x_i)\mathcal{R}_{\mathrm{m}\mathcal{R}} = (x_j)\mathcal{R}_{\mathrm{m}\mathcal{R}}$.*

Proof Note that if $\{x_{i_1}, \ldots, x_{i_m}\}$ is an edge in G, then $x_{i_1} \cdots x_{i_m} t \in \mathcal{R} \setminus \mathrm{m}\mathcal{R}$. Hence, $x_{i_1} \cdots x_{i_m} t$ is invertible in $\mathcal{R}_{\mathrm{m}\mathcal{R}}$. If $x_i \approx x_j$, then there is a subset $A \subset \{x_1, \ldots, x_n\} \setminus \{x_i, x_j\}$, such that $\{x_i\} \cup A$ and $\{x_j\} \cup A$ belong to $E(G)$. Write $A = \{x_{p_1}, \ldots, x_{p_{m-1}}\}$. Then $x_{p_1} \cdots x_{p_{m-1}} x_i t$ and $x_{p_1} \cdots x_{p_{m-1}} x_j t$ are invertible in the localization $\mathcal{R}_{\mathrm{m}\mathcal{R}}$. Therefore,

$$\frac{x_i}{1} = \frac{x_i x_{p_1} \cdots x_{p_{m-1}} t}{x_{p_1} \cdots x_{p_{m-1}} x_j t} \cdot \frac{x_j}{1},$$

which implies that $(x_i)\mathcal{R}_{\mathrm{m}\mathcal{R}} = (x_j)\mathcal{R}_{\mathrm{m}\mathcal{R}}$. If x_i and x_j are pivot equivalent, then there is a sequence of nodes $x_{i_1}, \ldots x_{i_r}$ such that

$$x_i = x_{i_1} \approx x_{i_2} \approx \cdots \approx x_{i_r} = x_j.$$

Hence by what we observed earlier,

$$(x_i)\mathcal{R}_{\mathrm{m}\mathcal{R}} = (x_{i_1})\mathcal{R}_{\mathrm{m}\mathcal{R}} = \cdots = (x_{i_r})\mathcal{R}_{\mathrm{m}\mathcal{R}} = (x_j)\mathcal{R}_{\mathrm{m}\mathcal{R}}.$$

$\square$

Remark 7.4 The converse of Lemma 7.3 is not true in general. Indeed, if $(x_i)\mathcal{R}_{\mathrm{m}\mathcal{R}} = (x_j)\mathcal{R}_{\mathrm{m}\mathcal{R}}$, then one can show that there are two subsets of $E(G)$, with associated square-free monomials $\{m_1, \ldots m_s\}$ and $\{m'_1, \ldots m'_s\}$ in $I(G)$, such that

$$x_i\, m_1 \cdots m_s = x_j\, m'_1 \cdots m'_s. \tag{8}$$

But we cannot conclude that x_i and x_j are pivot equivalent. For example, let G be a 3-uniform hypergraph with $V(G) = \{x, y, z, w, x_1, x_2, x_3\}$ and $E(G)$ the triangles in the simplicial complex illustrated in Fig. 1. Then one may directly verify that

$$w(xx_1x_2)(xx_1x_3)(xx_2x_3)(yzw) = x(xywt)(xzw)(x_1x_2x_3)^2. \tag{9}$$

Note that the expression in each parenthesis in (9) corresponds to an edge in G, hence it is invertible in $\mathcal{R}_{\mathrm{m}\mathcal{R}}$ after multiplying by the variable t. Therefore, $(w)\mathcal{R}_{\mathrm{m}\mathcal{R}} = (x)\mathcal{R}_{\mathrm{m}\mathcal{R}}$. However, x and w are not pivot equivalent. It would be interesting to find a combinatorial interpretation of (8) in graph-theoretical terms.

Fig. 1 The boundary of a tetrahedron attached to a union of three triangles
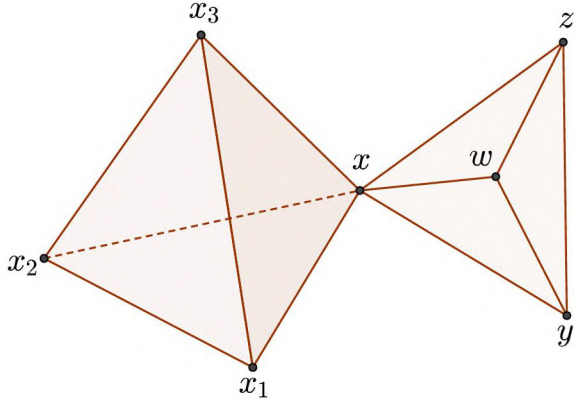
Now we consider the case that G has more than one properly connected component.

Theorem 7.5 *Let G be an m-uniform hypergraph with properly connected components. If c is the number of components and $j(I(G))$ is not zero, then*

$$j(I(G)) = m^c e(k[G]).$$

Proof Let $G_1, \ldots, G_c$ denote the connected components of G. Then by Proposition 5.3 and Theorem 7.2, we obtain $j(I(G)) = m^c e(k[G_1]) \cdots e(k[G_c])$. Therefore, the result follows from the main theorem of [26] which implies $e(k[G_1]) \cdots e(k[G_c]) = e(k[G])$ since $k[G_1] \otimes_k \cdots \otimes_k k[G_c] \simeq k[G]$. □

Below we also sketch a direct proof of Theorem 7.5 without using the multiplicativity formula in Proposition 5.3 and the main result of [26].

Proof Let $\mathcal{G}$ be the associated graded ring of R with respect to the edge ideal I of G. Then, as in the proof of Theorem 7.2,

$$j(I) = \lambda(\mathcal{G}_{\mathfrak{m}\mathcal{G}}) \cdot e(k[G]).$$

We need to show that $\mathcal{G}_{\mathfrak{m}\mathcal{G}}$ has length m^c. Recall that $\mathcal{G} = \mathcal{R}/I\mathcal{R}$, where $\mathcal{R}$ is the Rees algebra of I. Thus, $\mathcal{G}_{\mathfrak{m}\mathcal{G}} \simeq \mathcal{R}_{\mathfrak{m}\mathcal{R}}/I\mathcal{R}_{\mathfrak{m}\mathcal{R}}$. Now let $X_k \subset \{x_1, \ldots, x_n\}$ be the set of the nodes of the connected component G_k, so $\{x_1, \ldots, x_n\}$ is the disjoint union of $X_1, \ldots, X_c$. After a possible relabeling of the nodes, we may assume that $x_k \in X_k$ for $k = 1, \ldots, c$. Then Lemma 7.3 implies $(X_k)\mathcal{R}_{\mathfrak{m}\mathcal{R}} = (x_k)\mathcal{R}_{\mathfrak{m}\mathcal{R}}$ for $k = 1, \ldots, c$. Therefore,

$$\mathfrak{m}\mathcal{G}_{\mathfrak{m}\mathcal{G}} = (x_1, \ldots, x_c)\mathcal{G}_{\mathfrak{m}\mathcal{G}}.$$

Also $(x_k^m)\mathcal{G}_{\mathfrak{m}\mathcal{G}} = (0)$ for all $k = 1, \ldots, c$ as in the proof of Theorem 7.2. Thus, by the pigeonhole principle

$$\mathfrak{m}^{c(m-1)+1}\mathcal{G}_{\mathfrak{m}\mathcal{G}} = (0).$$

Furthermore, it can be readily seen that for $i = 0, \ldots, c(m-1)$ the ideal $\mathfrak{m}^i \mathcal{G}_\mathfrak{m} \mathcal{G}$ is minimally generated by monomials $x_1^{a_1} \cdots x_c^{a_c}$ of degree i such that the a_k are less than m. Therefore,

$$\lambda(\mathcal{G}_\mathfrak{m} \mathcal{G}) = \sum_{i=0}^{c(m-1)} \lambda\left(\mathfrak{m}^i \mathcal{G}_\mathfrak{m} \mathcal{G} / \mathfrak{m}^{i+1} \mathcal{G}_\mathfrak{m} \mathcal{G}\right)$$

is the number of all monomials $x_1^{a_1} \cdots x_c^{a_c}$ such that the a_k are less than m, which is m^c. $\qquad\square$

Example 7.6 Let G be the complete multipartite graph on n nodes of type $(q_1, \ldots, q_k)$. If k is at least 3, then by [17, Corollary 2.7] and Theorem 7.2 we obtain

$$j(I(G)) = 2e(k[G]) = 2^n - 2\sum_{i=1}^{k}\sum_{j=1}^{q_i}\binom{n-1}{j-1}.$$

The following result is an immediate consequence of Theorem 7.5 and Corollary 5.1

Corollary 7.7 *Let G be an m-uniform hypergraph on n nodes with properly connected components. If G has c connected components and $\mathrm{Vol}_{n-1}(F(G)) \neq 0$, equivalently, if the nodes in each connected component of G are pivot equivalent, then*

$$e(k[G]) = m^{c-1}\mathrm{Vol}_{n-1}(F(G)).$$

Remark 7.8 Note that in Theorem 7.2, if we do not assume G is properly connected then the statement fails, as the following example illustrates. Here G is a connected 3-uniform hypergraph with $V(G) = \{x_1, x_2, x_3, x_4, x_5, y_1, y_2, y_3\}$. The edge set $E(G)$ is given by the triangles in the simplicial complex represented in Fig. 2. Note that G
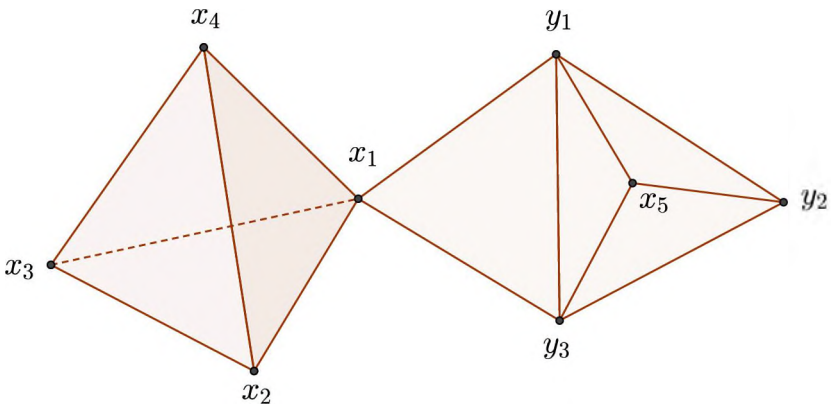
Fig. 2 The boundary of a tetrahedron attached to a union of four triangles

has 8 nodes and 8 edges, and the incidence matrix $M(G)$ is a square 8×8 matrix of full rank. A simple calculation provides

$$j(G) = \mathrm{Vol}_8(\mathrm{pyr}(F(G))) = \det M(G) = 6.$$

On the other hand, as in the proof of Proposition 8.2, one can see that the edge ring $k[G]$ is isomorphic to a polynomial ring over a field, and so $e(k[G]) = 1$, which shows that Theorem 7.2 fails for not properly connected hypergraphs. We can also calculate $j(G)$ directly as in the proof of Theorem 7.2. Recall that

$$j(G) = e(\Gamma_\mathrm{m}(\mathcal{G})) = \lambda(\mathcal{G}_{\mathrm{m}\mathcal{G}}) \cdot e(k[G]) = \lambda(\mathcal{G}_{\mathrm{m}\mathcal{G}}).$$

Let us show that the length of $\mathcal{G}_{\mathrm{m}\mathcal{G}} \simeq \mathcal{R}_{\mathrm{m}\mathcal{R}}/I\mathcal{R}_{\mathrm{m}\mathcal{R}}$ is 6. First, note that G has two pivot classes $\{x_1, x_2, x_3, x_4, x_5\}$ and $\{y_1, y_2, y_3\}$. Then by Lemma 7.3 we have $(x_i)\mathcal{R}_{\mathrm{m}\mathcal{R}} = (x_1)\mathcal{R}_{\mathrm{m}\mathcal{R}}$ for $i = 1, \ldots 5$, and $(y_j)\mathcal{R}_{\mathrm{m}\mathcal{R}} = (y_1)\mathcal{R}_{\mathrm{m}\mathcal{R}}$ for $j = 1, 2, 3$. Thus, $\mathrm{m}\mathcal{R}_{\mathrm{m}\mathcal{R}} = (x_1, y_1)\mathcal{R}_{\mathrm{m}\mathcal{R}}$. Using edges $\{x_1, y_1, y_3\}$ and $\{x_1, x_2, x_4\}$, we have $x_2 x_4 (x_1 y_1 y_3 t) = y_1 y_3 (x_1 x_2 x_4 t)$. Hence, we may write

$$(x_1^2)\mathcal{R}_{\mathrm{m}\mathcal{R}} = (x_2 x_4)\mathcal{R}_{\mathrm{m}\mathcal{R}} = (y_1 y_3)\mathcal{R}_{\mathrm{m}\mathcal{R}} = (y_1^2)\mathcal{R}_{\mathrm{m}\mathcal{R}}.$$

This implies that $\mathrm{m}^2 \mathcal{R}_{\mathrm{m}\mathcal{R}} = (x_1^2, x_1 y_1)\mathcal{R}_{\mathrm{m}\mathcal{R}}$, $\mathrm{m}^3 \mathcal{R}_{\mathrm{m}\mathcal{R}} = (x_1^3 \cdot y_1^3)\mathcal{R}_{\mathrm{m}\mathcal{R}}$ and $\mathrm{m}^4 \mathcal{R}_{\mathrm{m}\mathcal{R}} = (x_1^4, x_1^3 y_1)\mathcal{R}_{\mathrm{m}\mathcal{R}}$. Note that $(x_1^3)\mathcal{R}_{\mathrm{m}\mathcal{R}} = (x_1 x_2 x_3)\mathcal{R}_{\mathrm{m}\mathcal{R}} \subset I\mathcal{R}_{\mathrm{m}\mathcal{R}}$. Therefore, $\mathrm{m}^4 \mathcal{G}_{\mathrm{m}\mathcal{G}} = (0)$ and $\mathrm{m}^i \mathcal{G}_{\mathrm{m}\mathcal{G}}/\mathrm{m}^{i+1}\mathcal{G}_{\mathrm{m}\mathcal{G}}$ for $i = 0, 1, 2, 3$ have bases $\{1\}$, $\{x_1, y_1\}$, $\{x_1^2, x_1 y_1\}$, and $\{y_1^3\}$, respectively. Thus,

$$\lambda(\mathcal{G}_{\mathrm{m}\mathcal{G}}) = \sum_{i=0}^{3} \lambda\left(\mathrm{m}^i \mathcal{G}_{\mathrm{m}\mathcal{G}}/\mathrm{m}^{i+1}\mathcal{G}_{\mathrm{m}\mathcal{G}}\right) = 1 + 2 + 2 + 1 = 6.$$

8 The j-multiplicity of edge ideals and toric edge ideals

Let $I(G)$ be the edge ideal of an m-uniform hypergraph G on n nodes $x_1, \ldots, x_n$. As we mentioned in the previous section, the associated edge subring $k[G]$ can be regarded as a standard graded algebra over k. Therefore, we may define a homogeneous epimorphism of k-algebras

$$\phi : S = k[T_{i_1 \cdots i_m} \mid \{x_{i_1}, \ldots, x_{i_m}\} \in E(G)] \to k[G],$$

where the $T_{i_1 \cdots i_m}$ are indeterminates over k, by assigning $\phi(T_{i_1 \cdots i_m}) = x_{i_1} \cdots x_{i_m}$ for $\{x_{i_1}, \ldots, x_{i_m}\} \in E(G)$. Thus, one obtains a homogeneous isomorphism $k[G] \simeq S/I_G$, where $I_G = \ker(\phi)$ is a homogeneous prime ideal called the *toric edge ideal* of G. Indeed, the ideal I_G is generated by binomials, defining an affine toric variety [36].

Proposition 8.1 *Let G be an m-uniform hypergraph on n nodes with properly connected components. Let e denote the number of edges, p the number of pivot equivalence classes and c the number of connected components of G. Then*

$$\operatorname{ht} I_G = e - n + p - c.$$

Proof Recall that $\dim k[G] = \ell(I(G))$ by Remark 7.1. Thus, one can compute the height of the toric edge ideal of G as

$$\operatorname{ht} I_G = \dim S - \dim k[G] = e - \ell(I(G)).$$

If all connected components of G are properly connected, then $\ell(I(G)) = n - p + c$ by Proposition 6.1 and the result follows. □

Recall that if $j(I(G)) \neq 0$, then by Remark 5.2 the number of edges of G is at least the number of nodes of G. The following result deals with the extremal case and extends Proposition 5.6 to m-uniform hypergraphs.

Proposition 8.2 *Let G be an m-uniform hypergraph with properly connected components. Assume the number of edges of G is equal to the number of nodes of G. If G has c connected components and $j(I(G)) \neq 0$, then*

$$j(I(G)) = m^c.$$

Proof Since all connected components of G are properly connected and $j(I(G)) \neq 0$, by Proposition 6.2, each connected component of G admits only one pivot equivalence class. Then by Proposition 8.1, the toric edge ideal I_G has height zero. Thus, I_G is zero. Hence, $k[G]$ is isomorphic to a polynomial ring over a field, and thus, $e(k[G]) = 1$. Therefore, by Theorem 7.5 we obtain

$$j(I(G)) = m^c e(k[G]) = m^c.$$

□

Example 8.3 If G is the complete $(n-1)$-uniform hypergraph on n nodes, then $e = n$. In addition, G is properly connected and has only one pivot equivalence class. Therefore, by Proposition 8.2 we obtain $j(I(G)) = n - 1$, as in Example 3.3.

Recall that a *walk* w of length s in a simple graph G is a sequence of edges of the form

$$\{x_{i_0}, x_{i_1}\}, \{x_{i_1}, x_{i_2}\}, \ldots, \{x_{i_{s-1}}, x_{i_s}\}.$$

A walk w is called *closed* if the initial and the end nodes x_{i_0}, x_{i_s} are equal. If w is a closed walk of even length $2l$, then we call w a *monomial walk* and we define

$$T_w = T_{i_0 i_1} T_{i_2 i_3} \cdots T_{i_{2l-2} i_{2l-1}} - T_{i_1 i_2} T_{i_3 i_4} \cdots T_{i_{2l-1} i_{2l}} \in S,$$

which belongs to the toric edge ideal I_G. Indeed, the toric edge ideal I_G is generated by binomials of the form T_w associated to monomial walks in G [42]. More generally, one may define monomial walks in an m-uniform hypergraph G such that the toric
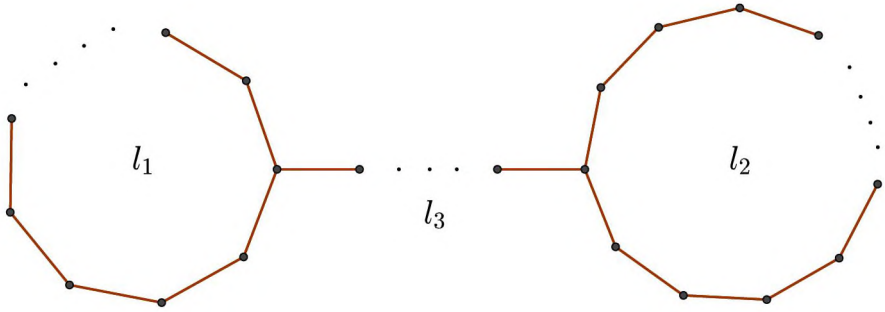
Fig. 3 A bicyclic graph of type 1

edge ideal I_G is generated by the associated binomials [28]. We say a monomial walk w is *nontrivial* if $T_w \neq 0$, and *minimal* if T_w is irreducible. For example, if G is unicyclic with an odd cycle, then it does not admit a nontrivial monomial walk; hence, I_G is zero as we observed in the proof of Proposition 8.2. Two monomial walks w and w' are called *equivalent* if $T_w = T_{w'}$.

A simple connected graph G is called *bicyclic* if the number of edges is one more than the number of nodes. For instance, if G is a simple graph obtained by connecting two disjoint cycles with a path, then G is a bicyclic graph known as a *bowtie* (Fig. 3). If G consists of two cycles with a common node, then we regard it as a bowtie graph where the length of the path between the two cycles is zero. The following result computes the j-multiplicity of the edge ideals of bicyclic graphs.

Proposition 8.4 *Let G be an m-uniform hypergraph with properly connected components. Assume the number of edges in G is one more than the number of nodes and G has c connected components. If $j(I(G)) \neq 0$, then there is a unique nontrivial minimal monomial walk w in G up to equivalence. Furthermore, if the length of w is $2l$, then*

$$j(I(G)) = m^c l.$$

In particular, if G is a bicyclic graph with an odd cycle, then $j(I(G))$ is the length of the unique nontrivial minimal monomial walk in G.

Proof Recall that by Proposition 6.2 $j(I(G)) \neq 0$ if and only if each connected component of G contains only one pivot equivalence class. Then we have ht $I_G = e - n + p - c = 1$ by Proposition 8.1. Therefore, I_G is a principal prime ideal generated by an irreducible homogeneous binomial T_w corresponding to a unique minimal monomial walk w in G up to equivalence. Hence, we obtain $e(k[G]) = e(S/I_G) = e(S/(T_w)) = \deg T_w$. Thus, by Theorem 7.5 we conclude that

$$j(I(G)) = m^c \cdot e(k[G]) = m^c \cdot \deg T_w.$$

Thus, the result follows as the degree of T_w is half the length of the monomial walk w. $\qquad\qquad\square$

Fig. 4 A bicyclic graph of type 2

l_1 $\vdots$ l_3 l_2
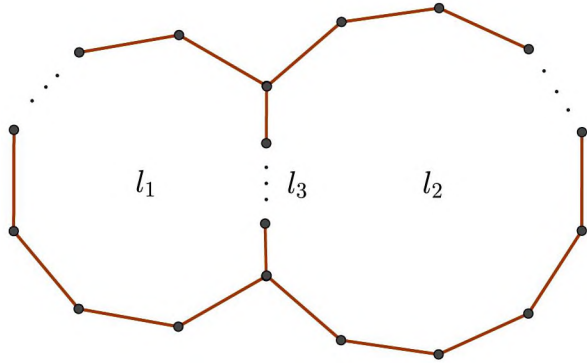
Example 8.5 Let G be a bicyclic graph, consisting of two cycles of lengths l_1 and l_2 connected by a path (Fig. 3) or attached along a path of length l_3 (Fig. 4). If both l_1 and l_2 are odd, then the length of the unique nontrivial minimal monomial walk in G is $l_1 + l_2 + 2l_3$ for the first type of graphs, and it is $l_1 + l_2 - 2l_3$ for the second type of graphs. Thus,

$$j(I(G)) = l_1 + l_2 \pm 2l_3.$$

If l_1 is odd and l_2 is even, then $j(I(G)) = l_2$, and if both l_1 and l_2 are even, then $j(I(G)) = 0$ by Proposition 5.5.

One may also obtain the following result as an immediate corollary of Proposition 8.4, Proposition 8.2 and Proposition 5.3.

Corollary 8.6 *Let G be a simple graph in which the connected components are unicyclic or bicyclic. If $j(I(G))$ is not zero, then*

$$j(I(G)) = 2^c l_1 \cdots l_k,$$

where c is the number of connected components of G and the l_i are half the length of the unique nontrivial minimal monomial walks in the bicyclic connected components of G.

Remark 8.7 Note that the toric edge ideal of the graphs as in the statement of Corollary 8.6 is complete intersections. Let G be an arbitrary m-uniform hypergraph with complete intersection toric edge ideal I_G, generated by a regular sequence of binomials $T_{w_1}, \ldots, T_{w_s}$. Then

$$e(k[G]) = e(S/(T_{w_1}, \ldots, T_{w_s})) = \deg T_{w_1} \cdots \deg T_{w_s}.$$

Therefore, if G has properly connected components and the j-multiplicity of the edge ideal of G is not zero, then by Theorem 7.5 we obtain

$$j(I(G)) = m^c \cdot e(k[G]) = m^c \deg T_{w_1} \cdots \deg T_{w_s} = m^c l_1 \cdots l_s,$$

where l_i is half the length of the monomial walk w_i for $i = 1, \ldots, s$. In particular, we recover Corollary 8.6 without using Proposition 5.3 and the volumes. For a study of simple graphs with complete intersection toric edge ideals, see [2, 14, 37].

9 Inequalities on the j-multiplicity of edge ideals

In this section, we explore the relations between the j-multiplicity of the edge ideals of hypergraphs and their subhypergraphs and we obtain general bounds for the j-multiplicity of edge ideals. Let G and H be hypergraphs. Then H is called a *subhypergraph* of G if $V(H)$ and $E(H)$ are subsets of $V(G)$ and $E(G)$, respectively. In Theorem 9.2 below, we prove a monotonicity property of the j-multiplicity, which will be useful in providing bounds for the j-multiplicity of edge ideals. We start with the following geometric observation.

Lemma 9.1 *Let A be any finite set of lattice points in $\mathbb{R}^n$ and $B \subset A$. Then the normalized volume of $\mathrm{conv}(B)$ in the affine span of B is no greater than the normalized volume of $\mathrm{conv}(A)$ in the affine span of A.*

Proof By induction, it is enough to assume that $|A| - |B| = 1$. Also, by choosing coordinates we may assume that the affine span of A is $\mathbb{R}^n$. Let $A \setminus B = \{a\}$. If the affine span of B is also $\mathbb{R}^n$, then clearly

$$\mathrm{Vol}_n(\mathrm{conv}(B)) \leq \mathrm{Vol}_n(\mathrm{conv}(A)).$$

Otherwise, the affine span of B is an affine hyperplane $L \subset \mathbb{R}^n$ and $\mathrm{conv}(A)$ is the pyramid over $\mathrm{conv}(B)$ with apex a. Then

$$\mathrm{Vol}_{n-1}(\mathrm{conv}(B)) \leq \mathrm{Vol}_n(\mathrm{conv}(A))$$

follows from (2) since the lattice distance from the affine span of B to a is a positive integer. □

Theorem 9.2 *Let G be an m-uniform hypergraph. If $j(I(G))$ is not zero and H is a subhypergraph of G, then*

$$j(I(H)) \leq j(I(G)).$$

Proof Let $A \subset \mathbb{R}^n$ consist of the origin and the lattice points corresponding to the edges of G. Then $j(I(G)) = \mathrm{Vol}_n(\mathrm{conv}(A))$ by Theorem 3.1. The set of nodes $V(H)$ defines a coordinate subspace of $\mathbb{R}^n$ which we identify with $\mathbb{R}^k$, where $k = |V(H)|$. Similarly, let $B \subset \mathbb{R}^k$ consist of the origin and the lattice points corresponding to the edges of H, and hence, $j(I(H)) = \mathrm{Vol}_k(\mathrm{conv}(B))$. If the affine span of B equals $\mathbb{R}^k$, then $j(I(H)) \leq j(I(G))$ by Lemma 9.1. Otherwise, $j(I(H)) = 0$ and the inequality obviously holds. □

Remark 9.3 The above argument easily carries over to the case of arbitrary monomial ideals I in $R = k[x_1, \ldots, x_n]_{(x_1, \ldots, x_n)}$ whose minimal monomial generators have exponents lying in a hyperplane (that is when dim $F(I) < n$). Namely, if $\mathcal{B}$ is a subset of the set of the minimal monomial generators of I and $X \subset \{x_1, \ldots, x_n\}$ is the set of variables appearing in $\mathcal{B}$, then the ideal $J \subset k[X]_{(X)}$ generated by $\mathcal{B}$ satisfies $j(J) \leq j(I)$. Note that the condition dim $F(I) < n$ is essential here as the following simple example shows. If $I = \langle x^3, xy, y^3 \rangle$ and $J = \langle x^3, y^3 \rangle$ in $R = k[x, y]_{(x,y)}$ then $j(J) > j(I)$.

Corollary 9.4 *Let G be an m-uniform hypergraph on n nodes. Then $j(I(G))$ is bounded above by the j-multiplicity of the edge ideal of the complete m-uniform hypergraph on n nodes mentioned in Example 3.3. In particular if G is a simple graph, then $j(I(G))$ is at most $2^n - 2n$.*

Let G be a simple graph with *odd tulgeity* τ_0, which is the maximum number of node-disjoint odd cycles in G. Let H be a subgraph of G consisting of τ_0 node-disjoint odd cycles in G. Then by Proposition 5.6 or Proposition 8.2, the j-multiplicity of $I(H)$ is 2^{τ_0}. Therefore, if $I(G)$ has nonzero j-multiplicity, then $j(I(G)) \geq 2^{\tau_0}$ by Theorem 9.2. On the other hand, if G is a multipartite graph of type $(q_1, \ldots, q_k)$, then by Theorem 9.2 $j(I(G))$ is bounded above by the j-multiplicity of the complete multipartite graph of type $(q_1, \ldots, q_k)$ as in Example 7.6. Therefore, we obtain the following corollary.

Corollary 9.5 *Let G be a simple multipartite graph of type $(q_1, \ldots, q_k)$ with n nodes and odd tulgeity τ_0. If the j-multiplicity of $I(G)$ is not zero, then*

$$2^{\tau_0} \leq j(I(G)) \leq 2^n - 2 \sum_{i=1}^{k} \sum_{j=1}^{q_i} \binom{n-1}{j-1}.$$

For a node x in G, we let $G - x$ denote the subhypergraph of G obtained by removing x and the edges containing it from G. We say that x is a *free node* if it is contained in only one edge in $E(G)$. For simple graphs, a free node is also known as a *whisker*. Recall that by Theorem 9.2, $j(I(G - x)) \leq j(I(G))$ for every node x in G. Below we note that equality holds for free nodes.

Proposition 9.6 *Let G be an m-uniform hypergraph containing a free node x. Then*

$$j(I(G)) = j(I(G - x)).$$

Proof If $x_i \in V(G) = \{x_1, \ldots, x_n\}$ is a free node, then removing x_i and the corresponding edge from G is equivalent to removing the unique vertex of the edge polytope $F(G)$ with z_i-coordinate being 1. Note that $F(G)$ is a pyramid with apex at this vertex and base $F(G - x_i)$. Since the base lies in the hyperplane $z_i = 0$, the height of the pyramid is one. Therefore, the normalized $(n - 1)$-volume of $F(G)$ equals the normalized $(n - 2)$-volume of the base $F(G - x_i)$. Then by Corollary 5.1, we obtain

$$j(I(G)) = m \cdot \mathrm{Vol}_{n-1}(F(G)) = m \cdot \mathrm{Vol}_{n-2}(F(G - x_i)) = j(I(G - x_i)).$$

$\square$

One could also prove Proposition 9.6 algebraically for simple graphs using toric edge ideals as follows.

Proof By Proposition 5.3, we may assume G is connected. We may further assume G contains an odd cycle, otherwise the statement is trivially true as both $j(I(G))$ and $j(I(G - x))$ are zero. Let α be the only edge in $E(G)$ containing x. Then α is not part of any nontrivial minimal monomial walk in G. Therefore, if we write $k[G] \simeq S/I_G$ as in Sect. 8, then α corresponds to a variable T_α in S not appearing in the generators of the toric edge ideal I_G. If we let $\bar{S} = S/(T_\alpha)$ and consider α as an element in $k[G]$, then we have the following homogenous isomorphisms of graded k-algebras,

$$k[G]/(\alpha) \simeq S/(I_G + (T_\alpha)) \simeq \bar{S}/I_{G-x} \simeq k[G - x].$$

Therefore, using the homogenous short exact sequence

$$0 \to k[G](-1) \xrightarrow{\alpha} k[G] \to k[G]/(\alpha) \simeq k[G - x] \to 0$$

we obtain $e(k[G]) = e(k[G - x])$. Now since both G and $G - x$ are connected and contain an odd cycle, by Theorem 7.2 we conclude

$$j(I(G)) = 2e(k[G]) = 2e(k[G - x]) = j(I(G - x)).$$

$\square$

The following result gives a lower bound for the j-multiplicity of the edge ideal of an m-uniform hypergraph in terms of the multiplicity of the associated edge subring.

Proposition 9.7 *Let G be an m-uniform hypergraph with c connected components, not necessarily properly connected. If $j(I(G))$ is not zero, then*

$$j(I(G)) \geq m^c \cdot e(k[G]).$$

Proof If G is a connected m-uniform hypergraph, not necessarily properly connected, then as in the proof of Theorem 7.2 we have

$$j(I(G)) = \lambda(\mathcal{G}_{m\mathcal{G}}) \cdot e(k[G])$$

when $j(I(G))$ is not zero. Note that $I\mathcal{R}_{m\mathcal{R}} \subset \mathfrak{m}^m \mathcal{R}_{m\mathcal{R}}$. Thus, $\mathfrak{m}^k \mathcal{G}_{m\mathcal{G}} = (\mathfrak{m}^k + I)\mathcal{R}_{m\mathcal{R}}/I\mathcal{R}_{m\mathcal{R}}$ is not zero for k less than m. Hence,

$$\lambda(\mathcal{G}_{m\mathcal{G}}) = \sum_{k \geq 1} \lambda\left(\mathfrak{m}^{k-1}\mathcal{G}_{m\mathcal{G}}/\mathfrak{m}^k \mathcal{G}_{m\mathcal{G}}\right) \geq m.$$

Therefore, $j(I(G))$ is greater than or equal to $m \cdot e(k[G])$. If G is not connected, then the desired inequality follows from Proposition 5.3 and the fact that the multiplicity of the edge subring is multiplicative over the connected components. □

Let G be an m-uniform hypergraph with properly connected components. Assume the toric edge ideal I_G is minimally generated by binomials $T_{w_1}, \ldots, T_{w_s}$. For a description of the minimal generators of the toric edge ideals of simple graphs, see [31]. Then as in Sect. 8 we may represent the edge subring $k[G]$ as $S/(T_{w_1}, \ldots, T_{w_s})$. Therefore,

$$e(k[G]) = e(S/(T_{w_1}, \ldots, T_{w_s})) \leq \deg T_{w_1} \cdots \deg T_{w_s}.$$

Hence, by Theorem 7.2 we obtain

$$j(I(G)) = m^c \cdot e(k[G]) \leq m^c \deg T_{w_1} \cdots \deg T_{w_s}.$$

Thus, we have the following result.

Proposition 9.8 *Let G be an m-uniform hypergraph with properly connected components. Then*

$$j(I(G)) \leq m^c l_1 \cdots l_s,$$

where the l_i are half the length of the monomial walks in G corresponding to a minimal generating set of I_G.

Let G be a simple connected graph on n nodes and e edges, such that the edge subring $k[G]$ is Cohen–Macaulay. See, for instance, [3] for a study of graphs with Cohen–Macaulay edge subring. Then Lemma 4.1 in [18] states that $\mathrm{Vol}_{n-1}(F(G))$ is at least $e - n + 1$ when G is not bipartite. Therefore, by Corollary 5.1 we obtain the following lower bound for the j-multiplicity of the edge ideal of G.

Proposition 9.9 *Let G be a simple connected graph on n nodes and e edges whose edge subring is Cohen–Macaulay. If $j(I(G))$ is not zero, then*

$$j(I(G)) \geq 2(e - n + 1).$$

10 The ε-multiplicity of edge ideals

We recall the notion of the ε-multiplicity as introduced in [22] and [40]. Let I be an arbitrary ideal in a Noetherian local ring R with maximal ideal $\mathfrak{m}$ and dimension n. Then the ε-multiplicity of I is defined as

$$\varepsilon(I) = n! \limsup_k \frac{\lambda_R(\Gamma_{\mathfrak{m}}(R/I^k))}{k^n} \in \mathbb{R}_{\geq 0}.$$

Similar to the j-multiplicity, the ε-multiplicity can be viewed as an extension of the Hilbert–Samuel multiplicity to arbitrary ideals, for if I is $\mathfrak{m}$-primary, then $\Gamma_\mathfrak{m}(R/I^k) = R/I^k$; therefore, $\varepsilon(I) = e(I)$. However, the ε-multiplicity exhibits a very different behavior than the j-multiplicity. For instance, the j-multiplicity is always a non-negative integer, while the ε-multiplicity could be an irrational real number [9]. In this section, we will compute the ε-multiplicity of the edge ideal of cycles and complete hypergraphs, which further highlights the differences of the two invariants. The vanishing of the ε-multiplicity of an ideal is captured by the analytic spread of the ideal. Indeed, as in the case of j-multiplicity, the ε-multiplicity of I is not zero if and only if the analytic spread of I is maximal [22,40]. In particular, by Proposition 6.1 we obtain the following result.

Proposition 10.1 *If G is an m-uniform hypergraph with properly connected components, then $\varepsilon(I(G)) \neq 0$ if and only if the nodes in each connected component of G are pivot equivalent. Recall that for simple graphs, this condition means that each connected component contains an odd cycle.*

Let I be a monomial ideal in $R = k[x_1, \ldots, x_n]_{(x_1,\ldots,x_n)}$. Let $L_i \subset \mathbb{R}^n$ be the coordinate hyperplane defined by $z_i = 0$ and $\pi_i : \mathbb{R}^n \to L_i$ the corresponding orthogonal projection. For the Newton polyhedron $P(I)$, define

$$\hat{P}(I) = \bigcap_{i=1}^{n} \pi_i^{-1}\left(\pi_i(P(I))\right), \quad \hat{F}(I) = \mathrm{cl}(\hat{P}(I) \setminus P(I)), \tag{10}$$

where $\mathrm{cl}(K)$ denotes the closure of K in $\mathbb{R}^n$. The following theorem by Jeffries and Montaño [20, Theorem 5.1] gives an interpretation of the ε-multiplicity of monomial ideals in terms of the volumes of the associated polytopes.

Theorem 10.2 *Let $I \subset R$ be a monomial ideal. Then $\varepsilon(I) = \mathrm{Vol}_n(\hat{F}(I))$.*

Note that since $\hat{P}(I) \setminus P(I)$ is bounded, $P(I)$ and $\hat{P}(I)$ coincide outside of a large enough ball. Therefore, $P(I)$ and $\hat{P}(I)$ have the same facet inequalities for their unbounded facets. In particular, since $P(I) = F(I) + \mathbb{R}^n_{\geq 0}$, the inequalities $z_i \geq 0$ for $i = 1, \ldots, n$ are among the facet inequalities for both $P(I)$ and $\hat{P}(I)$.

Proposition 10.3 *Let $G_{m,n}$ be the complete m-uniform hypergraph on n nodes. Then*

$$\varepsilon(I(G_{m,n})) = \frac{n - m}{n - 1} A(n - 1, m).$$

In particular, for the complete simple graph $G_{2,n}$ and for the complete $(n-1)$-uniform hypergraph $G_{n-1,n}$ we obtain

$$\varepsilon(I(G_{2,n})) = \frac{n - 2}{n - 1}(2^{n-1} - n), \quad \varepsilon(I(G_{n-1,n})) = \frac{1}{n - 1}.$$

Proof Denote $I_{m,n} = I(G_{m,n})$. Clearly, when $m = n$ we have $I_{n,n} = (x_1 \cdots x_n)$ and $\varepsilon(I_{n,n}) = 0$ which agrees with the formula in the statement. Thus, we may assume that $m > n$. Let $P = P(I_{m,n})$ be the Newton polyhedron of $I_{m,n}$ and $F = F(I_{m,n})$ its compact facet. Recall from Example 3.3 that F is given by $\sum_{j=1}^{m} z_j = m$. For every $i = 1, \ldots, n$ the projection $\pi_i(P)$ equals $P(I_{m-1,n-1})$ embedded in the coordinate hyperplane $z_i = 0$. This implies that $\pi_i^{-1}\pi_i(P)$ has a facet given by $\langle u_i, z \rangle \geq m - 1$, where $u_i = -e_i + \sum_{j=1}^{m} e_j$. Therefore, $\hat{P}(I_{m,n})$ is given by the facet inequalities $\langle u_i, z \rangle \geq m - 1$ and $z_i \geq 0$ for all $i = 1, \ldots, n$. Since these facets are unbounded, they are also the unbounded facets of P. This shows that $\hat{F}(I_{m,n})$ is a pyramid over F with apex $a = \left(\frac{m-1}{n-1}, \ldots, \frac{m-1}{n-1}\right)$. Consequently, by Example 3.3 and equation (2) we obtain

$$\varepsilon(I_{m,n}) = \mathrm{Vol}_n(\hat{F}(I_{m,n})) = \left(m - \frac{n(m-1)}{n-1}\right)\mathrm{Vol}_{n-1}(F) = \frac{n-m}{n-1}A(n-1,m).$$
(11)

$\square$

Proposition 10.4 *Let G be a cycle of length n. If n is even, then $\varepsilon(I(G)) = 0$. If n is odd, then*

$$\varepsilon(I(G)) = \frac{2}{n+1}.$$

Proof If n is even, then $\varepsilon(I(G)) = 0$ by Proposition 10.1, so assume $n = 2k + 1$ for $k \in \mathbb{N}$. To simplify notation, we set $P = P(I)$, $F = F(I)$, and let $\hat{P} = \hat{P}(I)$ and $\hat{F} = \hat{F}(I)$ as defined in (10). By Theorem 10.2, $\varepsilon(I(G)) = \mathrm{Vol}_n(\hat{F})$. In Proposition 10.7 below, we show that $\hat{F}$ is the pyramid over F with apex $a = \left(\frac{1}{k+1}, \ldots, \frac{1}{k+1}\right)$. Since F lies in the hyperplane $\sum_{j=1}^{m} z_j = 2$ and $\mathrm{Vol}_{n-1}(F) = 1$, equation (2) produces

$$\mathrm{Vol}_n(\hat{F}) = \left(2 - \frac{n}{k+1}\right)\mathrm{Vol}_{n-1}(F) = \frac{2}{n+1}.$$

$\square$

To show that $\hat{F}$ is a pyramid over F we first describe the facet inequalities of $\hat{P}$ in Lemma 10.5 below. Recall that the *circulant matrix* C_u generated by a vector $u = (u_0, \ldots, u_{n-1}) \in \mathbb{R}^n$ is the $n \times n$ matrix whose rows are obtained by the cyclic permutations of the entries of u. The associated polynomial $f_u(t) = u_0 + u_1 t + \cdots + u_{n-1}t^{n-1}$ of C_u gives a formula for the rank of C_u [19, Proposition 1.1]:

$$\mathrm{rank}(C_u) = n - \deg\left(\gcd(t^n - 1, f_u(t))\right).$$
(12)

Lemma 10.5 *The facets of $\hat{P}$ are defined by the inequalities $I_n z \geq 0$, $C_u z \geq 1$, where I_n is the identity matrix, 1 is the vector of 1's, and C_u is the circulant matrix generated by $u = e_1 + \sum_{i=1}^{k} e_{2i} \in \mathbb{R}^n$, where $n = 2k + 1$. The same inequalities define the unbounded facets of P.*

Fig. 5 A cycle with an omitted 5th node
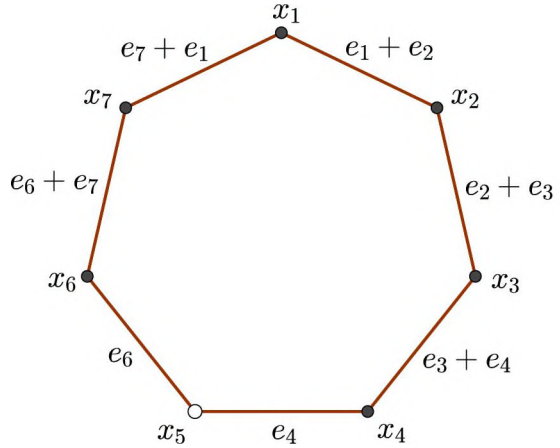
Proof First, let us describe the primitive normals to the facets of $F_i = \pi_i(F)$. By definition, F is an $(n-1)$-simplex lying in the hyperplane $\sum_{j=1}^{n} z_j = 2$ whose vertices are the rows of the incidence matrix of the cycle G. Then F_i is an $(n-1)$-simplex lying in L_i whose vertices are the rows of the incidence matrix of a "graph" G_i which is a cycle with omitted i-th node, so the rows corresponding to the edges with a missing node are two standard basis vectors, see Fig. 5 for an example.

Since F_i is a simplex, for every vertex $v \in F_i$ there is exactly one facet $F_i(v)$ not containing v. Here is a combinatorial way to produce a primitive normal to $F_i(v)$. (Note that its i-th entry can be arbitrary, so we may assume it is zero. Then it is unique up to sign). Removing the edge from G_i corresponding to v, we obtain a "graph" $G_i(v)$. Place 0 and 1 at the nodes of $G_i(v)$ in an alternating way starting with the 0 in i-th node and going both ways. This results in a vector $u(v) \in \mathbb{R}^n$ which is a primitive normal to $F_i(v)$. This process is illustrated in Fig. 6 with $n = 7, i = 5$, and v corresponding to the edge $\{x_1, x_2\}$.

Indeed, u is normal to $F_i(v)$ if and only if the linear function $\langle u, z \rangle$ takes the same value at all vertices of F_i, but v. Assume for simplicity that v corresponds to $\{x_1, x_2\}$ and $i = n = 2k + 1$. Then $v = e_1 + e_2$ and the remaining vertices are $e_2 + e_3, \ldots, e_{2k-1} + e_{2k}, e_{2k}, e_1$. Let $u = (u_1, \ldots, u_{2k+1})$. Then $\langle u, z \rangle$ takes the same value on the remaining vertices if and only if

$$u_2 + u_3 = u_3 + u_4 = \cdots = u_{2k-1} + u_{2k} = u_{2k} = u_1,$$

which implies $u_2 = u_4 = \cdots = u_{2k}$ and $u_3 = u_5 = \cdots = u_{2k-1}$, together with $u_{2k-1} = 0$ and $u_{2k} = u_1$. Since u is primitive, $u_1 = u_2 = u_4 = \cdots = u_{2k} = 1$ which justifies the combinatorial process of producing $u(v)$. The general case is similar.

Notice that the value of $\langle u(v), z \rangle$ at all vertices of F_i, but v equals 1. Furthermore, its value at v equals the sum of the two values placed at the nodes of v. These can be either both 1 or both 0. This shows that $u(v)$ is an inner normal to $\pi_i^{-1}(F_i)$ and, hence, to $\pi_i^{-1}(\pi_i(P))$ if and only if the two values are both 1. Thus, the primitive inner normals to the facets of $\pi_i^{-1}(\pi_i(P))$ are vectors obtained by a cyclic permutation of

Fig. 6 The vector $u(v) = (1, 1, 0, 1, 0, 1, 0)$ is normal to $F_5(v)$ for $v = \{x_1, x_2\}$
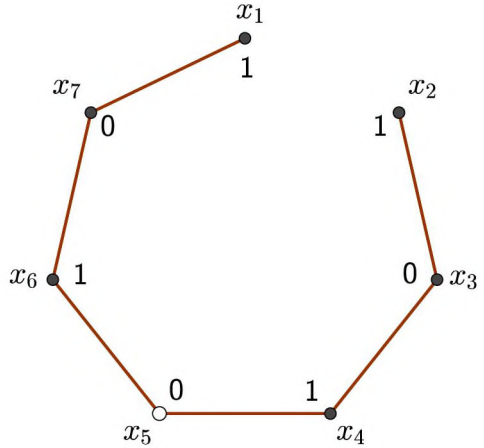
(1, 1, 0, 1, 0 ..., 1, 0) and every such vector is the primitive inner normal to a facet of $\pi_i^{-1}(\pi_i(P))$ for some i. Therefore, the facets of $\hat{P}$ are given by $C_u z \geq 1$ for $u = (1, 1, 0, 1, 0 ..., 1, 0)$, as stated.

Finally, we remark that all the facets of $\hat{P}$ are unbounded as the corresponding normals have at least one coordinate equal zero. Thus, the same inequalities describe the unbounded facets of P. □

Lemma 10.6 *Let C_u be the circulant matrix generated by $u = (1, 1, 0, 1, 0 ..., 1, 0)$ in $\mathbb{R}^n$ for $n = 2k + 1$. Then* $\operatorname{rank} C_u = n$.

Proof Let $f_u(t) = 1 + t + t^3 + \cdots + t^{2k-1}$ be the associated polynomial and let $g(t) = t^n - 1$. By (12), $\operatorname{rank} C_u = n - \deg(\gcd(g(t), f_u(t)))$. Note that $(t^2 - 1)f_u(t) - g(t) = t(t - 1)$. But neither $t = 0$ nor $t = 1$ is a root of $f(t)$; hence, $\gcd(g(t), f_u(t)) = 1$ and the statement follows. □

Proposition 10.7 *The polytope $\hat{F}$ is the pyramid over F with apex at $a = \left(\frac{1}{k+1}, \ldots, \frac{1}{k+1}\right)$.*

Proof Recall that F is the unique compact facet of P corresponding to the inequality $\sum_{j=1}^n z_j \geq 2$. Since $\hat{F} = \operatorname{cl}(\hat{P} \setminus P)$ lies in the other half space and the remaining facets inequalities for $\hat{P}$ and P are the same, we conclude that $\hat{F}$ is given by $C_u z \geq 1$ and $\sum_{j=1}^n z_j \leq 2$. (One can see that the inequalities $I_n z \geq 0$ are redundant. Indeed, given $1 \leq i \leq n$, add the two inequalities in $C_u z \geq 1$ with 1's at the i-th and at the two adjacent places to obtain $z_i + 2 \geq z_i + \sum_{j=1}^n z_j \geq 2$, which implies $z_i \geq 0$.) By Lemma 10.6, $a = \left(\frac{1}{k+1}, \ldots, \frac{1}{k+1}\right)$ is the unique solution to $C_u z = 1$ which implies that $\hat{F}$ is the pyramid over F with apex a. □

Remark 10.8 Unlike the j-multiplicity in Proposition 5.3, the ε-multiplicity of edge ideals is not multiplicative over the connected components of a graph. For instance, if G is the disjoint union of a 3-cycle and a 5-cycle, then by direct computation using Theorem 10.2 the ε-multiplicity of the edge ideal of G is $\frac{4}{9}$, while by Proposition 10.4

the ε-multiplicity of the edge ideals of the 3-cycle and the 5-cycle are $\frac{1}{2}$ and $\frac{1}{3}$, respectively. Furthermore, in contrast to Proposition 9.6 for j-multiplicity, the ε-multiplicity is not preserved after removal of a free node. For example, if G is a 3-cycle with a path of length 2 attached to one of its nodes, then the ε-multiplicity of $I(G)$ is indeed $\frac{1}{3}$, while after removing the free node the ε-multiplicity of the edge ideal is $\frac{1}{2}$. This example also shows that the ε-multiplicity may increase if we pass to a subgraph. Therefore, Theorem 9.2 does not hold true for the ε-multiplicity of edge ideals. However, since the ε-multiplicity is less than or equal to the j-multiplicity for an arbitrary ideal [40], the upper bounds in Corollary 9.4 and Corollary 9.5 are valid for the ε-multiplicity of the edge ideals as well.

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