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TORIC RESIDUE AND COMBINATORIAL DEGREE

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Abstract. Consider an n-dimensional projective toric variety X defined by a convex lattice polytope P. David Cox introduced the toric residue map given by a collection of $n+1$ divisors $(Z_0, \dots, Z_n)$ on X. In the case when the Z_i are $\mathbb{T}$-invariant divisors whose sum is $X \setminus \mathbb{T}$ the toric residue map is the multiplication by an integer number. We show that this number is the degree of a certain map from the boundary of the polytope P to the boundary of a simplex. This degree can be computed combinatorially. We also study radical monomial ideals I of the homogeneous coordinate ring of X. We give a necessary and sufficient condition for a homogeneous polynomial of semiample degree to belong to I in terms of geometry of toric varieties and combinatorics of fans. Both results have applications to the problem of constructing an element of residue one for semiample degrees.

Introduction

Toric residue is defined for every collection of $n+1$ divisors on an n-dimensional complete toric variety as long as they do not have a common point. It appears in a variety of contexts, e.g. in mirror symmetry [**BM**] or in sparse polynomial systems [**CaD**], [**CDS**]. Toric residues were first introduced by D. Cox in [**C2**] in the case when the divisors are ample and linearly equivalent. In [**CCD**] Cattani, Cox, and Dickenstein extended the definition to the general case and revealed a connection between the toric residue and the sum of local Grothendieck residues in the torus. They also provided an algorithm for computing the toric residue when the divisors are ample using Gröbner bases. Another approach was taken by D'Andrea and Khetan in [**AK**] where they compute the toric residue as a quotient of two determinants. One of the key ideas in both methods is reduction to a particular choice of $n+1$ $\mathbb{T}$-invariant divisors for which the toric residue is one. This motivates the problem of computing the toric residue for an arbitrary choice of $n+1$ $\mathbb{T}$-invariant divisors.

Let X be a projective toric variety of dimension n defined by a convex lattice polytope P. Consider $n+1$ effective $\mathbb{T}$-invariant divisors $Z_0, \dots, Z_n$ on X whose sum Z equals $X \setminus \mathbb{T}$ (or more generally the support of Z equals $X \setminus \mathbb{T}$). Then the toric residue map for $Z_0, \dots, Z_n$ has a nice combinatorial description: Notice that the irreducible components of the Z_i correspond to the facets of the polytope P. Thus, we get a "coloring" of the facets of P into $n+1$ colors: a facet has color i if the corresponding irreducible divisor appears in Z_i. Let us now pick a color $0 \le i \le n$ for each facet of the $n+1$ facets of the standard n-simplex Δ, so that all facets have different colors. Choose a continuous piecewise linear map $f : \partial P \to \partial \Delta$ that matches the colors. Such a map exists as long as $Z_0 \cap \cdots \cap Z_n$ is empty. The degree of f is independent of the choice of f and is called the combinatorial degree of the coloring of P (we give the precise definition in Section 1). We prove that the toric residue for $Z_0, \ldots, Z_n$ coincides with the combinatorial degree of the coloring defined by the Z_i (Theorem 2.2, Theorem 2.3). The combinatorial degree can be computed explicitly as a signed number of certain complete flags of faces of P (see Theorem 1.7).

In the second part of the paper we consider a radical monomial ideal I of the homogeneous coordinate ring $S = \bigoplus_{\alpha \in A_{n-1}(X)} S_\alpha$ of X (see [**C1**]). Let $z_1, \ldots, z_m$ be the minimal generating set of I. Consider a homogeneous polynomial $F \in S_\alpha$ of semiample degree α (i.e. the line bundles corresponding to α are generated by global sections). It follows from [**CCD**], Section 2 that if α is $\mathbb{Q}$-ample the polynomial F belongs to I as long as $Z_1 \cap \cdots \cap Z_m$ is empty, where Z_i is the zero set of z_i on X. In Theorem 3.3 we extend this observation. It is known that if α is semiample there exists a unique toric variety X' and a surjective morphism $\pi : X \to X'$ such that α is the pull-back of an ample class on X' (see [**M**], Theorem 1.2). We show that a generic polynomial F of semiample degree lies in I if and only if $\pi(Z_1) \cap \cdots \cap \pi(Z_m)$ is empty on X'. This condition also has a combinatorial interpretation (Theorem 3.3, part (4)).

Let us remark that it is still an open question how to compute the toric residue map for arbitrary non-ample divisors even when we know that the map is an isomorphism (e.g. for big and nef divisors). In the final section we apply our results to this problem. We obtain a combinatorial condition when it is possible to construct an element of toric residue one. More results in this direction are obtained in the joint work with A. Khetan [**KS**].

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1. Combinatorial Degree

1.1. **Polyhedral sets.** A polyhedral set X is a finite union of convex compact polyhedra intersecting in faces. We assume that they all are embedded in some Euclidean space E, thus X is a topological space with topology inherited from E. The dimension of X is the maximum of dimensions of the polyhedra it contains.

Example 1. *Boundary subdivision.* Let $P \subset E$ be a convex n-dimensional polytope. Then the boundary ∂P is a polyhedral set of dimension $n-1$. More generally, any polyhedral subdivision of ∂P is a polyhedral set of dimension $n-1$.

For the purposes of this paper it is enough to consider only polyhedral sets of Example 1.

Let X be a polyhedral set and $\mathcal{F}(X)$ the set of all faces of all polyhedra appearing in X. The set $\mathcal{F}(X)$ is a finite partially ordered set by inclusion.

Definition 1.1. Let X, Y be two polyhedral sets and $\psi : \mathcal{F}(X) \to \mathcal{F}(Y)$ a map that preserves the partial ordering. A continuous map $f_\psi : X \to Y$ is called a *characteristic map for* ψ if the image of every face $G \in \mathcal{F}(X)$ under f_ψ lies in $\psi(G)$.

For any map $\psi : \mathcal{F}(X) \to \mathcal{F}(Y)$ there exists a characteristic map $f_\psi : X \to Y$. One can construct a piecewise linear f_ψ using barycentric subdivisions of X and Y (see [**S**], Proposition 2.1).

It is easy to see that if f_ψ and f'_ψ are two characteristic maps for ψ then for any $0 \le t \le 1$ the map $f^t_\psi = (1-t)f_\psi + tf'_\psi$ is also characteristic for ψ.

Definition 1.2. Let X and Y be n-dimensional polyhedral sets and $\psi : \mathcal{F}(X) \to \mathcal{F}(Y)$ a map that preserves the partial ordering. Let f_ψ be any characteristic map for ψ. The induced map of the n-th homology groups

$$H_n(f_\psi) : H_n(X) \to H_n(Y) \tag{1.1}$$

is called the *combinatorial degree map* of ψ.

By the above the combinatorial degree map is independent of the choice of a characteristic map f_ψ. We denote it by $\operatorname{c.deg}\psi$.

Next we will see that the combinatorial degree map is invariant under subdivisions of X.

Definition 1.3. A polyhedral set X' is called a *subdivision* of a polyhedral set X if every polyhedron in X is a finite union of polyhedra in X', and the support of X equals the support of X'. We say that $\psi' : \mathcal{F}(X') \to \mathcal{F}(Y)$ *refines* $\psi : \mathcal{F}(X) \to \mathcal{F}(Y)$ if $G' \subset G$ implies $\psi'(G') \subset \psi(G)$ for every $G' \in \mathcal{F}(X')$ and $G \in \mathcal{F}(X)$.

Proposition 1.4. *Let X and Y be n-dimensional polyhedral sets. Let X' be a subdivision of X. Consider $\psi : \mathcal{F}(X) \to \mathcal{F}(Y)$ and $\psi' : \mathcal{F}(X') \to \mathcal{F}(Y)$ that refines ψ. Then* $\operatorname{c.deg}\psi' = \operatorname{c.deg}\psi$.

Proof. Let $f_{\psi'} : X' \to Y$ be any characteristic map for ψ'. Since ψ' refines ψ the map $f_{\psi'}$ considered as a map from X to Y is characteristic for ψ. Therefore $\operatorname{c.deg}\psi' = H_n(f_{\psi'}) = \operatorname{c.deg}\psi$. □

1.2. **Combinatorial degree for polytopes.** Now assume that X and Y are boundary subdivisions of some convex n-dimensional polytopes P and Q as in Example 1. In this case we have isomorphisms $H_{n-1}(X) \cong \mathbb{Z}$ and $H_{n-1}(Y) \cong \mathbb{Z}$ given by a choice of orientations of P and Q. The map (1.1) is then the multiplication by an integer number. This number will be called the *combinatorial degree* of ψ and denoted also by $\operatorname{c.deg}\psi$.

Remark 1.5. The combinatorial degree is a global analog of combinatorial coefficients, first introduced by O. Gelfond and A. Khovanskii in [**GKh1**]. Combinatorial coefficients appear in the Gelfond-Khovanskii residue formula and the Khovanskii product of roots formula for a polynomial system whose Newton polytopes have generic relative position (see [**GKh2**] and [**Kh**] for details).

The next theorem shows how to compute the combinatorial degree for polytopes. We will need the following definition.

Definition 1.6. Let X be a boundary subdivision of an n-dimensional convex oriented polytope P. Consider a complete flag in X (a maximal chain of elements of $\mathcal{F}(X)$):

$$\mathcal{X} :\ X_0 \subset \cdots \subset X_{n-1}, \quad \dim X_i = i.$$

For every $1 \le i \le n$ choose a vector e_i which begins at X_0 and points strictly inside X_i, where $X_n = P$. Define the *sign* of the flag $\mathcal{X}$ to be 1 if $(e_1, \dots, e_n)$ gives a positive oriented frame for P, and -1 otherwise.

Let X and Y be boundary subdivisions of n-dimensional convex oriented polytopes P and Q, and $\psi : \mathcal{F}(X) \to \mathcal{F}(Y)$. For complete flags $\mathcal{X} : X_0 \subset \cdots \subset X_{n-1}$ in X and $\mathcal{Y} : Y_0 \subset \cdots \subset Y_{n-1}$ in Y we will write $\psi(\mathcal{X}) = \mathcal{Y}$ if and only if $\psi(X_i) = Y_i$ for all $0 \le i \le n-1$.

Theorem 1.7. [S] *Fix any complete flag $\mathcal{Y}$ in Y. Then the combinatorial degree of ψ is equal to the sign of $\mathcal{Y}$ times the number of all complete flags $\mathcal{X}$ in X counted with signs, such that $\psi(\mathcal{X}) = \mathcal{Y}$:*

$$\mathrm{c.deg}\,\psi = \mathrm{sgn}\,\mathcal{Y} \sum_{\psi(\mathcal{X})=\mathcal{Y}} \mathrm{sgn}\,\mathcal{X}.$$

Now let P be an n-dimensional convex polytope in E, $\dim E = n$, and consider its *polar* polytope P° in the dual space E^*. Let $X = \partial P$ and $X^* = \partial(P^\circ)$. Recall that there is a one-to-one and order-reversing correspondence $G \mapsto G^*$ between $\mathcal{F}(X)$ and $\mathcal{F}(X^*)$ such that $\dim G^* = n - 1 - \dim G$.

Definition 1.8. Let $X = \partial P$, $Y = \partial Q$ for convex n-dimensional polytopes P and Q in E. Let $\psi : \mathcal{F}(X) \to \mathcal{F}(Y)$ be a map of partially ordered sets. Define the *dual map* $\psi^* : \mathcal{F}(X^*) \to \mathcal{F}(Y^*)$ by putting $\psi^*(G^*) = (\psi(G))^*$ for any $G \in \mathcal{F}(X)$.

It is readily seen that ψ^* preserves the partial order.

Proposition 1.9. *Let P, Q be convex n-dimensional polytopes in E, and $X = \partial P$, $Y = \partial Q$. Consider any $\psi : \mathcal{F}(X) \to \mathcal{F}(Y)$ and its dual $\psi^* : \mathcal{F}(X^*) \to \mathcal{F}(Y^*)$. Then* $\mathrm{c.deg}\,\psi = \mathrm{c.deg}\,\psi^*$.

Proof. This follows from Theorem 1.7. Indeed, every complete flag $\mathcal{X}$ in X corresponds to a unique complete flag $\mathcal{X}^*$ in X^* and $\mathrm{sgn}\,\mathcal{X} = \lambda\,\mathrm{sgn}\,\mathcal{X}^*$, where $\lambda = \pm 1$ and is the same for all $\mathcal{X}$.[1] Then for any fixed flag $\mathcal{Y}$ in Y

$$\mathrm{c.deg}\,\psi = \mathrm{sgn}\,\mathcal{Y} \sum_{\psi(\mathcal{X})=\mathcal{Y}} \mathrm{sgn}\,\mathcal{X} = \lambda\,\mathrm{sgn}\,\mathcal{Y}^* \sum_{\psi^*(\mathcal{X}^*)=\mathcal{Y}^*} \lambda\,\mathrm{sgn}\,\mathcal{X}^* = \mathrm{c.deg}\,\psi^*.$$

□

1.3. **Simplicial coloring.** Consider a convex n-dimensional polytope P.

Definition 1.10. Let $C_0, \ldots, C_n$ be closed subsets of ∂P, each set being the union of some facets of P. We say that $C = (C_0, \ldots, C_n)$ forms a *simplicial coloring* of (the boundary of) P if

(1) $C_0 \cup \cdots \cup C_n = \partial P$,
(2) $C_0 \cap \cdots \cap C_n = \emptyset$.

Let Δ denote the standard n-simplex:

$$\Delta = \{y = (y_0, \ldots, y_n) \in \mathbb{R}^{n+1} \mid y_0 + \cdots + y_n = 1,\ 0 \le y_i \le 1\}.$$

For each $1 \le k \le n$ the codimension k faces of Δ are

$$\Delta_{i_1 \ldots i_k} = \{y \in \Delta \mid y_{i_1} = \cdots = y_{i_k} = 0\}, \quad \text{where } 0 \le i_1 < \cdots < i_k \le n.$$

Then every simplicial coloring $C = (C_0, \ldots, C_n)$ defines a map

$$\psi_C : \mathcal{F}(\partial P) \to \mathcal{F}(\partial \Delta),$$

given by $\psi_C(G) = \Delta_{i_1 \ldots i_k}$ if G belongs to $C_{i_1} \cap \cdots \cap C_{i_k}$ with k maximal. Clearly ψ_C respects the partial order.

[1] In fact, $\lambda = (-1)^{\frac{n(n+1)}{2}}$.

Definition 1.11. Fix orientations of the polytope P and the simplex Δ. The *combinatorial degree of a simplicial coloring* $C = (C_0, \ldots, C_n)$ of P is the combinatorial degree $\mathrm{c.deg}\, \psi_C$ of the corresponding map $\psi_C : \mathcal{F}(\partial P) \to \mathcal{F}(\partial \Delta)$.

It follows from the definition that $\mathrm{c.deg}\, \psi_C$ is alternating on the order of the C_i.

Recall that the *normal fan* Σ_P of a polytope P in E is a complete fan in E^* whose cones are

$$\sigma_G = \{v \in E^* \mid \langle u, v\rangle \geq \langle u', v\rangle, \text{ for all } u \in P,\ u' \in G\}, \tag{1.2}$$

for every face G of P. Note that the 1-dimensional cones (rays) of Σ_P are generated by the inner normals to the facets of P. Thus, every coloring of P defines a "coloring" of the rays of Σ_P into $n+1$ colors. We arrive at the following definition.

Definition 1.12. Let Σ be a fan in $\mathbb{R}^n$. A decomposition of the set $\Sigma(1)$ of rays of Σ into a union of $n+1$ subsets

$$\Sigma(1) = \Lambda_0 \cup \cdots \cup \Lambda_n$$

is called a *coloring of* Σ. We say a coloring is *disjoint* if the union above is disjoint.

A coloring is called *simplicial* if no (maximal) cone of Σ contains rays of all $n+1$ colors, i.e. every set of rays $\{\rho_i \in \Lambda_i,\ 0 \leq i \leq n\}$ is not contained in any of the (maximal) cones of Σ.

Note that we allow multiple colors of a ray unless the coloring is disjoint.

Clearly, every simplicial coloring of a polytope P defines a simplicial coloring of its normal fan Σ_P. Conversely, if Σ is a *projective* fan (i.e. the normal fan of a polytope) then every simplicial coloring of Σ defines a simplicial coloring of any polytope whose normal fan is Σ. It is not hard to see that these colorings will all have the same combinatorial degree (for example one can deduce it from Theorem 1.7). We will call it the *combinatorial degree of the simplicial coloring of* Σ.

Applying Theorem 1.7 we obtain an explicit formula for the combinatorial degree of a coloring of a fan Σ in terms of complete flags of faces of Σ. For simplicity and for further purposes we will state this formula in the case when Σ is *simplicial* (i.e. all cones of Σ are simplicial) and the coloring is disjoint.

Fix $0 \leq i_1 < \cdots < i_k \leq n$. Then we say that a cone σ of Σ is $(i_1, \ldots, i_k)$*-colored* if for every $1 \leq j \leq k$ the cone σ contains a ray from Λ_{i_j}, and k is maximal.

Let σ be a maximal simplicial $(i_1, \ldots, i_n)$-colored cone for a simplicial disjoint coloring of Σ. We say it has *positive orientation* if a collection of generators $(e_1, \ldots, e_n)$, where e_j generates the ray from Λ_{i_j}, gives a positive oriented frame for $\mathbb{R}^n$.

Corollary 1.13. *Let Σ be a projective simplicial fan in $\mathbb{R}^n$ and*

$$\Sigma(1) = \Lambda_0 \sqcup \cdots \sqcup \Lambda_n$$

a disjoint simplicial coloring. Fix any k, $0 \leq k \leq n$. Then the combinatorial degree of the coloring is equal to $(-1)^k$ times the number of maximal $(0, \ldots, \widehat{k}, \ldots, n)$-colored cones, counted with orientations.

Now suppose that Σ' is a refinement of Σ with the same set of rays. Then any simplicial coloring of Σ is also a simplicial coloring of Σ'. The following theorem shows that these colorings will have the same combinatorial degree.

Theorem 1.14. *Let Σ and Σ' be projective fans such that $\Sigma' \to \Sigma$ is a refinement and $\Sigma(1) = \Sigma'(1)$. Consider a simplicial coloring of Σ. The induced coloring of Σ' is simplicial and has the same combinatorial degree.*

Proof. The fact that the induced coloring of Σ' is simplicial follows by definition.

Let P and P' be polytopes whose normal fans are Σ and Σ', respectively. Furthermore, let C (resp. C') be the simplicial coloring of P (resp. P') defined by the simplicial coloring of Σ (resp. Σ'). We need to show that $\text{c.deg}\, \psi_C = \text{c.deg}\, \psi_{C'}$.

We can assume that the origin lies in the interior of P and P' and so the fans Σ and Σ' consist of the cones over the proper faces of P° and $(P')^\circ$, respectively. By Proposition 1.9 it is enough to show that $\text{c.deg}\, \psi_C^* = \text{c.deg}\, \psi_{C'}^*$ for the dual maps.

Let X denote the boundary of P°. Projecting the proper faces of $(P')^\circ$ onto the faces of X along the rays of Σ we get a subdivision X' of X. It is clear that we get an order-preserving bijection between $\mathcal{F}(X')$ and the proper faces of $(P')^\circ$, and that the composition map $\psi' : \mathcal{F}(X') \to \mathcal{F}(\partial(\Delta^\circ))$ has the same combinatorial degree as $\psi_{C'}^*$.

It remains to show that $\psi' : \mathcal{F}(X') \to \mathcal{F}(\partial(\Delta^\circ))$ refines $\psi_C^* : \mathcal{F}(X) \to \mathcal{F}(\partial(\Delta^\circ))$. Indeed, ψ_C^* sends an element $G \in \mathcal{F}(X)$ to $\Delta_{i_1 \dots i_k}^*$ if and only if the cone over G is $(i_1, \dots, i_k)$-colored. Suppose $G' \in \mathcal{F}(X')$ is contained in G. Then the rays of the cone over G' are also rays of the cone over G and, hence, will have the same colors $(i_1, \dots, i_k)$ (all of them or fewer). Therefore, $\psi'(G') \subset \Delta_{i_1 \dots i_k}^*$. By Proposition 1.4 $\text{c.deg}\, \psi' = \text{c.deg}\, \psi_C^*$ which completes the proof. ∎

2. Toric residue

Let $X = X(\Sigma)$ be an n-dimensional complete toric variety determined by a complete rational fan Σ (see for example [**F**]). Following the notation of [**F**] we let N denote a lattice of rank n and $N_{\mathbb{R}}$ the real vector space $N \otimes \mathbb{R}$ which contains the fan Σ. Also let $M = \text{Hom}(N, \mathbb{Z})$ be the dual lattice and $M_{\mathbb{R}}$ the corresponding dual space $M \otimes \mathbb{R}$.

Let $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ be the set of rays of Σ. Each ray $\rho_i \in \Sigma(1)$ determines a $\mathbb{T}$-invariant irreducible divisor D_i on X. The variety X has the homogeneous coordinate ring $S = \mathbb{C}[x_1, \dots, x_r]$ graded by the Chow group $A_{n-1}(X)$ so that a monomial $x^a = \prod_{j=1}^r x_j^{a_j}$ has degree $\deg(x^a) = [\sum_{j=1}^r a_j D_j] \in A_{n-1}(X)$ (see [**C1**]). Denote by S_α the graded piece of S consisting of all polynomials of degree $\alpha \in A_{n-1}(X)$. As shown in [**C1**] S_α is canonically isomorphic to the global sections of the sheaf $\mathcal{O}_X(D)$ on X, where $\alpha = [D]$.

We will recall the definition of the toric residue [**C2, CCD**]. Consider $n+1$ homogeneous polynomials $F_i \in S_{\alpha_i}$, for $0 \le i \le n$. Their critical degree is defined to be

$$\rho = \sum_{i=0}^n \alpha_i - \sum_{j=1}^r \deg(x_j).$$

Then every polynomial H of degree ρ defines a meromorphic n-form on X:

$$\omega_F(H) = \frac{H\Omega}{F_0 \dots F_n},$$

where Ω is the Euler form [**CCD**]. We use F to denote the vector $(F_0, \dots, F_n)$.

Suppose that F_i do not vanish simultaneously on X. Then X has an open cover $\mathcal{U} = \{U_i, 0 \le i \le n\}$, where $U_i = \{x \in X \mid F_i(x) \ne 0\}$. Therefore $\omega_F(H)$ defines a Čech cohomology class $[\omega_F(H)] \in H^n(X, \widehat{\Omega}_X^n)$, relative to the cover $\mathcal{U}$. Here $\widehat{\Omega}_X^n$

denotes the sheaf of Zariski n-forms on X. This class $[\omega_F(H)]$ is alternating on the order of the F_i and is zero if H belongs to the ideal $\langle F_0,\dots,F_n\rangle$. The *toric residue map*

$$\mathrm{Res}_F : S_\rho/\langle F_0,\dots,F_n\rangle \to \mathbb{C},$$

is given by $\mathrm{Res}_F(H) = \mathrm{Tr}_X([\omega_F(H)])$, where Tr_X is the trace map on X.

As proved in [**CCD**], when X is simplicial the toric residue is in fact the sum of local Grothendieck residues:

Theorem 2.1. [**CCD**] *Let X be a complete simplicial toric variety. Let $F_0,\dots,F_n$ be homogeneous polynomials which do not vanish simultaneously on X and suppose the set $Z_{\hat{k}} = \{x \in X \mid F_i(x) = 0,\ 0 \le i \le n, i \ne k\}$ is finite, for some k. Then for any homogeneous polynomial H of the critical degree ρ*

$$\mathrm{Res}_F(H) = (-1)^k \sum_{x\in Z_{\hat{k}}} \mathrm{Res}_x\left(\frac{(H/F_k)\Omega}{F_0\dots\widehat{F}_k\dots F_n}\right).$$

We consider a special case when $F_i = z_i$ are monomials whose product is the product of the variables, $z_0\dots z_n = x_1\dots x_r$. In this case the critical degree ρ is zero and the toric residue is the multiplication by $\mathrm{Res}_z(1)$.

On the other hand, since the variables x_j correspond to the rays ρ_j of Σ, the monomials $z_0,\dots,z_n$ define a disjoint coloring of Σ:

$$\Sigma(1) = \Lambda_0 \sqcup \dots \sqcup \Lambda_n,$$

where the ray ρ_j lies in Λ_i if and only if x_j divides z_i. Clearly, this coloring is simplicial if and only if $z_0,\dots,z_n$ do not vanish simultaneously on X.

In the following theorem we show that $\mathrm{Res}_z(1)$ is equal to the combinatorial degree of the coloring defined by $z_0,\dots,z_n$.

Theorem 2.2. *Let X be an n-dimensional projective toric variety defined by a projective fan Σ. Let $z_0,\dots,z_n$ be monomials in the homogeneous coordinate ring $S = \mathbb{C}[x_1,\dots,x_r]$ such that*

(1) $z_0\dots z_n = x_1\dots x_r$,
(2) *$z_0,\dots,z_n$ do not vanish simultaneously on X.*

Then the toric residue $\mathrm{Res}_z(1)$ equals the combinatorial degree of the simplicial coloring of Σ defined by $z_0,\dots,z_n$.

Proof. First assume that X is simplicial, and so Σ is a simplicial fan. Let Z_i be the zero locus of z_i on X. Suppose the intersection $Z_{\hat{0}} = Z_1\cap\dots\cap Z_n$ is infinite. Since it is a union of orbits it must contain a 1-dimensional orbit O_τ, for some cone τ of dimension $n-1$. But this would imply that there are n distinct irreducible components $Z_i' \subset Z_i$, $1 \le i \le n$ which contain O_τ, i.e. the cone τ contains n distinct rays of Σ, which is impossible when Σ is simplicial. Therefore, $Z_{\hat{0}}$ is finite.

By Theorem 2.1 the toric residue is the sum of the Grothendieck residues over the points of $Z_{\hat{0}}$. But these points are the closed orbits O_σ of X that correspond to $(1,\dots,n)$-colored maximal cones σ of Σ. Therefore by Corollary 1.13 one only needs to check that the Grothendieck residue at O_σ is equal to ±1 depending on the orientation of σ, which is straightforward.

In the general case let $X' = X(\Sigma')$ be the toric variety determined by a simplicial refinement Σ' of Σ with the same set of rays. Then X and X' have the same homogeneous coordinate ring and the birational morphism $X' \to X$ maps every Z_i' defined by z_i on X' onto Z_i. By the functorial property of the trace map the toric residues $\mathrm{Res}_z(1)$ on X and on X' are equal. On the other hand, by Theorem 1.14 the combinatorial degrees of the colorings of Σ and Σ' are also equal, and the theorem follows. □

We will now consider a more general case when z_i are any (monic) monomials whose product is divisible by the product of the variables, $x_1 \dots x_r \,|\, z_0 \dots z_n$. In this case the quotient $z_0 \dots z_n / x_1 \dots x_r$ has the critical degree and it makes sense to consider the toric residue

$$\mathrm{Res}_z \left(\frac{z_0 \dots z_n}{x_1 \dots x_r} \right).$$

As before the monomials $z_0, \dots, z_n$ define a coloring of Σ which is now not necessarily disjoint. Theorem 2.2 implies the following more general statement.

Theorem 2.3. *Let X be an n-dimensional projective toric variety defined by a projective fan Σ. Let $z_0, \dots, z_n$ be monomials in the homogeneous coordinate ring $S = \mathbb{C}[x_1, \dots, x_r]$ such that*

(1) $x_1 \dots x_r \,|\, z_0 \dots z_n$,

(2) $z_0, \dots, z_n$ *do not vanish simultaneously on* X.

Then the toric residue

$$\mathrm{Res}_z \left(\frac{z_0 \dots z_n}{x_1 \dots x_r} \right)$$

equals the combinatorial degree of the simplicial coloring of Σ defined by $z_0, \dots, z_n$.

Proof. To reduce our theorem to Theorem 2.2 we choose monomials $z'_0, \dots, z'_n$ such that $z'_i | z_i$ and $z'_0 \dots z'_n = x_1 \dots x_r$. Then

$$\mathrm{Res}_{z'}(1) = \mathrm{Res}_z \left(\frac{z_0 \dots z_n}{x_1 \dots x_r} \right). \tag{2.1}$$

Indeed, the open sets $U'_i = \{z'_i = 0\}$, $0 \le i \le n$ form a covering $\mathcal{U}'$ of X. Also the covering $\mathcal{U}$ by the sets $U_i = \{z_i = 0\}$, $0 \le i \le n$ is a refinement of $\mathcal{U}'$. Therefore the cocycles

$$[\omega_{z'}(1)] \in \mathcal{Z}^n(\mathcal{U}', \widehat{\Omega}^n_X) \quad \text{and} \quad \left[\omega_z \left(\frac{z_0 \dots z_n}{x_1 \dots x_r} \right)\right] \in \mathcal{Z}^n(\mathcal{U}, \widehat{\Omega}^n_X)$$

define the same element in $H^n(X, \widehat{\Omega}^n_X)$.

Now it remains to show that the combinatorial degrees of the two colorings defined by $z_0, \dots, z_n$ and $z'_0, \dots, z'_n$ are the same. Let P be a polytope whose normal fan is Σ, and $C = (C_0, \dots, C_n)$ and $C' = (C'_0, \dots, C'_n)$ be the two colorings of P defined by $z_0, \dots, z_n$ and $z'_0, \dots, z'_n$, respectively. Consider $G \in \mathcal{F}(\partial P)$ and assume that G belongs to $C'_{i_1} \cap \dots \cap C'_{i_k}$ with k maximal, where $0 \le i_1 < \dots < i_k \le n$. Then G belongs to $C_{i_1} \cap \dots \cap C_{i_k}$ and hence $\psi_C(G) \subset \psi_{C'}(G)$. But this implies that ψ_C refines $\psi_{C'}$ and so they have the same combinatorial degree by Proposition 1.4. □

Remark 2.4. The statement in (2.1) also follows from the Global Transformation Law (Theorem 0.1 [**CCD**]).

3. Monomial ideal

In this section we study radical monomial ideals of the homogeneous coordinate ring S of X. We answer the following question: When does a homogeneous polynomial $F \in S_\alpha$ of semiample degree α belong to a radical monomial ideal I?

3.1. **Semiample class.** Let $D = \sum_{j=1} a_j D_j$ be a $\mathbb{T}$-invariant Cartier divisor on X. Then it determines a unique continuous function ϕ_D linear on each cone of Σ such that $\phi_D(v_j) = -a_j$, $1 \le j \le r$, where v_j is the first lattice point along the ray $\rho_j \in \Sigma(1)$. Note that if D and D' are linearly equivalent then $\phi_D - \phi'_D$ is a linear function on $N_{\mathbb{R}}$.

The divisor D also determines a rational convex polytope P_D in $M_{\mathbb{R}}$:

$$P_D = \{u \in M_{\mathbb{R}} \mid \langle u, v_j \rangle \ge -a_j,\ 1 \le j \le r\} = \{u \in M_{\mathbb{R}} \mid u \ge \phi_D\}.$$

Linearly equivalent divisors have the same polytope up to a parallel translation.

Definition 3.1. A Cartier divisor D is called *semiample* if the corresponding line bundle $\mathcal{O}_X(D)$ is generated by global sections. Also $\alpha = [D] \in A_{n-1}(X)$ is called *semiample class* if D is semiample.

A divisor D is semiample if and only if the function ϕ_D is convex ([F], Section 3.4).

In [**M**] A. Mavlyutov shows that if α is a semiample class on X then there is a unique toric variety X' and a surjective morphism $\pi : X \to X'$ such that α is the pull-back of an ample class α' on X'. Here we recall this construction.

Let $\alpha = [D]$ be a semiample class. First, with the polytope P_D we associate a rational fan Σ_D as follows. If $\dim P_D = n$ then $\Sigma_D \subset N_{\mathbb{R}}$ is the normal fan of P_D. In general let $L \subset N_{\mathbb{R}}$ be the orthogonal complement to the affine space spanned by P_D. Then for every face G of P_D the cone σ_G contains L (see (1.2)). Define Σ_D to be the fan in $N_{\mathbb{R}}/L$ whose cones are the images $[\sigma_G]$ under the quotient map $N_{\mathbb{R}} \to N_{\mathbb{R}}/L$. The fan Σ_D is rational with respect to the lattice $N/N \cap L$. Since Σ_D is the same for all representatives D of α we denote it by Σ_α.

The quotient map $N \to N/N \cap L$ extends to a map of fans

$$\tilde{\pi} : \Sigma \to \Sigma_\alpha.$$

Indeed, for any maximal cone $\sigma \in \Sigma$ the restriction of ϕ_D to σ defines a lattice point u which is a vertex of P_D (by convexity of ϕ_D). Then $\tilde{\pi}(\sigma) \subset [\sigma_u]$. The map $\tilde{\pi}$ induces the surjective morphism

$$\pi : X \to X', \quad \text{where } X' = X(\Sigma_\alpha).$$

Note that when $\dim P_D = n$ the map $\tilde{\pi} : \Sigma \to \Sigma_\alpha$ is a refinement. Also $\Sigma = \Sigma_\alpha$ and $\tilde{\pi} = id$ when α is an ample class.

3.2. **Monomial ideal.** Let us now consider a radical monomial ideal I of S. We denote by $z_1, \ldots, z_m$ the minimal generating set of I. Similar to the construction of Section 3 these monomials determine a coloring of rays of the fan Σ into m colors: a ray ρ_j has color i if and only if x_j divides z_i. Note that we allow multi-colored rays and not all the rays of Σ are colored, in general.

Recall that the *irrelevant ideal* of X is the ideal B in S generated by all the products $\prod_{\rho_j \notin \sigma} x_j$ where σ runs over all (maximal) cones of Σ (see [**C1**]). Similarly, we define the ideal B_α in S generated by all the products $\prod_{\tilde{\pi}(\rho_j) \notin \sigma} x_j$ where σ runs over all (maximal) cones of Σ_α and $\tilde{\pi} : \Sigma \to \Sigma_\alpha$ as above. Clearly, $B \subset B_\alpha$ and $B = B_\alpha$ if α is ample.

Remark 3.2. As mentioned in [**CCD**], Section 2 for any ample degree α we have $S_\alpha \subset B$. This is not true in general, but as we will see in the proof of the next theorem, $S_\alpha \subset B_\alpha$ when α is semiample.

The following is the main result of this section.

Theorem 3.3. *Consider a radical monomial ideal $I = \langle z_1, \dots, z_m \rangle$ of the homogeneous coordinate ring S of X. Let $\alpha \in A_{n-1}(X)$ be semiample. The following are equivalent*

(1) $S_\alpha \subset I$,
(2) $F \in I$, *where F is a generic section of α,*
(3) $B_\alpha \subset I$,
(4) *no (maximal) cone of Σ_α contains images of rays of Σ of all m colors,*
(5) $\pi(Z_1) \cap \cdots \cap \pi(Z_m) = \emptyset$, *where $\pi : X \to X'$ is the morphism determined by $\tilde{\pi} : \Sigma \to \Sigma_\alpha$, and Z_i is the zero locus of z_i on X.*

Proof. (1)⇔(2) is clear.

(2)⇔(3) Let $D = \sum_{j=1}^r a_j D_j$ be a representative of α and consider its polytope P_D. Since F is generic, it is a linear combination with non-zero complex coefficients of the monomials χ^u as u varies over the set $P_D \cap M$ ([**F**], Section 3.4). But I is a monomial ideal thus $F \in I$ if and only if every χ^u belongs to I.

In homogeneous coordinates $\chi^u = \prod_{j=1}^r x_j^{\langle u, v_j \rangle + a_j}$. From this you can see that every such monomial is divisible by some χ^w, where w is a vertex of P_D. Thus we can assume that u varies over the vertices of P_D.

When u is a vertex of P_D we have $\langle u, v_j \rangle = -a_j$ if $\tilde{\pi}(\rho_j) \in \sigma_u$ and $\langle u, v_j \rangle > -a_j$ if $\tilde{\pi}(\rho_j) \notin \sigma_u$, where σ_u is the cone of Σ_α corresponding to u. Therefore we can write

$$\chi^u = \prod_{\tilde{\pi}(\rho_j) \notin \sigma_u} x_j^{\langle u, v_j \rangle + a_j}, \tag{3.1}$$

where all exponents are positive. Therefore $\chi^u \in I$ for all vertices u if and only if for every maximal cone σ of Σ_α there exists k such that $z_k | \prod_{\tilde{\pi}(\rho_j) \notin \sigma} x_j$ (by (3.1) and the fact that in z_k every variable has exponent 0 or 1). By the definition of B_α the latter is equivalent to $B_\alpha \subset I$.

(4)⇔(5) Suppose there exists a maximal cone σ' in Σ_α which contains the images $\tilde{\pi}(\rho_{k_1}), \dots, \tilde{\pi}(\rho_{k_m})$, where ρ_{k_i} has color i. It can be readily seen that the morphism $\pi : X \to X'$ maps every orbit O_τ, $\tau \in \Sigma$ onto the orbit $O_{\tau'}$, where τ' is the smallest cone of Σ_α containing $\tilde{\pi}(\tau)$, and hence it maps the closure $V(\tau)$ onto the closure $V(\tau')$. It follows then that for every $0 \leq i \leq n$ the image $\pi(V(\rho_{k_i}))$ contains the closed orbit $O_{\sigma'}$. Since $V(\rho_{k_i})$ is an irreducible component of Z_i we get a contradiction. The other implication is similar.

(3)⇔(4) Before we give a proof let us take a closer look at the ideal B_α.

We say that a collection of rays $\{\rho_{j_1}, \dots, \rho_{j_s}\}$ of Σ is *primitive with respect to* Σ_α if its image under $\tilde{\pi}$ does not lie in any of the cones of Σ_α, but the image of every its proper subset does. The corresponding collection of variables $\{x_{j_1}, \dots, x_{j_s}\}$ will be also called primitive with respect to Σ_α. The following is a slight generalization of the Batyrev description of the irrelevant ideal of the homogeneous coordinate ring.

Lemma 3.4. *The irreducible components of the variety $V(B_\alpha)$ of the ideal B_α are the coordinate planes $V(x_{j_1}, \dots, x_{j_s})$ for each collections $\{x_{j_1}, \dots, x_{j_s}\}$ primitive with respect to Σ_α.*

Proof. By definition the variety $V(B_\alpha)$ is the union of $V(x_{\hat{\sigma}} : \sigma \in \Sigma_\alpha)$, where for every maximal $\sigma \in \Sigma_\alpha$ we pick a variable $x_{\hat{\sigma}}$ whose corresponding ray is mapped outside of σ. Since the image of every primitive collection $\{\rho_{j_1},\dots,\rho_{j_s}\}$ does not lie in any of the cones of Σ_α the corresponding collection $\{x_{j_1},\dots,x_{j_s}\}$ appears among the collections $\{x_{\hat\sigma}:\sigma\in\Sigma_\alpha\}$. Therefore,

$$\bigcup_{\substack{\{\rho_{j_1},\dots,\rho_{j_s}\}\\ \text{primitive}}} V(x_{j_1},\dots,x_{j_s})\subseteq V(B_\alpha).$$

On the other hand, every collection $\{x_{\hat\sigma}:\sigma\in\Sigma_\alpha\}$ contains a primitive one. (If a collection of rays whose image does not lie in any cone is not primitive it contains a proper subset whose image does not lie in any cone etc.) Thus,

$$V(x_{\hat\sigma}:\sigma\in\Sigma_\alpha)\subseteq V(x_{j_1},\dots,x_{j_s}),$$

for some primitive collection $\{\rho_{j_1},\dots,\rho_{j_s}\}$. The lemma follows. □

Now we will finish the proof of the theorem. We have two irreducible decompositions:

$$V(I)=\bigcup_{x_{k_i}|z_i} V(x_{k_1},\dots,x_{k_m})\quad\text{and}\quad V(B_\alpha)=\bigcup_{\substack{\{\rho_{j_1},\dots,\rho_{j_s}\}\\ \text{primitive}}} V(x_{j_1},\dots,x_{j_s})$$

Then $V(I)\subset V(B_\alpha)$ if and only if for every collection $\{x_{k_1},\dots,x_{k_m}\}$ with $x_{k_i}|z_i$, there exists a collection $\{x_{j_1},\dots,x_{j_s}\}$ primitive with respect to Σ_α such that

$$V(x_{k_1},\dots,x_{k_m})\subset V(x_{j_1},\dots,x_{j_s}),$$

i.e., $\{\rho_{j_1},\dots,\rho_{j_s}\}\subset\{\rho_{k_1},\dots,\rho_{k_m}\}$. But the latter is equivalent to (4). Indeed, if $\{\rho_{k_1},\dots,\rho_{k_m}\}$ does not contain any primitive collection then its image lies in some cone σ of Σ_α, i.e. σ contains images of rays of all m colors. The converse is also clear. □

4. Homogeneous polynomials of residue one

Let X be a complete n-dimensional toric variety and $F_0,\dots,F_n$ homogeneous polynomials of degrees $\alpha_0,\dots,\alpha_n$, respectively. Consider the corresponding toric residue map

$$\mathrm{Res}_F : S_\rho/\langle F_0,\dots,F_n\rangle\to\mathbb{C}.$$

As recently proved by Cox and Dickenstein [**CD**] this map is an isomorphism if the degrees α_i are semiample and have n-dimensional polytopes. Then the computation of the toric residue reduces to the problem of finding a homogeneous polynomial of residue one. This is an open problem. The results of the previous sections allow us to say something in this direction.

Assume X is projective defined by a projective fan Σ. Let $\Sigma_i=\Sigma_{\alpha_i}$ be the fan associated with the degree α_i and $\tilde\pi_i:\Sigma\to\Sigma_i$ the corresponding map as in Section 3.1. We say that a disjoint simplicial coloring $\Sigma(1)=\Lambda_0\sqcup\dots\sqcup\Lambda_n$ is *compatible* with $\Sigma_0,\dots,\Sigma_n$ if it satisfies the condition of Theorem 3.3 for each Σ_i, i.e. no cone of Σ_i contains images under $\tilde\pi_i$ of rays of all $n+1$ colors, for every i.

Corollary 4.1. *Let $\alpha_0,\dots,\alpha_n$ be semiample degrees on X and $\Sigma_0,\dots,\Sigma_n$ the corresponding fans. Fix any disjoint simplicial coloring of Σ compatible with $\Sigma_0,\dots,\Sigma_n$:*

$$\Sigma(1)=\Lambda_0\sqcup\dots\sqcup\Lambda_n.$$

Let $z_i = \prod_{\rho_j \in \Lambda_i} x_j$ *be squarefree monomials,* $0 \leq i \leq n$*. Then for any homogeneous olynomials* $F_0, \ldots, F_n$ *of degrees* $\alpha_0, \ldots, \alpha_n$ *there are homogeneous polynomials* A_{ij} *such that*

$$F_i = A_{i0} z_0 + \cdots + A_{in} z_n, \quad 0 \leq i \leq n.$$

Furthermore,

$$\mathrm{Res}_F(\det(A)) = \mathrm{Res}_z(1) = \mathrm{c.deg}(\Lambda), \tag{4.1}$$

where $\mathrm{c.deg}(\Lambda)$ *is the combinatorial degree of the coloring above.*

Proof. The existence of A_{ij} is a consequence of Theorem 3.3. The first statement in (4.1) follows from the Global Transformation Law (Theorem 0.1, [**CCD**]) and the second statement follows from Theorem 2.2. □

Notice that if the degrees are ample then $\Sigma = \Sigma_0 = \cdots = \Sigma_n$ and we obtain many different ways of constructing a polynomial of residue one. In fact, every choice of a disjoint simplicial coloring of Σ of combinatorial degree one gives rise to such a polynomial. This extends previously known constructions by D'Andrea and Khetan [**AK**] and Cattani, Cox and Dickenstein [**CCD**]. Also, when the degrees are the same ($\Sigma_0 = \cdots = \Sigma_n$) there are many ways of choosing a compatible coloring of combinatorial degree one. It is not true, however, that compatible colorings of combinatorial degree one exist for any collection of $n+1$ semiample degrees. The simplest counterexample is presented in Figure 1.

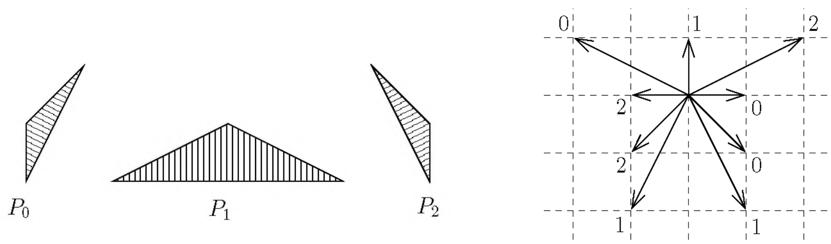

FIGURE 1.

Here P_0, P_1 and P_2 are the polytopes of semiample degrees α_0, α_1 and α_2 on the toric surface determined by the complete fan Σ on the right. The rays labeled with i, for $i = 0, 1, 2$, form the fan Σ_i corresponding to α_i. One can check that there are no compatible colorings of Σ of combinatorial degree one. In the forthcoming paper [**KS**] further methods are developed to deal with examples like that (in fact, all two-dimensional examples and a new class of n-dimensional examples are considered).

REFERENCES

[**AK**] C. D'Andrea, A. Khetan, *Macaulay style formulas for toric residues*, preprint math.AG/0307154.

[**BM**] V. Batyrev, E. Materov, *Toric Residues and Mirror Symmetry*, Moscow Math. J. **2** (2002), no. 3, 435–475.

[**CCD**] E. Cattani, D. Cox, A. Dickenstein, *Residues in Toric Varieties*, Compositio Math. **108** (1997), no. 1, 35–76.

[**CaD**] E. Cattani, A. Dickenstein, *A global view of residues in the torus*, J. Pure Appl. Algebra **117/118** (1997), 119–144.

[**CD**] D. Cox, A. Dickenstein, *A Codimension Theorem for Complete Toric Varieties*, preprint, math.AG/0310108.
[**CDS**] E. Cattani, A. Dickenstein, B. Sturmfels, *Residues and Resultants*, J. Math. Sci. Univ. Tokyo, **5** (1998), 119–148.
[**C1**] D. A. Cox, *The homogeneous coordinate ring of a toric variety*, J. Algebr. Geom. **4** (1995), 17–50.
[**C2**] D. A. Cox, *Toric residues*, Arkiv für Matematik **34** (1996) 73–96.
[**F**] W. Fulton, *Introduction to Toric Varieties*, Princeton Univ. Press, Princeton, 1993
[**GKh1**] O. A. Gelfond and A. G. Khovanskii, *Newton polyhedra and Grothendieck residues*, (in Russian) Dokl. Akad. Nauk, **350**, no. 3 (1996), 298–300.
[**GKh2**] O. A. Gelfond, A. G. Khovanskii, *Toric geometry and Grothendieck residues*, Moscow Math. J. **2** (2002), no. 1, 99–112.
[**KS**] A. Khetan, I. Soprounov, *Partition matrices for polytopes towards computing toric residue*, preprint math.AG/0406279
[**Kh**] A. G. Khovanskii, *Newton polyhedra, a new formula for mixed volume, product of roots of a system of equations*, Fields Inst. Comm., Vol. **24** (1999), 325–364.
[**M**] A. Mavlyutov, *The Hodge structure of semiample hypersurfaces and a generalization of the monomial-divisor mirror map*, Advances in algebraic geometry motivated by physics, Contemp. Math. 276 (2000).
[**S**] I. Soprounov, *Residues and tame symbols on toroidal varieties*, to appear in Compositio Math., math.AG/0203114.

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