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CRITERIA FOR STRICT MONOTONICITY OF THE MIXED VOLUME OF CONVEX POLYTOPES

FRÉDÉRIC BIHAN AND IVAN SOPRUNOV

ABSTRACT. Let $P_1,\dots,P_n$ and $Q_1,\dots,Q_n$ be convex polytopes in $\mathbb{R}^n$ such that $P_i \subset Q_i$. It is well-known that the mixed volume has the monotonicity property: $V(P_1,\dots,P_n) \leq V(Q_1,\dots,Q_n)$. We give two criteria for when this inequality is strict in terms of essential collections of faces as well as mixed polyhedral subdivisions. This geometric result allows us to characterize sparse polynomial systems with Newton polytopes $P_1,\dots,P_n$ whose number of isolated solutions equals the normalized volume of the convex hull of $P_1 \cup \dots \cup P_n$. In addition, we obtain an analog of Cramer's rule for sparse polynomial systems.

1. INTRODUCTION

The mixed volume is one of the fundamental notions in the theory of convex bodies. It plays a central role in the Brunn–Minkowski theory and in the theory of sparse polynomial systems. The mixed volume is the polarization of the volume form on the space of convex bodies in $\mathbb{R}^n$. More precisely, let $K_1,\dots,K_n$ be n convex bodies in $\mathbb{R}^n$ and $\mathrm{Vol}_n(K)$ the Euclidean volume of a body $K \subset \mathbb{R}^n$. Then the mixed volume of $K_1,\dots,K_n$ is

$$V(K_1,\dots,K_n) = \frac{1}{n!}\sum_{m=1}^{n}(-1)^{n+m}\sum_{i_1<\dots<i_m}\mathrm{Vol}_n(K_{i_1}+\dots+K_{i_m}), \tag{1.1}$$

where $K+L=\{x+y\in\mathbb{R}^n \mid x\in K, y\in L\}$ denotes the Minkowski sum of bodies K and L. It is not hard to see from the definition that the mixed volume is symmetric, multilinear with respect to Minkowski addition, invariant under translations, and coincides with the volume on the diagonal, i.e. $V(K,\dots,K)=\mathrm{Vol}_n(K)$. What is not apparent from the definition is that it satisfies the following *monotonicity property*, see [11, (5.25)]. If $L_1,\dots,L_n$ are convex bodies such that $K_i \subseteq L_i$ for $1\leq i\leq n$ then

$$V(K_1,\dots,K_n) \leq V(L_1,\dots,L_n).$$

The main goal of this paper is to give a geometric criterion for strict monotonicity in the class of convex polytopes. We give two equivalent criteria in terms of essential collections of faces and Cayley polytopes, see Theorem 3.3 and Theorem 4.4. The first criterion is especially simple when all L_i are equal (Corollary 3.7) which is the situation in our application to sparse polynomial systems. In the general case of convex bodies this is still an open problem, see [11, pp. 429–431] for special cases and conjectures.

The role of mixed volumes in algebraic geometry originates in the work of Bernstein, Kushnirenko, and Khovanskii, who gave a vast generalization of the classical Bézout formula for the intersection number of hypersurfaces in the projective space, see [1, 6, 7]. This beautiful result, which links algebraic geometry and convex geometry through toric varieties and sparse polynomial systems, is commonly known as the BKK bound. Consider an n-variate Laurent polynomial system $f_1(x) = \cdots = f_n(x) = 0$ over an algebraically closed field $\mathbb{K}$. The *support* $\mathcal{A}_i$ of f_i is the set of exponent vectors in $\mathbb{Z}^n$ of the monomials appearing with a non-zero coefficient in f_i. The *Newton polytope* P_i of f_i is the convex hull of $\mathcal{A}_i$. The BKK bound states that the number of isolated solutions of the system in the algebraic torus $(\mathbb{K}^*)^n = (\mathbb{K} \setminus \{0\})^n$ is at most $n!\,V(P_1, \ldots, P_n)$. Systems that attain this bound must satisfy a *non-degeneracy condition*, which means that certain subsystems (predicting solutions "at infinity") have to be inconsistent, see Theorem 5.1. However, the non-degeneracy condition may be hard to check. Let $\mathcal{A} = \cup_{i=1}^n \mathcal{A}_i$ be the total support of the system and choose an order of its elements, $\mathcal{A} = \{a_1, \ldots, a_\ell\}$. Then the system can be written in a matrix form

$$C\,x^A = 0, \tag{1.2}$$

where $C \in \mathbb{K}^{n\times\ell}$ is the matrix of coefficients, $A \in \mathbb{Z}^{n\times\ell}$ is the matrix of exponents whose columns are $a_1, \ldots, a_\ell$, and x^A is the transpose of $(x^{a_1}, \ldots, x^{a_\ell})$, see Section 5. The solution set of (1.2) in $(\mathbb{K}^*)^n$ does not change after left multiplication of C by a matrix in $\mathrm{GL}_n(\mathbb{K})$. Such an operation does not preserve the individual supports of (1.2) in general, but preserves the total support $\mathcal{A}$, see Remark 5.3. Furthermore, let $\bar{A} \in \mathbb{Z}^{(n+1)\times\ell}$ be the *augmented exponent matrix*, obtained by appending a first row of 1 to A. Then left multiplication of $\bar{A}$ by a matrix in $\mathrm{GL}_{n+1}(\mathbb{Z})$ with first row $(1, 0, \ldots, 0)$ corresponds to a monomial change of coordinates of the torus $(\mathbb{K}^*)^n$ and a translation of $\mathcal{A}$, hence, does not change the number of solutions of the system in $(\mathbb{K}^*)^n$, see Section 5.

Assume all Newton polytopes of a system (1.2) are equal to some polytope Q. Then the number of isolated solutions of (1.2) is at most $n!V(Q, \ldots, Q) = n!\,\mathrm{Vol}_n(Q)$, by the BKK bound. In Section 5 we characterize systems that reach (or do not reach) this bound. This characterization follows from the geometric criterion of Corollary 3.7, but it has a natural interpretation in terms of the coefficient matrix C and the augmented exponent matrix $\bar{A}$, see Theorem 5.5. In particular, it says that if Q has a proper face such that the rank of the corresponding submatrix of C (obtained by selecting the columns indexed by points of $\mathcal{A}$ which belong to that face) is strictly less than the rank of the corresponding submatrix of $\bar{A}$, then (1.2) has strictly less than $n!\,\mathrm{Vol}_n(Q)$ isolated solutions in $(\mathbb{K}^*)^n$. Naturally, this characterization is invariant under the actions on C and $\bar{A}$ described above.

Another consequence of Theorem 5.5 can be thought of as a generalization of Cramer's rule for linear systems. Linear systems occur when each P_i is contained in the standard unit simplex Δ in $\mathbb{R}^n$. The BKK bound for systems with all Newton polytopes equal to Δ is just $1 = n!\ \mathrm{Vol}_n(\Delta)$ and, by Cramer's rule, if all maximal minors of the coefficient matrix C are non-zero, then the system (1.2) has precisely one solution in $(\mathbb{K}^*)^n$. We generalize this to an arbitrary Newton polytope Q: If no maximal minor of C vanishes then the system (1.2) has the maximal number $n!\,\mathrm{Vol}_n(Q)$ of isolated solutions in $(\mathbb{K}^*)^n$, see Corollary 5.7.

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2. Preliminaries

In this section we recall necessary definitions and results from convex geometry and set up notation. In addition, we recall the notion of essential collections of polytopes for which we give several equivalent definitions, as well as define mixed polyhedral subdivisions and the combinatorial Cayley trick.

Throughout the paper we use $[n]$ to denote the set $\{1,\dots,n\}$.

Mixed Volume. For a convex body K in $\mathbb{R}^n$ the function $h_K : \mathbb{R}^n \to \mathbb{R}$, given by $h_K(u) = \max\{\langle u, x\rangle \mid x \in K\}$ is the *support function* of K. Here $\langle u, x\rangle$ is the standard scalar product in $\mathbb{R}^n$. For every $u \in \mathbb{R}^n$, we write $H_K(u)$ to denote the *supporting hyperplane* for K with outer normal u

$$H_K(u) = \{x \in \mathbb{R}^n \mid \langle u, x\rangle = h_K(u)\}.$$

Throughout the paper we use

$$K^u = K \cap H_K(u)$$

to denote the *face* of K corresponding to the supporting hyperplane $H_K(u)$. Since $H_K(u)$ and K^u are invariant under rescaling u by a non-zero scalar, we often assume that when $u \neq 0$, it lies in the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Clearly, for $u = 0$ we have $H_K(u) = \mathbb{R}^n$ and $K^u = K$.

Let $V(K_1,\dots,K_n)$ be the n-dimensional mixed volume of n convex bodies $K_1,\dots,K_n$ in $\mathbb{R}^n$, see (1.1). We have the following equivalent characterization.

Theorem 2.1. [11, Theorem 5.1.7] *Let $\lambda_1,\dots,\lambda_n$ be non-negative real numbers. Then $\mathrm{Vol}_n(\lambda_1 K_1 + \cdots + \lambda_n K_n)$ is a polynomial in $\lambda_1,\dots,\lambda_n$ whose coefficient of the monomial $\lambda_1\cdots\lambda_n$ equals $V(K_1,\dots,K_n)$.*

Essential Collections. Throughout the paper we use "collection" as a synonym for "multiset". Let $K_1,\dots,K_m$ be convex bodies in $\mathbb{R}^n$, not necessarily distinct. We say that a collection $\{K_1,\dots,K_m\}$ is *essential* if for any subset $I \subset [m]$ of size at most n we have

$$\dim \sum_{i\in I} K_i \geq |I|. \tag{2.1}$$

Note that every sub-collection of an essential collection is essential. Also $\{K,\dots,K\}$, where K is repeated m times, is essential if and only if $\dim K \geq m$.

The following well-known result asserts that essential collections of n convex bodies characterize positivity of the mixed volume.

Theorem 2.2. [11, Theorem 5.1.8] *Let $K_1, \dots, K_n$ be n convex bodies in $\mathbb{R}^n$. The following are equivalent:*

(1) $V(K_1, \dots, K_n) > 0$;
(2) *There exist segments $E_i \subset K_i$ for $1 \leq i \leq n$ with linearly independent directions;*
(3) $\{K_1, \dots, K_n\}$ *is essential.*

Another useful result is the inductive formula for the mixed volume, see [11, Theorem 5.1.7, (5.19)]. We present a variation of this formula for polytopes. Recall that a *polytope* $P \subset \mathbb{R}^n$ is the convex hull of finitely many points in $\mathbb{R}^n$. Furthermore, P is a *lattice* polytope if its vertices belong to the integer lattice $\mathbb{Z}^n \subset \mathbb{R}^n$.

Let K be a convex body and $Q_2, \dots, Q_n$ be polytopes in $\mathbb{R}^n$. Given $u \in \mathbb{S}^{n-1}$, let $V(Q_2^u, \dots, Q_n^u)$ denote the $(n-1)$-dimensional mixed volume of $Q_2^u, \dots, Q_n^u$ translated to the orthogonal subspace $u^\perp$. Then we have

$$V(K, Q_2 \dots, Q_n) = \frac{1}{n} \sum_{u \in \mathbb{S}^{n-1}} h_K(u) V(Q_2^u, \dots, Q_n^u). \tag{2.2}$$

Note that the above sum is finite, since there are only finitely many $u \in \mathbb{S}^{n-1}$ for which $\{Q_2^u, \dots, Q_n^u\}$ is essential. Namely, these u are among the outer unit normals to the facets of $Q_2 + \cdots + Q_n$.

Remark 2.3. There is a reformulation of (2.2) that is more suitable for lattice polytopes. It is not hard to see that $n! \operatorname{Vol}(P)$ is an integer for any lattice polytope. This implies that $n!V(P_1, \dots, P_n)$ is also an integer for any collection of lattice polytopes $P_1, \dots, P_n$. Recall that a vector $u \in \mathbb{Z}^n$ is *primitive* if the greatest common divisor of its components is 1. Given lattice polytopes $P, Q_2, \dots, Q_n$ we have

$$n!V(P, Q_2 \dots, Q_n) = \sum_{u \text{ primitive}} h_P(u)(n-1)!V(Q_2^u, \dots, Q_n^u), \tag{2.3}$$

where the $(n-1)$-dimensional mixed volume is normalized such that the volume of the parallelepiped spanned by a lattice basis for $u^\perp \cap \mathbb{Z}^n$ equals one. Note that the terms in the sum are non-negative integers, which, as above, equal zero for all but finitely many primitive $u \in \mathbb{Z}^n$.

Cayley Polytopes and Combinatorial Cayley Trick. Let $P_1, \dots, P_k \subset \mathbb{R}^n$ be convex polytopes. The associated *Cayley polytope*

$$\mathcal{C}(P_1, \dots, P_k)$$

is the convex hull in $\mathbb{R}^n \times \mathbb{R}^k$ of the union of the polytopes $P_i \times \{e_i\}$ for $i = 1, \dots, k$, where $\{e_1, \dots, e_k\}$ is the standard basis for $\mathbb{R}^k$.

Let $(x, y) = (x_1, \dots, x_n, y_1, \dots, y_k)$ be coordinates on $\mathbb{R}^n \times \mathbb{R}^k$ and let $\pi_1 : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ and $\pi_2 : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k$ be the projections defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$, respectively. Note that $\pi_2(\mathcal{C}(P_1, \dots, P_k))$ is the $(k-1)$-dimensional simplex Δ_{k-1} defined by $\sum_{i=1}^k y_i = 1$ and $y_i \geq 0$ for $1 \leq i \leq k$. Furthermore, for every $y \in \Delta_{k-1}$ we have

$$\pi_2^{-1}(y) \cap \mathcal{C}(P_1, \ldots, P_k) = (y_1 P_1 + \cdots + y_k P_k) \times \{y\}. \tag{2.4}$$

Note that when all $y_i > 0$ the preimage $\pi_2^{-1}(y) \cap \mathcal{C}(P_1, \ldots, P_k)$ has dimension equal to $\dim(P_1 + \cdots + P_k)$. This implies that

$$\dim \mathcal{C}(P_1, \ldots, P_k) = \dim(P_1 + \cdots + P_k) + k - 1. \tag{2.5}$$

If $\dim P_i \geq 1$ for $i = 1, \ldots, k$, then the Cayley polytope $\mathcal{C}(P_1, \ldots, P_k) \subset \mathbb{R}^{n+k}$, as well as the Minkowski sum $P_1 + \cdots + P_k$, is called *fully mixed.* The following result is an immediate consequence of (2.5).

Lemma 2.4. *Consider polytopes $P_1, \ldots, P_n \subset \mathbb{R}^n$. Then the following conditions are equivalent.*

(1) *The Cayley polytope $\mathcal{C}(P_1, \ldots, P_n)$ is a fully mixed $(2n-1)$-dimensional simplex.*
(2) *$P_1, \ldots, P_n$ are segments with linearly independent directions.*

Remark 2.5. Let $P_1, \ldots, P_n$ be polytopes in $\mathbb{R}^n$. From Theorem 2.2 and Lemma 2.4, we have $V(P_1, \ldots, P_n) > 0$ if and only if $\mathcal{C}(P_1, \ldots, P_n)$ contains a fully mixed $(2n-1)$-dimensional simplex $\mathcal{C}(E_1, \ldots, E_n)$.

Let $\tau_{\mathcal{C}}$ be any polyhedral subdivision of $\mathcal{C}(P_1, \ldots, P_k)$ with vertices in $\cup_{i=1}^k P_i \times \{e_i\}$. Consider any full-dimensional polytope $\mathcal{C}_\sigma$ of $\tau_{\mathcal{C}}$. Then it intersects each hyperplane $y_i = 1$ along a non-empty face $\sigma_i \times \{e_i\} \subset P_i \times \{e_i\}$ for $1 \leq i \leq k$, and it follows that $\mathcal{C}_\sigma = \mathcal{C}(\sigma_1, \ldots, \sigma_k)$. Therefore, $\tau_{\mathcal{C}}$ consists of the set of all the polytopes $\mathcal{C}(\sigma_1, \ldots, \sigma_k)$ together with their faces. Taking the image under π_1 of $\pi_2^{-1}(\frac{1}{k}, \ldots, \frac{1}{k}) \cap \mathcal{C}(P_1, \ldots, P_n)$ we obtain, by (2.4), the Minkowski sum $P_1 + \cdots + P_k$ (up to dilatation by $\frac{1}{k}$) together with a polyhedral subdivision by polytopes $\sigma_1 + \cdots + \sigma_k$, where $\sigma_i \subset P_i$ for $1 \leq i \leq k$. This defines a correspondence from the set of all polyhedral subdivisions of $\mathcal{C}(P_1, \ldots, P_k)$ with vertices in $\cup_{i=1}^k P_i \times \{e_i\}$ to a set of polyhedral subdivisions of $P_1 + \cdots + P_k$ which are called *mixed.* Note that $\tau_{\mathcal{C}}$ is uniquely determined by the corresponding mixed subdivision of $P_1 + \cdots + P_k$. This one-to-one correspondence is commonly called the *combinatorial Cayley trick* or simply the *Cayley trick*, see [8], [13] or [2], for instance.

A mixed polyhedral subdivision of $P_1 + \cdots + P_k$ is called *pure* if the corresponding subdivision of $\mathcal{C}(P_1, \ldots, P_k)$ is a triangulation. Let $\sigma_1 + \cdots + \sigma_k$ be a polytope in a pure mixed polyhedral subdivision of $P_1 + \cdots + P_k$. Then each σ_i is a simplex since $\sigma_i \times \{e_i\}$ is a face of the simplex $\mathcal{C}(\sigma_1, \ldots, \sigma_k)$. If furthermore $k = n$ and $\dim(\sigma_1 + \cdots + \sigma_n) = n$, then $\sigma_1 + \cdots + \sigma_n$ is fully mixed if and only if $\mathcal{C}(\sigma_1, \ldots, \sigma_n)$ is a fully mixed $(2n-1)$-dimensional simplex, equivalently, $\sigma_1, \ldots, \sigma_n$ are segments with linearly independent directions (see Lemma 2.4). The following result is well-known, see [9, Theorem 2.4] or [4, Theorem 6.7].

Lemma 2.6. *For convex polytopes $P_1, \ldots, P_n$ in $\mathbb{R}^n$, the quantity $n!\, V(P_1, \ldots, P_n)$ is equal to the sum of the Euclidean volumes of the fully mixed polytopes in any pure mixed polyhedral subdivision of $P_1 + \cdots + P_n$.*

3. First criterion

In this section we present our first criterion for strict monotonicity of the mixed volume and its corollaries.

Definition 3.1. Let K be a subset of a convex polytope A and let $F \subset A$ be a facet. We say K *touches* F when the intersection $K \cap F$ is non-empty.

We will often make use of the following proposition, which gives a criterion for strict monotonicity in a very special case, see [11, page 282].

Proposition 3.2. *Let $P_1, Q_1, \ldots, Q_n$ be convex polytopes in $\mathbb{R}^n$ and $P_1 \subseteq Q_1$. Then $V(P_1, Q_2, \ldots, Q_n) = V(Q_1, Q_2, \ldots, Q_n)$ if and only if P_1 touches every face Q_1^u for u in the set*

$$U = \{u \in \mathbb{S}^{n-1} \mid \{Q_2^u, \ldots, Q_n^u\} \text{ is essential}\}.$$

The above statement easily follows from (2.2) and the observation $h_{P_1}(u) \leq h_{Q_1}(u)$ with equality if and only if P_1 touches Q_1^u. See [11, Sec 5.1] for details.

Here is the first criterion for strict monotonicity.

Theorem 3.3. *Let $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$ be convex polytopes in $\mathbb{R}^n$ such that $P_i \subseteq Q_i$ for every $i \in [n]$. Given $u \in \mathbb{S}^{n-1}$ consider the set*

$$T_u = \{i \in [n] \mid P_i \textit{ touches } Q_i^u\}.$$

Then $V(P_1, \ldots, P_n) < V(Q_1, \ldots, Q_n)$ if and only if there exists $u \in \mathbb{S}^{n-1}$ such that the collection $\{Q_i^u \mid i \in T_u\} \cup \{Q_i \mid i \in [n] \setminus T_u\}$ is essential.

Proof. Assume that there exists $u \in \mathbb{S}^{n-1}$ such that the collection $\{Q_i^u \mid i \in T_u\} \cup \{Q_i \mid i \in [n] \setminus T_u\}$ is essential. Note that T_u is a proper subset of $[n]$, otherwise $\{Q_i^u \mid i \in T_u\}$ is a collection of n polytopes contained in translates of an $(n-1)$-dimensional subspace, hence, cannot be essential. Without loss of generality we may assume that $[n] \setminus T_u = \{1, \ldots, k\}$ for some $k \geq 1$. In other words, we assume the collection

$$\{Q_1, \ldots, Q_k, Q_{k+1}^u, \ldots, Q_n^u\} \tag{3.1}$$

is essential. Since P_i does not touch Q_i^u for $1 \leq i \leq k$ there is a hyperplane $H = \{x \in \mathbb{R}^n \mid \langle x, u \rangle = h_{Q_i}(u) - \varepsilon\}$ which separates P_i and Q_i^u. Let H_+ be the half-space containing P_i. Then the truncated polytope $\tilde{Q}_i = Q_i \cap H_+$ satisfies $P_i \subseteq \tilde{Q}_i \subset Q_i$. We claim that, after a possible renumbering of the first k of the Q_i, the collection

$$\{\tilde{Q}_2^u, \ldots, \tilde{Q}_k^u, Q_{k+1}^u, \ldots, Q_n^u\} \tag{3.2}$$

is essential. Indeed, since (3.1) is essential, by Theorem 2.2 there exist n segments $E_i \subset Q_i$ with linearly independent directions such that $E_i \subset Q_i^u$ for $k < i \leq n$. Replace the first k of the segments with their projections onto $\tilde{Q}_1^u, \ldots, \tilde{Q}_k^u$. By Lemma 3.4 below, after a possible renumbering of the first k segments, we obtain $n-1$ segments $E_2, \ldots, E_n$ with linearly independent directions such that $E_i \subset \tilde{Q}_i^u$ for $2 \leq i \leq k$ and $E_i \subset Q_i^u$ for $k < i \leq n$. By Theorem 2.2, the collection (3.2) is essential.

Now, by Proposition 3.2 and since P_1 does not touch Q_1^u, we obtain

$$V(P_1, \tilde{Q}_2, \ldots, \tilde{Q}_k, Q_{k+1}, \ldots, Q_n) < V(Q_1, \tilde{Q}_2, \ldots, \tilde{Q}_k, Q_{k+1}, \ldots, Q_n).$$

Finally, by monotonicity we have $V(P_1,\dots,P_n) \le V(P_1,\tilde{Q}_2,\dots,\tilde{Q}_k,Q_{k+1},\dots,Q_n)$ and $V(Q_1,\tilde{Q}_2,\dots,\tilde{Q}_k,Q_{k+1},\dots,Q_n) \le V(Q_1,\dots,Q_n)$. Therefore,

$$V(P_1,\dots,P_n) < V(Q_1,\dots,Q_n).$$

Conversely, assume $V(P_1,\dots,P_n) < V(Q_1,\dots,Q_n)$. Then, by monotonicity, for some $1 \le k \le n$ we have

$$V(P_1,\dots,P_{k-1},P_k,Q_{k+1},\dots,Q_n) < V(P_1,\dots,P_{k-1},Q_k,Q_{k+1},\dots,Q_n).$$

By Proposition 3.2 there exists $u \in \mathbb{S}^{n-1}$ such that $\{P_1^u,\dots,P_{k-1}^u,Q_{k+1}^u,\dots,Q_n^u\}$ is essential and $k \notin T_u$. By choosing a segment in Q_k not parallel to the orthogonal hyperplane $u^\perp$ (which exists since $P_k \subset Q_k$, but P_k does not touch Q_k^u) we see that

$$\{P_1^u,\dots,P_{k-1}^u,Q_k,Q_{k+1}^u,\dots,Q_n^u\}$$

is essential. It remains to notice that $P_i^u \subseteq Q_i^u$ for $i \in T_u$ and, hence, the collection $\{Q_i^u \mid i \in T_u\} \cup \{Q_i \mid i \in [n] \setminus T_u\}$ is essential as well. □

Lemma 3.4. *Let $\{v_1,\dots,v_k,v_{k+1},\dots,v_n\}$ be a basis for $\mathbb{R}^n$ where $v_{k+1},\dots,v_n$ belong to a hyperplane $H \subset \mathbb{R}^n$. Let π denote the orthogonal projection onto H. Then, after a possible renumbering of the first k vectors, the set $\{\pi(v_2),\dots,\pi(v_k),v_{k+1},\dots,v_n\}$ is a basis for H.*

Proof. Clearly, the set $\{\pi(v_1),\pi(v_2),\dots,\pi(v_k),v_{k+1},\dots,v_n\}$ spans H. Starting with the linearly independent set $\{v_{k+1},\dots,v_n\}$ we can extend it to a basis for H by appending $k-1$ vectors from $\{\pi(v_1),\dots,\pi(v_k)\}$. □

Remark 3.5. Note that if $Q_1,\dots,Q_n$ are n-dimensional then $\{Q_i^u \mid i \in T_u\} \cup \{Q_i \mid i \in [n] \setminus T_u\}$ is essential if and only if $\{Q_i^u \mid i \in T_u\}$ is essential. (This can be readily seen from (2.1).) In this case we can simplify the criterion of Theorem 3.3 as follows: $V(P_1,\dots,P_n) < V(Q_1,\dots,Q_n)$ if and only if there exists $u \in \mathbb{S}^{n-1}$ such that the collection $\{Q_i^u \mid i \in T_u\}$ is essential.

Remark 3.6. After the initial submission of our paper to arxiv.org we were informed by Maurice Rojas that a similar criterion for rational polytopes appeared in his paper [10, Corollary 9]. The proof of his criterion is algebraic and is based on the BKK bound. We are thankful to Maurice Rojas for pointing that out.

A particular instance of Theorem 3.3, especially important for applications to polynomial systems, is the case when $P_1,\dots,P_n$ are arbitrary polytopes and $Q_1,\dots,Q_n$ are equal to the same polytope Q. We will assume that Q is n-dimensional, otherwise $\{P_1,\dots,P_n\}$ is not essential and, hence, both $V(P_1,\dots,P_n)$ and $V(Q,\dots,Q) = \mathrm{Vol}_n(Q)$ are zero. Then the strict monotonicity has the following simple geometric interpretation.

Corollary 3.7. *Let $P_1,\dots,P_n$ be polytopes in $\mathbb{R}^n$ contained in an n-dimensional polytope Q. Then $V(P_1,\dots,P_n) < \mathrm{Vol}_n(Q)$ if and only if there is a proper face of Q of dimension t which is touched by at most t of the polytopes $P_1,\dots,P_n$.*

Proof. By Theorem 3.3 and Remark 3.5, we have $V(P_1,\dots,P_n) < \mathrm{Vol}_n(Q)$ if and only if there exists $u \in \mathbb{S}^{n-1}$ such that the collection $\{Q^u,\dots,Q^u\}$, where Q^u is repeated $|T_u|$ times, is essential. The last condition is equivalent to $\dim Q^u \ge |T_u|$.

This precisely means that the face Q^u is touched by at most $\dim Q^u$ of the polytopes. □

Note that in particular, if a vertex of Q does not belong to any of the polytopes $P_1, \dots, P_n$, then $V(P_1, \dots, P_n) < \mathrm{Vol}_n(Q)$.

Example 3.8. Let P_1, P_2 be convex polytopes in $\mathbb{R}^2$ and Q be the convex hull of their union. Then Corollary 3.7 shows that $V(P_1, P_2) < V(Q)$ if and only if either P_1 or P_2 does not touch some edge of Q.

Remark 3.9. One can obtain a more direct proof of Corollary 3.7 by modifying the proof of Theorem 2.6 in [12]. In the case when Q equals the convex hull of $P_1 \cup \dots \cup P_n$, several sufficient conditions for $V(P_1, \dots, P_n) = \mathrm{Vol}_n(Q)$ were found by Tianran Chen in [3] whose preprint appeared on arxiv.org shortly after ours. Chen studies other applications of Corollary 3.7 to sparse polynomial systems from the computational complexity point of view.

Example 3.10. Let $\mathcal{A}$ be a finite set in $\mathbb{R}^n$ with n-dimensional convex hull Q and choose a subset $\{a_1, \dots, a_n\} \subset \mathcal{A}$. Define $\mathcal{A}_i = (\mathcal{A} \setminus \{a_1, \dots, a_n\}) \cup \{a_i\}$ for $1 \leq i \leq n$ and let P_i be the convex hull of $\mathcal{A}_i$. Then Corollary 3.7 leads to $V(P_1, \dots, P_n) = \mathrm{Vol}_n(Q)$. Indeed, for any $I \subset [n]$ of size $n - t$ and any face $F \subset Q$ of dimension t we have $|\cup_{i \in I} \mathcal{A}_i| = |\mathcal{A}| - t$ and $|F \cap \mathcal{A}| \geq t + 1$. Therefore, the subsets $\cup_{i \in I} \mathcal{A}_i$ and $F \cap \mathcal{A}$ cannot be disjoint, i.e. every face F of dimension t is touched by the union of any $n - t$ of the P_i. This shows that every face F of dimension t is touched by at least $t + 1$ of the P_i and we can apply Corollary 3.7.

Going back to the statement of Corollary 3.7 it seems natural to ask: If there is a dimension t face of Q not touched by more than $n - t$ polytopes among $P_1, \dots, P_n$, can the inequality $V(P_1, \dots, P_n) < \mathrm{Vol}_n(Q)$ be improved? The answer is clearly no in the class of all polytopes: For any $\varepsilon > 0$, take $P_i = (1 - \varepsilon)Q$ for $1 \leq i \leq n$, then

$$V(P_1, \dots, P_n) = (1 - \varepsilon)^n \mathrm{Vol}_n(Q)$$

and no face of Q is touched by any of the P_i. However, one might expect an improvement in the class of lattice polytopes. We present such an improvement of Corollary 3.7 in Proposition 3.11.

Let $P \subset Q$ be lattice polytopes and $u \in \mathbb{Z}^n$ primitive. We can define the *lattice distance from P to Q in the direction of u* as the difference $h_Q(u) - h_P(u)$. Note that this is a non-negative integer.

Proposition 3.11. *Let $P_1, \dots, P_n$ be lattice polytopes in $\mathbb{R}^n$ contained in an n-dimensional lattice polytope Q. Suppose there exists a facet $Q^v \subset Q$, for some primitive $v \in \mathbb{Z}^n$, which is not touched by $P_1, \dots, P_m$, for some $m \geq 1$. Moreover, suppose that $\{P_1^v, \dots, P_{m-1}^v\}$ is essential. Then*

$$n! \,\mathrm{Vol}\, Q - n! \,V(P_1, \dots, P_n) \geq l_1 + \dots + l_m,$$

where $l_i \geq 1$ is the lattice distance from P_i to Q in the direction of v.

Proof. By the essentiality of $\{P_1^v, \dots, P_{m-1}^v\}$ and since $\dim Q^v = n - 1$ it follows that for any $1 \leq i \leq m$ the collection $\{P_1^v, \dots, P_{i-1}^v, Q^v, \dots, Q^v\}$, where Q^v is repeated $n - i$ times, is essential. Applying (2.3) from Remark 2.3 we obtain

$$n!V(P_1,\dots,P_{i-1},\underbrace{Q,\dots,Q}_{n-i+1}) - n!V(P_1,\dots,P_i,\underbrace{Q,\dots,Q}_{n-i}) =$$

$$\sum_{u \text{ primitive}} (h_Q(u)-h_{P_i}(u))\,(n-1)!V(P_1^u,\dots,P_{i-1}^u,Q^u,\dots,Q^u) \geq$$

$$(h_Q(v)-h_{P_i}(v))\,(n-1)!V(P_1^v,\dots,P_{i-1}^v,Q^v,\dots,Q^v) \geq h_Q(v)-h_{P_i}(v) = l_i,$$

where the last inequality follows from the fact that $\{P_1^v,\dots,P_{i-1}^v,Q^v,\dots,Q^v\}$ is essential and, hence, $(n-1)!V(P_1^v,\dots,P_{i-1}^v,Q^v,\dots,Q^v)$ is a positive integer. Summing up these inequalities for $1 \leq i \leq m$ we produce

$$n!\ \mathrm{Vol}\, Q - n!\, V(P_1,\dots,P_m,\underbrace{Q,\dots,Q}_{n-m}) \geq l_1+\dots+l_m.$$

Now the required inequality follows by the monotonicity of the mixed volume.

□

Note that the case of $m=1$ and $l_1=1$ recovers an instance of Corollary 3.7 with $t=n-1$. In this case the condition that $\{P_1^u,\dots,P_{m-1}^u\}$ is essential is void. We remark that in general this condition cannot be removed. Indeed, let $Q=\mathrm{conv}\{0,e_1,\dots,e_n\}$ be the standard n-simplex, Q^u one of its facets, and $P_i \subset Q$ for $1 \leq i \leq n$. Then if $P_1,\dots,P_m$ equal the vertex of Q not contained in Q^u, then

$$0 = n!V(P_1,\dots,P_n) = n!V_n(Q) - 1,$$

regardless of m. It would be interesting to obtain a more general statement than Proposition 3.11 which deals with smaller dimensional faces, rather than facets.

Remark 3.12. Given a polytope Q, Corollary 3.7 provides a characterization of collections $P_1,\dots,P_n$ such that $P_i \subset Q$ for $1 \leq i \leq n$ and $V(P_1,\dots,P_n)=\mathrm{Vol}_n(Q)$. Clearly, when Q and the P_i are lattice polytopes there are only finitely many such collections. Describing them explicitly is a hard combinatorial problem, in general. In the case when Q is the standard simplex we have $\mathrm{Vol}_n(Q)=1/n!$. On the other hand, since $n!V(P_1,\dots,P_n)$ is an integer when the P_i are lattice polytopes, Theorem 2.2 implies the following: For lattice polytopes $P_1,\dots,P_n$ contained in the standard simplex Q we have $V(P_1,\dots,P_n)=\mathrm{Vol}_n(Q)$ if and only if $\{P_1,\dots,P_n\}$ is essential. This is a particular case of a much deeper result by Esterov and Gusev, who described all collections of lattice polytopes $\{P_1,\dots,P_n\}$ satisfying $V(P_1,\dots,P_n)=1/n!$, see [5].

4. Second Criterion

In this section, we obtain another criterion for the strict monotonicity property (Theorem 4.4) based on mixed polyhedral subdivisions and the combinatorial Cayley trick. We first present a result about faces of Cayley polytopes which will be useful. Consider convex polytopes $P_1,\dots,P_k \subset \mathbb{R}^n$. Recall that the Cayley polytope $\mathcal{C}(P_1,\dots,P_k)$ is the convex hull in $\mathbb{R}^{n+k}$ of the union of the polytopes $P_i \times \{e_i\}$ for $1 \leq i \leq k$, where $\{e_1,\dots,e_k\}$ is the standard basis of $\mathbb{R}^k$ (see Section 2).

Lemma 4.1. *Let (u,v) be a vector in $\mathbb{R}^n \times \mathbb{R}^k$ with $v=(v_1,\dots,v_k)$. Then*

$$h_{\mathcal{C}(P_1,\dots,P_k)}(u,v) = \max\{h_{P_i}(u)+v_i \mid i \in [k]\}.$$

Moreover, $\mathcal{C}(P_1,\ldots,P_k)^{(u,v)}$ *is the convex hull in* $\mathbb{R}^n\times\mathbb{R}^k$ *of the union of the polytopes* $P_i^u\times\{e_i\}$ *for* i *in the set*

$$I=\{i\in[k]\mid h_{\mathcal{C}(P_1,\ldots,P_k)}(u,v)=h_{P_i}(u)+v_i\}.$$

Proof. Let $(x,y)\in\mathbb{R}^n\times\mathbb{R}^k$ be a point of $\mathcal{C}(P_1,\ldots,P_k)$. There exist $\alpha_i\in\mathbb{R}_{\geq 0}$ and points $x_i\in P_i$ for $i\in[k]$ such that $(x,y)=\sum_{i=1}^k\alpha_i(x_i,e_i)$ and $\sum_{i=1}^k\alpha_i=1$. Then $\langle(u,v),(x,y)\rangle=\sum_{i=1}^k\alpha_i(\langle u,x_i\rangle+v_i)$ is bounded above by $\max\{h_{P_i}(u)+v_i\mid i\in[k]\}$. Moreover, this bound is attained if and only if $\langle u,x_i\rangle+v_i=\max\{h_{P_i}(u)+v_i\mid i\in[k]\}$ for all i such that $\alpha_i>0$. Since $\langle u,x_i\rangle+v_i\leq h_{P_i}(u)+v_i\leq\max\{h_{P_i}(u)+v_i\mid i\in[k]\}$, the latter condition is equivalent to $x_i\in P_i^u$ with $i\in I$. □

Remark 4.2. The polytope $\mathcal{C}(P_1,\ldots,P_k)$ is contained in the hyperplane $\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^k\mid\sum_{i=1}^k y_i=1\}$. Therefore, $\mathcal{C}(P_1,\ldots,P_k)^{(u,v)}=\mathcal{C}(P_1,\ldots,P_k)$ for $u=0$ and $v=(\lambda,\ldots,\lambda)$ for any non-zero $\lambda\in\mathbb{R}$. Moreover, if $\dim(P_1+\cdots+P_k)=n$, then $\mathcal{C}(P_1,\ldots,P_k)^{(u,v)}=\mathcal{C}(P_1,\ldots,P_k)$ only if $u=0$ and $v=(\lambda,\ldots,\lambda)$ with $\lambda\in\mathbb{R}$, by (2.5).

Before stating the main result of this section, we need a technical lemma about regular polyhedral subdivisions (see, for example, [8] for a reference on this topic). Let $\mathcal{A}$ be a finite set in $\mathbb{R}^n$ and let A denote the convex hull of $\mathcal{A}$. A polyhedral subdivision τ of A with vertices in $\mathcal{A}$ is called *regular* if there exists a map $h:\mathcal{A}\to\mathbb{R}$ such that τ is obtained by projecting the lower faces of the convex-hull $\hat{A}$ of $\{(a,h(a))\mid a\in\mathcal{A}\}$ via the projection $\mathbb{R}^{n+1}\to\mathbb{R}^n$ forgetting the last coordinates. Here a lower face of $\hat{A}$ is a face of a facet of $\hat{A}$ with an inward normal vector with positive last coordinate. Intuitively, this is a face that can be seen from below in the direction of e_{n+1}.The union of the lower faces of $\hat{A}$ is the graph of a convex piecewise linear map $\hat{h}:A\to\mathbb{R}$ whose domains of linearity are the polytopes of τ. We say that h (respectively $\hat{h}$) *certifies the regularity* of τ and that τ is induced by h (respectively $\hat{h}$). Note that for h generic enough any $n+2$ points of $\{(a,h(a))\mid a\in\mathcal{A}\}$ are affinely independent (i.e. do not lie on a hyperplane), hence the induced subdivision τ is a triangulation.

Lemma 4.3. *Let* $\mathcal{A}_1,\mathcal{A}_2$ *be finite subsets of* $\mathbb{R}^n$ *and* A_1,A_2 *their convex hulls.*

(1) *If* $\mathcal{A}_1\subset\mathcal{A}_2$ *and if* τ_1 *is a regular polyhedral subdivision (respectively a regular triangulation) of* A_1 *with vertices in* $\mathcal{A}_1$*, then there exists a regular polyhedral subdivision (respectively a regular triangulation)* τ_2 *of* A_2 *with vertices in* $\mathcal{A}_2$ *such that* $\tau_1\subset\tau_2$*.*

(2) *Assume that the relative interiors of* A_1 *and* A_2 *do not intersect. Let* $\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2$ *and let* A *denote the convex hull of* $\mathcal{A}$*. If* τ_1 *is a regular polyhedral subdivision (respectively a regular triangulation) of* A_1 *with vertices in* $\mathcal{A}_1$ *and* τ_2 *is a regular polyhedral subdivision (respectively a regular triangulation) of* A_2 *with vertices in* $\mathcal{A}_2$*, then there exists a regular polyhedral subdivision (respectively a regular triangulation)* τ *of* A *with vertices in* $\mathcal{A}$ *such that* $\tau_1\cup\tau_2\subset\tau$*.*

Proof. (1) Consider a map $H:\mathcal{A}_2\to\mathbb{R}$ which vanishes on $\mathcal{A}_1$ and takes positive values on $\mathcal{A}_2\backslash\mathcal{A}_1$. Then, this map certifies the regularity of a polyhedral subdivision $\tilde{\tau}_2$

of A_2 with vertices in $\mathcal{A}_2$ and which contains A_1. Now consider a regular polyhedral subdivision τ_1 of A_1 with vertices in $\mathcal{A}_1$ whose regularity is certified by $h_1 : \mathcal{A}_1 \to \mathbb{R}$. Then, for $\epsilon > 0$ small enough, the function $h_2 : \mathcal{A}_2 \to \mathbb{R}$ defined by $h_2(a) = H(a)+\epsilon h_1(a)$ if $a \in \mathcal{A}_1$ and by $h_2(a) = H(a)$ otherwise certifies a regular polyhedral subdivision τ_2 of A_2 with vertices in $\mathcal{A}_2$ such that $\tau_1 \subset \tau_2$. Finally, if τ_1 is a triangulation and if the values of H on $\mathcal{A}_2 \setminus \mathcal{A}_1$ are generic enough, then τ_2 is a triangulation.

(2) Consider a regular polyhedral subdivision (respectively a regular triangulation) τ_i of A_i with vertices in $\mathcal{A}_i$ for $i = 1, 2$. Let $h_i : \mathcal{A}_i \to \mathbb{R}$ be a function certifying the regularity of τ_i. Since the relative interiors of the convex sets A_1 and A_2 do not intersect, there is a hyperplane which separates A_1 and A_2. This means that there exist $u \in \mathbb{R}^n$ and $c \in \mathbb{R}$ such that $A_1 \subset B_- = \{x \in \mathbb{R}^n \mid \langle u, x\rangle \le c\}$, $A_2 \subset B_+ = \{x \in \mathbb{R}^n \mid \langle u, x\rangle \ge c\}$ and A_1, A_2 are not both contained in the separating hyperplane $B_+ \cap B_-$. Let $H : \mathcal{A} \to \mathbb{R}$ be a piecewise linear map which vanishes on B_- and positive on $B_+ \setminus B_-$. For $\epsilon > 0$ consider $h : \mathcal{A} \to \mathbb{R}$ defined by $h(a) = H(a)+\epsilon h_i(a)$ if $a \in \mathcal{A}_i$. If ϵ is small enough and generic, the map h certifies the regularity of a polyhedral subdivision (respectively a regular triangulation) τ of A such that $\tau_1 \cup \tau_2 \subset \tau$. ☐

Recall that for convex polytopes $Q_1, \dots, Q_n$ in $\mathbb{R}^n$, we have $V(Q_1, \dots, Q_n) > 0$ if and only if the collection $\{Q_1, \dots, Q_n\}$ is essential, which is equivalent to the existence of a fully mixed $(2n-1)$-dimensional simplex $\mathcal{C}(E_1, \dots, E_n)$ contained in $\mathcal{C}(Q_1, \dots, Q_n)$, see Theorem 2.2 and Remark 2.5. We now describe a generalization of these equivalences. Consider convex polytopes $P_i \subseteq Q_i \subset \mathbb{R}^n$ for $1 \le i \le n$. For any non-zero vector $u \in \mathbb{R}^n$ define convex polytopes

$$B_{i,u} = \{x \in Q_i \mid \langle u, x\rangle \ge h_{P_i}(u)\},\ 1 \le i \le n.$$

Intuitively, $B_{i,u}$ is the part of Q_i lying on top of P_i if one looks in the direction of the vector u.

Theorem 4.4. *Let $P_1, \dots, P_n$ and $Q_1, \dots, Q_n$ be convex polytopes in $\mathbb{R}^n$ such that $P_i \subseteq Q_i$ for every $i \in [n]$. The following conditions are equivalent:*

(1) $V(P_1, \dots, P_n) < V(Q_1, \dots, Q_n)$,
(2) *there exists a fully mixed $(2n-1)$-dimensional simplex $\mathcal{C}(E_1, \dots, E_n)$ contained in the relative closure of $\mathcal{C}(Q_1, \dots, Q_n) \setminus \mathcal{C}(P_1, \dots, P_n)$,*
(3) *there exists a non-zero vector $u \in \mathbb{R}^n$ such that the collection $\{B_{1,u}, \dots, B_{n,u}\}$ is essential.*

Proof. First we note that if $\dim(Q_1 + \dots + Q_n) < n$, then none of the conditions (1), (2) and (3) holds. Indeed, if $\dim(Q_1 + \dots + Q_n) < n$, then $V(P_1, \dots, P_n) = V(Q_1, \dots, Q_n) = 0$. Moreover, $\dim \mathcal{C}(Q_1, \dots, Q_n) < 2n-1$ by (2.5) and, thus, $\mathcal{C}(Q_1, \dots, Q_n)$ cannot contain a $(2n-1)$-dimensional simplex. Finally, (3) does not hold since otherwise $Q_1 + \dots + Q_n$ would contain a fully mixed polytope which has dimension n. When $(P_1, \dots, P_n) = (Q_1, \dots, Q_n)$, we also conclude that none of the conditions (1), (2) and (3) holds for obvious reasons.

Assume now that $(P_1, \dots, P_n) \ne (Q_1, \dots, Q_n)$ and $\dim(Q_1 + \dots + Q_n) = n$. Write $(\mathbf{0}, \mathbf{1})$ for the vector (u, v) with $u = (0, \dots, 0) \in \mathbb{R}^n$ and $v = (1, \dots, 1) \in \mathbb{R}^n$. Then $\mathcal{C}(Q_1, \dots, Q_n)$ has dimension $2n-1$ and its affine span is a hyperplane orthogonal to $(\mathbf{0},\mathbf{1})$, see Remark 4.2. Consider a fully mixed simplex $\mathcal{C}_E = \mathcal{C}(E_1,\dots,E_n) \subset \mathcal{C}(Q_1,\dots,Q_n)$. Here $E_1,\dots,E_n$ are segments with linearly independent directions contained in $Q_1,\dots,Q_n$, respectively, and $\mathcal{C}_E$ is the convex hull of the union $\cup_{i=1}^n E_i \times \{e_i\}$, by Lemma 2.4. Since $\mathcal{C}(P_1,\dots,P_n)$ and $\mathcal{C}_E$ are convex sets contained in a hyperplane orthogonal to $(\mathbf{0},\mathbf{1})$, the simplex $\mathcal{C}_E$ is contained in the relative closure of $\mathcal{C}(Q_1,\dots,Q_n)\setminus\mathcal{C}(P_1,\dots,P_n)$ if and only if there is a vector $(u,v)\in\mathbb{R}^n\times\mathbb{R}^n$ such that (u,v) and $(\mathbf{0},\mathbf{1})$ are not collinear and

$$\mathcal{C}_E \subset \{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n \mid \langle u,x\rangle+\langle v,y\rangle \geq h_{\mathcal{C}(P_1,\dots,P_n)}(u,v)\}. \tag{4.1}$$

Note that the hyperplane defined by $\langle u,x\rangle+\langle v,y\rangle = h_{\mathcal{C}(P_1,\dots,P_n)}(u,v)$ is a supporting hyperplane of $\mathcal{C}(P_1,\dots,P_n)$ and (4.1) means that this hyperplane separates $\mathcal{C}(P_1,\dots,P_n)$ and $\mathcal{C}_E$. Recall that $\dim \mathcal{C}_E = 2n-1$ and thus $\mathcal{C}_E$ cannot be contained in this supporting hyperplane since otherwise it would be contained in the intersection of two distinct hyperplanes. Since C_E is the convex-hull of the union of the polytopes $E_i\times\{e_i\}$ for $1\leq i\leq n$, we see that (4.1) is equivalent to

$$E_i \subset \{x\in\mathbb{R}^n \mid \langle u,x\rangle \geq h_{\mathcal{C}(P_1,\dots,P_n)}(u,v)-v_i,\ 1\leq i\leq n\}. \tag{4.2}$$

By Lemma 4.1, we have $h_{\mathcal{C}(P_1,\dots,P_n)}(u,v)=\max\{h_{P_i}(u)+v_i \mid 1\leq i\leq n\}$. For $u=0$ equation (4.2) implies $v_1=\dots=v_n$, which contradicts the fact that (u,v) and $(\mathbf{0},\mathbf{1})$ are not collinear (note that $h_{P_i}(u)=0$ when $u=0$). Therefore, if (4.2) holds, then $u\neq 0$. Moreover, we get $E_i\subset B_{i,u}$ for $1\leq i\leq n$ since $h_{\mathcal{C}(P_1,\dots,P_n)}(u,v)-v_i\geq h_{P_i}(u)$. Consequently, (4.2) implies that $\{B_{1,u},\dots,B_{n,u}\}$ is essential, as the E_i have linearly independent directions (see Theorem 2.2.) We have proved the implication (2) $\Rightarrow$ (3).

Assume (3) holds and let $u\in\mathbb{R}^n$ be a non-zero vector such that $\{B_{1,u},\dots,B_{n,u}\}$ is essential. Let $E_i\subset B_{i,u}$, for $1\leq i\leq n$, be segments with linearly independent directions and choose $v=(v_1,\dots,v_n)\in\mathbb{R}^n$ such that $h_{P_1}(u)+v_1=\dots=h_{P_n}(u)+v_n$. Then, by Lemma 4.1, $h_{\mathcal{C}(P_1,\dots,P_n)}(u,v)=h_{P_i}(u)+v_i$ for $1\leq i\leq n$ and (4.2) follows from $E_i\subset B_{i,u}$, $1\leq i\leq n$. Since (4.2) is equivalent to (2), we conclude that (3) $\Rightarrow$ (2).

Now assume that (2) holds. Then, there exists a fully mixed $(2n-1)$-dimensional simplex $\mathcal{C}_E$ satisfying (4.1). It follows then from Lemma 4.3 that there exists a triangulation of $\mathcal{C}(Q_1,\dots,Q_n)$ with vertices in $\cup_{i=1}^n Q_i\times\{e_i\}$ which contains $\mathcal{C}_E$ and restricts to a triangulation of $\mathcal{C}(P_1,\dots,P_n)$. Indeed, we may apply part (2) of Lemma 4.3 to the set of vertices $\mathcal{A}_1$ of C_E and to the set of vertices $\mathcal{A}_2$ of $\mathcal{C}(P_1,\dots,P_n)$. Taking for τ_1 the trivial triangulation of C_E and for τ_2 any regular triangulation induced by a generic function h_2, we get the existence of a regular triangulation of the convex hull of $\mathcal{A}=\mathcal{A}_1\cup\mathcal{A}_2$ which contains C_E and τ_2. Applying part (1) of Lemma 4.3 to $\mathcal{A}\subset\mathcal{A}\cup V$ where V is the set of vertices of $C(Q_1,\dots,Q_n)$ gives a regular triangulation of $\mathcal{C}(Q_1,\dots,Q_n)$ as required. By the combinatorial Cayley trick, this corresponds to a pure mixed subdivision τ_Q of $Q_1+\dots+Q_n$ restricting to a pure mixed subdivision τ_P of $P_1+\dots+P_n$ and a fully mixed polytope E contained in $\tau_Q\setminus\tau_P$. Therefore, $V(P_1,\dots,P_n)<V(Q_1,\dots,Q_n)$ by Lemma 2.6. This proves the implication (2) $\Rightarrow$ (1).

Assume now that $V(P_1,\dots,P_n)<V(Q_1,\dots,Q_n)$. Then, as in the proof of Theorem 3.3, there exist $u\in\mathbb{S}^{n-1}$ and $1\leq k\leq n$ such that $\{P_1^u,\dots,P_{k-1}^u,Q_{k+1}^u,\dots,Q_n^u\}$ is essential and P_k does not touch Q_k^u. By choosing a segment in $B_{k,u}$ not parallel

to the hyperplane $u^\perp$, which exists since P_k does not touch Q_k^u, we conclude that $\{P_1^u, \dots, P_{k-1}^u, B_{k,u}, Q_{k+1}^u, \dots, Q_n^u\}$ is essential as well. It remains to note that P_i^u and Q_i^u are contained in $B_{i,u}$ for $i \neq k$, $1 \leq i \leq n$. We have proved the implication $(1) \Rightarrow (2)$.

Remark 4.5. Note that if P_i touches Q_i^u then $B_{i,u} = Q_i^u$ and if P_i does not touch Q_i^u then $\dim B_{i,u} = \dim Q_i$. Therefore, the condition (3) in the above theorem is equivalent to the condition in Theorem 3.3.

5. Polynomial systems

Consider a finite set $\mathcal{A} = \{a_1, \dots, a_\ell\} \subset \mathbb{Z}^n$ where $\ell = |\mathcal{A}|$. Let $(a_{1j}, \dots, a_{nj})$ be the coordinates of a_j for $1 \leq j \leq \ell$. Consider a Laurent polynomial system with coefficients in an algebraically closed field $\mathbb{K}$

$$f_1(x) = \cdots = f_n(x) = 0, \tag{5.1}$$

where $f_i(x) = \sum_{j=1}^{\ell} c_{ij} x^{a_j}$ and x^{a_j} stands for the monomial $x_1^{a_{1j}} \cdots x_n^{a_{nj}}$. We assume that no polynomial f_i is the zero polynomial. Call $\mathcal{A}_i = \{a_j \in \mathcal{A} \mid c_{ij} \neq 0\}$ the *individual support* of f_i. We may assume that for any $1 \leq j \leq \ell$ there exists i such that $c_{ij} \neq 0$. Then $\mathcal{A} = \cup_{i=1}^{n} \mathcal{A}_i$ is called the *total support* of the system (5.1). The *Newton polytope* P_i of f_i is the convex hull of $\mathcal{A}_i$ and the *Newton polytope* Q of the system (5.1) is the convex hull of $\mathcal{A}$.

The matrices

$$C = (c_{ij}) \in \mathbb{K}^{n\times \ell} \text{ and } A = (a_{ij}) \in \mathbb{Z}^{n\times \ell}$$

are the *coefficient* and *exponent* matrices of the system, respectively.

Choose $u \in \mathbb{S}^{n-1}$ and let $\mathcal{A}_i^u = P_i^u \cap \mathcal{A}_i$. Then the *restricted system* corresponding to u is the system

$$f_1^u(x) = \cdots = f_n^u(x) = 0,$$

where $f_i^u(x) = \sum_{j=1}^{\ell} c_{ij}^u x^{a_j}$ with $c_{ij}^u = c_{ij}$ if $a_j \in \mathcal{A}_i^u$ and $c_{ij}^u = 0$, otherwise. Finally, a system (5.1) is called *non-degenerate* if for every $u \in \mathbb{S}^{n-1}$ the corresponding restricted system is inconsistent.

The relation between mixed volumes and polynomial systems originates in the following fundamental result, known as the BKK bound, discovered by Bernstein, Kushnirenko, and Khovanskii, see [1, 6, 7].

Theorem 5.1. *The system* (5.1) *has at most* $n!\,V(P_1, \dots, P_n)$ *isolated solutions in* $(\mathbb{K}^*)^n$ *counted with multiplicity. Moreover, it has precisely* $n!\,V(P_1, \dots, P_n)$ *solutions in* $(\mathbb{K}^*)^n$ *counted with multiplicity if and only if it is non-degenerate.*

Remark 5.2. Systems with fixed individual supports and generic coefficients are non-degenerate. Moreover, the non-degeneracy condition is not needed if one passes to the toric compactification X_P associated with the polytope $P = P_1 + \cdots + P_n$. Namely, a system has at most $n!\,V(P_1, \dots, P_n)$ isolated solutions in X_P counted with multiplicity, and if it has a finite number of solutions in X_P then this number equals $n!\,V(P_1, \dots, P_n)$ counted with multiplicity.

There are two operations on (5.1) that preserve its number of solutions in the torus $(\mathbb{K}^*)^n$: Left multiplication of C by an element of $\mathrm{GL}_n(\mathbb{K})$ and left multiplication of the *augmented exponent matrix*

$$\bar{A} = \begin{pmatrix} 1 & \cdots & 1 \\ & A & \end{pmatrix} \in \mathbb{Z}^{(n+1)\times \ell}$$

by a matrix in $\mathrm{GL}_{n+1}(\mathbb{Z})$ whose first row is $(1,0,\dots,0)$. The first operation obviously produces an equivalent system. The second operation amounts to applying an invertible affine transformation with integer coefficients $a \mapsto b + \ell(a)$ on the total support $\mathcal{A} \subset \mathbb{Z}^n$. Here $b \in \mathbb{Z}^n$ is a translation vector and $\ell : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map whose matrix with respect to the standard basis belongs to $\mathrm{GL}_n(\mathbb{Z})$. A basic result of toric geometry says that there is a monomial change of coordinates $x \mapsto y$ of the torus $(\mathbb{K}^*)^n$ so that $x^a = y^{\ell(a)}$ for any $a \in (\mathbb{K}^*)^n$. Moreover, translating $\mathcal{A}$ by b amounts to multiplying each equation of (5.1) by the monomial x^b. Thus starting from (5.1), we get a system with same coefficient matrix, total support $b+\ell(\mathcal{A})$, and the same number of solutions in $(\mathbb{K}^*)^n$.

Remark 5.3. Consider a non-degenerate system with coefficient matrix C. While left multiplication of C by an invertible matrix does not preserve the individual supports and Newton polytopes in general, it preserves the total support of the system. Indeed, since $\mathcal{A}$ is the total support of the system, no column of C is zero, and thus no column of C can become zero after left multiplication by an invertible matrix.

Example 5.4. Assume that (5.1) has precisely $n!\,\mathrm{Vol}_n(Q)$ solutions in $(\mathbb{K}^*)^n$ counted with multiplicity and C has a non-zero maximal minor. Up to renumbering, we may assume that this minor is given by the first n columns of C. Left multiplication of (5.1) by the inverse of the corresponding submatrix of C gives an equivalent system with invidual supports $\mathcal{A}'_i \subset \mathcal{A}''_i = (\mathcal{A} \setminus \{a_1,\dots,a_n\}) \cup \{a_i\}$ for $1 \le i \le n$. Thus, this new system has precisely $n!\,\mathrm{Vol}_n(Q)$ solutions in $(\mathbb{K}^*)^n$ counted with multiplicity. By Theorem 5.1 this number of solutions is at most $n!\,V(P'_1,\dots,P'_n)$, where P'_i is the convex hull of $\mathcal{A}'_i$. On the other hand, by monotonicity of the mixed volume we have $V(P'_1,\dots,P'_n) \le V(P''_1,\dots,P''_n) \le \mathrm{Vol}_n(Q)$. We conclude that $V(P'_1,\dots,P'_n) = V(P''_1,\dots,P''_n) = \mathrm{Vol}_n(Q)$. The second equality is also a consequence of Corollary 3.7, see Example 3.10.

Theorem 5.5. *Assume* $\dim Q = n$. *If a system* (5.1) *has* $n!\,\mathrm{Vol}_n(Q)$ *isolated solutions in* $(\mathbb{K}^*)^n$ *counted with multiplicity, then for any proper face* F *of* Q *the submatrix* $C_{\mathcal{F}} \in \mathbb{K}^{n\times|\mathcal{F}|}$ *of* C *with columns indexed by* $\mathcal{F} = \{j \in [\ell]\,,\, a_j \in F \cap \mathcal{A}\}$ *satisfies*

$$\operatorname{rank} C_{\mathcal{F}} \ge \dim F + 1,$$

or equivalently,

$$\operatorname{rank} C_{\mathcal{F}} \ge \operatorname{rank} \bar{A}_{\mathcal{F}}. \tag{5.2}$$

Conversely, if (5.2) *is satisfied for all proper faces* F *of* Q *and if the system* (5.1) *is non-degenerate, then it has precisely* $n!\,\mathrm{Vol}_n(Q)$ *isolated solutions in* $(\mathbb{K}^*)^n$ *counted with multiplicity.*

Proof. First, note that for any proper face F of Q we have $\operatorname{rank} \bar{A}_{\mathcal{F}} = \dim F + 1$. Consider a proper face F of Q of codimension $s \geq 1$ and assume that $\operatorname{rank} C_{\mathcal{F}} \leq \dim F = n - s$. Then there exist an invertible matrix L and $I \subset [n]$ of size $|I| = s$ such that the submatrix of $C' = LC$ with rows indexed by I and columns indexed by $\mathcal{F}$ is the zero matrix. The matrix C' is the coefficient matrix of an equivalent system with the same total support, see Remark 5.3. Denote by $P'_1, \ldots, P'_n$ the individual Newton polytopes of this equivalent system. Then the polytopes P'_i for $i \in I$ do not touch the face F of Q, as I corresponds to the zero submatrix of C'. Since $\dim F = n - s$ and $|I| = s$, it follows then from Corollary 3.7 that $V(P'_1, \ldots, P'_n) < \operatorname{Vol}_n(Q)$. Theorem 5.1 applied to the system with coefficient matrix C' gives that it has at most $n! V(P'_1, \ldots, P'_n) < n! \operatorname{Vol}_n(Q)$ isolated solutions in $(\mathbb{K}^*)^n$ counted with multiplicity. The same conclusion holds for the equivalent system (5.1). Therefore, if (5.1) has $n! \operatorname{Vol}_n(Q)$ isolated solutions in $(\mathbb{K}^*)^n$ counted with multiplicity, then (5.2) is satisfied for all proper faces F of Q.

Conversely, assume that (5.1) is non-degenerate and that (5.2) is satisfied for all proper faces F of Q. Then (5.1) has $n!\ V(P_1, \ldots, P_n)$ isolated solutions in $(\mathbb{K}^*)^n$ counted with multiplicity by Theorem 5.1. Suppose that $V(P_1, \ldots, P_n) < \operatorname{Vol}_n(Q)$. Then by Corollary 3.7 there exists a proper face F of Q of codimension $s \geq 1$ and $I \subset [n]$ of size $|I| = s$ such that the polytopes P_i for $i \in I$ do not touch F. But then $\operatorname{rank} C_{\mathcal{F}} \leq n - s = \dim F$, which gives a contradiction. Thus $V(P_1, \ldots, P_n) = \operatorname{Vol}_n(Q)$ and (5.2) has $n! \operatorname{Vol}_n(Q)$ isolated solutions in $(\mathbb{K}^*)^n$ counted with multiplicity. □

As an immediate consequence of Theorem 5.5 from which we keep the notation, we get the following corollary.

Corollary 5.6. *Consider any Laurent polynomial system* (5.1) *with* $\dim Q = n$. *If there exists a proper face* F *of* Q *such that* $\operatorname{rank} C_{\mathcal{F}} < \operatorname{rank} \bar{A}_{\mathcal{F}}$, *then the system has either infinitely many solutions or strictly less than* $n! \operatorname{Vol}_n(Q)$ *solutions in* $(\mathbb{K}^*)^n$ *counted with multiplicity.*

Proof. Assume the existence of a proper face F of Q such that $\operatorname{rank} C_{\mathcal{F}} < \operatorname{rank} \bar{A}_{\mathcal{F}}$. If (5.1) has precisely $n! \operatorname{Vol}_n(Q)$ solutions in $(\mathbb{K}^*)^n$ counted with multiplicity, then it is non-degenerate by Theorem 5.1 and thus $\operatorname{rank} C_{\mathcal{F}} \geq \operatorname{rank} \bar{A}_{\mathcal{F}}$ by Theorem 5.5, a contradiction. □

A very nice consequence of Theorem 5.5 is the following result, which can be considered as a generalization of Cramer's rule to polynomial systems.

Corollary 5.7. *Assume that* $\dim Q = n$ *and that no maximal minor of* C *vanishes. Then the system* (5.1) *has the maximal number of* $n!\ \operatorname{Vol}_n(Q)$ *isolated solutions in* $(\mathbb{K}^*)^n$ *counted with multiplicity.*

Proof. Note that $\ell \geq n + 1$ since $\dim Q = n$ (recall that ℓ is the number of columns of C). Thus a maximal minor of C has size n and the fact that no maximal minor of C vanishes implies that for any $J \subset [\ell]$ the submatrix of C with rows indexed by $[n]$ and columns indexed by J has maximal rank. This rank is equal to n if $|J| \geq n$ or to $|J|$ if $|J| < n$. Since $|\mathcal{F}| \geq \dim F + 1 = \operatorname{rank} \bar{A}_{\mathcal{F}}$ for any face F of Q, we get that $\operatorname{rank} C_{\mathcal{F}} \geq \operatorname{rank} \bar{A}_{\mathcal{F}}$ for any proper face F of Q. Moreover, no restricted system is consistent for otherwise this would give a non-zero vector in the kernel of the corresponding submatrix of C. Thus (5.1) is non-degenerate and the result follows from Theorem 5.5. $\square$

When the polytope $Q = \operatorname{conv}\{0, e_1, \ldots, e_n\}$ is the standard simplex, the system (5.1) is linear and it has precisely $n!\,\operatorname{Vol}_n(Q) = 1$ solution in $(\mathbb{K}^*)^n$ if and only if no maximal minor of $C \in \mathbb{K}^{n\times(n+1)}$ vanishes, in accordance with Cramer's rule for linear systems.

6. Examples

We conclude with a few examples illustrating the results of the previous section.

Example 6.1. Let $\mathcal{A}_1 = \{(0,0),(1,2),(2,1)\}$ and $\mathcal{A}_2 = \{(2,0),(0,1),(1,2)\}$ be individual supports, and $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ the total support of a system. The Newton polytopes $P_1 = \operatorname{conv}\mathcal{A}_1$, $P_2 = \operatorname{conv}\mathcal{A}_2$, and $Q = \operatorname{conv}\mathcal{A}$ are depicted in Figure 1, where the vertices of P_1 and P_2 are labeled by $\{1,2,3\}$ and $\{4,5,2\}$, respectively. We use the labeling in Figure 1 to order the columns of the augmented matrix

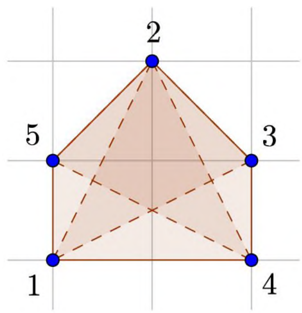

Figure 1. The mixed volume of the two triangles equals the volume of the pentagon.

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 0 \\ 0 & 2 & 1 & 0 & 1 \end{pmatrix}.$$

A general system with these supports has the following coefficient matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 \\ 0 & c_{22} & 0 & c_{24} & c_{25} \end{pmatrix},$$

where $c_{ij} \in \mathbb{K}$ are non-zero. One can see that each edge of Q is touched by both P_1 and P_2, hence, $V(P_1, P_2) = \operatorname{Vol}_2(Q)$, see Example 3.8. Also one can check that the rank conditions $\operatorname{rank} C_{\mathcal{F}} \geq \operatorname{rank} \bar{A}_{\mathcal{F}}$ are satisfied for every face F of Q. (In fact, both ranks equal 2 when F is an edge and 1 when F is a vertex.)

Example 6.2. Now we modify the previous example slightly, keeping $\mathcal{A}_1$ the same and changing one of the points in $\mathcal{A}_2$, so $\mathcal{A}_2 = \{(2,0),(0,1),(1,1)\}$, see Figure 2. The augmented exponent matrix and the coefficient matrix are as follows.

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 & 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{24} & c_{25} & c_{26} \end{pmatrix}.$$

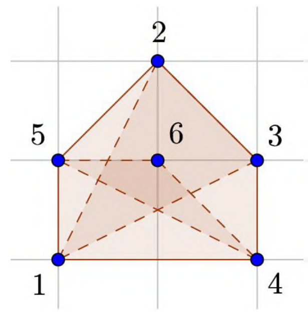

FIGURE 2. The mixed volume of the two triangles is less than the volume of the pentagon.

This time the edge of Q labeled by $\mathcal{F} = \{2,3\}$ is not touched by P_2 and, hence, $V(P_1, P_2) < \mathrm{Vol}_2(Q)$. Also, the rank condition for $\mathcal{F} = \{2,3\}$ fails: $\mathrm{rank}\, C_{\mathcal{F}} = 1$ and $\mathrm{rank}\, \bar{A}_{\mathcal{F}} = 2$.

Example 6.3. Consider a system defined by the following augmented exponent matrix and coefficient matrix

$$\bar{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 3 & 5 & 1 & -2 & 2 \\ 1 & 1 & -3 & 3 & 1 & -1 \\ 1 & 3 & 1 & 3 & -1 & 1 \end{pmatrix}.$$

Here $P_1 = P_2 = P_3 = Q$ which is a prism depicted in Figure 3. We label the vertices of Q using the order of the columns in $\bar{A}$. The submatrix of C corresponding

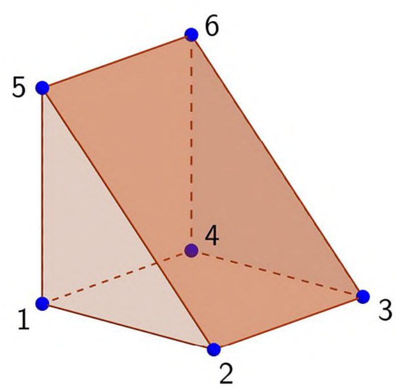

FIGURE 3. The Newton polytope of the system in Example 6.3.

to the edge F labeled $\{5,6\}$ has rank 1 which is less than $\dim F + 1$. Therefore the associated system has less than $3!\,\mathrm{Vol}_3(Q) = 3$ isolated solutions in $(\mathbb{C}^*)^3$. (In fact, it has two solutions.) In particular, this is a degenerate system.

In the following very particular situation, the rank condition (5.2) in Theorem 5.5 implies the non-degeneracy of the system.

Remark 6.4. Assume that $P_1 = P_2 = \cdots = P_n = Q$ with $\dim Q = n$ and any proper face F of Q is a simplex which intersects $\mathcal{A}$ only at its vertices. Assume furthermore that $\operatorname{rank} C_{\mathcal{F}} \geq \operatorname{rank} \bar{A}_{\mathcal{F}}$ for any proper face F of Q. Then (5.1) is non-degenerate and thus has precisely $n!\ \mathrm{Vol}_n(Q)$ solutions in $(\mathbb{K}^*)^n$ counted with multiplicity according to Theorem 5.1. Indeed, if F is a proper face of Q, then the corresponding restricted system has total support $F \cap \mathcal{A}$. If this restricted system is consistent, then there is a non-zero vector in the kernel of $C_{\mathcal{F}}$ and thus $\operatorname{rank} C_{\mathcal{F}} < |F \cap \mathcal{A}| = 1 + \dim F$ which gives a contradiction.

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