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CHARACTERIZATION OF SIMPLICES VIA THE BEZOUT INEQUALITY FOR MIXED VOLUMES

CHRISTOS SAROGLOU, IVAN SOPRUNOV, AND ARTEM ZVAVITCH

ABSTRACT. We consider the following Bezout inequality for mixed volumes:

$$V(K_1,\dots,K_r,\Delta[n-r])V_n(\Delta)^{r-1} \le \prod_{i=1}^{r} V(K_i,\Delta[n-1]) \text{ for } 2 \le r \le n.$$

It was shown previously that the inequality is true for any n-dimensional simplex Δ and any convex bodies $K_1,\dots,K_r$ in $\mathbb{R}^n$. It was conjectured that simplices are the only convex bodies for which the inequality holds for arbitrary bodies $K_1,\dots,K_r$ in $\mathbb{R}^n$. In this paper we prove that this is indeed the case if we assume that Δ is a convex polytope. Thus the Bezout inequality characterizes simplices in the class of convex n-polytopes. In addition, we show that if a body Δ satisfies the Bezout inequality for all bodies $K_1,\dots,K_r$ then the boundary of Δ cannot have *strict* points. In particular, it cannot have points with positive Gaussian curvature.

1. INTRODUCTION

It was noticed in [SZ] that the classical Bezout inequality in algebraic geometry [F, Sec. 8.4] together with the Bernstein–Kushnirenko–Khovanskii bound [B, Ku, Kh] produces a new inequality involving mixed volumes of convex bodies:

$$V(K_1,\dots,K_r,\Delta[n-r])V_n(\Delta)^{r-1} \le \prod_{i=1}^{r} V(K_i,\Delta[n-1]) \text{ for } 2 \le r \le n. \tag{1.1}$$

Here Δ is an n-dimensional simplex and $K_1,\dots,K_r$ are arbitrary convex bodies in $\mathbb{R}^n$. Throughout the paper $V_n(K)$ denotes the n-dimensional Euclidean volume of a body K and $V(K_1,\dots,K_n)$ denotes the n-dimensional mixed volume of bodies $K_1,\dots,K_n$. Furthermore, $K[m]$ indicates that the body K is repeated m times in the expression for the mixed volume.

In [SZ] it was conjectured that the Bezout inequality characterizes simplices, that is if Δ is a convex body such that (1.1) holds for all convex bodies $K_1,\dots,K_r$ then Δ is necessarily a simplex (see [SZ, Conjecture 1.2]). It was proved that Δ has to be indecomposable (see [SZ, Theorem 3.3]) which, in particular, confirms the conjecture in dimension $n=2$. In the present paper we prove this conjecture for the class of convex polytopes.

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Theorem 1.1. *Fix $2 \le r \le n$. Let Δ be a convex n-dimensional polytope in $\mathbb{R}^n$ satisfying (1.1) for all convex bodies $K_1, \dots, K_r$ in $\mathbb{R}^n$. Then Δ is a simplex.*

Although the above theorem covers a most natural class of convex bodies, in full generality the conjecture remains open. Going outside of the class of polytopes we show that if a convex body Δ satisfies (1.1) for all convex bodies $K_1, \dots, K_r$ in $\mathbb{R}^n$ then Δ cannot have strict points. We say a boundary point $x \in K$ is a *strict point* if x does not belong to any segment contained in the boundary of K.

Theorem 1.2. *Fix $2 \le r \le n$. Let Δ be an n-dimensional convex body in $\mathbb{R}^n$ satisfying (1.1) for all convex bodies $K_1, \dots, K_r$ in $\mathbb{R}^n$. Then Δ does not contain any strict points.*

In particular, we see that Δ cannot have points with positive Gaussian curvature.

Let us say a few words about the idea behind the proofs of Theorems 1.1 and 1.2. First, note that it is enough to prove the theorems in the case of $r = 2$ as this implies the general statement. Thus we are going to restate (1.1) for $r = 2$ as follows

$$V(L, M, K[n-2])V_n(K) \le V(L, K[n-1])V(M, K[n-1]), \tag{1.2}$$

where L and M are convex bodies and K is a polytope. The fact that there is equality in (1.2) when $L = K$ allows us to see this as a variational problem, by fixing an appropriate body M and using an appropriate deformation $L = K_t$ of K. In the case of Theorem 1.1, K_t is obtained from K by moving one of its facets along the direction of its normal unit vector. In the case of Theorem 1.2, K_t is obtained from K by cutting out a small cup in a neighborhood of a strict point.

2. Preliminaries

In this section we collect basic definitions and set up notation. As a general reference on the theory of convex sets and mixed volumes we use R. Schneider's book "Convex bodies: the Brunn-Minkowski theory" [Sch].

A *convex body* is a non-empty convex compact set. A *(convex) polytope* is the convex hull of a finite set of points. An n-dimensional polytope is called an n-*polytope* for short. For $x, y \in \mathbb{R}^n$ we write $\langle x, y\rangle$ for the inner product of x and y. We use $\mathbb{S}^{n-1}$ to denote the $(n-1)$-dimensional unit sphere and $B(x, \delta)$ to denote the closed Euclidean ball of radius $\delta > 0$ centered at $x \in \mathbb{R}^n$.

For a convex body K the function $h_K : \mathbb{S}^{n-1} \to \mathbb{R}$, $h_K(u) = \max\{\langle x, u\rangle \mid x \in K\}$ is the *support function* of K. For every $u \in \mathbb{S}^{n-1}$ we write $H_K(u)$ to denote the supporting hyperplane for K with outer normal u

$$H_K(u) = \{x \in \mathbb{R}^n : \langle x, u\rangle = h_K(u)\}.$$

Furthermore, we use K^u to denote the face $K \cap H_K(u)$ of K.

Let β be a subset of the boundary ∂K of a convex body K. The *spherical image* $\sigma(K, \beta)$ of β with respect to K is defined by

$$\sigma(K, \beta) = \{u \in \mathbb{S}^{n-1} : \exists x \in \beta, \text{ such that } \langle x, u\rangle = h_K(u)\}.$$

If Ω is a subset of $\mathbb{S}^{n-1}$ define the *inverse spherical image* $\tau(K, \Omega)$ of Ω with respect to K by

$$\tau(K, \Omega) = \big\{x \in \partial K : \exists u \in \Omega, \text{ such that } \langle x, u\rangle = h_K(u)\big\}.$$

The *surface area measure* $S(K,\cdot)$ of K (viewed as a measure on $\mathbb{S}^{n-1}$) is defined as

$$S(K,\Omega)=\mathcal{H}^{n-1}\big(\tau(K,\Omega)\big),\quad \text{for } \Omega \text{ a Borel subset of } \mathbb{S}^{n-1}.$$

Here $\mathcal{H}^{n-1}(\cdot)$ stands for the $(n-1)$-dimensional Hausdorff measure.

Let $V(K_1,\dots,K_n)$ denote the n-dimensional mixed volume of n convex bodies $K_1,\dots,K_n$ in $\mathbb{R}^n$. We write $V(K_1[m_1],\dots,K_r[m_r])$ for the mixed volume of the bodies $K_1,\dots,K_r$ where each K_i is repeated m_i times and $m_1+\dots+m_r=n$. In particular, $V(K[n])=V_n(K)$, the n-dimensional Euclidean volume of K.

Let $S(K_1,\dots,K_{n-1},\cdot)$ be the *mixed area measure* for bodies $K_1,\dots,K_{n-1}$ defined by

$$V(L,K_1,\dots,K_{n-1})=\frac{1}{n}\int_{\mathbb{S}^{n-1}} h_L dS(K_1,\dots,K_{n-1},\cdot)$$

for any compact convex set L. In particular, when the K_i are polytopes the mixed area measure $S(K_1,\dots,K_{n-1},\cdot)$ has finite support and for every $u\in\mathbb{S}^{n-1}$ we have

$$S(K_1,\dots,K_{n-1},u)=V(K_1^u,\dots,K_{n-1}^u), \tag{2.1}$$

where $V(K_1^u,\dots,K_{n-1}^u)$ is the $(n-1)$-dimensional mixed volume of the faces K_i^u translated the the subspace orthogonal to u, see [Sch, Sec 5.1].

Finally, for $u\in\mathbb{S}^{n-1}$ the orthogonal projection of a set $A\subset\mathbb{R}^n$ onto the subspace $u^\perp$ orthogonal to u is denoted by $A|u^\perp$.

3. Proof of Theorem 1.1

In this section we give a proof of Theorem 1.1. As mentioned in the introduction, it is enough to prove it for $r=2$ in which case we write the Bezout inequality as

$$V(L,M,K[n-2])V_n(K)\le V(L,K[n-1])V(M,K[n-1]). \tag{3.1}$$

We assume that L, M are arbitrary convex bodies and K is a polytope in $\mathbb{R}^n$.

We need to set up additional notation. Let K be defined by inequalities

$$K=\bigcap_{j=1}^{N}\{x\in\mathbb{R}^n:\langle x,u_j\rangle\le h_K(u_j)\},$$

where u_j are the outer normals to the facets of K (in some fixed order) and N is the number of facets of K. Denote by $K_{t,i}$ the polytope obtained by moving the i-th facet of K by t, that is

$$K_{t,i}=\bigcap_{\substack{j=1\\ j\ne i}}^{N}\{x\in\mathbb{R}^n:\langle x,u_j\rangle\le h_K(u_j)\}\bigcap\{x\in\mathbb{R}^n:\langle x,u_i\rangle\le h_K(u_i)+t\}.$$

By abuse of notation we let K_t denote $K_{t,N}$.

Lemma 3.1. *Let K and K_t be as above. Then there exists $\delta=\delta(K)$ such that the following supports are equal*

$$\operatorname{supp} S(K_t[r],K[n-1-r],\cdot)=\operatorname{supp} S(K,\cdot)$$

for any $0\le r\le n-1$ and any $t\in(-\delta,\delta)$.

Proof. By (2.1) it is enough to show that $V(K_t^u[r], K^u[n-1-r]) = 0$ if and only if $V_{n-1}(K^u) = 0$, that is K^u is not a facet of K. Indeed, by choosing δ small enough we can ensure that K_t has the same facet normals as K and so $\dim K_t^u = n-1$ whenever K^u is a facet of K. In this case $V(K_t^u[r], K^u[n-1-r]) > 0$.

Conversely, assume K^u is a face of K of dimension less than $n-1$. As before, for small enough t the face K_t^u also has dimension less than $n-1$. First, suppose K^u is not contained in the moving facet $F = K \cap H_K(u_N)$. Then $h_K(u) = h_{K_t}(u)$ and so $K^u \subseteq K_t^u$ for $t \geq 0$ and $K^u \supseteq K_t^u$ for $t < 0$. Then, by the monotonicity of the mixed volume, if $t \geq 0$ then

$$0 \leq V(K_t^u[r], K^u[n-1-r]) \leq V_{n-1}(K_t^u) = 0,$$

and so $V(K_t^u[r], K^u[n-1-r]) = 0$. The case $t < 0$ is similar.

Now suppose K^u is contained in the moving facet F. Then $K^u \subseteq H_K(u) \cap H_K(u_N)$ and $K_t^u \subseteq H_{K_t}(u) \cap H_{K_t}(u_N)$. This shows that K^u and K_t^u are contained in two affine $(n-2)$-dimensional subspaces which are translates of the same linear subspace of dimension $n-2$. Therefore, for any collection of line segments $(L_1, \dots, L_{n-1})$, where $L_i \subset K_t^u$ for $1 \leq i \leq r$ and $L_i \subset K^u$ for $r+1 \leq i \leq n-1$, the L_i have linearly dependent directions. The latter implies that $V(K_t^u[r], K^u[n-1-r]) = 0$ by [Sch, Theorem 5.1.7].

□

Proposition 3.2. *Let K, P be n-polytopes with the following properties:*

(1) $\operatorname{supp} S(P, \cdot) = \operatorname{supp} S(K, \cdot)$,
(2) *there exists a constant $\lambda > 0$ such that $V(L, P[n-1]) \leq \lambda V(L, K[n-1])$ for all convex bodies L,*
(3) $V(K, P[n-1]) = \lambda V_n(K)$.

Then,

$$S(P, \cdot) = \lambda S(K, \cdot) \ .$$

Proof. As before, let $\{u_1, \dots, u_N\}$ be the outer normals to the facets of K. By assumption (1) they are the outer normals to the facets of P as well. Fix $1 \leq i \leq N$ and let $L = K_{s,i}$ be the polytope obtained from K by moving its i-th facet by a small number $s \in (-\delta_i, \delta_i)$ as in Lemma 3.1.

By assumption (2), for any $s \in (-\delta_i, \delta_i)$ we have

$$V(K_{s,i}, P[n-1]) \leq \lambda V(K_{s,i}, K[n-1]).$$

Consider the function

$$F(s) = \lambda V(K_{s,i}, K[n-1]) - V(K_{s,i}, P[n-1]).$$

Then $F(s) \geq 0$ and $F(0) = 0$. Below we show that $F(s)$ is, in fact, linear on $(-\delta_i, \delta_i)$. But then $F(s)$ is identically zero on $(-\delta_i, \delta_i)$, which implies that

$$V(K_{s,i}, P[n-1]) = \lambda V(K_{s,i}, K[n-1]) \tag{3.2}$$

for all $s \in (-\delta_i, \delta_i)$. We claim that this also implies that

$$S(P, u_i) = \lambda S(K, u_i), \tag{3.3}$$

and since i is chosen arbitrarily and the supports of the two measures are equal, the statement of the proposition follows.

Now we show that $F(s)$ is linear and then prove that (3.2) implies (3.3). Since the polytopes P and K have the same set of facet normals $\{u_1, \dots, u_N\}$, we obtain:

$$\begin{aligned} nV(K_{s,i}, P[n-1]) &= \sum_{j=1}^{N} h_{K_{s,i}}(u_j) V_{n-1}(P^{u_j}) \\ &= \sum_{j=1}^{N} h_K(u_j) V_{n-1}(P^{u_j}) + (h_K(u_i) + s) V_{n-1}(P^{u_i}) \\ &= nV(K, P[n-1]) + sV_{n-1}(P^{u_i}) \\ &= n\lambda V_n(K) + sV_{n-1}(P^{u_i}). \end{aligned} \tag{3.4}$$

Similarly,

$$nV(K_{s,i}, K[n-1]) = nV_n(K) + sV_{n-1}(K^{u_i}). \tag{3.5}$$

Substituting (3.4) and (3.5) into the definition of $F(s)$ and using assumption (3), we see that $F(s) = \lambda s$ for some λ, that is $F(s)$ is linear.

It remains to show that (3.2) implies (3.3). Since $F(s)$ is identically zero we have $\lambda = 0$, which translates to

$$V_{n-1}(P^{u_i}) = \lambda V_{n-1}(K^{u_i}).$$

But that is precisely what (3.3) is stating, which completes the proof of the proposition.

□

Lemma 3.3. *Let K be an n-polytope satisfying (3.1) for all bodies L and for all $M = K_t$ where $t \in (-\delta, \delta)$ as in Lemma 3.1. Then*

$$S(K_t[r], K[n-1-r], \cdot) = \frac{V(K_t, K[n-1])^r}{V_n(K)^r} S(K, \cdot)$$

for all $0 \le r \le n-1$ and all $t \in (-\delta, \delta)$.

Proof. For $0 \le r \le n-1$, set P_r to be the polytope whose surface area measure equals $S(K_t[r], K[n-r-1], \cdot)$ and let $\lambda := V(K_t, K[n-1])/V_n(K)$. For each r the existence and uniqueness of P_r is ensured by the Minkowski Existence and Uniqueness Theorem (see [Sch, Sections 7.1, 7.2]). We need to prove that

$$S(P_r, \cdot) = \lambda^r S(K, \cdot), \qquad r = 0, 1, \dots, n-1. \tag{3.6}$$

Note that by Lemma 3.1, we have:

$$\operatorname{supp} S(P_r, \cdot) = \operatorname{supp} S(K, \cdot), \qquad r = 1, \dots, n-1. \tag{3.7}$$

We prove (3.6) by induction on r. The case $r = 0$ is trivial. For the case $r = 1$ we apply Proposition 3.2 with $P = P_1$. Indeed, by our assumption, (3.1) is satisfied for $M = K_t$ and becomes equality when $L = K$. Thus the conditions (1)–(3) of Proposition 3.2 hold and so $S(P_1, \cdot) = \lambda S(K, \cdot)$, as required.

Now assume (3.6) holds for $1 \le m \le r-1$. This is equivalent to the following:

$$V(L, P_m[n-1]) = \lambda^m V(L, K[n-1]), \tag{3.8}$$

for all convex bodies L and $1 \leq m \leq r-1$. Next fix a convex body $L \subset \mathbb{R}^n$ and apply the Aleksandrov-Fenchel inequality

$$\begin{aligned}
& V(L, P_{r-1}[n-1])^2 = V(L, K_t[r-1], K[n-r])^2 \\
= \ & V(K, K_t, K_t[r-2], K[n-r-1], L)^2 \\
\geq \ & V(K, K, K_t[r-2], K[n-r-1], L) V(K_t, K_t, K_t[r-2], K[n-r-1], L) \\
= \ & V(L, K_t[r-2], K[n-r+1]) V(L, K_t[r], K[n-r-1]) \\
= \ & V(L, P_{r-2}[n-1]) V(L, P_r[n-1]) \ ,
\end{aligned}$$

which, by (3.8) with $m = r-2$ and $m = r-1$, gives

$$\lambda^{2(r-1)} V(L, K[n-1])^2 \geq \lambda^{r-2} V(L, K[n-1]) V(L, P_r[n-1]).$$

Thus

$$V(L, P_r[n-1]) \leq \lambda^r V(K, P_r[n-1]). \tag{3.9}$$

Furthermore, using (3.8) for $m = r-1$, we get:

$$\begin{aligned}
V(K, P_r[n-1]) &= V(K, K_t[r], K[n-1-r]) \\
&= V(K_t, K_t[r-1], K[n-r]) \\
&= V(K_t, P_{r-1}[n-1]) \\
&= \lambda^{r-1} V(K_t, K[n-1]) \\
&= \frac{V(K_t, K[n-1])^{r-1}}{V_n(K)^{r-1}} V(K_t, K[n-1]) = \lambda^r V_n(K). \qquad (3.10)
\end{aligned}$$

Now, as in the case of $r = 1$, (3.7), (3.9), (3.10) together with Proposition 3.2, show that $S(P_r, \cdot) = \lambda^r S(K, \cdot)$, which completes the proof of the lemma. □

Now we are ready to prove the main theorem which implies Theorem 1.1.

Theorem 3.4. *Let K be an n-polytope in $\mathbb{R}^n$. Suppose that*

$$V(L, M, K[n-2]) V_n(K) \leq V(L, K[n-1]) V(M, K[n-1]) \tag{3.11}$$

holds for all convex bodies L and M in $\mathbb{R}^n$. Then K is a simplex.

Proof. Let K_t be the polytope obtained by moving one of the facets of K for t small enough. Then Lemma 3.3 with $r = n-1$ implies that the surface area measures of K_t and K are proportional, and hence, K_t is homothetic to K.

We may assume that one of the vertices of K not lying on the moving facet is at the origin, so $K_t = \lambda K$ for some $\lambda \neq 1$. For every vertex v in K, λv must be a vertex of λK. Therefore, the origin is the only vertex of K not lying on the moving facet. In other words, K is the cone over the moving facet. But since the facet was chosen arbitrarily, for every vertex v the polytope K is the convex hull of v and the facet not containing v. This implies that K is a simplex. □

4. Proof of Theorem 1.2

Recall that a boundary point $y \in \partial K$ is *strict* if it does not belong to any segment contained in ∂K. Note that points with positive Gaussian curvature and, more generally, regular exposed points are strict points (see [Sch] for the definitions). Clearly the boundary of a polytope does not contain any strict points, but there are other convex bodies having this property (for example, a cylinder).

As before it is enough to prove Theorem 1.2 in the case of $r = 2$. It follows from the theorem below.

Theorem 4.1. *Let K be a convex body whose boundary contains at least one strict point. Then there exist convex bodies L and M such that*

$$V(L, M, K[n-2])V_n(K) > V(L, K[n-1])V(M, K[n-1]). \tag{4.1}$$

Proof. First let us fix some notation. For $a > 0$ and $u \in \mathbb{S}^{n-1}$, define the closed half-spaces:

$$H_a^+(u) = \{x \in \mathbb{R}^n : \langle x, u\rangle \geq a\} \quad \text{and} \quad H_a^-(u) = \{x \in \mathbb{R}^n : \langle x, u\rangle \leq a\}.$$

Also set $H_a(u) := H_a^+(u) \cap H_a^-(u)$. With this notation, the supporting hyperplane of K whose unit normal vector is u, can be written as $H_{h_K(u)}(u)$.

Let y be a strict point of ∂K and u be a normal vector of K at y. Choose $v \in \mathbb{S}^{n-1}$, such that $y|v^\perp \in \operatorname{relint}(K|v^\perp)$, where $\operatorname{relint}(K|v^\perp)$ denotes the relative interior of the body $K|v^\perp$ in $v^\perp$. We claim that there exists $\varepsilon > 0$, such that

$$\Big(K \cap H^-_{h_K(u)-\varepsilon}(u)\Big)|v^\perp = K|v^\perp. \tag{4.2}$$

To see this, assume that (4.2) is not true for all $\varepsilon > 0$. This means that for any $\varepsilon > 0$, there exists a point $x_\varepsilon \in \partial K$, such that $x_\varepsilon|v^\perp \in \partial(K|v^\perp)$ and $x_\varepsilon \in H^+_{h_K(u)-\varepsilon}(u)$. Let x_0 be an accumulation point of the set $\{x_\varepsilon : \varepsilon > 0\}$. Then, by compactness, $x_0 \in \partial K$, $x_0|v^\perp \in \partial(K|v^\perp)$, and $x_0 \in H_{h_K(u)}(u)$ (because $x_0 \in H^+_{h_K(u)}(u)$ and $x_0 \in K$). Note that, since $x_0|v^\perp \in \partial(K|v^\perp)$ and $y|v^\perp \in \operatorname{relint}(K|v^\perp)$, we have $x_0 \neq y$. It follows that the segment $[x_0, y]$ is contained in a supporting hyperplane of K, thus $[x_0, y] \subseteq \partial K$, which contradicts the assumption that y is strict. Hence, (4.2) holds for some $\varepsilon > 0$.

Next, set $K_\varepsilon := K \cap H^-_{h_K(u)-\varepsilon}(u)$. Clearly, $h_{K_\varepsilon} \leq h_K$. We claim that there exists an open subset $\beta \subset \partial K \setminus \partial K_\varepsilon$, such that $y \in \beta$ and

$$h_{K_\varepsilon}(u) < h_K(u), \qquad \text{for all } u \in \sigma(K, \beta). \tag{4.3}$$

Suppose not. Then for any δ-neighborhood $\beta_\delta = (\partial K \setminus \partial K_\varepsilon) \cap B(y, \delta)$ of y there exists a unit vector $u_\delta \in \sigma(K, \beta_\delta)$ such that $h_K(u_\delta) = h_{K_\varepsilon}(u_\delta)$. In other words, there exist points $y_\delta \in \beta_\delta$ and $x_\delta \in \partial K_\varepsilon$ lying in the same hyperplane $H_K(u_\delta)$. But then, by compactness, there exist a point $x \in \partial K_\varepsilon$ and a unit vector u, which is normal for K at y and at x. This shows again that the points y and x of K lie in the same supporting hyperplane $H_K(u)$, thus $[y, x]$ is a boundary segment of K, which contradicts our assumption. Therefore, (4.3) holds for some open set $\beta \subseteq \partial K \setminus \partial K_\varepsilon$.

Note, furthermore, that $\tau(K, \sigma(K, \beta)) \supseteq \beta$, thus $\mathcal{H}^{n-1}\big(\tau(K, \sigma(K, \beta))\big) > 0$, which shows that

$$S(K, \sigma(K, \beta)) > 0. \tag{4.4}$$

Now we are ready to exhibit examples of compact convex sets L and M satisfying (4.1). Set $L = [-v, v]$ and $M = K_\varepsilon$. Then, by (5.3.23) in [Sch, p. 294] and applying (4.2) we obtain

$$V(L, M, K[n-2]) = V(K_\varepsilon|v^\perp, K|v^\perp[n-2]) = V_{n-1}(K|v^\perp) = V(L, K[n-1]).$$

On the other hand, by (4.3) and (4.4), we have:

$$\begin{aligned} V(M, K[n-1]) = V(K_\varepsilon, K[n-1]) &= \frac{1}{n}\int_{S^{n-1}} h_{K_\varepsilon} dS(K, \cdot) \\ &< \frac{1}{n}\int_{S^{n-1}} h_K dS(K, \cdot) = V_n(K). \end{aligned}$$

This shows that

$$V(L, M, K[n-2])V_n(K) > V(L, K[n-1])V(M, K[n-1]),$$

as asserted. □

Remark 4.2. One might ask the following: If K is a convex body whose boundary contains at least one strict point x, is it true that ∂K has an open neighborhood that does not contain any line segments, i.e. K is strictly convex in a neighborhood of x? If yes, this would simplify the proof of Theorem 4.1 considerably. The following simple 3-dimensional example shows, however, that this is not the case. Take K equal to

$$\{x \in \mathbb{R}^3 : x_3 \leq 1\} \bigcap \operatorname{conv}\Big(\{(0, x_2, x_3) \in \mathbb{R}^3 : x_3 = x_2^2\} \cup \{(x_1, 0, x_3) \in \mathbb{R}^3 : x_3 = x_1^2\}\Big).$$

Then the origin is a strict point of the boundary of K, but no neighborhood of the origin is strictly convex.

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