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Wulff Shapes and a Characterization of Simplices via a Bezout Type Inequality

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WULFF SHAPES AND A CHARACTERIZATION OF SIMPLICES VIA A BEZOUT TYPE INEQUALITY

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ABSTRACT. Inspired by a fundamental theorem of Bernstein, Kushnirenko, and Khovanskii we study the following Bezout type inequality for mixed volumes

$$V(L_1,\dots,L_n)V_n(K)\le V(L_1,K[n-1])V(L_2,\dots,L_n,K).$$

We show that the above inequality characterizes simplices, i.e. if K is a convex body satisfying the inequality for all convex bodies $L_1,\dots,L_n\subset\mathbb{R}^n$, then K must be an n-dimensional simplex. The main idea of the proof is to study perturbations given by Wulff shapes. In particular, we prove a new theorem on differentiability of the support function of the Wulff shape, which is of independent interest.

In addition, we study the Bezout inequality for mixed volumes introduced in [SZ]. We introduce the class of weakly decomposable convex bodies which is strictly larger than the set of all polytopes that are non-simplices. We show that the Bezout inequality in [SZ] characterizes weakly indecomposable convex bodies.

1. Introduction

One of the fundamental results in the theory of Newton polytopes is the Bernstein-Kushnirenko-Khovanskii theorem which expresses algebraic-geometric information such as the degree of an algebraic variety and intersection numbers in terms of convex-geometric invariants such as volumes and mixed volumes [B, Kh, Ku]. There are many consequences and applications of this beautiful relation in both algebraic and convex geometry (see [KaKh] and references therein). Among them is a recent geometric inequality, called the Bezout inequality for mixed volumes, first introduced in [SZ]. The name comes from interpreting the classical Bezout inequality in algebraic geometry in terms of the mixed volumes with the help of the Bernstein-Kushnirenko-Khovanskii result. In the present paper we study another inequality of this type: For any convex bodies $L_1,\dots,L_n$ in $\mathbb{R}^n$ and any n-simplex K one has

$$V(L_1,\dots,L_n)V_n(K)\le V(L_1,K[n-1])V(L_2,\dots,L_n,K). \tag{1.1}$$

Here $V_n(K)$ denotes the n-dimensional Euclidean volume and $V(K_1,\dots,K_n)$ denotes the n-dimensional mixed volume of bodies $K_1,\dots,K_n$. We write $K[m]$ to indicate that the body K is repeated m times in the expression for the mixed volume.

The inequality (1.1) has a direct proof based on the monotonicity of the mixed volume presented in Section 4. We also give an algebraic-geometric interpretation of (1.1) at the end of this section. Our main result is that (1.1), in fact, characterizes n-simplices in the class of all convex bodies in $\mathbb{R}^n$. In other words, the only convex body $K \subset \mathbb{R}^n$ which satisfies (1.1) for all convex bodies $L_1, \dots, L_n$ is the n-simplex. It turns out that a slightly stronger statement is true.

Theorem 1.1. *Let K be a convex body in $\mathbb{R}^n$. Then K satisfies*

$$V(L_1, \dots, L_{n-1}, K)V_n(K) \le V(L_1, K[n-1])V(L_2, \dots, L_{n-1}, K, K) \tag{1.2}$$

for all convex bodies $L_1, \dots, L_{n-1}$ in $\mathbb{R}^n$ if and only if K is an n-simplex.

The question of characterization of n-simplices has played an essential role in a number of major results of modern convex geometry, including Ball's reverse isoperimetric inequality [Ba] and Zhang's projection inequality [Zh]. Also simplices are conjecturally the extreme bodies for several open problems. For example, a variant of Bourgain's slicing conjecture [Bo1, Bo2] characterizes simplices as convex bodies with a maximal isotropic constant. The Mahler conjecture states that simplices have the minimal volume product among all convex bodies. We refer the reader to [Ma1, Ma2, RZ, Sch] and [Kl] for connections between the slicing problem, the Mahler conjecture, and characterizations of simplices. To the best of the authors' knowledge, (1.1) and (1.2) are the first characterizations of simplices via an inequality rather than an equality, as presented in the above mentioned classical results and open problems.

Let us briefly describe the method of the proof of Theorem 1.1. The "if" part follows from (1.1). For the "only if" part, if K is a polytope, the claim follows from the result of [SSZ], as we show in Section 4.1. In the remaining case, we observe that the problem can actually be seen as an extremal (variational) problem. Indeed, assume that K satisfies (1.2) for all convex bodies $L_1, \dots, L_{n-1} \subset \mathbb{R}^n$. Let K_t be a convex perturbation of K where $t \in (-\delta, \delta)$, for some small $\delta > 0$, and $K_0 = K$. Then the function

$$F(t) := V(K_t, M, K[n-2])V_n(K) - V(K_t, K[n-1])V(M, K[n-1])$$

is non-positive in $(-\delta, \delta)$ and equals 0 at $t = 0$, where M is any compact convex set, independent of t. To arrive at a contradiction, one needs to construct K_t and M such that the right derivative of F at $t = 0$ (if it exists) is strictly positive. The main idea is to use a more general construction for the perturbation body K_t than the one used in [SSZ]. Namely, given a continuous function $f : \mathbb{S}^{n-1} \to \mathbb{R}$, define K_t as

$$K_t = \bigcap_{u \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : \langle x, u \rangle \le h_K(u) + tf(u)\},$$

where h_K is the support function of K.

The above construction goes back to Aleksandrov's 1938 paper [Al1] (see also [Al2, Sch]) and utilizes the notion of the Wulff shape. This notion was introduced by G. Wulff [Wu] who studied the equilibrium shape of a droplet or a crystal of fixed volume. In the recent years Wulff shapes have been widely used in the study of extremal problems of convex bodies. We refer to [BLYZ1, BLYZ2, KM, HLYZ, HuLYZ, Lu, Sa, SW] for recent results or [Ga, Sch, Se] for expository work on the subject. A core lemma of Aleksandrov [Sch, Lemma 7.5.3] (see also (3) below) provides a differentiability property of the volume of K_t. In our main result in Section 3 we prove a differentiability property of h_{K_t} (Theorem 3.5):

$$\frac{dh_{K_t}(u)}{dt}\Big|_{t=0} = f(u)$$

S_K-almost everywhere on $\mathbb{S}^{n-1}$, where S_K is the surface area measure of K. This can be thought of as a local version of Aleksandrov's lemma.

There is a close relationship between (1.1) and the Bezout inequality for mixed volumes studied in [SZ, SSZ]. Recall that it says that

$$V(L_1, L_2, K[n-2])V_n(K) \leq V(L_1, K[n-1])V(L_2, K[n-1]). \tag{1.3}$$

A more general version of this inequality is discussed in Section 5. Observe that (1.1) implies (1.3). It was conjectured in [SZ] that (1.3) characterizes simplices in the class of all convex bodies. This conjecture was confirmed in [SSZ] for the class of convex n-polytopes. Although the general case remains open, the results of this paper provide additional information regarding the conjecture, which we collect in Section 5. For example, Theorem 1.1 gives an affirmative answer to the conjecture when $n = 3$. Furthermore, in Section 5 we introduce the notion of weakly decomposable bodies generalizing the classical notion of decomposable bodies. The class of weakly decomposable bodies is quite large; we show that it is strictly larger than the set of all polytopes that are non-simplices. In fact, we currently do not have examples of convex bodies which are not weakly decomposable, besides simplices. We show that if K satisfies (1.3) for any L_1, L_2 in $\mathbb{R}^n$ then K cannot be weakly decomposable, see Theorem 5.7.

Theorem 1.1 shows that the inequality (1.1) may not hold when K is not an n-simplex. Thus, one can ask if (1.1) can be relaxed so that it holds for arbitrary $L_1, \ldots, L_n$ and K in $\mathbb{R}^n$. In Section 6 we prove that

$$V(L_1, \ldots, L_n)V_n(K) \leq nV(L_1, K[n-1])V(L_2, \ldots, L_n, K)$$

for all convex sets $L_1, \ldots, L_n$ and K in $\mathbb{R}^n$ and show that this inequality cannot be improved. In addition, we discuss isomorphic versions of (1.2) and (1.3).

We now turn to an algebraic-geometric interpretation of (1.1). Recall that the Newton polytope of a polynomial $f \in \mathbb{C}[t_1, \ldots, t_n]$ is the convex hull in $\mathbb{R}^n$ of the exponent vectors appearing in f. Fix $1 \leq r \leq n$. A polynomial system $f_1(t) = \cdots = f_r(t) = 0$ for $t \in (\mathbb{C} \setminus \{0\})^n$ with generic coefficients and fixed Newton polytopes $P_1, \ldots, P_r$ defines an $(n-r)$-dimensional algebraic set X in $(\mathbb{C} \setminus \{0\})^n$. We may assume that each P_i intersects every coordinate hyperplane $x_i = 0$, as translations of the Newton polytopes do not change the set X. By definition, the degree $\deg X$ of X is the number of intersection points of X with a generic affine subspace of dimension r. According to the Bernstein-Kushnirenko-Khovanskii theorem, this number can be computed as

$$\deg X = n!V(P_1, \ldots, P_r, \Delta[n-r]),$$

where $\Delta = \operatorname{conv}\{0, e_1, \ldots, e_n\}$ is the standard n-simplex (the Newton polytope of a linear polynomial). Here $e_1, \ldots, e_n$ denote the standard basis vectors in $\mathbb{R}^n$. In particular, when $r = 1$, X is a hypersurface and $\deg X$ coincides with the degree of the polynomial defining X. When $r = n$, X consists of isolated points whose number equals $\deg X$.

Now the algebraic-geometric interpretation of (1.1) is as follows. Let X be the hypersurface defined by $f_1(t) = 0$ and Y be defined by a polynomial system $f_2(t) = \cdots = f_n(t) = 0$ in $(\mathbb{C} \setminus \{0\})^n$. As before we assume that the Newton polytopes $P_1, \dots, P_n$ are fixed and the coefficients of the f_i are generic. Then $\deg X = n!V(P_1, \Delta[n-1])$, $\deg Y = n!V(P_2, \dots, P_n, \Delta)$, and $\deg(X \cap Y) = n!V(P_1, \dots, P_n)$. Also note that $V_n(\Delta) = 1/n!$. The inequality (1.1) then turns into a classical Bezout inequality

$$\deg(X \cap Y) \leq \deg X \deg Y,$$

see, for instance, Proposition 8.4 of [F] and examples therein.

2. Preliminaries

2.1. **Basic definitions.** In this section we introduce notation and collect basic facts from classical theory of convex bodies that we use in the paper. As a general reference on the theory we use R. Schneider's book "Convex bodies: the Brunn-Minkowski theory" [Sch].

We write $\langle x, y \rangle$ for the inner product of x and y in $\mathbb{R}^n$. Next, $\mathbb{S}^{n-1}$ denotes the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ and $B(x, \delta)$ denotes the closed Euclidean ball of radius $\delta > 0$ centered at $x \in \mathbb{R}^n$. A *spherical cap* $U(u, r)$ of radius $r > 0$ centered at $u \in \mathbb{S}^{n-1}$ is the intersection $\mathbb{S}^{n-1} \cap B(u, r)$.

For $A \subset \mathbb{R}^n$ its *dimension* $\dim A$ is the dimension of the smallest affine subspace of $\mathbb{R}^n$ containing A. A *convex body* is a convex compact set with non-empty interior. Note that convex bodies in $\mathbb{R}^n$ are n-dimensional. A *(convex) polytope* is the convex hull of a finite set of points. An n-dimensional polytope is called an *n-polytope* for short. An *n-simplex* is the convex hull of $n+1$ affinely independent points in $\mathbb{R}^n$.

For a convex body K the function $h_K : \mathbb{S}^{n-1} \to \mathbb{R}$, $h_K(u) = \max\{\langle x, u \rangle \mid x \in K\}$ is the *support function* of K. For every convex body K we write K^u to denote the face corresponding to an outer normal vector $u \in \mathbb{S}^{n-1}$, i.e.

$$K^u = \{x \in K : \langle x, u \rangle = h_K(u)\}.$$

If Ω is a subset of $\mathbb{S}^{n-1}$ define the *inverse spherical image* $\tau(K, \Omega)$ of Ω with respect to K by

$$\tau(K, \Omega) = \big\{x \in \partial K : \exists u \in \Omega, \text{ such that } \langle x, u \rangle = h_K(u)\big\}.$$

The *surface area measure* $S_K(\cdot)$ of K is a measure on $\mathbb{S}^{n-1}$ such that

$$S_K(\Omega) = \mathcal{H}^{n-1}\big(\tau(K, \Omega)\big),$$

for any Borel $\Omega \subset \mathbb{S}^{n-1}$. Here $\mathcal{H}^{n-1}(\cdot)$ stands for the $(n-1)$-dimensional Hausdorff measure.

Let $V_n(K)$ be the Euclidean volume of $K \subset \mathbb{R}^n$. We will often use the classical formula connecting the volume of a convex body, the support function, and the surface area measure:

$$V_n(K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(u) dS_K(u). \tag{2.1}$$

The Minkowski sum of two sets $K, L \subset \mathbb{R}^n$ is defined as $K + L = \{x + y : x \in K \text{ and } y \in L\}$. A classical theorem of Minkowski says that if $K_1, K_2, \dots, K_n$ are convex compact sets in $\mathbb{R}^n$ and $\lambda_1, \dots, \lambda_n \geq 0$, then $V_n(\lambda_1 K_1 + \lambda_2 K_2 + \cdots + \lambda_n K_n)$ is a homogeneous polynomial in $\lambda_1, \dots, \lambda_n$ of degree n. The coefficient of $\lambda_1 \cdots \lambda_n$ is called the *mixed volume* of $K_1, \dots, K_n$ and is denoted by $V(K_1, \dots, K_n)$. We will also write $V(K_1[m_1], \dots, K_r[m_r])$ for the mixed volume of $K_1, \dots, K_r$ where each K_i is repeated m_i times and $m_1 + \cdots + m_r = n$. We summarize the main properties of the mixed volume below:

- $V(K, \dots, K) = V(K[n]) = V_n(K)$;
- the mixed volume is symmetric in $K_1, \dots, K_n$;
- the mixed volume is multilinear: For any $\lambda, \mu \geq 0$
$$V(\lambda K + \mu L, K_2, \dots, K_n) = \lambda V(K, K_2, \dots, K_n) + \mu V(L, K_2, \dots, K_n);$$
- the mixed volume is translation invariant: For any $a \in \mathbb{R}^n$
$$V(K_1 + a, K_2, \dots K_n) = V(K_1, K_2, \dots, K_n);$$
- the mixed volume is monotone: If $K \subset L$ then
$$V(K, K_2, \dots, K_n) \leq V(L, K_2, \dots, K_n).$$

We will need the following classical inequalities for the mixed volumes: The Minkowski First inequality

$$V(K, L[n-1]) \geq V_n(K)^{1/n} V_n(L)^{(n-1)/n}, \tag{2.2}$$

and the Aleksandrov–Fenchel inequality

$$V(K_1, K_2, K_3, \dots, K_n) \geq \sqrt{V(K_1, K_1, K_3, \dots, K_n) V(K_2, K_2, K_3, \dots, K_n)}. \tag{2.3}$$

Let $S(K_1, \dots, K_{n-1}, \cdot)$ be the *mixed area measure* for bodies $K_1, \dots, K_{n-1}$ defined by

$$V(L, K_1, \dots, K_{n-1}) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L(u) dS(K_1, \dots, K_{n-1}, u) \tag{2.4}$$

for any compact convex set L. In particular,

$$V(L, K[n-1]) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_L(u) dS_K(u) \tag{2.5}$$

and

$$S(K, \dots, K, \cdot) = S(K[n-1], \cdot) = S_K(\cdot).$$

The identity (2.4) implies that the invariance properties of mixed volumes are inherited by mixed area measures. More specifically, $S(K_1, \dots, K_{n-1}, \cdot)$ is translation invariant, symmetric, and multilinear with respect to $K_1, \dots, K_{n-1}$. We also note that if the K_i are polytopes the mixed area measure $S(K_1, \dots, K_{n-1}, \cdot)$ has finite support and for every $u \in \mathbb{S}^{n-1}$ we have

$$S(K_1, \dots, K_{n-1}, \{u\}) = V(K_1^u, \dots, K_{n-1}^u), \tag{2.6}$$

where $V(K_1^u, \dots, K_{n-1}^u)$ is the $(n-1)$-dimensional mixed volume of the faces K_i^u translated the the subspace orthogonal to u, see [Sch, Sec 5.1]. We will need a slightly more general statement.

Lemma 2.1. *Let $K_1, \dots, K_{n-1}$ be convex bodies in $\mathbb{R}^n$. Then*

$$S(K_1, \dots, K_{n-1}, \{u\}) = V(K_1^u, \dots, K_{n-1}^u).$$

Proof. We use an alternative definition of the mixed area measure given in [Sch, (5.21), page 281]:

$$S(K_1, \dots, K_{n-1}, \cdot) = \frac{1}{(n-1)!}\sum_{k=1}^{n-1}(-1)^{n-1-k}\sum_{i_1<\dots<i_k} S_{K_{i_1}+\dots+K_{i_k}}(\cdot).$$

By the definition of the surface area measure

$$S(K_{i_1} + \dots + K_{i_k}, \{u\}) = V_{n-1}((K_{i_1} + \dots + K_{i_k})^u),$$

where as before $(K_{i_1} + \dots + K_{i_k})^u$ denotes the face of $K_{i_1} + \dots + K_{i_k}$ corresponding to u. Note that $(K_{i_1} + \dots + K_{i_k})^u = K_{i_1}^u + \dots + K_{i_k}^u$. Therefore,

$$S(K_1, \dots, K_{n-1}, \{u\}) = \frac{1}{(n-1)!}\sum_{k=1}^{n-1}(-1)^{n-1-k}\sum_{i_1<\dots<i_k} V_{n-1}(K_{i_1}^u + \dots + K_{i_k}^u).$$

The right hand side of the above equation is the well-known formula for the mixed volume $V(K_1^u, \dots, K_{n-1}^u)$, see [Sch, Lemma 5.1.4]. $\square$

Finally, we will need the following formula relating the mixed volumes and projections, which is a particular case of [Sch, Theorem 5.3.1]. For $u \in \mathbb{S}^{n-1}$ the orthogonal projection of a set $A \subset \mathbb{R}^n$ onto the subspace $u^\perp$ orthogonal to u is denoted by $A|u^\perp$. Let $I = [0, u]$ be a unit segment for some $u \in \mathbb{S}^{n-1}$ and $K_2, \dots, K_n$ convex bodies in $\mathbb{R}^n$. Then

$$V(I, K_2, \dots, K_n) = \frac{1}{n}V(K_2|u^\perp, \dots, K_n|u^\perp), \tag{2.7}$$

where in the right-hand side is the $(n-1)$-dimensional mixed volume of convex bodies in the orthogonal subspace $u^\perp$.

3. Differentiation of Wulff Shapes.

The notion of the Wulff shape is the main tool in constructing the perturbation K_t. We refer to Section 7.5 of [Sch] for details. We start with the definition of the Wulff shape and several properties that we need later.

Definition 3.1. Consider a non-negative function $g : \mathbb{S}^{n-1} \to \mathbb{R}$. The *Wulff shape* $W(g)$ of g is defined as

$$W(g) = \bigcap_{u \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : \langle x, u\rangle \le g(u)\}.$$

Note that when g is the support function h_K of a convex body K then $W(h_K) = K$.

Proposition 3.2. *Consider non-negative functions $g_1, g_2 : \mathbb{S}^{n-1} \to \mathbb{R}$ and $\lambda \in (0, 1)$. Then*

$$W((1-\lambda)g_1 + \lambda g_2) \supseteq (1-\lambda)W(g_1) + \lambda W(g_2).$$

Proof. Indeed consider $x \in W(g_1)$ and $y \in W(g_2)$. Then

$$\langle x, u\rangle \le g_1(u) \text{ and } \langle x, u\rangle \le g_2(u)$$

for all $u \in \mathbb{S}^{n-1}$. Thus

$$(1-\lambda)x + \lambda y \in \{z \in \mathbb{R}^n : \langle z, u\rangle \le (1-\lambda)g_1(u) + \lambda g_2(u)\}$$

for all $u \in \mathbb{S}^{n-1}$, which completes the proof. □

Consider a convex body $K \subset \mathbb{R}^n$, containing the origin in its interior. Given a continuous function $f : \mathbb{S}^{n-1} \to \mathbb{R}$, we define the *perturbation* $K_t(f)$ of K as

$$K_t(f) = W(h_K + tf), \tag{3.1}$$

where $t \in (a, b)$ such that $h_K(u) + tf(u) > 0$ for all $u \in \mathbb{S}^{n-1}$. Note that $K_t(f)$ is also a convex body containing the origin in its interior. To shorten our notation we write K_t instead of $K_t(f)$ when it is clear what function f is being considered.

The following proposition shows that the support function of the perturbation K_t is concave with respect to t.

Proposition 3.3. *The support function $h_{K_t}(u)$ is concave with respect to $t \in (a, b)$ for all $u \in \mathbb{S}^{n-1}$.*

Proof. Consider $\lambda \in [0, 1]$ and $t, s \in (a, b)$. Then, using Proposition 3.2, we get

$$\begin{aligned} K_{\lambda t+(1-\lambda)s} &= W\left(\lambda h_K + (1-\lambda)h_K + (\lambda t + (1-\lambda)s)f\right) \\ &\subseteq \lambda W(h_K + tf) + (1-\lambda)W(h_K + sf). \end{aligned}$$

□

Aleksandrov's lemma [Sch, Lemma 7.5.3] states that if $h : \mathbb{S}^{n-1} \to (0, \infty)$ and $f : \mathbb{S}^{n-1} \to \mathbb{R}$ are continuous functions, then

$$\frac{d}{dt} W(h + tf)|_{t=0} = \int_{\mathbb{S}^{n-1}} f(u)dS_{W(h)}(u). \tag{3.2}$$

In particular, when $h = h_K$ one has

$$\frac{d}{dt} V(K_t)|_{t=0} = \int_{\mathbb{S}^{n-1}} f(u)dS_K(u).$$

Aleksandrov's lemma (3.2) follows from the following fact (see the proof of Lemma 7.5.3 in [Sch]), which is going to be crucial for the proof of Theorem 3.5 below.

Proposition 3.4. *Let $h : \mathbb{S}^{n-1} \to (0, \infty)$ and $f : \mathbb{S}^{n-1} \to \mathbb{R}$ be continuous functions. Then,*

$$\frac{d}{dt} V(W(h + tf), W(h), \dots, W(h))|_{t=0} = \frac{1}{n} \int_{\mathbb{S}^{n-1}} f(u)dS_{W(h)}(u).$$

In particular, when $h = h_K$

$$\frac{d}{dt} V(K_t(f), K, \dots, K)|_{t=0} = \frac{1}{n} \int_{\mathbb{S}^{n-1}} f(u)dS_K(u).$$

Now we go back to the support function $h_{K_t}(u)$. By concavity established in Proposition 3.3, the one-sided derivatives of $h_{K_t}(u)$ exist at every t. In particular, for $t = 0$ we have the following theorem.

Theorem 3.5.

$$\frac{dh_{K_t}(u)}{dt}\Big|_{t=0} = f(u)$$

S_K-almost everywhere on $\mathbb{S}^{n-1}$.

Proof. We need to show that

$$\lim_{t\to 0+} \frac{h_{K_t}(u) - h_K(u)}{t} = f(u) \quad \text{and} \quad \lim_{t\to 0-} \frac{h_{K_t}(u) - h_K(u)}{t} = f(u), \tag{3.3}$$

S_K-almost everywhere on $\mathbb{S}^{n-1}$. Indeed, it follows from the definition of the Wulff shape that

$$\lim_{t\to 0+} \frac{h_{K_t}(u) - h_K(u)}{t} \le f(u)$$

for all $u \in \mathbb{S}^{n-1}$. Assume that there is a set of positive S_K-measure where we have a strict inequality, then

$$\int_{\mathbb{S}^{n-1}} \lim_{t\to 0+} \frac{h_{K_t}(u) - h_K(u)}{t} dS_K(u) < \int_{\mathbb{S}^{n-1}} f(u) dS_K(u).$$

On the other hand, Proposition 3.4, together with the Dominating Convergence theorem gives

$$\frac{1}{n}\int_{\mathbb{S}^{n-1}} f(u)dS_K(u) = \frac{1}{n}\lim_{t\to 0+}\int_{\mathbb{S}^{n-1}} \frac{h_{K_t}(u) - h_K(u)}{t} dS_K(u)$$

$$= \frac{1}{n}\int_{\mathbb{S}^{n-1}} \lim_{t\to 0+} \frac{h_{K_t}(u) - h_K(u)}{t} dS_K(u) < \frac{1}{n}\int_{\mathbb{S}^{n-1}} f(u)dS_K(u).$$

which gives a contradiction. We use similar logic to prove the second part of (3.3). $\square$

Remark 3.6. In view of Theorem 3.5, one might ask if $\frac{dh_{K_t}(u)}{dt}\big|_{t=0} = f(u)$ holds for all $u \in \mathbb{S}^{n-1}$. We note that the answer to this question is negative, in general. Indeed, let $K = \text{conv}\{0, e_1, e_2\}$ be the standard triangle in $\mathbb{R}^2$. Let $f : \mathbb{S}^1 \to [0,1]$ be a continuous function such that $f(e_2) = 1$ and f is zero outside of a small neighborhood of e_2. Then $K_t = K$, for $t \ge 0$ and thus $\lim\limits_{t\to 0+} \frac{h_{K_t} - h_K}{t} = 0$. Moreover, if $-1 < t < 0$ then $h_{K_t}(e_1) \le 1 + t$ and thus $\lim\limits_{t\to 0-} \frac{h_{K_t}(e_2) - h_K(e_2)}{t} \ge 1$. Therefore, the derivative $\frac{dh_{K_t}(e_2)}{dt}\big|_{t=0}$ does not exist.

4. Proof of Theorem 1.1

We begin with the "if" part of the statement. Let K be an n-simplex and $L_1, \dots, L_n$ convex bodies in $\mathbb{R}^n$. Then

$$V(L_1, \dots, L_n)V_n(K) \leq V(L_1, K[n-1])V(L_2, \dots, L_n, K). \tag{4.1}$$

Indeed, since this inequality is invariant under dilations and translations of L_1 we may assume that L_1 is contained in K and intersects every facet of K. Then, by (2.5), we have

$$V(L_1, K[n-1]) = \frac{1}{n} \int\limits_{\mathbb{S}^{n-1}} h_{L_1}(u) dS_K(u) = \frac{1}{n} \sum_{i=0}^{n} h_{L_1}(u_i) V_{n-1}(K^{u_i}), \tag{4.2}$$

where $\{u_0, \dots, u_n\}$ is the set of the outer unit normals to the facets of K. But $h_{L_1}(u_i) = h_K(u_i)$ since $L_1 \subseteq K$ and it intersects each facet of K. Therefore, the last expression in (4.2) equals $V_n(K)$. Now (4.2) follows from the monotonicity property of the mixed volume,

$$V(L_1, \dots, L_n) \leq V(L_2, \dots, L_n, K).$$

Finally, we note that (1.2) is a particular case of (4.1).

To prove the "only if" part we treat the following three cases separately: (1) K is a polytope, (2) K has an infinite number of facets, and (3) K has a finite number of facets but is not a polytope. Then Theorem 1.1 follows from Theorems 4.1, 4.2, and 4.4 below.

4.1. The case of polytopes.

Suppose K is a polytope which satisfies

$$V(L_1, \dots, L_{n-1}, K)V_n(K) \leq V(L_1, K[n-1])V(L_2, \dots, L_{n-1}, K, K) \tag{4.3}$$

for any bodies $L_1, \dots, L_{n-1}$. In particular, when $L_3 = \cdots = L_{n-1} = K$ we obtain

$$V(L_1, L_2, K[n-2])V_n(K) \leq V(L_1, K[n-1])V(L_2, K[n-1]). \tag{4.4}$$

Now the fact that K is an n-simplex follows from [SSZ, Theorem 3.4] which we formulate next.

Theorem 4.1. *Let K be an n-polytope in $\mathbb{R}^n$. Suppose (4.4) holds for all convex bodies L_1 and L_2 in $\mathbb{R}^n$. Then K is a simplex.*

Theorem 4.1 follows also from the results about weakly decomposable convex bodies in Section 5 below.

4.2. K has an infinite number of facets.

The following result implies Theorem 1.1 in this case.

Theorem 4.2. *Let K be a convex body in $\mathbb{R}^n$ with infinitely many facets. Then there exist convex bodies L_1 and L_2 in $\mathbb{R}^n$ such that the inequality*

$$V(L_1, L_2, K[n-2])V_n(K) \leq V(L_1, K[n-1])V(L_2, K[n-1]) \tag{4.5}$$

is false.

Proof. Since K has infinitely many facets, for any $\varepsilon > 0$, there exists a facet $K^u \subset K$ such that $V_{n-1}(K^u) \le \varepsilon$, otherwise the surface area of K would be infinite which contradicts the definition of a convex body. Set $\varepsilon_0 = V_{n-1}(K^u) \le \varepsilon$. Since K has non-empty interior it contains a ball of radius $\delta > 0$, independent of ε_0. Cut this ball in half by a hyperplane orthogonal to u and let M be the half whose facet has outer normal u. Furthermore, let $K_t = K_t(f)$ be a perturbation as in (3.1) for some continuous $f : \mathbb{S}^{n-1} \to [0,1]$ and $t \in (-a, a)$ which we choose below.

Now assume (4.5) holds for any L_1, L_2. Put $L_1 = K_t$ and $L_2 = M$ and consider the function

$$F(t) = V(K_t, M, K[n-2])V_n(K) - V(K_t, K[n-1])V(M, K[n-1]).$$

Then $F(0) = 0$ and $F(t) \le 0$ on $(-a, a)$. Our goal is to show that for small enough ε the right-sided derivative of $F(t)$ at $t = 0$ is positive, which provides a contradiction. We deal with each summand in the definition of $F(t)$ separately. Using (2.4) we get

$$\begin{aligned}
&\lim_{t\to 0+} \frac{1}{t}\big(V(K_t, M, K[n-2]) - V(K, M, K[n-2])\big) \\
&\qquad = \frac{1}{n} \lim_{t\to 0+} \frac{1}{t} \int_{\mathbb{S}^{n-1}} (h_{K_t}(x) - h_K(x))\, dS(M, K[n-2], x) \\
&\qquad \ge \frac{1}{n} \lim_{t\to 0+} \frac{1}{t} \int_{\{u\}} (h_{K_t}(x) - h_K(x))\, dS(M, K[n-2], x) \\
&\qquad = \frac{1}{n} f(u) V(M^u, K^u[n-2]),
\end{aligned}$$

where the last equality follows from and (2.6) and Theorem 3.5 (note that since $S_K(\{u\}) = \varepsilon_0 > 0$, the derivative of $h_{K_t}(u)$ at $t = 0$ exists). Setting $f(u) = 1$ and using the Minkowski inequality (2.2) for the mixed volume $V(M^u, K^u[n-2])$ we get

$$\lim_{t\to 0+} \frac{1}{t}\big(V(K_t, M, K[n-2]) - V(K, M, K[n-2])\big) \ge \frac{1}{n} V_{n-1}(M^u)^{\frac{1}{n-1}} V_{n-1}(K^u)^{\frac{n-2}{n-1}}.$$

Recall that $V_{n-1}(K^u) = \varepsilon_0$ and $V_{n-1}(M^u)$ is positive, independent of ε_0. Thus, we can write

$$\lim_{t\to 0+} \frac{1}{t}\big(V(K_t, M, K[n-2]) - V(K, M, K[n-2])\big) \ge C_1 \varepsilon_0^{\frac{n-2}{n-1}}, \tag{4.6}$$

for some $C_1 > 0$, independent of ε_0.

Now we turn to the second summand in $F(t)$. This is straightforward from Proposition 3.4:

$$\lim_{t\to 0^+} \frac{1}{t} \left(V(K_t, K[n-1]) - V_n(K)\right) = \frac{d}{dt} V(K_t, K[n-1])|_{t=0} = \frac{1}{n} \int_{\mathbb{S}^{n-1}} f(x) dS_K(x). \tag{4.7}$$

We can choose f to be positive on the facet K^u and zero outside. Then the integral above equals $C_2\varepsilon_0$ for some $C_2 > 0$, independent of ε_0. Combining (4.6) and (4.7) we obtain

$$\lim_{t\to 0^+} \frac{F(t)}{t} \ge C_1 V_n(K) \varepsilon_0^{\frac{n-2}{n-1}} - C_2 V(M, K[n-1]) \varepsilon_0 > 0,$$

for small enough ε_0 and, consequently, for small enough ε. This completes the proof of the theorem. $\square$

4.3. **K has finitely many facets and is not a polytope.** Although the methods in this case are similar to the ones used in the previous case, the proof is more involved.

Let $\mathrm{supp}(S_K)$ be the support of the measure S_K, i.e. the largest (closed) subset of $\mathbb{S}^{n-1}$ for which every open neighborhood of every point of the set has positive measure. We start with the following observation.

Proposition 4.3. *Let K be a convex body in $\mathbb{R}^n$ which satisfies*

$$V(L_1, L_2, K[n-2])V_n(K) \leq V(L_1, K[n-1])V(L_2, K[n-1]) \tag{4.8}$$

for all L_1, L_2. Then $\dim K^u > 0$ for every $u \in \mathrm{supp}(S_K)$.

Proof. The proof is similar to the one of [SSZ, Theorem 4.1]. Assume there exists $u \in \mathrm{supp}(S_K)$ such that $K^u = \{y\}$, for some $y \in \partial K$. For $\varepsilon > 0$ define the truncated body

$$K_\varepsilon = \{x \in K : \langle x, u\rangle \leq h_K(u) - \varepsilon\}.$$

Since $K^u = \{y\}$, it follows that there exist $v \in \mathbb{S}^{n-1}$ and $\varepsilon > 0$ such that the projections of K and K_ε are the same:

$$K_\varepsilon|v^\perp = K|v^\perp. \tag{4.9}$$

A proof of this statement can be found in the proof of [SSZ, Theorem 4.1].

Let $L_1 = [0, v]$ and $L_2 = K_\varepsilon$. Then, from (2.7) and (4.9), we get

$$V(L_1, L_2, K[n-2]) = \frac{1}{n}V(K_\varepsilon|v^\perp, K|v^\perp[n-2]) = \frac{1}{n}V(K|v^\perp) = V(L_1, K[n-1]).$$

Therefore, it suffices to show that $V_n(K) > V(L_2, K[n-1])$; this will contradict our assumption (4.8). Note that $h_{K_\varepsilon} \leq h_K$ and

$$V(K) - V(L_2, K[n-1]) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} (h_K(v) - h_{K_\varepsilon}(v))dS_K(v),$$

by (2.1) and (2.5). Therefore it is enough to show that there exists a Borel set $\Omega \subseteq \mathbb{S}^{n-1}$, such that $S_K(\Omega) > 0$ and $h_K(v) > h_{K\varepsilon}(v)$, for all $v \in \Omega$. To this end we claim the following.

Claim. There exists an open set $U \subseteq \mathbb{S}^{n-1}$, such that U contains u and $\tau(K, U) \subseteq K \setminus K_\varepsilon$. To see this, assume that this is not the case. Then, there is a sequence of open spherical caps $U_N = U(u, r_N)$ of radii $r_N \to 0$, as $N \to \infty$, such that $\tau(K, U_N)$ contains a point $x_N \in K_\varepsilon$. This implies that there exists $u_N \in U_N$ such that $u_N \in K^{u_N}$. Since $u_N \to u$, by compactness there exists $x \in \partial K \cap K_\varepsilon$, such that $x \in K^u$. This contradicts our assumption $K^u = \{y\}$, since $x \neq y$ as $y \notin K_\varepsilon$.

An immediate reformulation of the previous claim states that there exists an open set U in $\mathbb{S}^{n-1}$ containing u, such that $K^v \subseteq K \setminus K_\varepsilon$, for all $v \in U$. In other words, $h_K(v) > h_{K_\varepsilon}(v)$, for all $v \in U$. However, since $u \in \mathrm{supp}(S_K)$, we have $S_K(U) > 0$, which completes the proof. $\square$

Suppose K satisfies the conditions of Proposition 4.3. Then we can choose an open subset $U \subseteq \mathbb{S}^{n-1}$ satisfying the following three conditions:

(i) $V = U \cap \mathrm{supp}(S_K) \neq \varnothing$ (and, hence, $S_K(V) > 0$),

(ii) V contains no normals to facets of K,

(iii) there exists $\ell > 0$ such that for any $u \in V$ the face K^u contains a segment of length at least ℓ.

Indeed, we can ensure (i) and (ii) are satisfied since we can exclude the finite set of unit normals to the facets of K from the support of S_K. For part (iii), choose V' satisfying (i) and (ii) and consider the sequence of subsets

$$V_i = \{u \in V' : K^u \text{ contains a segment of length at least } 1/i\}, \quad i \in \mathbb{N}.$$

By Proposition 4.3, V_i is nonempty for i large enough and $\cup_{i\geq 1} V_i = V'$. Therefore, there exists $m \geq 1$ such that V_m has positive S_K-measure. Put $\ell = 1/m$ and $V = V_m$.

Our next step is to construct a sequence of perturbations $K_t(f_N)$ for certain functions f_N. We begin by constructing a nested sequence $U_N \subset U$ of spherical caps centered at points of V.

Let $U_1 = U(u_0, \rho_1) \subset U$ for some $u_0 \in V$ and $\rho_1 > 0$. Consider a continuous function $f_1 : \mathbb{S}^{n-1} \to [0,1]$ such that f_1 is zero on $\mathbb{S}^{n-1} \setminus U_1$, positive on U_1, and takes values at least $\frac{1}{2}$ on $U_1' := U(u_0, \frac{\rho_1}{2})$. By Theorem 3.5 there exists $u_1 \in U_1' \cap V$ such that $h_{K_t(f_1)}$ has derivative equal to $f_1(u_1)$. Define $U_2 = U(u_1, \rho_2) \subset U_1$ where $\rho_2 \leq \frac{\rho_1}{2}$ and continue the process. We obtain a sequence of spherical caps $U_N = U(u_{N-1}, \rho_N)$ with centers at $u_{N-1} \in V$ and radii $\rho_N \to 0$, as $N \to \infty$, and a sequence of functions $f_N : \mathbb{S}^{n-1} \to [0,1]$ satisfying:

(a) f_N is positive on U_N,

(b) f_N is zero on $\mathbb{S}^{n-1} \setminus U_N$,

(c) $\frac{d}{dt} h_{K_t(f_N)}|_{t=0} = f_N(u_N) \geq \frac{1}{2}$.

We are ready for the main result in the case when K is not a polytope and has at most finitely many facets.

Theorem 4.4. *Let K be a convex body in $\mathbb{R}^n$ with finitely many facets which is not a polytope. Then there exist convex bodies $L_1, \dots, L_{n-1}$ in $\mathbb{R}^n$ such that the inequality*

$$V(L_1, \dots, L_{n-1}, K) V_n(K) \leq V(L_1, K[n-1]) V(L_2, \dots, L_{n-1}, K, K) \tag{4.10}$$

is false.

Proof. Suppose K satisfies (4.10) for all $L_1, \dots, L_{n-1}$. In particular, it satisfies the conditions of Proposition 4.3 and, hence, there exist a set $V \subset \mathbb{S}^{n-1}$, a sequence of points $u_N \in V$, and perturbations $K_{t,N} = K_t(f_N)$ as constructed above. Choose a ball B of radius $\delta > 0$ contained in K and cut it in half by a hyperplane orthogonal to u_N. Let M_N be the half of B whose facet has outer normal u_N. We set $L_1 = K_{t,N}$ and $L_i = M_N$ for $2 \leq i \leq n-1$. Then (4.10) produces

$$V(K_{t,N}, M_N[n-2], K) V_n(K) \leq V(K_{t,N}, K[n-1]) V(M_N[n-2], K, K). \tag{4.11}$$

For every $N \geq 1$, define the function

$$F_N(t) = V(K_{t,N}, M_N[n-2], K) V_n(K) - V(K_{t,N}, K[n-1]) V(M_N[n-2], K, K). \tag{4.12}$$

Note that $F_N(0) = 0$ and if (4.11) holds then $F_N(t) \leq 0$ for all $t \in (-a, a)$. As in the proof of Theorem 4.2 we show that for N large enough the right-sided derivative of $F_N(t)$ at $t = 0$ is positive, which gives a contradiction.

For the first summand in $F(t)$ we have

$$\begin{aligned} V(K_{t,N}, M_N[n-2], K) &- V(K, M_N[n-2], K) \\ &= \frac{1}{n} \int\limits_{\mathbb{S}^{n-1}} \left(h_{K_{t,N}}(x) - h_K(x)\right) dS(M_N[n-2], K, x) \\ &\geq \frac{1}{n} \int\limits_{\{u_N\}} \left(h_{K_{t,N}}(x) - h_K(x)\right) dS(M_N[n-2], K, x) \\ &= \frac{1}{n} \left(h_{K_{t,N}}(u_N) - h_K(u_N)\right) V(M_N^{u_N}[n-2], K^{u_N}), \end{aligned}$$

where the last equality follows Lemma 2.1. We claim that the mixed volume is bounded below by a positive constant C independent of N:

$$V(M_N^{u_N}[n-2], K^{u_N}) \geq C > 0.$$

Indeed, by condition (iii) in the definition of U, the face K^{u_N} contains a segment I of length at least $\ell > 0$. Also, by construction, the facet $M_N^{u_N}$ is an $(n-1)$-dimensional disc D_δ of radius $\delta > 0$. Thus,

$$V(M_N^{u_N}[n-2], K^{u_N}) \geq V(D_\delta[n-1], I) = \frac{\ell}{n} V_{n-1}(D_\delta) = C > 0.$$

Therefore, for the first summand of $F_N(t)$ we have

$$\begin{aligned} \lim_{t\to 0^+} \frac{1}{t} \big(V(K_{t,N}, &M_N[n-2], K) - V(K, M_N[n-2], K)\big) \\ &\geq C \lim_{t\to 0^+} \frac{h_{K_{t,N}}(u_N) - h_K(u_N)}{t} = C f_N(u_N) \geq \frac{C}{2}. \end{aligned}$$

The last equality follows from Theorem 3.5 and the last inequality follows from condition (c) in the definition of f_N.

For the second summand of $F_N(t)$ we use Proposition 3.4 as before:

$$\lim_{t\to 0^+} \frac{1}{t} \left(V(K_{t,N}, K[n-1]) - V_n(K)\right) = \frac{d}{dt} V(K_{t,N}, K[n-1])|_{t=0} = \frac{1}{n} \int\limits_{\mathbb{S}^{n-1}} f_N(x) dS_K(x).$$

Bringing the two summands together we obtain

$$\lim_{t\to 0^+} \frac{F_N(t)}{t} \geq \frac{C}{2} V_n(K) - \frac{1}{n} V(M_N[n-2], K, K) \int\limits_{\mathbb{S}^{n-1}} f_N(x) dS_K(x).$$

Recall that f_N is bounded and vanishes outside of a spherical cap of radius $\rho_N \to 0$. By choosing N large enough we can ensure that the integral of f_N is smaller than any given number. On the other hand, $V(M_N[n-2], K, K) \leq V_n(K)$ since $M_N \subset K$, for any $N \geq 1$. Therefore, it is enough to take N large enough so $\int\limits_{\mathbb{S}^{n-1}} f_N(x) dS_K(x) < nC/2$ to obtain

$$\lim_{t\to 0^+} \frac{F_N(t)}{t} > 0.$$

This completes the proof of the theorem. $\square$

5. Weakly decomposable convex bodies

In this section we discuss the impact of the results of Section 4 on the conjecture about the Bezout inequality for mixed volumes formulated in [SZ]. Recall this conjecture:

Conjecture 5.1. *Fix an integer $2 \le r \le n$ and let $K \subset \mathbb{R}^n$ be a convex body satisfying the Bezout inequality*

$$V(L_1, \dots, L_r, K[n-r])V_n(K)^{r-1} \le \prod_{i=1}^{r} V(L_i, K[n-1]) \tag{5.1}$$

for all convex bodies $L_1, \dots, L_r$ in $\mathbb{R}^n$. Then K is an n-simplex.

We then introduce a class of weakly decomposable convex bodies which generalizes the classical notion of decomposable bodies. We show that every polytope which is not a simplex is weakly decomposable and there are many weakly decomposable bodies which are not polytopes. Our main result of this section (Theorem 5.7) asserts that a convex body K satisfying (5.1) for all convex bodies $L_1, \dots, L_r$ cannot be weakly decomposable, which improves the result of [SZ, Theorem 3.3].

5.1. **Impact of results of Section 4 on Conjecture 5.1.** The special case of the inequality (5.1) with $r = 2$ was already considered in the introduction, see (1.3). As we mentioned there, (1.1) implies (1.3). In fact, (1.1) implies (5.1) for any $2 \le r \le n$. Indeed, (5.1) with $r = n$ can be obtained from (1.1) by successive iterations. Clearly the case of $r = n$ implies all the other cases.

Next, observe that if Conjecture 5.1 is true for $r = 2$ then it is true for any $2 \le r \le n$. In dimension three the conjecture (with $r = 2$) says that if a convex body $K \subset \mathbb{R}^3$ satisfies

$$V(L_1, L_2, K)V_3(K) \le V(L_1, K, K)V(L_2, K, K) \tag{5.2}$$

for any convex bodies L_1, L_2 in $\mathbb{R}^3$ then K is a 3-simplex. This is precisely the statement of Theorem 1.1 with $n = 3$. Therefore, Conjecture 5.1 is true in the three-dimensional case. In the general case, the results of Theorem 4.1, Proposition 4.3, and Theorem 4.2 provide with the following new information regarding Conjecture 5.1.

Corollary 5.2. *Let $K \subset \mathbb{R}^n$ for $n \ge 4$ be a convex body which is not a polytope and which satisfies (5.1) for all convex bodies $L_1, \dots, L_r$ in $\mathbb{R}^n$. Then K has at most finitely many facets and for every u in the support of the surface area measure S_K, the face K^u is positive dimensional.*

5.2. **Weakly decomposable bodies.** Recall that a convex body is called *decomposable* if $K = L + M$, for some compact convex sets L, M which are not homothetic to K. It was shown in [SZ] that decomposable convex bodies do not satisfy the Bezout inequality (5.1) for any $2 \le r \le n$. We generalize the definition of decomposability as follows.

Definition 5.3. A convex body K in $\mathbb{R}^n$ is called *weakly decomposable* if there exists a convex set M, not homothetic to K, such that the surface area measure S_{K+M} of $K + M$ is absolutely continuous with respect to the surface area measure S_K of K. A convex body which is not weakly decomposable is called *weakly indecomposable.*

The following proposition justifies the terminology.

Proposition 5.4. *Every decomposable convex body is weakly decomposable.*

To prove this we need the following lemma.

Lemma 5.5. *Let M and L be compact convex sets in $\mathbb{R}^n$. Then*

i) $S_{L+M} = \sum_{r=0}^{n-1} \binom{n-1}{r} S(M[r], L[n-r], \cdot)$,

ii) S_{L+M} *is absolutely continuous with respect to* S_L *if and only if the mixed area measure* $S(M[r], L[n-r], \cdot)$ *is absolutely continuous with respect to* S_L *for all* $0 \leq r < n$.

Proof. Part i) is an immediate consequence of the invariance properties of mixed area measures mentioned in Section 2 (see also [Sch, (5.18)]) and part ii) follows immediately from part i). □

Proof of Proposition 5.4. Let $K = L + M$ be a decomposable convex body. To see that it is weakly decomposable, notice that by part i) of Lemma 5.5

$$S_{K+M} = S_{L+2M} = \sum_{r=0}^{n-1} \binom{n-1}{r} 2^r S(M[r], L[n-r], \cdot),$$

which is absolutely continuous with respect to

$$S_K = \sum_{r=0}^{n-1} \binom{n-1}{r} S(M[r], L[n-r], \cdot).$$

□

There are many convex bodies that are not decomposable. For example, every polytope whose 2-dimensional faces are simplices is indecomposable, see [Sch, Corollary 3.2.17]. The class of weakly decomposable bodies, however, is much larger than that of decomposable bodies. The next proposition shows that it includes all convex polytopes.

Proposition 5.6. *The only weakly indecomposable polytopes are simplices.*

Proof. In [SSZ, Lemma 3.1], we showed that if K is a polytope which is not a simplex, then there exists a convex body M not homothetic to K (obtained from K by moving a facet of K along the direction of its outer normal) such that $S(K[r], M[n-r], \cdot)$ is absolutely continuous with respect to S_K, for all $0 \leq r < n$. Combing this result with Lemma 5.5, we obtain the required statement. □

The following theorem, which is the main result of this section, is a generalization of both facts that decomposable convex bodies and polytopes that are not simplices do not satisfy the Bezout inequality (5.1) for any $2 \leq r \leq n$, see Theorem 4.1 and [SZ, Theorem 3.3].

Theorem 5.7. *Let $K \subset \mathbb{R}^n$ be a convex body satisfying*

$$V(L_1, \ldots, L_r, K[n-r]) V_n(K)^{r-1} \leq \prod_{i=1}^{r} V(L_i, K[n-1]) \tag{5.3}$$

for all convex bodies $L_1, \dots, L_r$ in $\mathbb{R}^n$, where $2 \le r \le n$. Then K is weakly indecomposable.

Proof. It is enough to prove the theorem when $r = 2$ as the general case follows from it. Assume that K satisfies

$$V(L_1, L_2, K[n-2])V_n(K) \le V(L_1, K[n-1])V(L_2, K[n-1]) \tag{5.4}$$

for all L_1, L_2 and there exists a convex body M such that S_{K+M} is absolutely continuous with respect to S_K. We need to show that M is homothetic to K. The proof combines Theorem 3.5 and the argument in the proof of [SSZ, Lemma 3.3].

We use induction to show that

$$S(M[r], K[n-1-r], \cdot) = \lambda^r S_K \tag{5.5}$$

holds for $0 \le r < n$, where $\lambda = V(M, K[n-1])/V_n(K)$. The case $r = 0$ is trivial. Assume (5.5) holds for all $0 \le r \le m$. We need to show that (5.5) holds for $r = m+1$. First, we claim that for any convex body L we have

$$V(L, M[m+1], K[n-m-2]) \le \lambda^{m+1} V(L, K[n-1]), \tag{5.6}$$

with equality when $L = K$. If $m = 0$, the claim follows immediately from our assumption (5.4). Therefore we may assume $m \ge 1$. By (5.5) with $r = m$, for any convex body L we have

$$V(L, M[m], K[n-1-m]) = \lambda^m V(L, K[n-1]). \tag{5.7}$$

Applying the Aleksandrov–Fenchel inequality (2.3) to (5.7), we obtain

$$\sqrt{V(L, M[m+1], K[n-2-m])V(L, M[m-1], K[n-m])} \le \lambda^m V(L, K[n-1]).$$

Then, applying (5.5) with $r = m-1$ to the second factor in the left-hand side of the above inequality, we obtain (5.6). Furthermore, if $L = K$, (5.7) produces

$$\begin{aligned} V(K, M[m+1], K[n-m-2]) &= V(M, M[m], K[n-1-m]) = \lambda^m V(M, K[n-1]) \\ &= \lambda^{m+1} V_n(K) = \lambda^{m+1} V(K, K[n-1]). \end{aligned}$$

Thus there is equality in (5.6) when $L = K$.

Next, for a continuous function $f : \mathbb{S}^{n-1} \to \mathbb{R}$ and sufficiently small $|t|$, define

$$F(t) = V(K_t, M[m+1], K[n-m-2]) - \lambda^{m+1} V(K_t, K[n-1]),$$

where $K_t = K_t(f)$ as in (3.1). By assumption and Lemma 5.5, the mixed area measure $S(M[m+1], K[n-r-2], \cdot)$ is absolutely continuous with respect to S_K. Therefore, by Theorem 3.5, $F(t)$ is differentiable at $t = 0$ with

$$F'(0) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} f(u) dS(M[m+1], K[n-m-2], u) - \lambda^{m+1} \frac{1}{n} \int_{\mathbb{S}^{n-1}} f(u) dS_K(u).$$

Since $F(0) = 0$ and $F(t) \le 0$ for small $|t|$, it follows that $F'(0) = 0$ or equivalently,

$$\int_{\mathbb{S}^{n-1}} f(u) dS(M[m+1], K[n-m-2], u) = \lambda^{m+1} \int_{\mathbb{S}^{n-1}} f(u) dS_K(u).$$

But f is arbitrary, which implies that

$$S(M[m+1], K[n-m-2], \cdot) = \lambda^{m+1} S_K$$

and, hence, (5.5) holds for $r=m+1$. Finally, using (5.5) with $r=n-1$ we see that S_M is proportional to S_K. Therefore, by Minkowski's Uniqueness Theorem (see [Sch, Theorem 8.1.1]) M is homothetic to K, as required. $\square$

In light of Proposition 5.6 it seems plausible that the only weakly indecomposable convex bodies are simplices. Then Theorem 5.7 would imply that Conjecture 5.1 is true. In fact, one can construct many weakly decomposable convex bodies that are not convex polytopes. For example, one can start with a polytope and fix one of its facets. Then any convex body which coincides with the polytope in a neighborhood of the fixed facet will be weakly decomposable, as the same argument as in the proof of Proposition 5.6 applies. Still a complete description of the class of weakly decomposable bodies is open.

6. Isomorphic versions of inequalities (1.1) and (1.2).

As follows from Theorem 1.1, the inequality (1.1) may not hold when K is not an n-simplex. It is natural to ask if (1.1) can be relaxed so that it holds for arbitrary convex sets $L_1,\dots,L_n$ and K in $\mathbb{R}^n$. It turns out that such an inequality is obtained by introducing the constant n in the right-hand side:

$$V(L_1,\dots,L_n)V_n(K)\le nV(L_1,K[n-1])V(L_2,\dots,L_n,K). \tag{6.1}$$

To show this, we follow the idea of Jian Xiao who used Diskant's inequality to prove that for any convex bodies K,L in $\mathbb{R}^n$ one has

$$V_n(K)L\subseteq nV(L,K[n-1])K, \tag{6.2}$$

up to a translation, see [Xi, Section 3.1]. Then, (6.1) follows from (6.2) by the monotonicity of the mixed volume.

The inequality (6.1) is, in fact, sharp. For example, one can take L_1 to be a unit segment $L_1=[0,u]$ for some $u\in\mathbb{S}^{n-1}$ and K to be a cylinder $K=L_1\times K'$ for some $(n-1)$-dimensional convex body K' in the orthogonal hyperplane $u^\perp$. Then $V_n(K)=V_{n-1}(K')=nV(L_1,K[n-1])$ by (2.7). Choose any $(n-1)$-dimensional convex bodies $L_2,\dots,L_n$ in $u^\perp$. From the monotonicity of the mixed volume we see that $V(L_2\dots,L_n,K')=0$ as all of the sets are contained in some $(n-1)$-dimensional ball. Now, since $K=L_1+K'$, by the linearity and symmetry properties

$$V(L_2,\dots,L_n,K)=V(L_2\dots,L_n,L_1)+V(L_2\dots,L_n,K')=V(L_1,L_2\dots,L_n),$$

which provides equality in (6.1). Note that both sides of the equality are positive. Thus, one can use an approximation argument to show that the constant n in (6.1) cannot be improved for the class of convex bodies as well.

Next we turn to an isomorphic version of (1.2). From (6.1) we have

$$V(L_1,\dots,L_{n-1},K)V_n(K)\le nV(L_1,K[n-1])V(L_2,\dots,L_{n-1},K,K). \tag{6.3}$$

However, we do not expect this inequality to be sharp. Although we do not have a better estimate in general, we can show that for any *zonoids* $L_1,\dots,L_{n-1}$ and any K in $\mathbb{R}^n$ one has

$$V(L_1,\dots,L_{n-1},K)V_n(K)\le (n-1)V(L_1,K[n-1])V(L_2,\dots,L_{n-1},K,K), \tag{6.4}$$

and in this class the inequality is sharp. (See [Sch, p. 191] for the definition of zonoids.) Indeed, similar to the proof of [SZ, Theorem 5.6], it is enough to show that (6.4) holds when $L_1, \dots, L_{n-1}$ are orthogonal segments. This is a particular case of the Loomis-Whitney type inequalities in [AAGJV, Theorem 1.7] (see also [FGM, GHP]). Note that (6.4) becomes equality, for example, when $L_1 = [0, e_1], \dots, L_{n-2} = [0, e_{n-1}]$ and $K = \mathrm{conv}\{K', e_n\}$, where K' is the unit $(n-1)$-dimensional cube in $e_n^\perp$.

An isomorphic version of (5.1) was first studied in [SZ]. It was shown that there exists a constant $c_{n,r} > 0$ such that

$$V(L_1, \dots, L_r, K[n-r]) V_n(K)^{r-1} \le c_{n,r} \prod_{i=1}^{r} V(L_i, K[n-1]) \tag{6.5}$$

holds for arbitrary convex bodies $L_1, \dots, L_r$ and K in $\mathbb{R}^n$, see [SZ, Theorem 5.7]. Since then, several new results on estimating the constant $c_{n,r}$ have been obtained, see [AFO, BGL, Xi]. Moreover, a generalization of (6.5) also appeared in [Xi]. In particular, [BGL, Theorem 1.5] provides an isomorphic version of inequality (1.3)

$$V(L_1, L_2, K[n-2]) V_n(K) \le 2V(L_1, K[n-1]) V(L_2, K[n-1]).$$

This inequality becomes equality, for example, when $L_1 = [0, e_1]$, $L_2 = [0, e_2]$, and $K = \mathrm{conv}\{K', e_3, \dots, e_n\}$, where K' is the unit square in the span of $\{e_1, e_2\}$.

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