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Closed Range Composition Operators on Hilbert Function Spaces

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1. Introduction

Let φ be an analytic self-map of the unit disk $\mathbb{D}$. The *composition operator with symbol* φ is defined by

$$C_\varphi(f) = f \circ \varphi\,,$$

for any function f that is analytic on $\mathbb{D}$. Littlewood in 1925 proved a subordination principle, which in operator theory language, says that composition operators are bounded in the *Hardy space* H^2, the Hilbert space of analytic functions on $\mathbb{D}$ with square summable power series coefficients. This is the first setting by which many properties of the composition operator such as boundedness, compactness, and closed range have been studied. It is natural to study these properties on other function spaces.

Let $\mathcal{H}$ be a Hilbert space of analytic functions on $\mathbb{D}$ with inner product $\langle \cdot\,, \cdot \rangle$. We say that $\mathcal{H}$ is a *Hilbert function space* if all point evaluations are bounded linear functionals. By the Riesz representation theorem, for each $z \in \mathbb{D}$, there exists a unique element K_z of $\mathcal{H}$, called the *reproducing kernel* at z, such that for each $f \in \mathcal{H}$, $f(z) = \langle f, K_z \rangle$. For each $z \in \mathbb{D}$ we have

$$|f(z)| \le \|f\|\,\|K_z\| = \|f\|\, K_z(z)^{1/2}, \tag{1}$$

where $\|\cdot\|$ denotes the norm in $\mathcal{H}$. In particular for each $z, w \in \mathbb{D}$ we have

$$| < K_w, K_z > | = |K_w(z)| \leq K_w(w)^{1/2} \, K_z(z)^{1/2} \, . \tag{2}$$

For each $w \in \mathbb{D}$, the *normalized reproducing kernel* in $\mathcal{H}$ is

$$k_w(z) = \frac{K_w(z)}{||K_w||} \, , \quad z \in \mathbb{D} \, . \tag{3}$$

If C_φ is a bounded operator on $\mathcal{H}$, then by Theorem 1.4 in [11] we have

$$C_\varphi^*(K_z) = K_{\varphi(z)} \tag{4}$$

and hence

$$||K_{\varphi(z)}|| \leq ||C_\varphi|| \, ||K_z|| \, . \tag{5}$$

Cima, Thomson and Wogen in [9] were the first to study closed range composition operators in H^2. Their results were in terms of the boundary behavior of φ. Next, Zorboska in [27] studied closed range composition operators in H^2 and in the weighted Hilbert Bergman space in terms of properties of φ inside $\mathbb{D}$. Since then several authors studied this problem in different Banach spaces of analytic functions, see for example [14, 2–4,18,25].

In this paper we continue the study of closed range composition operators on the Bergman space A^2, the Hardy space H^2 and the Dirichlet space $\mathcal{D}$. We will define and discuss properties of these spaces as well as other preliminary work in Section 2. In Section 3 we develop general machinery that can be useful in studying closed range composition operators in any Hilbert function space $\mathcal{H}$ with reproducing kernel K_z. Given $\varepsilon > 0$, let

$$\Lambda_\varepsilon = \Lambda_\varepsilon(\mathcal{H}) = \left\{ z \in \mathbb{D} : ||K_{\varphi(z)}|| \, > \, \varepsilon ||K_z|| \right\}$$

and $G_\varepsilon = G_\varepsilon(\mathcal{H}) = \varphi(\Lambda_\varepsilon)$. In Section 3 we give a necessary condition for G_ε to intersect each pseudohyperbolic disk in $\mathbb{D}$, see Proposition 3.2. It is useful in our results in Section 4 and in Section 6.

In Section 4 we build on known results about closed range composition operators on A^2 provided in [2] and [25]. It is well known that μ is a Carleson measure on $\mathbb{D}$ if and only if the Berezin symbol of μ is bounded; our first main result in Section 4 is an analog of this for Carleson measures that satisfy the reverse Carleson condition, see Theorem 4.1. We use this to provide necessary and sufficient conditions for the pull-back measure of area measure on $\mathbb{D}$ to satisfy the reverse Carleson condition, see Theorems 4.2 and 4.3.

Akeroyd and Ghatage in [2] showed that C_φ is closed range in $\mathcal{H} = A^2$ if and only if for some $\varepsilon > 0$ the set G_ε above satisfies the reverse Carleson condition. This means that G_ε intersects every pseudohyperbolic disk, of some fixed radius $r \in (0, 1)$, in a set that has area comparable to the area of each pseudohyperbolic disk. In Theorem 4.4 we show that in fact this is equivalent to G_ε merely having non-empty intersection with each pseudohyperbolic disk. Moreover we provide an analog of [9, Theorem 2] in A^2 in terms of the pull-back measure of normalized area measure on $\mathbb{D}$; we also show that C_φ is closed range on A^2 if and only if for all $w \in \mathbb{D}$, $||C_\varphi K_w|| \asymp ||K_w||$. Lastly we provide a condition that makes it easy to check whether C_φ is closed range on A^2, see (e) of Theorem 4.4.

In Section 5 we revisit closed range composition operators on H^2. The first main result of this section is not new. It is a combination of [25, Theorem 5.4] and a result in Luery's thesis [18, Theorem 5.2.1]. Our new short proof uses pseudohyperbolic disks. Zorboska proved in [27] that for univalent symbols, C_φ is closed range on A^2 if and only if it is closed range on H^2. We extend Zorboska's result to include all symbols $\varphi \in \mathcal{D}$. Akeroyd and Ghatage show in [2] that the only univalent symbols that give rise to a closed range C_φ

on A^2 are given by conformal automorphisms. We provide an example of a closed range C_φ on H^2 and on A^2 where the symbol φ is an outer function. Note that all other known such examples involve inner functions.

In Section 6 we apply our techniques to the Dirichlet space. Nevanlinna type counting functions in A^2 and in H^2 were instrumental in determining closed range. In comparison Luecking showed in [17] that the Dirichlet space analog does not determine closed range. We provide a new Nevanlinna type counting function in $\mathcal{D}$ and introduce three Carleson measures and study their properties, see Propositions 6.1 and 6.2 and Corollary 6.1. We end the paper with a conjecture about closed range composition operators in $\mathcal{D}$.

Throughout this paper C denotes a positive and finite constant which may change from one occurrence to the next but will not depend on the functions involved. Given two quantities $A = A(z)$ and $B = B(z)$, $z \in \mathbb{D}$, we say that A is *equivalent* to B and write $A \asymp B$ if, $C\,A \leq B \leq C\,A$.

2. Preliminaries

For each $p \in \mathbb{D}$ let α_p denote the Mobius transformation exchanging 0 and p,

$$\alpha_p(z) = \frac{p - z}{1 - \bar{p}z}\,, \tag{6}$$

and Aut($\mathbb{D}$) denote the set of all Mobius transformations of $\mathbb{D}$. The *pseudo-hyperbolic distance* ρ between two points w, z in $\mathbb{D}$ defined by

$$\rho(z, w) = |\alpha_z(w)| = |\frac{z - w}{1 - \overline{w}z}|\,, \tag{7}$$

is Mobius invariant, that is for all $z, w \in \mathbb{D}$ and $\gamma \in \mathrm{Aut}(\mathbb{D})$,

$$\rho(\gamma(z), \gamma(w)) = \rho(z, w)\,,$$

and satisfies a strong form of the triangle inequality; given $z, w, \zeta \in \mathbb{D}$,

$$\rho(z, w) \leq \frac{\rho(z, \zeta) + \rho(\zeta, w)}{1 + \rho(z, \zeta)\rho(\zeta, w)}\,. \tag{8}$$

Moreover it has the following important property:

$$1 - \rho(z, w)^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \overline{w}z|^2}\,. \tag{9}$$

The *pseudo-hyperbolic disk* $D(z, r)$ centered at z with radius $r \in (0, 1)$ is

$$D(z, r) = \{w : \rho(w, z) < r\}\,. \tag{10}$$

As mentioned in [12, page 39], the function $f(x, y) = \frac{x+y}{1+xy}$ attains a maximum value in the rectangle $[0, r] \times [0, s]$ at the point (r, s). It now follows by (8) that if $z \in D(\zeta, r)$ and $w \in D(\zeta, s)$ then

$$\rho(z, w) \leq \frac{r + s}{1 + rs}\,. \tag{11}$$

Let A denote area measure on $\mathbb{D}$ normalized by the condition $A(\mathbb{D}) = 1$. By [26, Proposition 4.5] if $r \in (0, 1)$ is fixed and $z \in D(z_0, r)$ then

$$A(D(z, r)) \asymp (1 - |z|^2)^2 \asymp A(D(z_0, r)) \asymp (1 - |z_0|^2)^2\,, \tag{12}$$

and

$$|1-\bar{z}_0 z| \asymp 1-|z|^2 \asymp 1-|z_0|^2 \,. \tag{13}$$

The *Bergman space* A^2 is the Hilbert space of analytic functions f on $\mathbb{D}$ that are square-integrable with respect to the area measure A that is,

$$||f||_{A^2}^2 = \int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty \,. \tag{14}$$

By [26, Theorem 4.28] an equivalent norm on A^2 is given by

$$||f||_{A^2}^2 \asymp |f(0)|^2 + \int_{\mathbb{D}} (1-|z|^2)^2 |f'(z)|^2 dA(z) \,. \tag{15}$$

As was mentioned in the Introduction, the *Hardy space* H^2 is the Hilbert space of analytic functions on $\mathbb{D}$ with square summable power series coefficients. Moreover if $f = \sum a_n z^n \in H^2$ then $||f||_{H^2}^2 = \sum |a_n|^2$ and by the Littlewood–Paley identity an equivalent norm in H^2 is given by

$$||f||_{H^2}^2 = |f(0)|^2 + \int_{\mathbb{D}} (1-|z|^2)|f'(z)|^2 dA(z) \,. \tag{16}$$

The *Dirichlet space* $\mathcal{D}$ is the space of analytic functions on $\mathbb{D}$ such that

$$||f||_{\mathcal{D}}^2 = |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA(z) < \infty \,.$$

By [1, Proposition 2.18] the reproducing kernel in the Bergman space A^2 is

$$K_w(z, A^2) := K_w(z) = \frac{1}{(1-\overline{w}z)^2} \quad \text{and} \quad ||K_w|| = (1-|w|^2)^{-1} \;\; ; \tag{17}$$

similarly the reproducing kernel in the Hardy space H^2 is

$$K_w(z, H^2) := K_w(z) = \frac{1}{1-\overline{w}z} \quad \text{and} \quad ||K_w||^2 = (1-|w|^2)^{-1} \,; \tag{18}$$

the reproducing kernel of $\mathcal{D}_0 = \{f \in \mathcal{D} : f(0) = 0\}$ is

$$K_w(z, \mathcal{D}_0) := K_w(z) = \log \frac{1}{1-\overline{w}z} \quad \text{and} \quad ||K_w||^2 = \log \frac{1}{1-|w|^2} \tag{19}$$

and the reproducing kernel of $\mathcal{D}$ is

$$K_w(z, \mathcal{D}) := K_w(z) = 1 + \log \frac{1}{1-\overline{w}z} \quad \text{and} \quad ||K_w||^2 = 1 + \log \frac{1}{1-|w|^2} \,. \tag{20}$$

3. On Hilbert function spaces

Let $\mathcal{H}$ be a Hilbert function space with reproducing kernels K_w, $w \in \mathbb{D}$. For $\varepsilon > 0$, let

$$\Lambda_\varepsilon = \Lambda_\varepsilon(\mathcal{H}) = \left\{ z \in \mathbb{D} : ||K_{\varphi(z)}|| \; > \; \varepsilon ||K_z|| \right\} \tag{21}$$

and

$$G_\varepsilon = G_\varepsilon(\mathcal{H}) = \varphi(\Lambda_\varepsilon)\,. \tag{22}$$

By (17) if $\mathcal{H} = A^2$,

$$\Lambda_\varepsilon(A^2) = \Lambda_\varepsilon = \left\{z \in \mathbb{D} : 1 - |z|^2 \; > \; \varepsilon\,(1 - |\varphi(z)|^2)\right\}\,, \tag{23}$$

and by (18) if $\mathcal{H} = H^2$ then $\Lambda_\varepsilon(H^2) = \Lambda_{\varepsilon^2}(A^2)$. The sets $G_\varepsilon(A^2)$ were used in [2,3] and in [25] to study closed range composition operators on the Bergman space. Moreover by (19) if $\mathcal{H} = \mathcal{D}_0$, then

$$\Lambda_\varepsilon = \{z \in \mathbb{D} : (1 - |z|^2)^{\varepsilon^2} \; > \; 1 - |\varphi(z)|^2\}\,. \tag{24}$$

Definition 3.1. We say that $H \subseteq \mathbb{D}$ is hyperbolically dense if there exists $r \in (0,1)$ such that every point of $\mathbb{D}$ is within pseudo-hyperbolic distance r of H.

For $r \in (0,1)$ and $w \in \mathbb{D}$ we define

$$\Delta_w = \Delta_w(r) = \varphi^{-1}(D(w,r))\,. \tag{25}$$

Then for $\varepsilon > 0$, the set G_ε, defined in (22), is hyperbolically dense if and only if there exists $r \in (0,1)$ such that for each $w \in \mathbb{D}$, $\Delta_w(r) \cap \Lambda_\varepsilon \neq \emptyset$.

Definition 3.2. We say that the reproducing kernels satisfy the nearness property if for each $r \in (0,1)$ there exist $\delta > 0$ and $c_r > 0$ such that for each $w, \zeta \in \mathbb{D}$ with $|w| \geq c_r$, $|\zeta| \geq c_r$ and $\rho(\zeta, w) \leq r$, we have that $| < K_\zeta, K_w > | \geq \delta \,||K_w||.||K_\zeta||$ and

$$||K_w|| \asymp ||K_\zeta||\,. \tag{26}$$

Next we show that nearness property is quite common.

Proposition 3.1. *The reproducing kernel functions for the Hardy space H^2, the Bergman space A^2, the Dirichlet space $\mathcal{D}$ and of $\mathcal{D}_0$ satisfy the nearness property.*

Proof. By (9), given $r \in (0,1)$, $\rho(z,w) \leq r$ if and only if

$$\frac{1 - r^2}{(1 - |z|^2)\,(1 - |w|^2)} \leq \frac{1}{|1 - \overline{w}z|^2}. \tag{27}$$

Therefore by (17) and (18) it is immediate that the reproducing kernels of A^2 and H^2 satisfy the nearness property.

Next, by taking logarithms of both sides of (27) and using the inequality of arithmetic and geometric means we see that, given $r \in (0,1)$, if $\rho(z,w) \leq r$ then

$$\sqrt{2}\sqrt{\log\frac{1}{1-|z|^2}}\sqrt{\log\frac{1}{1-|w|^2}} + \log(1 - r^2) \leq 2\log\frac{1}{|1 - \overline{w}z|}$$

and letting K_z denote the reproducing kernel in $\mathcal{D}_0$ we obtain

$$\frac{\sqrt{2}}{2} - \frac{\log\frac{1}{1-r^2}}{2||K_w||\,||K_z||} \leq \frac{| < K_w, K_z > |}{||K_w\,||K_z||}. \tag{28}$$

By (19), if $|w| > \sqrt{(e-1)/e}$ then $||K_w|| > 1$ and by (28) we may find $\delta > 0$ and $c_r > 0$ such that if $|z| > c_r$ and $|w| > c_r$ then

$$\delta \leq \frac{\sqrt{2}}{2} - \frac{\log \frac{1}{1-r^2}}{2||K_z||} \leq \frac{| < K_w, K_z > |}{||K_w\, ||K_z||}, \tag{29}$$

and the reproducing kernels for $\mathcal{D}_0$ and hence of $\mathcal{D}$ satisfy the nearness property as well. □

The proof above can be easily modified to show that the reproducing kernel functions in all weighted Bergman spaces satisfy the nearness property.

In the remainder of this section we will assume that $\mathcal{H}$ contains all constant functions, therefore by (1), for all $z \in \mathbb{D}$ we have $||K_z|| > 0$. Moreover we will assume that $\mathcal{H}$ satisfies the following property.

(S) The map $z \mapsto K_z$ is continuous on $\mathbb{D}$.

It is easy to see that property (S) is valid on H^2, A^2, $\mathcal{D}_0$ and $\mathcal{D}$. Below we provide a necessary condition for G_ε to be hyperbolically dense.

Proposition 3.2. *Let $\mathcal{H}$ be a Hilbert function space satisfying condition* (S) *and containing all constant functions. Suppose that C_φ is a bounded operator on $\mathcal{H}$ and that the reproducing kernels of $\mathcal{H}$ satisfy the nearness property. If there exists $\varepsilon > 0$ such that G_ε is hyperbolically dense, then for every $w \in \mathbb{D}$,*

$$||C_\varphi K_w|| \asymp ||K_w|| \asymp \sup_{z\in\mathbb{D}} | < C_\varphi K_w, \frac{K_z}{||K_z||} > | \,. \tag{30}$$

Proof. Fix $\varepsilon > 0$ such that G_ε is hyperbolically dense. Then there exists $r \in (0,1)$ such that given $w \in \mathbb{D}$, we may choose $z_w \in \Lambda_\varepsilon$ such that $\rho(w, \varphi(z_w)) \leq r < 1$. By the Cauchy–Schwarz inequality and (4) we obtain

$$\begin{aligned} ||C_\varphi K_w|| &\geq \sup_{z\in\mathbb{D}} | < C_\varphi K_w, \frac{K_z}{||K_z||} > | \\ &\geq | < C_\varphi K_w, \frac{K_{z_w}}{||K_{z_w}||} > | \\ &= \frac{| < K_w, C_\varphi^* K_{z_w} > |}{||K_{z_w}||} \\ &= \frac{| < K_w, K_{\varphi(z_w)} > |}{||K_{z_w}||} \,. \end{aligned} \tag{31}$$

By (5) and (21), $||K_{z_w}|| \asymp ||K_{\varphi(z_w)}||$. Therefore by (31), the nearness property of the reproducing kernels of $\mathcal{H}$ and since C_φ is a bounded operator, there exists $c_r > 0$ such that if $|w| > c_r$ then

$$C_1\, ||K_w|| \geq ||C_\varphi K_w|| \;\geq\; C\, \frac{||K_w|||.||K_{\varphi(z_w)}||}{||K_{z_w}||} \asymp ||K_w|| \,. \tag{32}$$

Moreover by (31) and (32), if $|w| \geq c_r$

$$C_1 ||K_w|| \geq \sup_{z\in\mathbb{D}} | < C_\varphi K_w, \frac{K_z}{||K_z||} > | \geq C_2 ||K_w|| \tag{33}$$

and (30) is valid if $|w| \geq c_r$.

Next suppose that there exists a sequence $(w_n) \in \mathbb{D}$ with $|w_n| \leq c_r$ such that

$$\frac{||C_\varphi K_{w_n}||}{||K_{w_n}||} \to 0 \,. \tag{34}$$

Without loss of generality we may assume that there exists $|w_0| < c_r$ such that $w_n \to w_0$. By assumption $\mathcal{H}$ satisfies condition (S). Then $||K_{w_n}|| \to ||K_{w_0}||$ and $||C_\varphi K_{w_n}|| \to ||C_\varphi K_{w_0}||$. By (34) we conclude that, $C_\varphi K_{w_0} = 0$ and so $K_{w_0} = 0$. This is a contradiction, therefore if $|w| \leq c_r$ the first set of inequalities in (30) follows. Lastly, suppose that there exists a sequence $(u_n) \in \mathbb{D}$ with $|u_n| \leq c_r$ such that

$$\frac{\sup_{z\in\mathbb{D}} |< C_\varphi K_{u_n}, \frac{K_z}{||K_z||} > |}{||K_{u_n}||} \to 0\,. \tag{35}$$

As before, without loss of generality we may assume that there exists $|u| < c_r$ such that $u_n \to u$ and by (35) we have that for all $z \in \mathbb{D}$, $C_\varphi K_u(z) = 0$ and so $K_u = 0$. This is a contradiction, therefore if $|w| \leq c_r$ the second set of inequalities in (30) follows as well. □

Remark 3.1. A close examination of the proof of Proposition 3.2 above reveals that under its hypotheses, there exists $r \in (0,1)$ such that if $w \in \mathbb{D}$ is outside a certain compact neighborhood of 0, then for each $z \in \Delta_w(r) \cap \Lambda_\varepsilon$

$$||K_w|| \asymp ||C_\varphi K_w|| \asymp |< C_\varphi K_w, \frac{K_z}{||K_z||} > |\,. \tag{36}$$

4. On Bergman spaces

Carleson measures play an important role in determining when a composition operator C_φ is bounded or compact in several Banach spaces of analytic functions. We shall see that they are also important in determining when C_φ is closed range.

Definition 4.1. Let μ be a finite positive Borel measure on $\mathbb{D}$. We say that μ is a (Bergman space) Carleson measure on $\mathbb{D}$ if there exists $c > 0$ such that for all $f \in A^2$

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \leq c \int_{\mathbb{D}} |f(z)|^2 \, dA(z).$$

By [26, Theorem 7.4], given $0 < r < 1$, μ is a Carleson measure if and only if there exists $c_r > 0$ such that for all $w \in \mathbb{D}$,

$$\mu(D(w,r)) \leq c_r \, A(D(w,r))\,. \tag{37}$$

By (17), the normalized reproducing kernel in A^2

$$k_w(z) = \frac{1 - |w|^2}{(1 - \overline{w}z)^2}\,.$$

Let $0 < r < 1$, $w \in \mathbb{D}$. Then by making a change of variables

$$\begin{aligned}\int_{\mathbb{D}\setminus D(w,r)} |k_w(z)|^2 \, dA(z) &= \int_{\mathbb{D}\setminus D(w,r)} |\alpha_w'(z)|^2 \, dA(z) \\ &= 1 - \int_{D(w,r)} |\alpha_w'(z)|^2 \, dA(z) \\ &= 1 - |D(0,r)|\,.\end{aligned}$$

We conclude that

$$\limsup_{r\to 1}\sup_{w\in\mathbb{D}} \int_{\mathbb{D}\setminus D(w,r)} |k_w(z)|^2\, dA(z) = 0\,. \tag{38}$$

In fact below we show that this is valid for any positive Carleson measure. We provide a more general version that is crucial in the proof of Theorem 5.1.

Proposition 4.1. *Let μ be a positive Carleson measure on $\mathbb{D}$, and α, β be such that $\alpha+\beta = 2$. Then*

$$\limsup_{r\to 1}\sup_{w\in\mathbb{D}} \int_{\mathbb{D}\setminus D(w,r)} \frac{(1-|w|^2)^\alpha\,(1-|\zeta|^2)^\beta}{|1-\overline{w}\zeta|^4}\, d\mu(\zeta) = 0\,. \tag{39}$$

Proof. For a fixed $0<\rho<1$ we can cover $\mathbb{D}$ with pseudohyperbolic disks of radius ρ that do not intersect too often. In fact as shown in [6, Lemma 3.5] there exist $(s_n)\subset\mathbb{D}$, and a positive integer M that depends only on ρ, such that $\mathbb{D} = \cup_{n=1}^\infty D(s_n,\rho)$ and each $\zeta\in\mathbb{D}$ is in at most M of the pseudohyperbolic disks $D(a_n,(\rho+1)/2)$, $n\in\mathbb{N}$.

Fix $r\in(0,1)$ and $w\in\mathbb{D}$. Similarly to the above, with $\rho=1/2$, there exist $(w_n)\subset\mathbb{D}$ such that

$$\mathbb{D}\setminus\overline{D(w,r)} = \cup_{n=1}^\infty D(w_n,\frac{1}{2})\,. \tag{40}$$

Moreover similarly to the argument in [6, Lemma 3.5], for each $n,m\in\mathbb{N}$ and each $\zeta\in\mathbb{D}$

$$|\alpha_\zeta(w_n)-\alpha_\zeta(w_m))| \geq [1-(\rho+1)^2/4]\,\rho/3 = 7/96\,.$$

We conclude that there exists a positive integer M, independent of $r\in(0,1)$ and $w\in\mathbb{D}$, such that each $\zeta\in\mathbb{D}\setminus\overline{D(w,r)}$ is in at most M of the pseudohyperbolic disks $D(w_n,3/4)$, $n\in\mathbb{N}$.

For each $w\in\mathbb{D}$ let

$$\delta_n(w) \quad = \quad \sup_{z\in D(w_n,\frac{1}{2})} \frac{(1-|w|^2)^\alpha\,(1-|z|^2)^\beta}{|1-\overline{w}z|^4}\,. \tag{41}$$

Then by [6, Lemma 3.4], or [26, Proposition 4.13], applied to the function $\frac{1-|w|^2}{(1-\overline{w}z)^2}$ and by (12), there is a constant C such that

$$\delta_n(w)|D(w_n,\frac{3}{4})| \leq C \int_{D(w_n,\frac{3}{4})} \frac{(1-|w|^2)^\alpha\,(1-|\zeta|^2)^\beta}{|1-\overline{w}\zeta|^4}\, dA(\zeta)\,. \tag{42}$$

Let

$$I = \int_{\mathbb{D}\setminus D(w,r)} \frac{(1-|w|^2)^\alpha\,(1-|\zeta|^2)^\beta}{|1-\overline{w}\zeta|^4}\, d\mu(\zeta)\,.$$

By (40) and (41) and since μ is a Carleson measure

$$I \leq \sum_{n=1}^\infty \int_{D(w_n,\frac{1}{2})} \frac{(1-|w|^2)^\alpha\,(1-|\zeta|^2)^\beta}{|1-\overline{w}\zeta|^4}\, d\mu(\zeta)$$

$$\leq \sum_{n=1}^{\infty} \delta_n(w)\mu(D(w_n, \frac{1}{2}))$$

$$\leq \sum_{n=1}^{\infty} \delta_n(w)\,|D(w_n, \frac{3}{4})|\,\frac{\mu(D(w_n, \frac{1}{2}))}{|D(w_n, \frac{1}{2})|}$$

$$\asymp \sum_{n=1}^{\infty} \delta_n(w)\,|D(w_n, \frac{3}{4})|\,.$$

Next, by (12) and (42)

$$I \leq C \sum_{n=1}^{\infty} \int_{D(w_n, \frac{3}{4})} \frac{(1-|w|^2)^{\alpha}\,(1-|\zeta|^2)^{\beta}}{|1-\overline{w}\zeta|^4}\,dA(\zeta)$$

$$\asymp \sum_{n=1}^{\infty} \int_{D(w_n, \frac{3}{4})} \frac{(1-|w|^2)^2}{|1-\overline{w}\zeta|^4}\,dA(\zeta)\,. \tag{43}$$

Lastly, as shown above, each $\zeta \in \mathbb{D} \setminus \overline{D(w,r)}$ is in at most M of the pseudohyperbolic disks $D(w_n, 3/4)$, $n \in \mathbb{N}$. Therefore

$$I \;\leq\; C \int_{\mathbb{D}\setminus D(w,r)} \frac{(1-|w|^2)^2}{|1-\overline{w}\zeta|^4}\,dA(\zeta)$$

for each $0 < r < 1$ and $w \in \mathbb{D}$. By (38) the result now is clear. □

Definition 4.2. Let μ be a finite positive Carleson measure on $\mathbb{D}$. We say that μ satisfies the reverse Carleson condition if there exists $r \in (0,1)$ such that for all $w \in \mathbb{D}$,

$$A(D(w,r)) \asymp \mu(D(w,r))\,. \tag{44}$$

We say that a set $G \subset \mathbb{D}$ *satisfies the reverse Carleson condition* if the Carleson measure $\chi_G(z)\,dA(z)$ satisfies the reverse Carleson condition; Luecking in [15] showed that this is equivalent to

$$\int_{\mathbb{D}} |f(z)|^2\,dA(z) \leq C \int_{G} |f(z)|^2\,dA(z), \tag{45}$$

for all $f \in A^2$. Moreover he showed that Carleson type squares can be used in place of the pseudohyperbolic disks. But proofs are easier using pseudohyperbolic disks.

Let μ be a finite positive measure. The *Berezin symbol of* μ is

$$\tilde{\mu}(w) = \int_{\mathbb{D}} |k_w(z)|^2\,d\mu(z)\,, \quad w \in \mathbb{D}\,. \tag{46}$$

It has played an important role in determining properties of Toeplitz operators and composition operators such as boundedness and compactness, see for example [7]. We shall see below that it is important in questions of closed range of composition operators as well.

It is well known that μ is a Carleson measure if and only if the Berezin symbol of μ is bounded above, see for example [26, Theorem 7.5]. Below we show that the reverse Carleson measure analog of this holds as well.

Theorem 4.1. *A Carleson measure μ satisfies the reverse Carleson condition if and only if $\tilde{\mu}$ is bounded above and below on $\mathbb{D}$.*

Proof. First assume that the Carleson measure μ satisfies the reverse Carleson condition. By (12) and (13), there exists $r \in (0,1)$, for each $w \in \mathbb{D}$ we obtain,

$$C \geq \int_{\mathbb{D}} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\, d\mu(z) \geq \int_{D(w,r)} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\, d\mu(z) \asymp \frac{\mu(D(w,r))}{A(D(w,r))} \asymp 1 \tag{47}$$

and therefore $\tilde{\mu}$ is bounded above and below on $\mathbb{D}$.

Next assume that $\tilde{\mu}$ is bounded above and below on $\mathbb{D}$. By Proposition 4.1

$$\lim_{r\to 1} \sup_{w\in\mathbb{D}} \int_{\mathbb{D}\setminus D(w,r)} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\, d\mu(z) = 0\,. \tag{48}$$

We conclude, by (12) and (13), that there exists $\delta > 0$ such that if $r > 1-\delta$ and $w \in \mathbb{D}$,

$$\begin{aligned} C &\geq \frac{\mu(D(w,r))}{A(D(w,r))} \\ &\asymp \int_{D(w,r)} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\, d\mu(z) \\ &\geq \int_{\mathbb{D}} \frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}\, d\mu(z) - \frac{1}{2}\inf_{w\in\mathbb{D}} \tilde{\mu}(w) \\ &= \tilde{\mu}(w) - \frac{1}{2}\inf_{w\in\mathbb{D}} \tilde{\mu}(w) \\ &\geq \frac{1}{2}\inf_{w\in\mathbb{D}} \tilde{\mu}(w) \geq C \end{aligned} \tag{49}$$

and the conclusion follows. □

The *pull-back measure* of normalized area measure in $\mathbb{D}$ under the map φ is defined on Borel subsets of $\mathbb{D}$ by

$$\mu_\varphi(E) = A(\varphi^{-1}(E))\,. \tag{50}$$

The Berezin symbol of $\tilde{\mu}_\varphi$ is,

$$\tilde{\mu}_\varphi(w) = \int_{\mathbb{D}} |k_w(z)|^2\, d\mu_\varphi(z) = \int_{\mathbb{D}} |k_w(\varphi(z))|^2\, dA(z) = ||C_\varphi k_w||^2\,. \tag{51}$$

Therefore since C_φ is a bounded operator on A^2 (see for example [24, Section 1.4, Exercise 5]),

$$\tilde{\mu}_\varphi(w) \leq ||C_\varphi||^2\, ||k_w||^2 = ||C_\varphi||^2\,. \tag{52}$$

We conclude that $\tilde{\mu}_\varphi$ is a bounded function on $\mathbb{D}$, and μ_φ is a Carleson measure on $\mathbb{D}$.

Proposition 4.2. *The following are equivalent:*

(a) *The Carleson measure μ_φ satisfies the reverse Carleson condition.*

(b) *There exists $0 < r < 1$ such that for all $w \in \mathbb{D}$,*

$$\int_{\Delta_w(r)} |C_\varphi K_w(z)|\, dA(z) \asymp 1\,. \tag{53}$$

(c) *There exists $0 < r < 1$ such that for all $w \in \mathbb{D}$,*

$$(1-|w|^2)^2 \int_{\Delta_w(r)} |C_\varphi K_w(z)|^2\, dA(z) \asymp 1\,. \tag{54}$$

Proof. By (9) and for any $w \in \mathbb{D}$, the absolute value of the normalized reproducing kernel in A^2 is

$$|K_w(z)| = \frac{1}{|1-\overline{w}z|^2} = \frac{1-\rho(z,w)^2}{(1-|z|^2)\,(1-|w|^2)}\,, \tag{55}$$

and by (12), if $z \in D(w,r)$ then

$$|K_w(z)| \asymp \frac{1-\rho(z,w)^2}{A(D(w,r)} \asymp \frac{1}{A(D(w,r)}\,. \tag{56}$$

Therefore by (12) and (25), if $n = 1, 2$ then μ_φ satisfies the reverse Carleson condition if and only if there exists $0 < r < 1$ such that for $w \in \mathbb{D}$,

$$\begin{aligned}\int_{\Delta_w(r)} |C_\varphi K_w(z)|^n\, dA(z) &= \int_{D(w,r)} |K_w(z)|^n\, d\mu_\varphi(z)\\ &\asymp \frac{\mu_\varphi(D(w,r))}{A(D(w,r))^n}\\ &\asymp \frac{1}{(1-|w|^2)^{2n-2}}\end{aligned} \tag{57}$$

and the result follows. □

The result below is an immediate corollary of Theorem 4.1 applied to the Carleson measure μ_φ.

Theorem 4.2. *The Carleson measure μ_φ satisfies the reverse Carleson condition if and only if $\tilde{\mu}_\varphi$ is bounded above and below on $\mathbb{D}$. Equivalently, for all $w \in \mathbb{D}$,*

$$||C_\varphi K_w|| \asymp ||K_w||\,. \tag{58}$$

In the theorem below we provide another equivalent condition for μ_φ to satisfy the reverse Carleson condition that is essential in the proof of our main result of this section. We single it out so that later we can make clear the analogy between μ_φ and the pull-back measure of Lebesgue measure on the unit circle $\mathbb{T}$ used in [9].

Theorem 4.3. *The Carleson measure μ_φ satisfies the reverse Carleson condition if and only if there exists $C > 0$ such that for all $w \in \mathbb{D}$,*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)\,(1-|w|^2)\,|K_w(\varphi(z))| \geq C\,. \tag{59}$$

Proof. First assume that μ_φ satisfies the reverse Carleson condition. By Proposition 4.2 there exist $0 < r < 1$ and $C > 0$ such that for all $w \in \mathbb{D}$

$$\int_{\Delta_w(r)} |C_\varphi K_w(z)|\, dA(z) \;\geq\; C\,. \tag{60}$$

We claim that there exist $\varepsilon > 0$ and $C' > 0$ such that for all $w \in \mathbb{D}$

$$\int_{\Delta_w(r)\cap\Lambda_\varepsilon} |C_\varphi K_w(z)|\, dA(z) \;\geq\; C'\,. \tag{61}$$

Suppose that (61) is not valid. Then there exists a sequence $w_n \in \mathbb{D}$ such that

$$\lim_{n\to\infty} \int_{\Delta_{w_n}(r)\cap\Lambda_{1/n}} |C_\varphi K_{w_n}(z)|\, dA(z) = 0\,. \tag{62}$$

Moreover by Theorem 4.2, (12) and (4), for each $n \in \mathbb{N}$,

$$\begin{aligned}
\int_{\Delta_{w_n}(r)\setminus\Lambda_{1/n}} |C_\varphi K_{w_n}(z)|\, dA(z) &\leq C \int_{\Delta_{w_n}(r)\setminus\Lambda_{1/n}} ||K_{\varphi(z)}||\; ||K_{w_n}||\, dA(z) \\
&\leq C\,\frac{1}{n} \int_{\Delta_{w_n}(r)} ||K_z||\; ||K_{w_n}||\, dA(z) \\
&\asymp \frac{1}{n} \int_{\Delta_{w_n}(r)} ||C_\varphi K_z||\; ||K_{w_n}||\, dA(z) \\
&\asymp \frac{1}{n} \int_{D(w_n,r)} ||K_z||\; ||K_{w_n}||\, d\mu_\varphi(z) \\
&\asymp \frac{1}{n}\,\frac{\mu_\varphi(D(w_n,r))}{A(D(w_n,r))} \\
&\asymp \frac{1}{n}
\end{aligned}$$

and

$$\lim_{n\to\infty} \int_{\Delta_{w_n}(r)\setminus\Lambda_{1/n}} |C_\varphi K_{w_n}(z)|\, dA(z) = 0\,. \tag{63}$$

Therefore (62) and (63) together contradict (60) and we conclude that (61) is valid. Then by (12) and since μ_φ satisfies the reverse Carleson condition,

$$\begin{aligned}
C' &\leq \sup_{z\in\Delta_w(r)\cap\Lambda_\varepsilon} (1-|z|^2)\,(1-|w|^2)\,|K_w(\varphi(z))|\,\frac{A(\Delta_w(r)\cap\Lambda_\varepsilon)}{(1-|z|^2)\,(1-|w|^2)} \\
&\asymp \sup_{z\in\Delta_w(r)\cap\Lambda_\varepsilon} (1-|z|^2)\,(1-|w|^2)\,|K_w(\varphi(z))|\,\frac{A(\Delta_w(r)\cap\Lambda_\varepsilon)}{(1-|z|^2)\,(1-|\varphi(z)|^2)} \\
&\leq \frac{1}{\varepsilon} \sup_{z\in\Delta_w(r)\cap\Lambda_\varepsilon} (1-|z|^2)\,(1-|w|^2)\,|K_w(\varphi(z))|\,\frac{\mu_\varphi(D(w,r)\cap G_\varepsilon)}{(1-|\varphi(z)|^2)^2}
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\varepsilon} \sup_{z\in\Delta_w(r)} (1-|z|^2)\,(1-|w|^2)\,|K_w(\varphi(z))| \,\frac{\mu_\varphi(D(w,r))}{A(D(w,r))} \\
&\leq C\,\frac{1}{\varepsilon} \sup_{z\in\mathbb{D}}(1-|z|^2)\,(1-|w|^2)\,|K_w(\varphi(z))|\,. \qquad (64)
\end{aligned}
$$

Next, if (59) holds then by the Cauchy–Schwarz inequality,

$$
\begin{aligned}
C &\leq \sup_{z\in\mathbb{D}} \frac{|< C_\varphi K_w, K_z >|}{||K_w||\,||K_z||} \\
&\leq \frac{||C_\varphi K_w||}{||K_w||}\,,
\end{aligned}
$$

and the conclusion follows by Theorem 4.2. □

The *Nevanlinna counting function* for φ is defined for $w \in \mathbb{D}$ by

$$
N_\varphi(w) = \sum_{\varphi(z)=w} (1-|z|^2)\,,
$$

where it is understood that if w is not a value of φ then $N_\varphi(w) = 0$. By Littlewood's inequality [24, page 187], it is easy to see that $\frac{N_\varphi(w)}{1-|w|^2}\,dA(w)$ is a Carleson measure.

Proposition 4.3. *Let φ be a non-constant analytic self-map of $\mathbb{D}$. Then for each $\varepsilon > 0$, φ is boundedly valent on $\Lambda_\varepsilon \cap \{|z| > 1/2\}$.*

Proof. Fix $\varepsilon > 0$ and recall that $\Lambda_\varepsilon = \{z \in \mathbb{D} : 1-|z|^2 \ > \varepsilon\,(1-|\varphi(z)|^2)\}$. By [24, Section 10.4] it is easy to see that if $w = \varphi(z)$ with $z \in \Lambda_\varepsilon$ and $|z| > 1/2$ then

$$
\varepsilon\,(1-|w|^2)\eta_{\varphi,\varepsilon}(w) \ \leq \ N_\varphi(w) \leq C\,(1-|w|^2)\,, \qquad (65)
$$

where $\eta_{\varphi,\varepsilon}(w)$ denotes the cardinality of $\varphi^{-1}(w) \cap \Lambda_\varepsilon$ counting multiplicities. Thus φ is boundedly valent on $\Lambda_\varepsilon \cap \{|z| > 1/2\}$ and the result is proved. □

The *Nevanlinna counting function* for A^2 is defined for $w \in \mathbb{D}$ by

$$
N_{2,\varphi}(w) = \sum_{\varphi(z)=w} (1-|z|^2)^2\,, \qquad (66)
$$

where it is understood that if w is not a value of φ then $N_{2,\varphi}(w) = 0$. By [23, Section 6.3], it is easy to see that $\frac{N_{2,\varphi}(w)}{(1-|w|^2)^2}\,dA(w)$ is a Carleson measure.

The authors of [9, Theorem 2] launched the study of closed range composition operators on H^2 by using the pull-back measure of Lebesgue measure on $\mathbb{T}$. Below is the main result of this section. We provide an analog of their result, on A^2, where we use the pull-back measure of normalized area measure on $\mathbb{D}$. Not all equivalences below are new. In particular, (f) $\Leftrightarrow$ (g) was proved in [2, Theorem 2.4] and $(a) \Leftrightarrow (g)$ is the case for $p = 2$ and $\alpha = 2$ in [25, Theorem 5.4]. Notice that while condition (f) requires G_ε to intersect each $D(w,r)$ at an area that is comparable to the area of $D(w,r)$, by condition (b) this is equivalent to each $D(w,r)$ merely intersecting G_ε. Moreover notice that condition (e) makes it easy to check whether C_φ is closed range on A^2.

Remark 4.1. By Lemma 2.1 and Lemma 2.2 in [2], to prove that C_φ is closed range on A^2 we may assume that $\varphi(0) = 0$ and show that C_φ is closed range on the invariant subspace of A^2 of functions vanishing at the origin. This will be useful in the proof of (d) $\Rightarrow$ (f) below, where we use the equivalent norm of A^2 given in (15).

Theorem 4.4. *The following statements are equivalent.*

(a) *The Carleson measure* $\frac{N_{2,\varphi}(\zeta)}{(1-|\zeta|^2)^2}\, dA(\zeta)$ *satisfies the reverse Carleson condition.*

(b) *There exists* $\varepsilon > 0$ *such that* G_ε *is hyperbolically dense.*

(c) *The Carleson measure* μ_φ *satisfies the reverse Carleson condition.*

(d) *For all* $w \in \mathbb{D}$, $||C_\varphi K_w|| \asymp ||K_w||$.

(e) *There exists* $C > 0$ *such that for all* $w \in \mathbb{D}$

$$C \le \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)\,(1-|w|^2)}{|1-\overline{w}\varphi(z)|^2}\,.$$

(f) *There exists* $\varepsilon > 0$ *such that* $G_\varepsilon = \{\varphi(z) : (1-|z|^2) > \varepsilon\,(1-|\varphi(z)|^2)\}$ *satisfies the reverse Carleson condition.*

(g) C_φ *is closed range on* A^2.

Proof. We will show that (a) ⇒ (b) ⇒ (c) ⇔ (e) ⇔ (d) ⇒ (f) ⇔ (g) ⇔ (a).

We start by showing (a) ⇒ (b). By assumption we may choose $0 < r < 1$ such that

$$\int\limits_{D(w,r)} \frac{N_{2,\varphi}(\zeta)}{(1-|\zeta|^2)^2}\, dA(\zeta) \asymp A(D(w,r))\,, \quad w \in \mathbb{D}\,. \tag{67}$$

Fix $w \in \mathbb{D}$, recall (25) and let

$$\delta_w(r) = \sup_{z\in\Delta_w(r)} \frac{1-|z|^2}{1-|\varphi(z)|^2}\,.$$

By (67) and since $\frac{N_\varphi(\zeta)}{1-|\zeta|^2}\, dA(\zeta)$ is a Carleson measure it is easy to see that

$$\begin{aligned} A(D(w,r)) &\le C\,\delta_w(r) \int\limits_{D(w,r)} \frac{N_\varphi(\zeta)}{1-|\zeta|^2}\, dA(\zeta) \\ &\le C\,\delta_w(r)\, A(D(w,r))\,. \end{aligned}$$

It follows that there exists an $\varepsilon > 0$ such that if $w \in \mathbb{D}$ then $\delta_w(r) \ge \varepsilon$. We conclude that there exists $0 < r < 1$ such that for all $w \in \mathbb{D}$, there exists $z_w \in \Delta_w(r) \cap \Lambda_\varepsilon$ or equivalently that G_ε is hyperbolically dense and (b) holds.

Next we show that (b) ⇒ (c). This is an immediate consequence of Proposition 3.2 and Theorem 4.2. But we will provide below another proof that may be of independent interest. By our assumption we may choose $\varepsilon > 0$ such that G_ε is hyperbolically dense. It follows that there exists $0 < r < 1$ with the property that given $w \in \mathbb{D}$ there exists $z_w \in \Delta_w(r)$ satisfying

$$1 - |z_w| \ge \varepsilon\,(1-|w|)\,. \tag{68}$$

We claim that

$$\varphi\left(D(z_w, \frac{1}{2})\right) \subset D\left(w, \frac{r+\frac{1}{2}}{1+\frac{r}{2}}\right)\,. \tag{69}$$

Indeed if $\rho(z, z_w) < 1/2$ then by the Invariant Schwarz Lemma, see [24, page 60], $\varphi(z) \in D(\varphi(z_w), 1/2)$. Moreover since $z_w \in \Delta_w(r)$, we have that $w \in D(\varphi(z_w), r)$. Therefore by (11) we obtain

$$\rho\,(\varphi(z), w) \;\le\; \frac{\frac{1}{2}+r}{1+\frac{r}{2}} =: r'$$

and (69) holds. Thus by (12) and (68), for every $w \in \mathbb{D}$ and since $z_w \in \Delta_w(r')$ we obtain

$$\begin{aligned}\mu_\varphi((D(w,r'))) &= A(\Delta_w(r')) \\ &\geq A(D(z_w, \frac{1}{2})) \\ &\geq \frac{1}{4}\,(1-|z_w|^2)^2 \\ &\geq \frac{1}{4}\,\varepsilon\,(1-|w|^2)^2 \\ &\asymp A(D(w,r'))\,. \end{aligned} \tag{70}$$

We conclude that μ_φ satisfies the reverse Carleson condition and (c) holds.

By Theorem 4.2 and Theorem 4.3, (c), (d) and (e) are equivalent.

Next we show that (d) $\Rightarrow$ (f). If $r \in (0,1)$ and $w \in \mathbb{D}$ let

$$I(w,r) := (1-|w|^2)^2 \int\limits_{\mathbb{D}\setminus\Delta_w(r)} |(C_\varphi K_w)'(z)|^2\,(1-|z|^2)^2\,dA(z)\,.$$

Then by the Schwarz–Pick lemma,

$$\begin{aligned} I(w,r) &= 4(1-|w|^2)^2 \int\limits_{\mathbb{D}\setminus\Delta_w(r)} \frac{|w|^2\,(1-|z|^2)^2\,|\varphi'(z)|^2}{|1-\overline{w}\varphi(z)|^6}\,dA(z) \\ &\leq 16\,(1-|w|^2)^2 \int\limits_{\mathbb{D}\setminus\Delta_w(r)} \frac{1}{|1-\overline{w}\varphi(z)|^4}\,dA(z) \\ &= 16\,(1-|w|^2)^2 \int\limits_{\mathbb{D}\setminus\Delta_w(r)} |K_w(\varphi(z))|^2\,dA(z) \\ &= 16\,(1-|w|^2)^2 \int\limits_{\mathbb{D}\setminus D(w,r)} |K_w(z)|^2\,d\mu_\varphi(z)\,. \end{aligned}$$

By Theorem 4.2 our hypothesis (d) is equivalent to $\tilde{\mu}_\varphi$ being bounded above and below on $\mathbb{D}$. Then by the proof of Theorem 4.1 we see that (48) is valid. We conclude that

$$\lim_{r\to 1}\sup_{w\in\mathbb{D}} I(w,r) = 0$$

or equivalently that

$$\lim_{r\to 1}\sup_{w\in\mathbb{D}}(1-|w|^2)^2 \int\limits_{\mathbb{D}\setminus\Delta_w(r)} |(C_\varphi K_w)'(z)|^2\,(1-|z|^2)^2\,dA(z) = 0\,. \tag{71}$$

Therefore there exists $r \in (0,1)$ such that for all $w \in \mathbb{D}$

$$\begin{aligned}\widehat{I}(w,r) &:= (1-|w|^2)^2 \int\limits_{\Delta_w(r)} |(C_\varphi K_w)'(z)|^2\,(1-|z|^2)^2\,dA(z) \\ &\asymp \tilde{\mu}_\varphi(w) - (1-|w|^2)^2 \int\limits_{\mathbb{D}\setminus\Delta_w(r)} |(C_\varphi K_w)'(z)|^2\,(1-|z|^2)^2\,dA(z)\end{aligned}$$

$$\geq \frac{1}{2}\,\tilde{\mu}_\varphi(w) \geq C\,, \tag{72}$$

where (15) and (51) were used in the second line of the display above and the hypothesis (d), which is equivalent to $\tilde{\mu_\varphi}$ being bounded below on $\mathbb{D}$, was used in the third line of the display above.

Next, if $r \in (0,1)$, $\varepsilon > 0$ and $w \in \mathbb{D}$, let $J(w,r,\varepsilon)$ denote the expression

$$(1-|w|^2)^2 \int\limits_{\Delta_w(r)\setminus\Lambda_\varepsilon\cap\varphi^{-1}\{|\zeta|>1/2\}} |(C_\varphi K_w)'(z)|^2\,(1-|z|^2)^2\,dA(z)\,.$$

Then,

$$J(w,r,\varepsilon) \;\leq\; 4\,\varepsilon\,(1-|w|^2)^2 \int\limits_{\Delta_w(r)\setminus\Lambda_\varepsilon\cap\varphi^{-1}\{|\zeta|>1/2\}} \frac{|\varphi'(z)|^2\,(1-|z|^2)\,(1-|\varphi(z)|^2)}{|1-\overline{w}\varphi(z)|^6}\,dA(z)\,,$$

and by making a non-univalent change of variables as in [24, 10.3] we obtain

$$J(w,r,\varepsilon) \leq 4\,\varepsilon\,(1-|w|^2)^2 \int\limits_{D(w,r)\setminus G_\varepsilon\cap\{|\zeta|>1/2\}} \frac{(1-|\zeta|^2)\,N_\varphi(\zeta)}{|1-\overline{w}\zeta|^6}\,dA(\zeta)\,.$$

By (12), (13) and [24, 10.4] and for all $w \in \mathbb{D}$ we obtain

$$\begin{aligned} J(w,r,\varepsilon) &\leq C\,\varepsilon\,(1-|w|^2)^2 \int\limits_{D(w,r)} \frac{(1-|\zeta|^2)^2}{|1-\overline{w}\zeta|^6}\,dA(\zeta) \\ &\leq C\,\varepsilon\,. \end{aligned} \tag{73}$$

Therefore, by (72) and (73), there exist $r \in (0,1)$ and $\varepsilon > 0$, such that if $w \in \mathbb{D}$ and $\widehat{J}(w,r,\varepsilon)$ denotes the expression

$$(1-|w|^2)^2 \int\limits_{\Delta_w(r)\cap\Lambda_\varepsilon\cap\varphi^{-1}\{|\zeta|>1/2\}} |(C_\varphi K_w)'(z)|^2\,(1-|z|^2)^2\,dA(z)$$

then

$$\widehat{J}(w,r,\varepsilon) \;\geq\; C - J(w,r,\varepsilon) \geq C\,. \tag{74}$$

Therefore by making once again a non-univalent change of variables as in [24, 10.3] and by [23, 6.3] we conclude that if $w \in \mathbb{D}$ then

$$\begin{aligned} C &\leq (1-|w|^2)^2 \int\limits_{\Delta_w(r)\cap\Lambda_\varepsilon\cap\varphi^{-1}\{|\zeta|>1/2\}} \frac{(1-|z|^2)^2\,|\varphi'(z)|^2}{|1-\overline{w}\varphi(z)|^6}\,dA(z) \\ &= (1-|w|^2)^2 \int\limits_{D(w,r)\cap G_\varepsilon\cap\{|\zeta|>1/2\}} \frac{N_{2,\varphi}(\zeta)}{|1-\overline{w}\zeta|^6}\,dA(\zeta) \\ &\leq C\,(1-|w|^2)^2 \int\limits_{D(w,r)\cap G_\varepsilon} \frac{(1-|\zeta|^2)^2}{|1-\overline{w}\zeta|^6}\,dA(\zeta) \end{aligned} \tag{75}$$

and by (12), (13)

$$C \le \frac{A(D(w,r)\cap G_\varepsilon)}{A(D(w,r))}\,.$$

By Theorem 1 in [2], (f) and (g) are equivalent. Lastly by Theorem 5.4 in [25] (a) and (g) are equivalent. This completes the proof of the theorem. □

Remark 4.2. The *Bloch space* $\mathcal{B}$, is the Banach space of analytic functions on $\mathbb{D}$ such that

$$||f||_{\mathcal{B}} = \sup_{z\in\mathbb{D}} |f'(z)|\,(1-|z|^2) < \infty\,.$$

For $\varepsilon > 0$, let

$$\Lambda_\varepsilon = \{z \in \mathbb{D} : \quad \frac{|\varphi'(z)|\,(1-|z|^2)}{1-|\varphi(z)|^2} > \varepsilon\}$$

and $F_\varepsilon = \varphi(\Lambda_\varepsilon)$. By [13] and [8] or [3], C_φ is closed range on the Bloch space if and only if there exists and $\varepsilon > 0$ such that F_ε is hyperbolically dense (this is condition (iii) in [3, Theorem 2.2]). Moreover in [3, Corollary 2.3] it was shown to be equivalent to $||C_\varphi(\alpha_p)||_{\mathcal{B}} \asymp 1$, for $p \in \mathbb{D}$. The similarity of this with our results on closed range operators on A^2 is interesting.

5. On Hardy spaces

Let φ be an analytic self-map of $\mathbb{D}$ and recall from Section 4 that the *Nevanlinna counting function*, $N_\varphi(\zeta)$, for H^2 is

$$N_\varphi(\zeta) = \sum_{\varphi(z)=\zeta} (1-|z|^2)\,,\ \zeta \in \mathbb{D}\,. \tag{76}$$

Let $f \in H^2$. Then by (16) and by making a non-univalent change of variables as done in [24, page 186] we can easily see that

$$\begin{aligned}||C_\varphi f||^2_{H^2} &\asymp |f(\varphi(0))|^2 + \int_{\mathbb{D}} |f'(\varphi(z)|^2\,|\varphi'(z)|^2\,(1-|z|^2)\,dA(z)\\ &= |f(\varphi(0))|^2 + \int_{\mathbb{D}} |f'(\zeta)|^2\,N_\varphi(\zeta)\,dA(\zeta)\,.\end{aligned}$$

Therefore recalling that the reproducing kernel for H^2 is

$$K_w(z) = \frac{1}{1-\overline{w}z}\,,$$

we obtain for $w \in \mathbb{D}$ that

$$||C_\varphi K_w||^2_{H^2} \asymp |K_w(\varphi(0))|^2 + \int_{\mathbb{D}} |K'_w(\zeta|^2\,N_\varphi(\zeta)dA(\zeta)\,. \tag{77}$$

Below is the first main theorem of this section. The result is not new. In particular the equivalence (a) ⇔ (b) follows by [25, Theorem 5.4]. Moreover Luery in her thesis [18] proved that (b) ⇔ (c) using the Aleksandrov–Clark measures. Our new short proof below uses pseudohyperbolic disks. We may assume below that

$\varphi(0) = 0$, and prove the result for the invariant subspace of H^2 of functions vanishing at the origin, see for example Remark 3.1 in [25].

Theorem 5.1. *Let φ be an analytic self-map of $\mathbb{D}$. The following statements are equivalent.*
(a) *The Carleson measure $\frac{N_\varphi(\zeta)}{1-|\zeta|^2}\, dA(\zeta)$ satisfies the reverse Carleson condition.*
(b) *C_φ is closed range on H^2.*
(c) *For all $w \in \mathbb{D}$, $||C_\varphi K_w|| \asymp ||K_w||$.*

Proof. By [25, Theorem 5.4] we have (a) $\Leftrightarrow$ (b). The implication (b) $\Rightarrow$ (c) is trivial.
Lastly we need to show that (c) $\Rightarrow$ (a). Assume that (c) holds. Then by (77)

$$\int_{\mathbb{D}} |K_w'(\zeta)|^2\, N_\varphi(\zeta)\, dA(\zeta) \asymp \frac{|w|^2}{1-|w|^2}$$

or

$$\gamma_w := (1-|w|^2) \int_{\mathbb{D}} \frac{1}{|1-\overline{w}\,\zeta|^4}\, N_\varphi(\zeta)\, dA(\zeta) \asymp 1 \, .$$

By Proposition 4.1, for the Carleson measure $\frac{N_\varphi(\zeta)}{1-|\zeta|^2}\, dA(\zeta)$ with $\alpha = \beta = 1$, we obtain

$$\lim_{r\to 1} \sup_{w\in\mathbb{D}} \int_{\mathbb{D}\setminus D(w,r)} \frac{1-|w|^2}{|1-\overline{w}\zeta|^4}\, N_\varphi(\zeta)\, dA(\zeta) = 0\, . \tag{78}$$

Therefore, if $\varepsilon = (1/2)\, \inf_{w\in\mathbb{D}} \gamma_w > 0$, there exists $\delta > 0$ such that if $r > 1-\delta$, and $w \in \mathbb{D}$,

$$\begin{aligned}
C &\geq (1-|w|^2) \int_{D(w,r)} \frac{1}{|1-\overline{w}\zeta|^4}\, N_\varphi(\zeta)\, dA(\zeta) \\
&= (1-|w|^2) \left(\int_{\mathbb{D}} \frac{1}{|1-\overline{w}\zeta|^4}\, N_\varphi(\zeta)\, dA(\zeta) - \int_{\mathbb{D}\setminus D(w,r)} \frac{1}{|1-\overline{w}\zeta|^4}\, N_\varphi(\zeta)\, dA(\zeta) \right) \\
&= \gamma_w - \int_{\mathbb{D}\setminus D(w,r)} \frac{1-|w|^2}{|1-\overline{w}\zeta|^4}\, N_\varphi(\zeta)\, dA(\zeta) \\
&\geq \gamma_w - \varepsilon = \gamma_w - (1/2)\, \inf_{w\in\mathbb{D}} \gamma_w \\
&\geq (1/2)\, \inf_{w\in\mathbb{D}} \gamma_w > 0\, .
\end{aligned}$$

By (12) and (13) we conclude that if $w \in \mathbb{D}$ then

$$\begin{aligned}
1 &\asymp \frac{1}{(1-|w|^2)^2} \int_{D(w,r)} \frac{N_\varphi(\zeta)}{1-|\zeta|^2}\, dA(\zeta) \\
&\asymp \frac{1}{A(D(w,r))} \int_{D(w,r)} \frac{N_\varphi(\zeta)}{1-|\zeta|^2}\, dA(\zeta)\, .
\end{aligned} \tag{79}$$

This shows that $\frac{N_\varphi(\zeta)}{1-|\zeta|^2}\, dA(\zeta)$ satisfies the reverse Carleson condition and (a) is proved. □

It is a consequence of the Schwarz–Pick Lemma that, for all $z \in \mathbb{D}$, $1 - |z|^2 \leq C\,(1 - |\varphi(z)|^2)$. Recalling the definitions of the Nevanlinna counting functions for A^2 and H^2 given in (66) and (76), if $\zeta \in \mathbb{D}$ then

$$\frac{N_{2,\varphi}(\zeta)}{(1 - |\zeta|^2)^2} \leq C\,\frac{N_\varphi(\zeta)}{1 - |\zeta|^2}\,.$$

Now the following is an immediate corollary of Theorems 4.4 and 5.1. It was first proved by Zorboska in [27, Corollary 4.2].

Corollary 5.1. *If C_φ is closed range on A^2 then it is closed range on H^2.*

By [27, Corollary 4.3] the converse of the above holds, if φ is univalent. We next provide a more general condition that guarantees this. Recall that $\eta_\varphi(w)$ denotes the cardinality of $\varphi^{-1}(w)$ counting multiplicities.

Corollary 5.2. *Suppose that $\eta_\varphi(\zeta)\,dA(\zeta)$ is a Carleson measure. Then C_φ is closed range on H^2 if and only if C_φ is closed range on A^2.*

Proof. First, assume that C_φ is closed range on H^2. By Theorem 5.1, $\frac{N_\varphi(\zeta)}{1-|\zeta|^2}\,dA(\zeta)$ satisfies the reverse Carleson condition. Therefore we may choose $0 < r < 1$ such that for each $w \in \mathbb{D}$,

$$\int\limits_{D(w,r)} \frac{N_\varphi(\zeta)}{1 - |\zeta|^2}\,dA(\zeta) \asymp |D(w,r)|\,. \tag{80}$$

Given $w \in \mathbb{D}$, recall (25) and let

$$\delta_w(r) = \sup_{z \in \Delta_w(r)} \frac{1 - |z|^2}{1 - |\varphi(z)|^2}\,.$$

By (80) and since by assumption $\eta_\varphi(\zeta)\,dA(\zeta)$ is a Carleson measure, we obtain

$$\begin{aligned} A(D(w,r)) &\leq C\,\delta_w(r) \int\limits_{D(w,r)} \eta_\varphi(\zeta)\,dA(\zeta) \\ &\leq C\,\delta_w(r)\,A(D(w,r)) \end{aligned} \tag{81}$$

and therefore, there exists $\varepsilon > 0$ such that for all $w \in \mathbb{D}$, $\delta_w(r) \geq \varepsilon$. We conclude that G_ε is hyperbolically dense and, by Theorem 4.4, C_φ is closed range on A^2. The other direction follows by Corollary 5.1. □

An analytic function f on $\mathbb{D}$ is said to belong to the *Hardy space* H^1 if

$$\sup_{0<r<1} \frac{1}{2\pi} \int\limits_0^{2\pi} |f(re^{i\theta})|\,d\theta \; < \infty\,.$$

Let φ be an analytic self-map of $\mathbb{D}$. The *pull-back* measure of the Lebesgue measure m on the unit circle $\mathbb{T}$ under the map φ is defined on E, a Borel subset of $\mathbb{T}$, by

$$m_\varphi(E) = m(\varphi^{-1}(E))\,.$$

By [22, page 50] (or [20]), m_φ is a measure that is absolutely continuous with respect to Lebesgue measure m on $\mathbb{T}$ and its Radon–Nikodym derivative is bounded. Moreover if E is a Borel subset of $\mathbb{T}$ then

$$m_\varphi(E) \leq \int_E \frac{1-|\varphi(0)|^2}{|1-\varphi(0)e^{it}|^2}\, dt\,.$$

Proposition 5.1. *Suppose φ is a non-constant analytic self-map of $\mathbb{D}$ such that $\mathbb{T} \subseteq \varphi(\mathbb{T})$. If $\varphi' \in H^1$, then C_φ is closed range on H^2.*

Proof. By hypothesis $\mathbb{T} \subseteq \varphi(\mathbb{T})$, hence if F is a measurable subset of $\mathbb{T}$, then $F = \varphi(E)$ where $E = \varphi^{-1}(F)$. Now if E_n is a sequence of measurable subsets of $\mathbb{T}$ such that $m(E_n) \to 0$ as $n \to \infty$ and since $\varphi' \in H^1$,

$$\lim_{n\to\infty} \int_{E_n} |\varphi'(t)|\, dt = 0\,.$$

Equivalently if $m_\varphi(\varphi(E_n)) \to 0$ then $m(\varphi(E_n)) \to 0$. Therefore the Lebesgue measure m is absolutely continuous with respect to m_φ. By [9, Theorem 2] C_φ is closed range on H^2. □

By [2, Theorem 2.5], the only univalent self-maps of $\mathbb{D}$ that induce a closed range composition operator C_φ on A^2 and, by [27, Corollary 4.2], on H^2 are the conformal automorphisms of $\mathbb{D}$. Two examples of non-inner functions that give rise to closed range composition operators on H^2 are given in [9, page 218]. Moreover all known examples of closed range composition operators on A^2 involve inner functions. So it is natural to ask if there are any such examples with non-inner functions. Next we provide an example of a boundedly valent outer function φ such that C_φ is closed range on H^2 and on A^2. The main idea regarding its construction was developed in a conversation with J. Akeroyd. We are grateful to him for allowing us to include it here.

If an analytic function f maps $\mathbb{D}$ conformally onto a bounded domain G then by Caratheodory's theorem (see [21, Theorem 2.6]) f has a continuous injective extension to $\overline{\mathbb{D}}$. Moreover by [21, Proposition 6.2, Theorem 6.8] if $f(\mathbb{T})$ is a rectifiable curve then $f' \in H^1$.

Let G denote the half annulus defined by

$$\left\{w : \frac{1}{2} < |w| < 1\,, \mathrm{Im}(w) > 0\right\}\,,$$

F the Riemann mapping of $\mathbb{D}$ onto G and $\varphi = F^3$. Then φ is three-valent, $\varphi(\mathbb{T}) \supseteq \mathbb{T}$ and by the discussion above it is easy to see that $F' \in H^1$ and therefore $\varphi' \in H^1$. Moreover since $|\varphi(z)| > 1/2$ for all $z \in \mathbb{D}$, φ is an outer function, see for example [10, Remark 3.2.5]. By Proposition 5.1 C_φ is closed range on H^2 which, by Corollary 5.2 is equivalent to C_φ being closed range on A^2.

6. On Dirichlet spaces

In this section we will assume that C_φ is a bounded operator on $\mathcal{D}$. It is well known that this is equivalent to the measure $\eta_\varphi(w)\, dA(w)$ being a Carleson measure on $\mathbb{D}$, see for example [5,19,14]. Moreover if φ is an analytic self-map of $\mathbb{D}$ with $\varphi(0) = 0$, then C_φ is closed range on $\mathcal{D}$ if and only if C_φ is closed range on $\mathcal{D}_0 = \{f \in \mathcal{D} : f(0) = 0\}$. If $\varphi(0) \neq 0$, letting $\psi = \alpha_{\varphi(0)} \circ \varphi$, then C_φ is closed range on $\mathcal{D}$ if and only if C_ψ is closed range on $\mathcal{D}_0$. Therefore in this section we shall assume that $\varphi(0) = 0$ and consider C_φ on $\mathcal{D}_0$. Moreover, for a non-constant symbol φ the composition operator C_φ is injective; therefore it is closed range if and only if it is bounded below.

Recall the notation introduced in Section 3, where for $\varepsilon > 0$ we let

$$\Lambda_\varepsilon = \left\{ z \in \mathbb{D} : \log \frac{1}{1-|\varphi(z)|^2} > \varepsilon^2 \log \frac{1}{1-|z|^2} \right\} \tag{82}$$

and $G_\varepsilon = \varphi(\Lambda_e)$.

Define the *Nevanlinna type counting function* $\widehat{N_\varphi}(\zeta)$ for $\zeta \in \mathbb{D}$ by

$$\widehat{N_\varphi}(\zeta) = \sum_{\varphi(z)=\zeta} \left(\log \frac{1}{1-|z|^2} \right)^{-1}, \tag{83}$$

where it is understood that if ζ is not in $\varphi(\mathbb{D})$ then $\widehat{N_\varphi}(\zeta) = 0$. By Schwarz's Lemma if $z, \zeta \in \mathbb{D}$ are such that $\varphi(z) = \zeta$ then

$$\log \frac{1}{1-|\zeta|^2} \left(\log \frac{1}{1-|z|^2} \right)^{-1} \leq 1$$

and therefore

$$\log \frac{1}{1-|\zeta|^2} \widehat{N_\varphi}(\zeta) \leq \eta_\varphi(\zeta) . \tag{84}$$

Since $\eta_\varphi(\zeta)\, dA(\zeta)$ is a Carleson measure it is clear that the measure $\log \frac{1}{1-|\zeta|^2} \widehat{N_\varphi}(\zeta)\, dA(\zeta)$ is also a Carleson measure.

If g is a non-negative measurable function on $\mathbb{D}$ then by making a non-univalent change of variables similar to [23, page 398] we obtain

$$\int_{\mathbb{D}} g(\varphi(z))|\varphi'(z)|^2 \frac{\log \frac{1}{1-|\varphi(z)|^2}}{\log \frac{1}{1-|z|^2}} dA(z) = \int_{\mathbb{D}} g(\zeta) \log \frac{1}{1-|\zeta|^2} \widehat{N_\varphi}(\zeta) dA(\zeta). \tag{85}$$

Proposition 6.1. *The following three statements are equivalent:*

(a) *The measure* $\log \frac{1}{1-|\zeta|^2} \widehat{N_\varphi}(\zeta)\, dA(\zeta)$ *satisfies the reverse Carleson condition.*

(b) *There exists an* $\varepsilon > 0$ *such that the measure* $\chi_{G_\varepsilon}(\zeta) \log \frac{1}{1-|\zeta|^2} \widehat{N_\varphi}(\zeta)\, dA(\zeta)$ *satisfies the reverse Carleson condition.*

(c) *There exists an* $\varepsilon > 0$ *such that the measure* $\chi_{G_\varepsilon}(\zeta)\eta_\varphi(\zeta)dA(\zeta)$ *satisfies the reverse Carleson condition.*

Proof. First, we assume (a). We may then choose $r \in (0,1)$ so that for each $w \in \mathbb{D}$,

$$\int_{D(w,r)} \log \frac{1}{1-|\zeta|^2} \widehat{N_\varphi}(\zeta)\, dA(\zeta) \asymp A(D(w,r)) . \tag{86}$$

Moreover for each $\varepsilon > 0$, $w \in \mathbb{D}$, (84) and since $\eta_\varphi(\zeta)\, dA(\zeta)$ is a Carleson measure,

$$\begin{aligned}\int_{D(w,r)\setminus G_\varepsilon} \log \frac{1}{1-|\zeta|^2} \widehat{N_\varphi}(\zeta)\, dA(\zeta) &\leq \varepsilon^2 \int_{D(w,r)} \eta_\varphi(\zeta)\, dA(\zeta) \\ &\leq \varepsilon^2\, C\, A(D(w,r))\, ;\end{aligned}$$

therefore by (86) we may choose $\varepsilon > 0$ small enough such that

$$\int_{D(w,r)\cap G_\varepsilon} \log\frac{1}{1-|\zeta|^2}\,\widehat{N_\varphi}(\zeta)\,dA(\zeta) \asymp A(D(w,r))$$

and (b) follows. By (84), it is immediate that (b) $\Rightarrow$ (c).

Lastly assume that (c) holds, and let $r \in (0,1)$ be such that for all $w \in \mathbb{D}$,

$$\int_{D(w,r)\cap G_\varepsilon} \eta_\varphi(\zeta)\,dA(\zeta) \asymp A(D(w,r))\,.$$

Then, by (25) and (85), for each $w \in \mathbb{D}$

$$\begin{aligned}\int_{D(w,r)} \log\frac{1}{1-|\zeta|^2}\,\widehat{N_\varphi}(\zeta)\,dA(\zeta) &\ge \int_{D(w,r)\cap G_\varepsilon} \log\frac{1}{1-|\zeta|^2}\,\widehat{N_\varphi}(\zeta)\,dA(\zeta)\\ &\ge \int_{\Delta_w(r)\cap\Lambda_\varepsilon} \frac{\log\frac{1}{1-|\varphi(z)|^2}}{\log\frac{1}{1-|z|^2}}\,|\varphi'(z)|^2\,dA(z)\\ &\ge \varepsilon^2 \int_{D(w,r)\cap G_\varepsilon} \eta_\varphi(\zeta)\,dA(\zeta)\\ &\asymp A(D(w,r))\end{aligned}$$

and (a) follows. $\square$

The following is an immediate corollary of Theorem 4.1.

Corollary 6.1. *Each of the following three statements is equivalent to* (a), (b), (c) *of the proposition above.*

(1)

$$\inf_{w\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|w|^2)^2}{|1-\overline{w}\zeta|^4}\,\log\frac{1}{1-|\zeta|^2}\,\widehat{N_\varphi}(\zeta)\,dA(\zeta) \ge C\,.$$

(2) *There exists $\varepsilon > 0$ such that*

$$\inf_{w\in\mathbb{D}} \int_{G_\varepsilon} \frac{(1-|w|^2)^2}{|1-\overline{w}\zeta|^4}\,\log\frac{1}{1-|\zeta|^2}\,\widehat{N_\varphi}(\zeta)\,dA(\zeta) \ge C\,.$$

(3) *There exists $\varepsilon > 0$ such that*

$$\inf_{w\in\mathbb{D}} \int_{G_\varepsilon} \frac{(1-|w|^2)^2}{|1-\overline{w}\zeta|^4}\,\eta_\varphi(\zeta)\,dA(\zeta) \ge C\,.$$

Proposition 6.2. *The following three statements are equivalent:*

(A) *For all $f \in \mathcal{D}$,*

$$\int_{\mathbb{D}} |f'(z)|^2\,dA(z) \le C \int_{\mathbb{D}} |f'(\zeta)|^2\,\log\frac{1}{1-|\zeta|^2}\,\widehat{N_\varphi}(\zeta)\,dA(\zeta)\,.$$

(B) *There exists* $\varepsilon > 0$ *such that for all* $f \in \mathcal{D}$,

$$\int_{\mathbb{D}} |f'(z)|^2 \, dA(z) \le C \int_{G_\varepsilon} |f'(\zeta)|^2 \, \log \frac{1}{1-|\zeta|^2} \, \widehat{N_\varphi}(\zeta) \, dA(\zeta) \, .$$

(C) *There exists* $\varepsilon > 0$ *such that for all* $f \in \mathcal{D}$,

$$\int_{\mathbb{D}} |f'(z)|^2 \, dA(z) \le C \int_{G_\varepsilon} |f'(\zeta)|^2 \, \eta_\varphi(\zeta) \, dA(\zeta) \, .$$

Proof. First assume that (A) holds. If (B) is not valid then we can find a sequence $f_n \in \mathcal{D}$ with $f_n(0) = 0$ and $||f_n||_{\mathcal{D}} = 1$ for all n, and

$$\lim_{n\to\infty} \int_{G_{\frac{1}{n}}} |f_n'(\zeta)|^2 \, \log \frac{1}{1-|\zeta|^2} \, \widehat{N_\varphi}(\zeta) \, dA(\zeta) = 0 \, . \tag{87}$$

Note that by (82) and (85)

$$\begin{aligned} \int_{\mathbb{D}\setminus G_{\frac{1}{n}}} |f_n'(\zeta)|^2 \, \log \frac{1}{1-|\zeta|^2} \, \widehat{N_\varphi}(\zeta) \, dA(\zeta) &\le \int_{\mathbb{D}\setminus \Lambda_{\frac{1}{n}}} |(f_n \circ \varphi)'(z)|^2 \, \frac{\log \frac{1}{1-|\varphi(z)|^2}}{\log \frac{1}{1-|z|^2}} \, |\varphi'(z)|^2 \, dA(z) \\ &\le \frac{1}{n^2} \int_{\mathbb{D}\setminus \Lambda_{\frac{1}{n}}} |(f_n \circ \varphi)'(z)|^2 \, dA(z) \\ &\le \frac{1}{n^2} \, . \end{aligned} \tag{88}$$

By (87) and (88) we conclude that

$$\lim_{n\to\infty} \int_{\mathbb{D}} |f_n'(\zeta)|^2 \, \log \frac{1}{1-|\zeta|^2} \, \widehat{N_\varphi}(\zeta) \, dA(\zeta) = 0$$

and thusly (A) is not valid. We have shown that (A) $\Rightarrow$ (B). By (84), the implication (B) $\Rightarrow$ (C) is clear.

Finally, if (C) holds, then by (87) and (88) and for each $f \in \mathcal{D}$,

$$\begin{aligned} \int_{\mathbb{D}} |f'(\zeta)|^2 \, \log \frac{1}{1-|\zeta|^2} \, \widehat{N_\varphi}(\zeta) \, dA(\zeta) &\ge \int_{G_\varepsilon} |f'(\zeta)|^2 \, \log \frac{1}{1-|\zeta|^2} \, \widehat{N_\varphi}(\zeta) \, dA(\zeta) \\ &\ge \int_{\Lambda_\varepsilon} |(f \circ \varphi)'(z)|^2 \, \frac{\log \frac{1}{1-|\varphi(z)|^2}}{\log \frac{1}{1-|z|^2}} \, |\varphi'(z)|^2 \, dA(z) \\ &\ge \varepsilon^2 \int_{G_\varepsilon} |f'(\zeta)|^2 \, \eta_\varphi(\zeta) \, dA(\zeta) \\ &\ge C \, \varepsilon^2 \int_{\mathbb{D}} |f'(\zeta)|^2 \, dA(\zeta) \end{aligned}$$

and (A) follows. □

Proposition 3.2 is an essential ingredient of the following corollary.

Corollary 6.2. *Assume that the measure* $\log \frac{1}{1-|\zeta|^2} \widehat{N_\varphi}(\zeta)\,dA(\zeta)$ *satisfies the reverse Carleson condition. Then the following hold.*

(a) *There exists an* $\varepsilon > 0$ *such that* G_ε *is hyperbolically dense.*

(b) *There exists* $r \in (0,1)$ *such that for all* $w \in \mathbb{D}$,

$$A(\varphi^{-1}(D(w,r))) \geq C\, A(D(w,r))^{1/\varepsilon^2}\,.$$

(c) *For every* $w \in \mathbb{D}$, $||C_\varphi K_w||_{\mathcal{D}} \asymp ||K_w||_{\mathcal{D}}$.

Proof. For each $w \in \mathbb{D}$ and $r \in (0,1)$ we define

$$\delta_w(r) = \sup_{z \in \Delta_w(r)} \log \frac{1}{1-|\varphi(z)|^2} \left(\log \frac{1}{1-|z|^2} \right)^{-1}.$$

By our assumption and since $\eta_\varphi(\zeta) dA(\zeta)$ is a Carleson measure, there exists $r \in (0,1)$ such that for all $w \in \mathbb{D}$,

$$\begin{aligned} A(D(w,r)) &\leq \int_{D(w,r)} \log \frac{1}{1-|\zeta|^2}\, \widehat{N_\varphi}(\zeta)\, dA(\zeta) \\ &\leq \delta_w(r) \int_{D(w,r)} \eta_\varphi(\zeta)\, dA(\zeta) \\ &\leq C\, \delta_w(r)\, A(D(w,r))\,, \end{aligned} \tag{89}$$

and $\delta_w(r) \geq C$. This is equivalent to G_ε being hyperbolically dense for some $\varepsilon > 0$ and (a) follows.

Next, by (a) we know that G_ε is hyperbolically dense. Therefore there exists $r \in (0,1)$ such that for all $w \in \mathbb{D}$ there exists $z_w \in \Delta_w(r) \cap \Lambda_\varepsilon$. By (69) and (82) we obtain

$$\begin{aligned} A(\varphi^{-1}(D(w,r))) &\geq C\,(1-|z_w|^2)^2 \\ &\geq C\,(1-|\varphi(z_w)|^2)^{2/\varepsilon^2} \\ &\geq C\, A(D(w,r))^{1/\varepsilon^2} \end{aligned}$$

and (b) follows.

Lastly, (c) is an immediate consequence of (a) and Proposition 3.2. □

Let (a), (b), (c), (A), (B), (C) be the statements in Proposition 6.1 and in Proposition 6.2. Moreover recall the three statements in Corollary 6.1 which are equivalent to (a), (b) and (c). An immediate consequence of [16, Theorem 4.3] is that (A) ⇒ (a), (B) ⇒ (b) and (C) ⇒ (c).

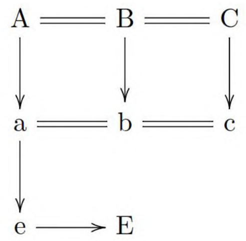
Consider the following three statements:

(e) There exists an $\varepsilon > 0$ such that G_ε is hyperbolically dense.

(E) For all $w \in \mathbb{D}$, $||C_\varphi K_w||_{\mathcal{D}} \asymp ||K_w||_{\mathcal{D}}$.

(CR) The operator C_φ is closed range on $\mathcal{D}$.

With the diagram below we summarize the results in this section.



Jovovic and MacCluer showed in [14] that if C_φ is closed range in $\mathcal{D}$ then $\eta_\varphi(\zeta)\, dA(\zeta)$ satisfies the reverse Carleson condition. Moreover, Luecking showed in [17] that the converse is false. Therefore by Theorem 4.1 the condition

$$\inf_{w\in\mathbb{D}} \int\limits_{\mathbb{D}} \frac{(1-|w|^2)^2}{|1-\overline{w}\zeta|^4}\, \eta_\varphi(\zeta)\, dA(\zeta) \geq C$$

is a necessary but not a sufficient condition for C_φ to be closed range on $\mathcal{D}$.

It is clear that each of (A), (B), (C) imply (CR) and that (CR) implies (E). We conjecture the following.

Conjecture. *C_φ is closed range in $\mathcal{D}$ if and only if there exists $\varepsilon > 0$ such that the measure $\chi_{G_\varepsilon}(\zeta)\eta_\varphi(\zeta)dA(\zeta)$ satisfies the reverse Carleson condition, equivalently*

$$\inf_{w\in\mathbb{D}} \int\limits_{G_\varepsilon} \frac{(1-|w|^2)^2}{|1-\overline{w}\zeta|^4}\, \eta_\varphi(\zeta)\, dA(\zeta) \geq C\,,$$

that is (CR) $\Leftrightarrow$ (c).

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