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COMBINATORIAL CONSTRUCTION OF TORIC RESIDUES

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ABSTRACT. The toric residue is a map depending on $n+1$ divisors on a complete toric variety of dimension n. It appears in a variety of contexts such as sparse polynomial systems, mirror symmetry, and GKZ hypergeometric functions. In this paper we investigate the problem of finding an explicit element whose toric residue is equal to one. Such an element is shown to exist if and only if the associated polytopes are essential. We reduce the problem to finding a collection of partitions of the lattice points in the polytopes satisfying a certain combinatorial property. We use this description to solve the problem when $n = 2$ and for any n when the polytopes of the divisors share a complete flag of faces. The latter generalizes earlier results when the divisors were all ample.

1. INTRODUCTION

Toric residues are fundamental invariants of sparse polynomial systems. They were first studied by Cox [13] who defined the residue of $n+1$ sections of an ample line bundle on a toric variety X. The definition was extended by Cattani, Cox, and Dickenstein to sections of $n+1$ arbitrary line bundles [4]. There are numerous applications to sparse resultants and resultant or subresultant complexes [7, 14], mixed Hodge structures [2], and mirror symmetry [3].

The related notion of global residue in the torus, a sum of Grothendiek local residues, was studied by Gelfond, Khovanskii, and Soprounov [17, 18]. Cattani, Cox and Dickenstein [4] showed that the global residue could always be computed as an instance of the toric residue. Applications of the toric and global residue include GKZ hypergeometric systems [6, 9, 10] and computations on sparse polynomial systems such as counting the number of real roots and computing elementary symmetric functions on the roots [8, 17].

Given $n+1$ arbitrary sparse Laurent polynomials $f_0, \dots, f_n$ in n affine variables, let $P_0, \dots, P_n$ be their corresponding Newton polytopes. The Minkowski sum $P = P_0 + \cdots + P_n$ determines a toric variety X, and each P_i corresponds to a semi-ample divisor class. In the homogeneous coordinate ring S of X each f_i can be homogenized to a polynomial F_i of degree α_i corresponding to the divisor class of P_i. The toric residue Res_F is a linear function on homogeneous polynomials of a certain critical degree corresponding to the interior of P which vanishes on the ideal of the F_i.

In many cases of interest, for example when all of the P_i are full dimensional, the ideal of the F_i has codimension 1 in the critical degree. Hence knowing a single element of non-zero residue will allow a full computation of the residue map. More generally, we show in Section 3 that there is an element of non-zero residue whenever the polytopes form an essential family.

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The goal of this paper is a general framework for the construction of specific elements whose residue we can compute. The construction depends only on the combinatorics and affine geometry of the polytopes P_i.

Theorem 1.1. *Let X be a complete toric variety of dimension n. Fix $n+1$ semi-ample degrees $\alpha_0, \dots, \alpha_n$ on X and let $P_0, \dots, P_n$ be their polytopes. Let*

$$P_i \cap \mathbb{Z}^n = M_{i0} \sqcup \cdots \sqcup M_{in}, \quad 0 \le i \le n,$$

be a collection of partitions of the lattice points of the P_i such that

1. *for any lattice point $u \in M_{ij}$, at least one vertex of the minimal face of P_i containing u lies in M_{ij},*
2. *for any permutation ε of $\{0, \dots, n\}$:*

$$\sum_{i=0}^{n} M_{\varepsilon(i)i} \subset \operatorname{int}\Big(\sum_{i=0}^{n} P_i\Big).$$

Given a collection of Laurent polynomials $f_0, \dots, f_n$ supported on $P_0, \dots, P_n$

$$f_i = \sum_{u \in P_i \cap \mathbb{Z}^n} c_u t^u, \quad 0 \le i \le n$$

define polynomials

$$f_{ij} = \sum_{u \in M_{ij}} c_u t^u, \quad 0 \le i, j \le n.$$

Then $h = \det(f_{ij})$ is a Laurent polynomial supported on $\operatorname{int}(\sum_{i=0}^{n} P_i)$. The toric residue $\operatorname{Res}_F(H)$ of the corresponding homogeneous polynomial H of critical degree for the homogenized $F_0, \dots, F_n$ is an integer that depends only on the combinatorics of the P_i and the partitions of their lattice points.

Using this theorem we are able to find an element of residue ± 1, i.e. find an appropriate collection of partitions, in two important cases. The first is when P_i share a complete flag of faces. This will generalize earlier results of D'Andrea and Khetan when all of the α_i were ample degrees. The second application is a complete analysis when $n = 2$. We show that, except for one degenerate family of supports, we can always find a collection of partitions yielding an element of residue ± 1.

The proof of the theorem makes use of some very elegant combinatorics. Starting with a partition of the lattice points we will show that there are induced colorings of the faces of the polytope $P = \sum P_i$. Moreover, the matrix will yield a canonical coloring of the facets of the barycentric refinement of P. Such a facet coloring will allow us to reduce the computation to that of the residue of a monomial with respect to a monomial ideal. By an earlier theorem of Soprounov [19], the residue is the *combinatorial degree* of the coloring which can be computed by counting the number of flags of certain colors.

The paper is organized as follows. Section 2 provides the definitions of the toric residue and some basic properties. Section 3 proves the existence of elements of non-zero residue if and only if the polytopes are essential. Section 4 introduces facet colorings of polytopes and their connection to the toric residue of monomials. The residue for general polynomials is reduced to the monomial case via the Global Transformation Law. Section 5 and Section 6 discuss the relationships between partitions, colorings, and residue matrices used to complete the proof of Theorem 1.1. Section 7 uses the previous results to give an explicit element of residue 1 when the polytopes F_i share a complete flag of faces. Section 8 is a complete analysis when X is of dimension 2. Finally, Section 9 discusses progress in dimensions three and higher.

2. Preliminaries

We begin by setting up the notation and reviewing some basic definitions and facts about toric varieties and toric residues. For details and proofs we refer the reader to [4, 12, 13, 15].

2.1. **Toric residue.** Consider an n-dimensional complete toric variety X determined by a rational complete fan $\Sigma \subset \mathbb{R}^n$. Let $\Sigma(1)$ denote the set of 1-dimensional cones (rays) of Σ. Each ray $\rho \in \Sigma(1)$ determines a $\mathbb{T}$-invariant irreducible divisor D_ρ on X. As introduced by Cox in [12] the variety X has the homogeneous coordinate ring $S = \mathbb{C}[x_\rho : \rho \in \Sigma(1)]$ graded by the Chow group $A_{n-1}(X)$ so that a monomial $x^a = \prod_\rho x_\rho^{a_\rho}$ has degree

$$\deg(x^a) = \Big[\sum_{\rho\in\Sigma(1)} a_\rho D_\rho \Big] \in A_{n-1}(X).$$

Denote by S_α the graded piece of S consisting of all polynomials of degree $\alpha \in A_{n-1}(X)$.

Let $D = \sum_\rho a_\rho D_\rho$ be a representative of $\alpha \in A_{n-1}(X)$. It defines a continuous piecewise linear function ψ_D on the support $|\Sigma|$ such that $\psi_D(v_\rho) = -a_\rho$ for all $\rho \in \Sigma(1)$, where v_ρ denotes the primitive generator of ρ (see [15, Section 3.3]). It also determines a convex polytope

$$P_D = \{u \in \mathbb{R}^n : \langle u, v_\rho\rangle \geq -a_\rho, \ \rho \in \Sigma(1)\} = \{u \in \mathbb{R}^n : u \geq \psi_D \text{ on } |\Sigma|\}.$$

To every lattice point u of P_D we can assign a monomial χ^u in S of degree α:

$$\chi^u = \prod_{\rho\in\Sigma(1)} x_\rho^{\langle u,v_\rho\rangle + a_\rho}, \quad u \in P_D \cap \mathbb{Z}^n.$$

One can check that this map is a bijection. Furthermore, given a Laurent polynomial $f(t) = \sum_u c_u t^u$ supported in P_D its P_D*-homogenization* is the homogeneous polynomial

$$F = \sum_{u\in P_D\cap\mathbb{Z}^n} c_u \chi^u = \sum_{u\in P_D\cap\mathbb{Z}^n} c_u \prod_{\rho\in\Sigma(1)} x_\rho^{\langle u,v_\rho\rangle + a_\rho} \in S_\alpha. \tag{2.1}$$

Notice that if f is supported in the interior of P_D then the P_D-homogenization is divisible by the product of all the variables x_ρ, $\rho \in \Sigma(1)$. It is easy to see that if D and D' are linearly equivalent then $\psi_D - \psi_{D'}$ is a linear function, and P_D and $P_{D'}$ are the same up to a translation. Therefore, P_D-homogenization is independent of the choice of the representative D of the divisor class α. In what follows the *polytope of* α will mean the polytope of any representative of α and will be denoted by P_α.

Recall the construction of the Euler form Ω from [4]. Let $(e_1, \ldots, e_n)$ be a basis for $\mathbb{Z}^n$ and for every subset $I \subset \Sigma(1)$ of size n denote

$$\det(\eta_I) = \det(\langle e_i, v_\rho\rangle : 1 \leq i \leq n, \rho \in I), \quad dx_I = \wedge_{\rho\in I}\, dx_\rho, \quad \hat{x}_I = \prod_{\rho\notin I} x_\rho.$$

Then the *Euler form* on X is the sum over all size n subsets $I \subset \Sigma(1)$:

$$\Omega = \sum_{|I|=n} \det(\eta_I)\hat{x}_I dx_I.$$

Now we recall the definition of the toric residue [13, 4]. Consider $n+1$ homogeneous polynomials $F_i \in S_{\alpha_i}$, for $0 \le i \le n$. Their critical degree is defined to be

$$\nu = \sum_{i=0}^{n} \alpha_i - \sum_{\rho} \deg(x_\rho).$$

Then for every polynomial H of degree ν consider a meromorphic n-form on X:

$$\omega_F(H) = \frac{H\Omega}{F_0 \cdots F_n},$$

where Ω is the Euler form. We use F to denote the list $(F_0, \dots, F_n)$. Suppose that the F_i do not vanish simultaneously on X. Then X has an open cover $\mathcal{U}$ by the $n+1$ sets $U_i = \{x \in X : F_i(x) \neq 0\}$ and $\omega_F(H)$ defines a Čech cohomology class $[\omega_F(H)] \in H^n(X, \widehat{\Omega}^n_X)$ relative to the cover $\mathcal{U}$. Here $\widehat{\Omega}^n_X$ denotes the sheaf of Zariski n-forms on X. One can check that the class $[\omega_F(H)]$ is alternating in the order of the F_i and is zero if H belongs to the ideal of $F_0, \dots, F_n$. Therefore, $[\omega_F(H)]$ depends on the equivalence class of H modulo the ideal $\langle F_0, \dots, F_n\rangle$. The *toric residue map*

$$\mathrm{Res}_F^X : S_\nu / \langle F_0, \dots, F_n\rangle_\nu \to \mathbb{C},$$

is given by $\mathrm{Res}_F^X(H) = \mathrm{Tr}_X([\omega_F(H)])$, where Tr_X is the trace map on X. When there is no danger of confusion we will write $\mathrm{Res}_F(H)$ instead of $\mathrm{Res}_F^X(H)$.

2.2. **Semi-ample degrees.** Let X be a complete n-dimensional toric variety defined by a complete fan Σ in $\mathbb{R}^n$. Recall that a $\mathbb{T}$-Cartier divisor D on X is called *semi-ample* if the corresponding line bundle $\mathcal{O}(D)$ is generated by global sections. Equivalently, D is semi-ample if and only if the corresponding piecewise linear function ψ_D is convex [15, Section 3.4]. Consider the (generalized) normal fan Σ_D of the polytope P_D of D, i.e. a complete fan whose cones are

$$\sigma_\Gamma = \{v \in (\mathbb{R}^n)^* : \langle u, v\rangle \ge \langle u', v\rangle, \text{ for all } u \in P_D, u' \in \Gamma\},$$

for every face Γ of P_D. It follows that if D is semi-ample then Σ refines Σ_D. Indeed, by the convexity of ψ_D for any maximal cone $\sigma \in \Sigma$ the restriction of ψ_D to σ defines a a vertex u of P_D. Then $\sigma \subset \sigma_u$, $\sigma_u \in \Sigma_D$. We will say that a degree $\alpha = [D] \in A_{n-1}(X)$ is *semi-ample* if D is semi-ample.

Consider a collection of $n+1$ semi-ample degrees $\alpha_0, \dots, \alpha_n$ on X. Let $P_0, \dots, P_n$ be their polytopes (defined up to translations) and $\Sigma_0, \dots, \Sigma_n$ the normal fans of the polytopes. By above Σ refines each Σ_i and, thus, refines the minimal common refinement of the Σ_i, which is the normal fan Σ_P of the Minkowski sum $P = \sum_{i=0}^n P_i$ by [16, Chapter 5, Theorem 4.8].

Now let $\pi : X' \to X$ be a birational morphism defined by a refinement $\Sigma' \to \Sigma$. If D is a $\mathbb{T}$-Cartier divisor on X then the pull-back $\pi^*(D)$ has the same piecewise linear function ψ_D and the same polytope P_D. It follows that $\alpha' = \pi^*(\alpha)$ is semi-ample on X' if α is semi-ample on X. Also if F is a homogeneous polynomial in S_α and f the corresponding Laurent polynomial supported in P_α then the pull-back $F' = \pi^*(F)$ is the P_α-homogenization of f in the homogeneous coordinate ring S' of X', and hence $F' \in S'_{\alpha'}$.

Next we will see how the toric residue Res_F^X behaves under the birational morphism $\pi : X' \to X$.

Proposition 2.1. *Let X be a complete n-dimensional toric variety defined by a complete fan Σ. Let $\pi : X' \to X$ be a birational morphism induced by a refinement $\Sigma' \to \Sigma$. Suppose $\alpha_0, \dots, \alpha_n$ are semi-ample degrees with polytopes $P_0, \dots, P_n$ and consider $n+1$ polynomials $F_i \in S_{\alpha_i}$ not vanishing simultaneously on X. Then the polynomials $F_i' = \pi^*(F_i) \in S'_{\alpha_i'}$ do not vanish simultaneously on X'. Furthermore, let g be any Laurent polynomial supported in the interior of $P = \sum_{i=0}^n P_i$, and G (resp. G') be the P-homogenization of g in S (resp. S'). Then the homogeneous polynomials $H = G/x_{\Sigma(1)}$ and $H' = G'/x_{\Sigma'(1)}$ are of critical degree for the F_i and the F_i', respectively, and satisfy*

$$\mathrm{Res}_F^X(H) = \mathrm{Res}_{F'}^{X'}(H').$$

Here $x_{\Sigma(1)}$ denotes the product of the homogeneous variables $\prod_{\rho\in\Sigma(1)} x_\rho$.

Proof. First the sets $U_i' = \{x \in X' : F_i'(x) \neq 0\}$ form a covering of X' since it is the pull-back of the covering $\mathcal{U}$ of X. In particular, the F_i' do not vanish simultaneously on X'.

Now let Ω and Ω' be the Euler forms on X and X', respectively. We have

$$\pi^*\big(\Omega/x_{\Sigma(1)}\big) = \Omega'/x_{\Sigma'(1)},$$

since both are rational extensions of the $\mathbb{T}$-invariant regular n-form $\frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n}$ on the torus, where the t_i are affine coordinates. Therefore

$$\pi^*(\omega_F(H)) = \pi^*\Big(\frac{G\,\Omega/x_{\Sigma(1)}}{F_0\cdots F_n}\Big) = \frac{G'\,\Omega'/x_{\Sigma'(1)}}{F_0'\cdots F_n'} = \omega_{F'}(H').$$

Since $\mathrm{Tr}_X = \mathrm{Tr}_{X'} \circ \pi^*$ both $\omega_F(H)$ and $\omega_{F'}(H')$ have the same toric residue. The proposition follows. □

3. Residues and essential polytopes

Definition 3.1. A collection of polytopes $P_0, \dots, P_n$ is said to be *essential* if for every $I \subsetneq \{0, \dots, n\}$ the dimension of the polytope $\sum_{i\in I} P_i$ is at least $|I|$. Given a toric variety X of dimension n, a collection of semi-ample degrees $\alpha_0, \dots, \alpha_n$ is called *essential* if the corresponding polytopes $P_0, \dots, P_n$ are essential.

The goal of this section is to prove that the toric residue is not identically zero if and only if the degrees α_i are essential.

Theorem 3.2. *Consider degrees $\alpha_0, \dots, \alpha_n$ on a complete toric variety X. The toric residue with respect to polynomials $F_0, \dots, F_n$, viewed as a rational function in the coefficients of the F_i, is identically zero if and only if the α_i are not essential. For essential α_i there is a polynomial H of critical degree and homogeneous of degree 1 in the coefficients of each F_i such that $\mathrm{Res}_F(H) = 1$.*

Proof. The first implication is that for the non-essential degrees the toric residue is identically 0. By Proposition 2.1, we can refine X to a simplicial variety without changing the toric residue. So assume X is simplicial. In this case the toric residue $\mathrm{Res}_F(H)$ is the sum of the Grothendieck local residues of any H/F_k with respect to the common zeros of the remaining F_i [4, Theorem 0.4].

Suppose there exists a proper subset I such that $\sum_{i\in I} P_i$ has dimension less than $|I|$. Let X_I be the toric variety corresponding to $P_I = \sum_{i\in I} P_i$, and $\pi : X \to X_I$ the morphism defined by the natural map of fans $\Sigma_X \to \Sigma_{F_I}$. The polynomial F_i for $i \in I$ is the pullback of a polynomial of semi-ample degree on X_I with polytope F_i. Clearly, generic polynomials supported on the F_i, $i \in I$, do not have a common zero on X_I since $|I| > \dim X_I$. Thus the corresponding $\{F_i : i \in I\}$ do not have a common zero on X for generic coefficients. Extend I to a subset of size n, without loss of generality we take it to be $\{1, \ldots, n\}$. For generic coefficients $F_1, \ldots, F_n$ do not have a common root. In particular there are no local residues in the sum. So for generic coefficients the toric residue is 0. Since, Res_F is a rational function of the coefficients of the F_i it must be identically zero.

For the converse, the main tool is the following dual Koszul complex of sheaves with respect to $F = (F_0, \ldots, F_n)$ which appears in numerous places including [4], [11], and [14]. Given a subset $I \subset \{0, \ldots, n\}$ let $\alpha_I = \sum_{i \in I} \alpha_i$. We have an exact sequence of sheaves

$$0 \to \mathcal{O}(-\beta_0) \to \bigoplus_{i=0}^{n} \mathcal{O}(\alpha_i - \beta_0) \to \cdots \to \bigoplus_{|I|=p} \mathcal{O}(\alpha_I - \beta_0) \to \cdots \to \mathcal{O}(\nu) \to 0,$$

where as before ν is the critical degree for F and $\beta_0 = \sum_\rho \deg(x_\rho)$.

One can take the Čech cohomology double complex and then pass to a spectral sequence. The E_1 terms of this spectral sequence are:

$$E_1^{p,q} = \bigoplus_{|I|=p} H^q(X, \mathcal{O}(\alpha_I - \beta_0)).$$

Because the α_i are essential, a result of [11] gives us:

$$E_1^{p,q} = 0 \text{ when } p + q > n, \text{ except for } E_1^{n+1,0} = S_\nu.$$

As a consequence there is a unique top differential $d_{n+1} : E_{n+1}^{0,n} \to E_{n+1}^{n+1,0}$ which must be an isomorphism since the spectral sequence is exact. Moreover,

$$E_{n+1}^{0,n} = E_1^{0,n} = H^n(X, \mathcal{O}(-\beta_0))$$

and $E_{n+1}^{n+1,0}$ is a quotient of S_ν.

So we have an induced map $S_\nu \to H^n(X, \mathcal{O}(-\beta_0))$ which is the composition of the projection onto $E_{n+1}^{n+1,0}$ and the inverse of the isomorphism d_{n+1}. We also have an isomorphism $\mathcal{O}(-\beta_0)) \to \widehat{\Omega}^n$ sending a local section s to $s \cdot \Omega$ where Ω is the Euler form. Finally there is the trace isomorphism $\mathrm{Tr}_X : H^n(X, \Omega^n) \to \mathbb{C}$. Composing all of these maps we obtain a map $S_\nu \to \mathbb{C}$. The maps are illustrated via the diagram below:

$$\begin{array}{ccccc} S_\nu = E_1^{n+1,0} & & & & \\ \downarrow & \searrow & & & \\ E_{n+1}^{n+1,0} & \underset{\simeq}{\xleftarrow{d_{n+1}}} & H^n(X, \mathcal{O}(-\beta_0)) \cong H^n(X, \widehat{\Omega}^n) & \xrightarrow{\mathrm{Tr}_X} & \mathbb{C}. \end{array}$$

We will prove that this composition is precisely the toric residue map. In that case if we started with a differential form $\omega \in H^n(X, \Omega^n)$ such that $\mathrm{Tr}_X(\omega) = 1$, it would correspond to an element of $H^n(X, \mathcal{O}(-\beta_0))$ which is mapped to an element $h \in E_{n+1}^{n+1,0}$. Let H be any element of S_ν lifting h. From the above constructions it would follow that $\mathrm{Res}_F(H) = 1$ and the toric residue is not identically zero as desired.

Moreover, by a theorem of Weyman [16, Chapter 3, Theorem 4.11], the differential d_{n+1}, and therefore the element H above, can be lifted up to a (non-unique) map $H^n(X, \mathcal{O}(-\beta_0)) \to S_\nu$ which is polynomial of degree 1 in the coefficients of each F_i.

To prove that the residue map coincides with the one constructed above we compute the cohomology terms using the Čech resolutions given by the open cover $U_i = \{x \in X : F_i(x) \neq 0\}$. More generally, given $J \subset \{0, \dots, n\}$ define $U_J = \cap_{j\in J} U_j$. In this way we have the E_0 terms of our spectral sequence

$$E_0^{p,q} = \bigoplus_{|I|=n+1-p} \bigoplus_{|J|=q+1} \mathcal{O}(\alpha_{\hat{I}} - \beta_0)(U_J),$$

where $\hat{I}$ denotes the complement of I. In terms of the cover, given a polynomial $H \in S_\nu$ we have

$$\frac{H}{F_0 \cdots F_n} \in \mathcal{O}(-\beta_0)(U_{\{0,\dots,n\}}) = E_0^{0,n}.$$

The residue map is defined to be the trace of the cohomology class of this latter element (after multiplying by the Euler form). So it is enough to show that $d_{n+1}([\frac{H}{F_0\cdots F_n}]) = [H] \in E_{n+1}^{n+1,0}$. To compute this differential we start with $\frac{H}{F_0\cdots F_n} \in E_0^{0,n}$ and map it via d_1 to $E_0^{1,n}$. This can be lifted via the Čech differential d_0 to an element of $E_0^{1,n-1}$ which is further mapped to $E_0^{2,n-1}$ and lifted to $E_0^{2,n-2}$ and so on. At the end we obtain an element of $E_0^{n,0}$ which is mapped via d_1 to $E_0^{n+1,0}$.

Let e_{IJ} be the basis of $E_0^{p,q}$ and $F_I = \prod_{i\in I} F_i$. We have the following lemma.

Lemma 3.3. *In the above mapping and lifting process, a valid choice for the element in* $E_0^{n-p,p}$ *is* $\sum_{|I|=p+1} \frac{H}{F_I} e_{II}$.

Proof. The base case $p = n$ is our starting element. For the inductive step we need to show that $d_1(\sum_{|I|=p+1} \frac{H}{F_I}) e_{II} = d_0(\sum_{|I'|=p} \frac{H}{F_{I'}} e_{I'I'})$.

However, by the definitions of the Koszul and Čech morphisms it is easy to see that both of the above elements are:

$$\sum_{I=\{i_0,\dots,i_p\}} \sum_{j=0}^{p} (-1)^j \frac{H}{F_{I_j}} e_{I_j I},$$

where $I_j = I \setminus \{i_j\}$. ☐

Therefore we get the element $\sum_{i=0}^n \frac{H}{F_i} e_{ii} \in E_0^{n,0}$. The final Koszul differential is multiplication by F_i in each factor so we are left with $(H, H, \dots, H) \in \sum_{i=0}^n \mathcal{O}(\nu)(U_i)$ which corresponds to the global section $H \in H^0(X, \mathcal{O}(\nu))$. So H is a valid lifting of the image of the class of $\frac{H}{F_0\cdots F_n}$ under d_{n+1} completing the proof. ☐

4. Facet coloring and toric residues for monomials

The theorem from the previous section guaranteed the existence of an element of toric residue one but was completely nonconstructive. This section and the next one provide the framework for an explicit combinatorial construction of such elements. Here we recall the definition of the facet coloring of a polytope and the relation between the combinatorial degree of a facet coloring and the toric residue for monomial ideals. We will also obtain the Generalized Global Transformation Law that allows us to reduce the computation of the toric residue for semi-ample degrees to the monomial case.

4.1. **Facet coloring.** Consider an n-dimensional polytope P in $\mathbb{R}^n$. We let ∂P denote the boundary of P and $\mathcal{F}(\partial P)$ the partially ordered set (poset) by inclusion of all proper faces of P. We also let $2^{[n+1]}$ denote the set of all subsets of $[n+1]=\{0,\dots,n\}$. It will be convenient for us to equip $2^{[n+1]}$ with the inverse partial order $<$, i.e. $J<J'$ if and only if $J\supset J'$ for $J,J'\in 2^{[n+1]}$.

Definition 4.1. A map of posets $C:(\mathcal{F}(\partial P),\subset)\to(2^{[n+1]},<)$ is called a *coloring of P into $n+1$ colors* (or simply *coloring*). The image $C(\Gamma)$ is called the *set of colors* of a face $\Gamma\in\mathcal{F}(\partial P)$. We will also say that Γ *is colored by* $C(\Gamma)$. A coloring is called *simplicial* if every face $\Gamma\in\mathcal{F}(\partial P)$ is colored by a non-empty proper subset of $[n+1]$.

The poset $(2^{[n+1]},<)$ can be identified with the poset of faces of the standard n-simplex:

$$\Delta=\{y=(y_0,\dots,y_n)\in\mathbb{R}^{n+1}\,:\,y_0+\dots+y_n=1,\,0\le y_i\le 1\}.$$

Indeed, each non-empty proper subset $\{j_1,\dots,j_k\}\subset[n+1]$ defines the codimension k face of Δ:

$$\Delta_{j_1\dots j_k}=\{y\in\Delta\,:\,y_{j_1}=\dots=y_{j_k}=0\}$$

Therefore, any simplicial coloring is, in fact, a map of posets $C:\mathcal{F}(\partial P)\to\mathcal{F}(\partial\Delta)$.

Fix orientations of P and Δ. Given a simplicial coloring C consider a continuous piecewise linear map $f_C:\partial P\to\partial\Delta$ such that $f_C(\Gamma)\subset C(\Gamma)$ for any $\Gamma\in\mathcal{F}(\partial P)$. One can show that such a map f_C always exists and the topological degree $\deg f_C$ does not depend on the choice of f_C (see [19]). We call it the *combinatorial degree* $\operatorname{cdeg}(C)$ of the simplicial coloring C. The combinatorial degree is alternating in the ordering of the elements of $[n+1]$ as every such ordering defines an orientation of the corresponding simplex Δ.

We have the following property of the combinatorial degree. Let C, C' be two simplicial colorings of P. We say that C' *refines* C if $C'(\Gamma)\subset C(\Gamma)$ for any $\Gamma\in\mathcal{F}(\partial P)$.

Proposition 4.2. [19] *Let C, C' be two simplicial colorings of P. If C' refines C then* $\operatorname{cdeg}(C)=\operatorname{cdeg}(C')$.

The combinatorial degree can be computed explicitly as a signed number of certain complete flags of faces of P. To state the precise formula we will need the following definition. Consider a complete flag F of faces of P

$$F:\ P^0\subset P^1\subset\dots\subset P^{n-1}\subset P^n=P,\qquad \dim P^j=j.$$

For every $1\le j\le n$ choose a vector e_j that begins at P^0 and points strictly inside P^j. Define the *sign* of the flag to be $\operatorname{sgn}F=1$ if $(e_1,\dots,e_n)$ gives a positive oriented frame for P, and $\operatorname{sgn}F=-1$ otherwise. It is easy to see that the sign is independent of the choice of the e_i.

Theorem 4.3. *Let C be a simplicial coloring of an n-dimensional polytope $P\subset\mathbb{R}^n$. Fix any permutation ε on the elements of $[n+1]$. Then the combinatorial degree of C equals the sign of ε times the number of complete flags*

$$P^0\subset P^1\subset\dots\subset P^{n-1}\subset P^n=P,$$

counted with signs, such that for every $1\le k\le n$ the face P^{k-1} is colored by $\{\varepsilon(k),\dots,\varepsilon(n)\}$.

Proof. This is a particular case of [18], Theorem 2.2. □

In particular, this theorem says that the combinatorial degree is zero unless for every $0 \leq k \leq n$ there is a facet in P colored by $\{k\}$, for every $0 \leq k < l \leq n$ there is a codimension two face in P colored by $\{k, l\}$, and so on.

One way to define a coloring $C : \mathcal{F}(\partial P) \to 2^{[n+1]}$ is to give the colors to every facet of P and then extend it by taking intersections, i.e. if $\Gamma = \bigcap_\nu Q_\nu$ for some facets Q_ν then $C(\Gamma) = \bigcup_\nu C(Q_\nu)$ (remember we have the reversed order in the target). The coloring obtained in this way is called a *facet coloring*. In the present paper we will only be interested in facet colorings.

Next, let P be a polytope in $\mathbb{R}^n$. Then one can consider the poset of all flags of faces of P (chains in $\mathcal{F}(\partial P)$). The partial order is defined as follows: If $F = \{\Gamma_1 \subset \cdots \subset \Gamma_k\}$ and $F' = \{\Gamma'_1 \subset \cdots \subset \Gamma'_l\}$ are two flags of faces of P then $F < F'$ if and only if $\{\Gamma'_1, \ldots, \Gamma'_l\}$ is a subset of $\{\Gamma_1, \ldots, \Gamma_k\}$. This poset can be realized as the poset of faces of a simple polytope $\hat{P}$ whose normal fan is the barycentric subdivision of the normal fan of P. Indeed, one can easily see that there is a 1-1 order preserving correspondence between codimension k faces of $\hat{P}$ and length k flags of faces of P. In particular, facets of $\hat{P}$ correspond to flags of faces of length one, i.e. to faces of P.

Later on we will be concerned with facet colorings not of the polytope P itself, but the polytope $\hat{P}$ associated with it. By the above, to define a facet coloring of $\hat{P}$ we need to assign a non-empty proper subset of $[n+1]$ to every facet of $\hat{P}$, hence, to every face of P. Therefore, any map $C : \mathcal{F}(\partial P) \to 2^{[n+1]}$ defines a facet coloring $\hat{C} : \mathcal{F}(\partial\hat{P}) \to 2^{[n+1]}$. (We should warn the reader, however, the map C may not be a map of posets, in general.) Clearly, for every flag $\Gamma_1 \subset \cdots \subset \Gamma_k$ the union $\bigcup_i C(\Gamma_i)$ is the set of colors of the face of $\hat{P}$ corresponding to this flag. We thus say that a *flag* $\Gamma_1 \subset \cdots \subset \Gamma_k$ *is colored by* $\bigcup_i C(\Gamma_i)$. Furthermore, $\hat{C}$ is simplicial if and only if for any flag $\Gamma_1 \subset \cdots \subset \Gamma_k$ the union $\bigcup_i C(\Gamma_i)$ is proper.

4.2. **Toric residue for monomials.** Let X be a projective toric variety of dimension n defined by a lattice polytope P, and let Σ denote the normal fan of P.

Consider a collection of $n+1$ (monic) monomials $z_0, \ldots, z_n$ in the homogeneous coordinate ring $S = \mathbb{C}[x_\rho : \rho \in \Sigma(1)]$ of X. Assume that the product of the variables $\prod_\rho x_\rho$ divides the product of the monomials $z_0 \cdots z_n$. Then the quotient $z_0 \cdots z_n / \prod_\rho x_\rho$ has critical degree with respect to $z_0, \ldots, z_n$.

On the other hand, since the variables x_ρ correspond to the facet normals of P, any collection of monomials $z = (z_0, \ldots, z_n)$ with $\prod_\rho x_\rho \,|\, z_0 \cdots z_n$ defines a facet coloring of P:

$$C_z : \mathcal{F}(\partial P) \to 2^{[n+1]}, \quad C_z(Q_\rho) = \{i \in [n+1] \,:\, x_\rho | z_i\},$$

where Q_ρ is the facet of P whose inner normal generates ρ. Conversely, any facet coloring C of P defines a collection of squarefree monomials in S whose product is divisible by the product of the variables: $z_i = \prod_{C(Q_\rho) \ni i} x_\rho$.

If $z_0, \ldots, z_n$ do not vanish simultaneously on X then the corresponding coloring C_z is simplicial. Indeed, if C_z is not simplicial then there is a vertex u of P which is colored by $\{0, \ldots, n\}$, i.e. $u \in Q_0 \cap \cdots \cap Q_n$ for some facets Q_i, such that Q_i contains i as one of its colors. But this implies that the corresponding point x_u on X lies on the irreducible divisors $D_{\rho_0}, \ldots, D_{\rho_n}$, where each D_{ρ_i} is a component of the zero locus of z_i on X, a contradiction.

The next theorem asserts that the combinatorial degree of C_z equals the toric residue of the quotient $z_0 \cdots z_n / \prod_\rho x_\rho$.

Theorem 4.4. [19] *Let X be an n-dimensional projective toric variety defined by a lattice polytope P. Let $z_0, \dots, z_n$ be monomials in the homogeneous coordinate ring S such that*

(1) $\prod_\rho x_\rho \,|\, z_0 \cdots z_n$,
(2) $z_0, \dots, z_n$ *do not vanish simultaneously on* X.

Then

$$\mathrm{Res}_z(z_0 \cdots z_n / \prod_\rho x_\rho) = \mathrm{cdeg}(C_z)$$

where C_z is the simplicial coloring of P defined by $z_0, \dots, z_n$.

4.3. **Reduction to toric residue for monomials.** To reduce the computation of the toric residue for arbitrary polynomials to the case of monomials we will need the following generalized version of the Global Transformation Law [4].

Theorem 4.5. *Let $F_j \in S_{\alpha_j}$ and $G_j \in S_{\beta_j}$ for $0 \le j \le n$. Suppose*

$$\sum_{j=0}^{n} B_{ij} F_j = \sum_{j=0}^{n} A_{ij} G_j, \quad 0 \le i \le n,$$

where B_{ij} and A_{ij} are homogeneous of degree $\gamma_i - \alpha_j$ and $\gamma_i - \beta_j$ respectively for some fixed degrees $\gamma_0, \dots, \gamma_n$. Assume that neither $F_0, \dots, F_n$ nor $G_0, \dots, G_n$ vanish simultaneously on X. Let $\alpha = \sum_i \alpha_i$, $\beta = \sum_i \beta_i$, $\gamma = \sum_i \gamma_i$, and $\nu_0 = \sum_\rho \deg(x_\rho)$. Then for any $H \in S_{\alpha+\beta-\gamma-\nu_0}$, the polynomials $H \det A$ and $H \det B$ are of critical degree for F and G respectively, and

$$\mathrm{Res}_F(H \det A) = \mathrm{Res}_G(H \det B). \tag{4.1}$$

Proof. For any $H \in S_{\alpha+\beta-\gamma-\nu_0}$ the degree of $H \det A$ is $\alpha - \nu_0$, which is the critical degree for $F_0, \dots, F_n$. Consider the $n+1$ homogeneous polynomials $K_i = \sum_{j=0}^n B_{ij} F_j$. According to the Global Transformation Law (Theorem 0.1, [4])

$$\mathrm{Res}_K((H \det A) \det B) = \mathrm{Res}_F(H \det A).$$

On the other hand, $K_i = \sum_{j=0}^n A_{ij} G_j$ and $H \det B$ has critical degree for $G_0, \dots, G_n$. Therefore,

$$\mathrm{Res}_K((H \det B) \det A) = \mathrm{Res}_G(H \det B),$$

again by the Global Transformation Law. The theorem follows. □

Our reduction is then based on the following assertion.

Corollary 4.6. *Let X be an n-dimensional projective toric variety. Let $F_j \in S_{\alpha_j}$ be homogeneous polynomials not vanishing simultaneously on X. Suppose $y_0, \dots, y_n$ and $z_0, \dots, z_n$ are squarefree monomials such that*

(1) $y_0 \cdots y_n = z_0 \cdots z_n / \prod_\rho x_\rho$,
(2) $y_i F_i = \sum_{j=0}^n A_{ij} z_j$ *for some* $A_{ij} \in S_{\alpha_i + \deg(y_i) - \deg(z_j)}$, $0 \le i \le n$,
(3) $z_0, \dots, z_n$ *do not vanish simultaneously on* X.

Then we have

$$\operatorname{Res}_F(\det A) = \operatorname{Res}_z(y_0 \cdots y_n) = \operatorname{cdeg}(C_z), \tag{4.2}$$

where C_z is the simplicial facet coloring defined by $z_0, \dots, z_n$.

Proof. The first statement in (4.2) follows from Theorem 4.5 and the second statement follows from Theorem 4.4. □

5. Partition matrix for polytopes and residue matrix

5.1. **Partition matrix.** Let P be a lattice polytope in $\mathbb{R}^n$. Consider any partition of the set of vertices of P into $n+1$ disjoint (possibly empty) subsets:

$$\operatorname{Vert}(P) = V_0 \sqcup \cdots \sqcup V_n. \tag{5.1}$$

Extend this partition to a partition of the set of lattice points of P by adding to V_i lattice points in the relative interior of faces containing a vertex from V_i:

$$P \cap \mathbb{Z}^n = M_0 \sqcup \cdots \sqcup M_n. \tag{5.2}$$

Any such extension (5.2) will be called an *induced partition* of $P \cap \mathbb{Z}^n$ defined by the *vertex partition* (5.1).

Now consider $n+1$ lattice polytopes $P_0, \dots, P_n$ in $\mathbb{R}^n$. For each polytope P_i fix an (ordered) vertex partition

$$\operatorname{Vert}(P_i) = V_{i0} \sqcup \cdots \sqcup V_{in}.$$

We say that these partitions are *compatible* if for any permutation ε of $\{0, \dots, n\}$

$$\sum_{i=0}^{n} V_{\varepsilon(i)i} \subset \operatorname{int}\Big(\sum_{i=0}^{n} P_i\Big), \tag{5.3}$$

where $\operatorname{int}(P)$ denotes the relative interior of P.

Definition 5.1. Let $P_0, \dots, P_n$ be lattice polytopes in $\mathbb{R}^n$. Then subsets $M_{ij} \subset P_i \cap \mathbb{Z}^n$, $0 \le i, j \le n$, form a *partition matrix* for $P_0, \dots, P_n$ if

$$P_i \cap \mathbb{Z}^n = M_{i0} \sqcup \cdots \sqcup M_{in}, \quad 0 \le i \le n$$

is a collection of induced partitions defined by a compatible collection of vertex partitions

$$\operatorname{Vert}(P_i) = V_{i0} \sqcup \cdots \sqcup V_{in}, \quad 0 \le i \le n.$$

Remark 5.2. It is not hard to see that the compatibility condition on the V_{ij} (5.3) implies the same condition on any induced partitions:

$$\sum_{i=0}^{n} M_{\varepsilon(i)i} \subset \operatorname{int}\Big(\sum_{i=0}^{n} P_i\Big). \tag{5.4}$$

Example 5.3. Consider three polygons P_0, P_1 and P_2 in Figure 5.1. We partition their lattice points in accordance with the labels: the set M_{ij} consists points of P_i labeled with j $(0 \le i, j \le 2)$.

Clearly these are induced partitions. To show that they are compatible it is enough to check that for any linear functional $v \neq 0$ any three vertices u_0, u_1 and u_2 that minimize v on P_0, P_1 and P_2, respectively, will not have all different labels.

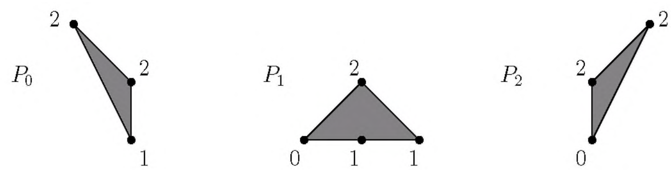

FIGURE 5.1.

5.2. **Coloring matrices.** Let P be a polytope in $\mathbb{R}^n$. Recall that every vector v in the dual space $(\mathbb{R}^n)^*$ defines a face P^v of P on which v restricted to P attains its minimal value.

Definition 5.4. Let M be a partition matrix for $P_0, \dots, P_n$. Define a map from $(\mathbb{R}^n)^*$ to the set of $(0,1)$-matrices of dimension $(n+1) \times (n+1)$:

$$\mathcal{M} : (\mathbb{R}^n)^* \to \mathrm{Mat}(n+1, \{0,1\})$$

where the value of $\mathcal{M}$ at $v \in (\mathbb{R}^n)^*$ is the matrix M^v whose (i,j)-th entry is

$$M^v_{ij} = \begin{cases} 1, & \text{if } M_{ij} \cap P^v_i \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases}$$

The matrix M^v is called the *coloring matrix* of v.

Informally speaking, the coloring matrix M^v "encodes" the partitions of the lattice points of the P_i restricted to the corresponding faces P^v_i.

The compatibility condition implies that for any non-zero v the coloring matrix M^v has permanent zero. Indeed, if the permanent is non-zero then there exists a permutation ε of $\{0, \dots, n\}$ such that $M^v_{\varepsilon(i)i} = 1$ for all $0 \le i \le n$. By the definition of M^v this implies that for each i there is a point u_i in $M_{\varepsilon(i)i}$ that lies on the face P^v_i. But then the sum $u_0 + \cdots + u_n$ gives a point on the face P^v of the Minkowski sum $P = \sum_i P_i$, which contradicts the compatibility condition (5.4).

The following statement is known as the Frobenius-König Theorem (it is also equivalent to Hall's Marriage Theorem [1]).

Theorem 5.5. *Let A be a $(0,1)$-matrix of dimension $n \times n$ with zero permanent. Then A has a submatrix of zeroes of dimension $r \times s$ for some positive r, s such that $r + s = n + 1$.*

By the above theorem for every non-zero v the $(n+1) \times (n+1)$ matrix M^v has a zero submatrix (not unique, in general) of dimension $r \times s$ with $r + s = n + 2$. The rows (resp. columns) of the submatrix are indexed by a subset of $\{0, \dots, n\}$ which we denote by I^v (resp. J^v). We thus have $|I^v| + |J^v| = n + 2$ for all non-zero v.

Now consider a polytope P whose normal fan Σ is a common refinements of the normal fans of $P_0, \dots, P_n$. Clearly, M^v is the same for all v in the intersection of the cones of the P^v_i. Therefore, $\mathcal{M}$ is constant on the cones of Σ. Since cones of Σ correspond to faces of P we arrive at the following definition.

Definition 5.6. Let M be a partition matrix for $P_0, \dots, P_n$. Let P be a polytope whose normal fan is a common refinements of the normal fans of $P_0, \dots, P_n$. Given a face Γ of P define its *coloring matrix* M^Γ to be the coloring matrix of any $v \in \sigma_\Gamma$, where σ_Γ is the cone of Γ.

We will need the following simple observation. Let Γ_1, Γ_2 be faces of P. Then

$$\text{if} \quad \Gamma_1 \subset \Gamma_2 \quad \text{then} \quad (M_{ij}^{\Gamma_2} = 0) \Rightarrow (M_{ij}^{\Gamma_1} = 0). \tag{5.5}$$

5.3. **Residue from a partition matrix.** Consider a projective n-dimensional toric variety X defined by a projective fan Σ. Let $\alpha_0, \dots, \alpha_n$ be $n+1$ semi-ample degrees on X and let $P_0, \dots, P_n$ be their polytopes.

Definition 5.7. Consider a collection of $n+1$ homogeneous polynomials $F = (F_0, \dots, F_n)$ of degrees $\alpha_0, \dots, \alpha_n$:

$$F_i = \sum_{u \in P_i \cap \mathbb{Z}^n} c_u \chi^u, \quad F_i \in S_{\alpha_i}, \quad 0 \le i \le n.$$

Given a partition matrix M for $P_0, \dots, P_n$ define the *residue matrix* M_F of F to be the matrix whose entries are the homogeneous polynomials

$$F_{ij} = \sum_{u \in M_{ij}} c_u \chi^u, \quad F_{ij} \in S_{\alpha_i}, \quad 0 \le i, j \le n.$$

The determinant $\det(M_F)$ is a homogeneous polynomial of degree $\alpha = \alpha_0 + \cdots + \alpha_n$. Since the α_i are semi-ample, α is also semi-ample and its polytope is the Minkowski sum $\sum_i P_i$. As follows from the definition of homogeneous coordinates (see (2.1)) a monomial χ^u of degree α is divisible by all the variables if and only if the corresponding lattice point u lies in the interior of the polytope of α. Therefore, by the compatibility condition (5.4) every monomial in $\det(M_F)$ is divisible by all the variables, and hence the quotient $\det(M_F)/\prod_\rho x_\rho$ is a homogeneous polynomial of critical degree $\alpha - \sum_\rho \deg(x_\rho)$.

Proposition 5.8. *Let $\alpha_0, \dots, \alpha_n$ be semi-ample degrees on X with polytopes $P_0, \dots, P_n$. Fix a partition matrix M for $P_0, \dots, P_n$. For every coloring matrix M^ρ, $\rho \in \Sigma(1)$, make any choice of an $r \times s$ zero submatrix with $r + s = n + 2$ and let its rows and columns be indexed by subsets I^ρ and J^ρ of $\{0, \dots, n\}$, respectively. Define squarefree monomials*

$$y_i = \prod_{I^\rho \not\ni i} x_\rho, \quad z_j = \prod_{J^\rho \ni j} x_\rho, \quad 0 \le i, j \le n.$$

Then for any homogeneous polynomials $F_0, \dots, F_n$ of degrees $\alpha_0, \dots, \alpha_n$

(1) $y_0 \cdots y_n = z_0 \cdots z_n / \prod_\rho x_\rho$,

(2) $y_i F_i = \sum_{j=0}^n A_{ij} z_j$ *for some* $A_{ij} \in S_{\alpha_i + \deg(y_i) - \deg(z_j)}$, $0 \le i \le n$.

Moreover, A_{ij} can be chosen so that

$$\det(M_F)/\prod_\rho x_\rho = \det(A),$$

where M_F is the residue matrix defined by the partition matrix M.

Proof. (1) For every $\rho \in \Sigma(1)$ the variable x_ρ appears in the product $z_0 \cdots z_n$ with multiplicity $|J^\rho|$ and in $y_0 \cdots y_n$ with multiplicity $n+1-|I^\rho| = |J^\rho|-1$ since $|I^\rho|+|J^\rho| = n+2$.

(2) For every $0 \le i \le n$ we have

$$F_i = \sum_{u \in P_i \cap \mathbb{Z}^n} c_u \chi^u.$$

We need to show that every monomial $y_i\chi^u$ is divisible by at least one of $z_0, \ldots, z_n$. Since every monomial χ^u is divisible by a vertex monomial we can assume that u is a vertex of F_i. Recall that in homogeneous coordinates

$$\chi^u = \prod_\rho x_\rho^{\langle u, v_\rho \rangle + a_\rho},$$

where $D = \sum_\rho a_\rho D_\rho$ is a representative of α_i (see (2.1)). Therefore x_ρ divides χ^u if and only if $\rho \notin \sigma_u$, where σ_u is the cone of Σ_i corresponding to u.

The vertex u is contained in M_{ij} for some $0 \le j \le n$. We show that z_j divides $y_i\chi^u$. Indeed, take any x_ρ with J^ρ containing j. If $i \in I^\rho$ then $M_{ij}^\rho = 0$. From the definition of M^ρ it follows that $P_i^{v_\rho}$ does not contain the vertex u, i.e. $\rho \notin \sigma_u$ and so $x_\rho | \chi^u$ by above. If $i \notin I^\rho$ then $x_\rho | y_i$ by the definition of y_i.

The above argument shows that $y_i F_{ij} = A_{ij} z_j$ for some homogeneous polynomial A_{ij}. Taking the determinant we obtain $y_0 \cdots y_n \det(M_F) = z_0 \cdots z_n \det(A)$. Now the last statement follows from part (1). □

The above proposition shows that given a partition matrix M, any choice of zero submatrices in M^ρ, for $\rho \in \Sigma(1)$, defines a collection of squarefree monomials $y_0, \ldots, y_n$ and $z_0, \ldots, z_n$ that satisfy the conditions (1) and (2) of Corollary 4.6. If the facet coloring C_z defined by the monomials $z_0, \ldots, z_n$ is simplicial the condition (3) of Corollary 4.6 is satisfied and that would imply the result of Theorem 1.1, namely that the residue of M_F equals the combinatorial degree of C_z. However, there are examples of $P_0, \ldots, P_n$ when the condition (3) fails no matter how one chooses a partition matrix and zero submatrices. To avoid this obstruction we are going to change the variety X by taking the barycentric refinement of its fan $\hat{\Sigma} \to \Sigma$. This gives a birational morphism $\hat{X} \to X$ which allows us to transfer our construction to the variety $\hat{X}$ (see Proposition 2.1). The advantage of this is that for any partition matrix M there is a canonical choice of a zero submatrix in every coloring matrix M^ρ, for $\rho \in \hat{\Sigma}$, which guarantees that the corresponding monomials $\hat{z}_0, \ldots, \hat{z}_n$ do not vanish simultaneously on $\hat{X}$.

6. Canonical colorings

Let $P_0, \ldots, P_n$ be $n+1$ lattice polytopes in $\mathbb{R}^n$ and $\Sigma_0, \ldots, \Sigma_n$ their normal fans. Let P be any polytope whose normal fan Σ is a common refinement of the Σ_i. Given a partition matrix we will define a canonical facet coloring of a polytope $\hat{P}$ whose normal fan $\hat{\Sigma}$ is the barycentric refinement of Σ. We will then prove that this coloring is simplicial. This will allow us to define monomials $\hat{z}_0, \ldots, \hat{z}_n$ on the toric variety corresponding to $\hat{P}$ that satisfy all the conditions of Corollary 4.6 and thus obtain our main result (Theorem 1.1).

6.1. **Canonical coloring.** Let M be a partition matrix for polytopes $P_0, \ldots, P_n$ and let the polytopes P and $\hat{P}$ be as above. As mentioned in Section 4.1, to define a facet coloring of $\hat{P}$ it suffices to assign a subset $C(\Gamma) \subset \{0, \ldots, n\}$ to every face Γ of P. We will start by describing all possible candidates for $C(\Gamma)$, so called admissible colorings of Γ.

Definition 6.1. Let Γ be a face of P and M^Γ its coloring matrix. A subset $J \subset \{0, \ldots, n\}$ is called an *admissible coloring of* Γ if M^Γ contains an $r \times s$ zero submatrix with $r+s = n+2$ whose columns are indexed by J.

It turns out that the set of admissible colorings of a face possesses very nice properties.

First, for any flag of faces of F we have the reversed inclusion of the corresponding sets of admissible colorings:

$$\text{If} \quad \Gamma_1 \subset \cdots \subset \Gamma_k \quad \text{then} \quad \mathcal{J}_1 \supset \cdots \supset \mathcal{J}_k, \tag{6.1}$$

where $\mathcal{J}_i$ is the set of admissible colorings of Γ_i. Indeed, (5.5) implies that every zero submatrix in the coloring matrix of Γ_i is also a zero submarix of the coloring matrix of Γ_{i-1}.

Second, let M^Γ be the coloring matrix of a face $\Gamma \subset F$. (In what follows we will only use that M^Γ is an $(n+1)\times(n+1)$ matrix with $(0,1)$-entries and zero permanent.) Denote by $\mathcal{B}$ the set of all zero submatrices B in M^Γ of dimension $r \times s$ such that $r+s = n+2$. This set is non-empty by Theorem 5.5. For $B \in \mathcal{B}$ we let $I(B)$ (resp. $J(B)$) denote the subset in $\{0,\dots,n\}$ of indices of rows (resp. columns) of B. We have the following lemma.

Lemma 6.2. *Let $B_1, B_2 \in \mathcal{B}$. Then there is $B \in \mathcal{B}$ such that either $J(B) = J(B_1) \cup J(B_2)$ or $J(B) = J(B_1) \cap J(B_2)$.*

Proof. Let B' be the submatrix whose rows are indexed by $I(B_1) \cap I(B_2)$ and whose columns are indexed by $J(B_1) \cup J(B_2)$. Clearly B' is a zero submatrix. Similarly, let B'' be the zero submatrix with rows indexed by $I(B_1) \cup I(B_2)$ and columns indexed by $J(B_1) \cap J(B_2)$. Denote $r_i = |I(B_i)|$, $r_\cap = |I(B_1) \cap I(B_2)|$, and $r_\cup = |I(B_1) \cup I(B_2)|$. By the inclusion/exclusion formula $r_\cup + r_\cap = r_1 + r_2$. Similarly, $s_\cup + s_\cap = s_1 + s_2$, where $s_i = |J(B_i)|$, $s_\cap = |J(B_1) \cap J(B_2)|$, and $s_\cup = |J(B_1) \cup J(B_2)|$. Summing up these two equations we obtain

$$(r_\cap + s_\cup) + (r_\cup + s_\cap) = (r_1 + s_1) + (r_2 + s_2) = 2(n+2).$$

Therefore, either $r_\cap + s_\cup \geq n+2$ or $r_\cup + s_\cap \geq n+2$. In other words, either B' or B'' contains a zero submatrix B with $r+s = n+2$, as required. □

Remark 6.3. The above lemma means that if J_1 and J_2 are two admissible colorings of Γ then either $J_1 \cap J_2$ or $J_1 \cup J_2$ is also an admissible coloring. As follows from the proof, a slightly stronger statement is true: If $J_1 \cup J_2$ is not an admissible coloring then any single color can be removed from $J_1 \cap J_2$ and the remaining set will still be an admissible coloring of Γ.

Lemma 6.4. *Let $\mathcal{B}$ be as above and consider the partially ordered by inclusion set*

$$\mathcal{J} = \{J(B) \subset \{0,\dots,n\} : B \in \mathcal{B}\}.$$

Let $\mathcal{J}_\cup$ be the set of maximal elements, and $\mathcal{J}_\cap$ the set of minimal elements of $\mathcal{J}$. Then the subsets

$$c = \bigcup_{J \in \mathcal{J}_\cap} J \quad \text{and} \quad C = \bigcap_{J \in \mathcal{J}_\cup} J$$

belong to $\mathcal{J}$ and satisfy $c \subset C$.

Proof. To prove $C \in \mathcal{J}$ we show that $J_1 \cap \cdots \cap J_k \in \mathcal{J}$ for any $J_i \in \mathcal{J}_\cup$, $1 \leq i \leq k$. We proceed by induction. The case $k = 1$ is trivial. Assume $J = J_1 \cap \cdots \cap J_k \in \mathcal{J}$ and let $J_{k+1} \in \mathcal{J}_\cup$. If $J \cap J_{k+1} \in \mathcal{J}$ we are done, otherwise $J \cup J_{k+1} \in \mathcal{J}$ by Lemma 6.2. Since J_{k+1} is maximal we have $J \subset J_{k+1}$, i.e. $J = J \cap J_{k+1} = J_1 \cap \cdots \cap J_k \cap J_{k+1} \in \mathcal{J}$. Similar arguments show that $c \in \mathcal{J}$.

To show $c \subset C$ it is enough to notice that for any $J \in \mathcal{J}_\cap$ and any $J' \in \mathcal{J}_\cup$ we have $J \subset J'$. Indeed, either $J \cap J' \in \mathcal{J}$ and so $J = J \cap J' \subset J'$ by minimality of J, or $J \cup J' \in \mathcal{J}$ and so $J \subset J \cup J' = J'$ by maximality of J'. □

The above lemma supplies us with two canonical coloring of a face Γ:

Definition 6.5. Let M^Γ be the coloring matrix of a face $\Gamma \subset P$ and $\mathcal{J}_\Gamma$ the set of all admissible colorings of Γ. Maximal (minimal) elements of $\mathcal{J}_\Gamma$ are called *maximal (minimal) colorings* of Γ. The union $c(\Gamma)$ of minimal colorings is called the *minimal canonical coloring* of Γ. The intersection $C(\Gamma)$ of maximal colorings is called the *maximal canonical coloring* of Γ.

Example 6.6. Consider the three polygons P_0, P_1 and P_2 from Example 5.3. Let Γ be the horizontal edge of the Minkowski sum $P = P_0 + P_1 + P_2$. Then it has the coloring matrix

$$M^\Gamma = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

We get $\mathcal{J}_\Gamma = \{\{2\}\}$ and $c(\Gamma) = C(\Gamma) = \{2\}$. Next let Γ' be the edge of P with $45°$ slope. (It is the sum of the highest vertex of P_0 and the two edges of P_1 and P_2 of slope $45°$.) Its coloring matrix is

$$M^{\Gamma'} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

This time we have $\mathcal{J}_{\Gamma'} = \{\{1\}, \{0,1\}\}$ and $c(\Gamma') = \{1\}$, $C(\Gamma') = \{0,1\}$.

To obtain less trivial example we need to consider the case $n = 3$. Here is an example of a coloring matrix whose set of admissible colorings has more than one maximal (and minimal) element.

$$M^\Gamma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Indeed, the set of admissible colorings is $\mathcal{J} = \{\{1\}, \{3\}, \{1,3\}, \{0,1,3\}, \{1,2,3\}\}$. Therefore, $c = \{1\} \cup \{3\} = \{1,3\}$ and $C = \{0,1,3\} \cap \{1,2,3\} = \{1,3\}$.

According to the discussion in Section 4.1 the maps $c : \Gamma \mapsto c(\Gamma)$ and $C : \Gamma \mapsto C(\Gamma)$ defined above give rise to two facet coloring $\hat{c}$ and $\hat{C}$ of $\hat{P}$ which we call the minimal and maximal canonical facet colorings of $\hat{P}$, respectively.

It is easy to see that under $\hat{C}$ no facet of $\hat{P}$ gets all the colors. Indeed, for any face $\Gamma \subset P$ its coloring matrix M^Γ cannot contain zero rows, thus every maximal coloring $J \in \mathcal{J}_\Gamma$ is a proper subset of $\{0, \dots, n\}$. The next theorem shows that an even stronger statement is true: no face of $\hat{P}$ gets all the colors.

Theorem 6.7. *The maximal and minimal canonical facet colorings of $\hat{P}$ are simplicial and have the same combinatorial degree.*

Proof. Recall from Section 4.1 that to prove $\hat{C}$ simplicial we need to show that for any maximal flag of faces of P

$$\Gamma_0 \subset \cdots \subset \Gamma_{n-1}, \quad \dim \Gamma_i = i$$

the union $\bigcup_{i=0}^{n-1} C(\Gamma_i)$ is a proper subset of $\{0,\dots,n\}$. We will prove by induction that for any $n-1 \geq k \geq 0$

$$\bigcup_{i=k}^{n-1} C(\Gamma_i) \subset J_k, \tag{6.2}$$

for some maximal coloring J_k of Γ_k. For $k=0$ this implies the statement of the theorem, since J_0 is a proper subset of $\{0,\dots,n\}$. The base $k=n-1$ is clear since $C(\Gamma_{n-1})$ is the intersection of maximal elements of $\mathcal{J}_{n-1}$. For the inductive step assume that (6.2) is true for some maximal $J_k \in \mathcal{J}_k$. By (6.1) $\mathcal{J}_k \subset \mathcal{J}_{k-1}$, thus there exists a maximal element $J_{k-1} \in \mathcal{J}_{k-1}$ such that $J_k \subset J_{k-1}$. Also $C(\Gamma_{k-1}) \subset J_{k-1}$, by definition. This together with (6.2) gives

$$\bigcup_{i=k-1}^{n-1} C(\Gamma_i) \subset J_{k-1},$$

as required.

By Lemma 6.4 $c(\Gamma) \subset C(\Gamma)$ for any face $\Gamma \subset P$. Therefore, $\hat{c}$ is also simplicial. Finally, $\operatorname{cdeg}(c) = \operatorname{cdeg}(C)$ follows from Proposition 4.2. □

When the maximal canonical coloring of a face consists of a single element we can say more about admissible colorings of this face:

Lemma 6.8. *Suppose a face $\Gamma \subset P$ is maximally canonically colored by a single color $C(\Gamma) = \{k\}$. Then this is the only admissible coloring of Γ. Moreover, any face containing Γ is also singly canonically colored by $\{k\}$ while every subface of Γ is canonically colored by a set containing k.*

Proof. Suppose $J_1,\dots,J_s$, $s \geq 2$, are maximal colorings of Γ such that $\{k\} = J_1 \cap \dots \cap J_s$, but $\{k\} \subsetneq J_1 \cap \dots \cap J_{s-1}$. By the proof of Lemma 6.4 $J = J_1 \cap \dots \cap J_{s-1}$ is an admissible coloring of Γ. By the remark after Lemma 6.2 Γ can either be colored by $J \cup J_s$ or else by $J \cap J_s$ with any single color removed. The first is a coloring strictly larger than J_s which is impossible since J_s is maximal. The second is empty since $J \cap J_s$ is already a single color. Both are contradictions. Thus, the unique maximal coloring of Γ is $\{k\}$ which is therefore the only admissible coloring.

If $\Gamma \subset \Gamma'$ then $\mathcal{J}_\Gamma \supset \mathcal{J}_{\Gamma'}$ by (6.1) and, hence, $\mathcal{J}_{\Gamma'} = \mathcal{J}_\Gamma = \{k\}$. If $\Gamma \supset \Gamma''$ then $\{k\}$ is an admissible coloring of Γ''. But $\{k\}$ can be appended to any coloring of Γ''. Thus k is contained in every maximal coloring of Γ'', i.e. in the maximal canonical coloring of Γ''. □

6.2. **Main theorem.** We now turn back to residues and prove the result of Theorem 1.1. As before X is a complete n-dimensional toric variety defined by a fan Σ and $\alpha_0,\dots,\alpha_n$ are semi-ample degrees with polytopes $P_0,\dots,P_n$. We can assume that X is projective and take P to be the polytope of an ample divisor on X (If X is not projective it can be dominated birationally by a projective toric variety. This will not affect the toric residue computation by Proposition 2.1.) We also let $\hat{P}$ denote a polytope whose normal fan is the barycentric subdivision of Σ.

Let M be a partition matrix for $P_0,\dots,P_n$. According to Section 6.1 M produces a map $C : \mathcal{F}(\partial P) \to 2^{[n+1]}$ which assigns to every proper face Γ of P its maximal canonical coloring $C(\Gamma)$. The induced canonical facet coloring $\hat{C}$ of $\hat{P}$ is simplicial by Theorem 6.7. The next theorem says that for any $F_0,\dots,F_n$ of degrees $\alpha_0,\dots,\alpha_n$ the determinant of the residue matrix M_F (see Definition 5.7) gives an element whose residue is the combinatorial degree of $\hat{C}$.

Theorem 6.9. *Let X be a complete toric variety of dimension n. Let $\alpha_0, \dots, \alpha_n$ be semi-ample degrees and $P_0, \dots, P_n$ their polytopes. Consider a partition matrix M for $P_0, \dots, P_n$. For any collection of homogeneous polynomials $F_0, \dots, F_n$ of degrees $\alpha_0, \dots, \alpha_n$ consider the corresponding residue matrix M_F. Then the residue of $\det(M_F)/\prod_\rho x_\rho$ is equal to the combinatorial degree of the canonical facet coloring of $\hat{P}$:*

$$\mathrm{Res}_F\left(\det(M_F)/\prod_\rho x_\rho\right) = \mathrm{cdeg}(\hat{C}).$$

Proof. First notice that we can work on the variety $\hat{X}$ defined by the polytope $\hat{P}$. Indeed, let $\pi : \hat{X} \to X$ be the birational morphism defined by the barycentric refinement $\hat{\Sigma} \to \Sigma$ and $\pi^* : S \to \hat{S}$ the induced homomorphism of homogeneous coordinate rings. Then each polynomial $\hat{F}_i = \pi^*(F_i)$ is of semi-ample degree $\hat{\alpha}_i = \pi^*(\alpha_i)$ and by Proposition 2.1

$$\mathrm{Res}_F^X(H) = \mathrm{Res}_{\hat{F}}^{\hat{X}}(\hat{H}),$$

where $H = \det(M_F)/\prod_\rho x_\rho$ and $\hat{H} = \pi^*(\det(M_F))/\prod_{\hat{\rho}} x_{\hat{\rho}}$, for $\hat{\rho} \in \hat{\Sigma}(1)$.

Since the degrees $\hat{\alpha}_i$ have the same polytopes P_i we did not change the partition matrix and the pullback $\pi^*(M_F)$ is the residue matrix $M_{\hat{F}}$ for the $\hat{F}_i$. Therefore we can apply Proposition 5.8 for the canonical facet coloring of $\hat{P}$ to obtain squarefree monomials $\hat{y}_0, \dots, \hat{y}_n$ and $\hat{z}_0, \dots, \hat{z}_n$ in $\hat{S}$ which satisfy

(1) $\hat{y}_0 \cdots \hat{y}_n = \hat{z}_0 \cdots \hat{z}_n / \prod_{\hat{\rho}} x_{\hat{\rho}}$,
(2) $\hat{y}_i \hat{F}_i = \sum_{j=0}^n \hat{A}_{ij} \hat{z}_j$ for some $\hat{A}_{ij} \in \hat{S}_{\alpha_i + \deg(\hat{y}_i) - \deg(\hat{z}_j)}$, $0 \le i \le n$,
(3) $\hat{z}_0, \dots, \hat{z}_n$ do not vanish simultaneously on $\hat{X}$,
(4) $\det(M_{\hat{F}})/\prod_{\hat{\rho}} x_{\hat{\rho}} = \det \hat{A}$.

(Part (3) follows since the $\hat{z}_i$ define the canonical facet coloring $\hat{C}$ of $\hat{P}$ which is simplicial according to Theorem 6.7.) By Corollary 4.6

$$\mathrm{Res}_{\hat{F}}(\det \hat{A}) = \mathrm{cdeg}(\hat{C}),$$

which completes the proof. □

7. Locally Unmixed Degrees

In this section we consider the special case when the $n+1$ polytopes share a complete flag of faces. An essential family of degrees with such collection of polytopes is called locally unmixed. We show that for any family of locally unmixed degrees one can write an explicit partition matrix yielding an element of residue ± 1 (Theorem 7.3).

Definition 7.1. Polytopes $P_0, \dots, P_m \subset \mathbb{R}^n$ are said to *share a complete flag* if for each P_i there is a complete flag of faces:

$$P_i^0 \subset P_i^1 \subset \cdots \subset P_i^{n-1}, \quad \dim P_i^j = j,$$

such that the sums of the corresponding entries $P^j = \sum_{i=0}^m P_i^j$ form a complete flag of faces of $P = \sum_{i=0}^m P_i$:

$$P^0 \subset P^1 \subset \cdots \subset P^{n-1} \subset P^n = P, \quad \dim P^j = j.$$

An immediate consequence of the above definition is that if I is any non-empty subset of $\{0,\dots,m\}$ we can similarly define $P_I = \sum_{i\in I} P_i$ such that the $P_I^j = \sum_{i\in I} P_i^j$ also form a complete flag of faces of P_I:

$$P_I^0 \subset P_I^1 \cdots \subset P_I^{n-1}.$$

Definition 7.2. Let X be a complete toric variety of dimension n. An essential family of semi-ample degrees $\alpha_0,\dots,\alpha_n$ is said to be *locally unmixed* if the corresponding polytopes $P_0,\dots,P_n$ share a complete flag.

Note that the P_i themselves may be only $n-1$ dimensional, although at least two of them must be n-dimensional since the family is essential.

Theorem 7.3. *Let $\alpha_0,\dots,\alpha_n$ be locally unmixed degrees on X. Define partitions:*

$$M_{ij} = \{u \in P_i^j \cap \mathbb{Z}^n \quad \text{with} \quad u \notin P_i^{j-1} \cap \mathbb{Z}^n\}.$$

This is a compatible collection of partitions and the corresponding residue matrix gives an element of residue ± 1 for any homogeneous polynomials $F_i \in S_{\alpha_i}$ not vanishing simultaneously on X.

Before we begin the proof let us illustrate the partition using the following 3-dimensional example.

Example 7.4. Consider four 3-dimensional polytopes P_0, P_1, P_2 and P_3 as in Figure 7.1. They share a complete flag of faces. Indeed, each of them has a face with inner normal $(0,0,-1)$ and this face has an edge along the vector $(1,0,0)$. Then for each $0 \le i \le 3$ set M_{i0} consists of the vertex of the flag (point marked as "0"), set M_{i1} consists of the other lattice points on the edge of the flag (lattice points marked as "1"), set M_{i2} consists of the lattice points on the face of the flag, but not on the edge (points marked as "2"), and the rest of the lattice points constitute M_{i3} (points marked as "3").

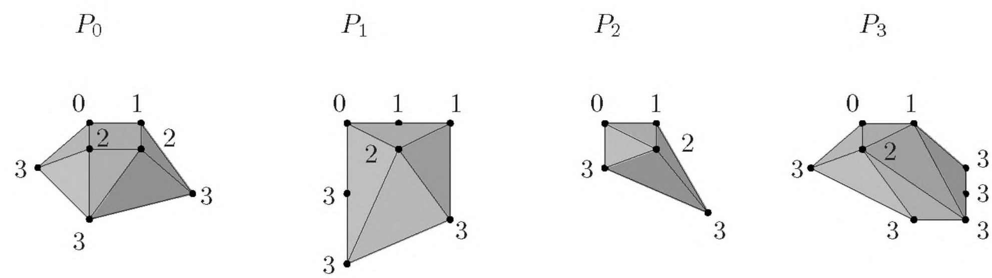

FIGURE 7.1.

We start with a simple lemma.

Lemma 7.5. *Let P be a polytope of dimension n and P' a polytope such that $P+P'$ is also a polytope of dimension n. For any facet Q of P there is a unique face Γ' of P' such that $Q+\Gamma'$ is a proper face (in fact a facet) of $P+P'$. Hence, if $u \in P$ is a point in the relative interior of Q and $u' \in P'$ is not on the corresponding face Γ' of P' then $u+u'$ is in the interior of $P+P'$.*

Proof. Let $\mathbb{R}^n$ be the affine span of P. Since $P+P'$ is also n-dimensional, we must have $P' \subset \mathbb{R}^n$ and $P+P' \subset \mathbb{R}^n$. For any facet Q of P there is a unique linear functional (up to scaling) $v_Q \in (\mathbb{R}^n)^*$ minimized on Q in P. Let Γ' be the unique maximal face of P' on which v_Q is minimized. The Minkowski sum $Q+\Gamma'$ is the facet of $P+P'$ on which v_Q is minimized and conversely any face of $P+P'$ with Q a summand must minimize v_Q and so must be $Q+\Gamma'$.

For the second statement, note that Q is the only face of P containing u. By the first part, every face Γ'' of P' such that $Q+\Gamma''$ is contained in a proper face of $P+P'$ must have $\Gamma'' \subset \Gamma'$. As a consequence if u' is not on Γ', hence not on any such Γ'', $u+u'$ is not contained in any proper face of $P+P'$. □

Proof of Theorem 7.3. The lattice point partitions M_{ij} are induced from the vertex partitions obtained from the same rule restricted to the vertices of the P_i. To show that M is a partition matrix we must show that:

$$\sum_{i=0}^{n} M_{\varepsilon(i)i} \subset \operatorname{int}\Big(\sum_{i=0}^{n} P_i\Big)$$

for any permutation ε of $\{0,\dots,n\}$. We will show by induction that

$$\sum_{i=0}^{j} M_{\varepsilon(i)i} \subset \operatorname{int}\Big(\sum_{i=0}^{j} P^j_{\varepsilon(i)}\Big)$$

for $j=0,\dots,n$. The case $j=n$ is our desired result. Let $I(j)=\{\varepsilon(0),\dots,\varepsilon(j)\} \subset \{0,\dots,n\}$. Hence, the right hand side is $P^j_{I(j)}$, a polytope of dimension j.

The case $j=0$ is trivial. For the induction, we assume

$$\sum_{i=0}^{j-1} M_{\varepsilon(i)i} \subset \operatorname{int}\big(P^{j-1}_{I(j-1)}\big).$$

Next, $P^{j-1}_{I(j-1)}$ is a facet of $P^j_{I(j-1)}$ (the case $j=n$ requires that $P^n_{I(n-1)}$ is actually n-dimensional), so we apply Lemma 7.5. Any point in $\sum_{i=0}^{j-1} M_{\varepsilon(i)i}$ lies in the interior of $P^{j-1}_{I(j-1)}$, and any point in $M_{\varepsilon(j)j}$ does not lie on the associated face $P^{j-1}_{\varepsilon(j)}$ of $P^j_{\varepsilon(j)}$. Therefore, by Lemma 7.5, any point in $\sum_{i=0}^{j} M_{\varepsilon(i)i}$ is in the (relative) interior of $P^j_{I(j-1)}+P^j_{\varepsilon(j)}=P^j_{I(j)}$ as desired.

To show that the combinatorial degree of the maximal canonical coloring of $\hat{P}$ is ±1 we apply Theorem 4.3. Recall that a face of codimension k of $\hat{P}$ is a flag of k faces $\Gamma_{i_1} \subset \Gamma_{i_2} \subset \cdots \subset \Gamma_{i_k}$ of P. We show that there is only one complete flag of faces of $\hat{P}$ colored $(\{n\},\{n,n-1\},\dots,\{n,\dots,1\})$, namely $(P^{n-1},(P^{n-1},P^{n-2}),\dots,(P^{n-1},\dots,P^0))$.

To do this we prove a few simple lemmas:

Lemma 7.6. *The maximal canonical coloring of the face $P^j \subset P$ for $j<n$ is $\{j+1,\dots,n\}$.*

Proof. The polytope P^j is the Minkowski sum of $P^j_0,\dots,P^j_n$, and each P^j_i contains precisely all of the lattice points in M_{ik} for $k=0,\dots,j$. Thus, the corresponding coloring matrix for P^j has all 1's in columns $0,\dots,j$ and all 0's in columns $j+1,\dots,n$. It follows immediately that the only maximal coloring is $\{j+1,\dots,n\}$, as desired. □

Lemma 7.7. *The maximal canonical coloring of any proper subface of P^j other than P^{j-1} contains some color k with $k < j$.*

Proof. Let Γ be a proper subface of P^j. We decompose Γ as the Minkowski sum $\Gamma_0 + \cdots + \Gamma_n$ where each Γ_i is a subface of P_i^j. If $j = n$ and if Γ were a counterexample to the lemma it would have to be colored just $\{n\}$. The last column of its coloring matrix is 0. Consequently Γ_i contains no points of M_{in} and so is entirely contained in P_i^{n-1}. So we can reduce to the case $j < n$ and assume that each Γ_i is a *proper* subface of P_i^j.

Now assume $\Gamma \neq P^{j-1}$. We show that we can take k to be the smallest number such that for all i, $P_i^k \not\subseteq \Gamma_i$. If $P_i^{j-1} \subset \Gamma_i$ for some i then as Γ_i is a proper face of P_i^j we must have $\Gamma_i = P_i^{j-1}$. This is a facet of P_i^j, so repeated applications of Lemma 7.5 show that every other summand $\Gamma_{i'} = P_{i'}^{j-1}$ and so $\Gamma = P^{j-1}$, a contradiction. Therefore, $k \leq j-1$.

By hypothesis, for some i, $P_i^{k-1} \subset \Gamma_i$ but $P_i^k \not\subseteq \Gamma_i$. In particular $\Gamma_i \cap M_{ik} = \emptyset$. If $\Gamma_{i'} \cap M_{i'k} \neq \emptyset$ for some $i' \neq i$, then another application of Lemma 7.5 shows that $\Gamma_i + \Gamma_{i'}$ contains a point in the relative interior of $P_i^k + P_{i'}^k$ and so must contain the entire face. But this would imply Γ_i contains P_i^k, a contradiction. Therefore, in the coloring matrix of Γ coming from M, the entire column C_k is 0 and so k is part of the canonical maximal coloring of Γ as desired. □

Our desired result now follows by induction. By Lemma 7.6, P^{n-1} is colored just $\{n\}$ and by Lemma 7.7 it is the only such face of P (facet of P). Inductively, the face of P given by the flag of faces $(P^{n-1}, P^{n-2}, \ldots, P^j)$ in P is colored $\{n, \ldots, j+1\}$. For the next step we must add a subface of P^j to the flag with j the only new color. But by Lemma 7.6 and Lemma 7.7, the only such subface is P^{j-1}. □

8. Dimension Two

In this section we prove that matrices whose determinant have residue ± 1 can be found for almost all essential, two-dimensional families of degrees.

Recall the definition of essential in Definition 3.1. In the special case $n = 2$, essential means that no P_i is zero dimensional, and while some or all of the P_i may be one dimensional line segments, no two such are parallel line segments. We will show we can always find a residue matrix that gives an element of residue ± 1 in all but one exceptional case.

Definition 8.1. Degrees $\alpha_0, \alpha_1, \alpha_2$ are *exceptional* if for two of them, α_i and α_j, the corresponding polygons P_i and P_j are 1-dimensional, and the third α_k is an ample divisor on the toric variety defined by $P_i + P_j$.

Theorem 8.2. *Let $\alpha_0, \alpha_1, \alpha_2$ be an essential, non exceptional family of degrees on a toric surface X. There exists a partition matrix for the α_i which yields an element of residue ± 1 for every set of $F_i \in S_{\alpha_i}$ without a common root.*

Note that the codimension 1 theorem for the critical degrees has been proved by Cox and Dickenstein [11] when all α_i are full dimensional. Such a case, of course, will never be exceptional. It is, however possible for the critical degree to be of codimension 1, in which case the residue map is an isomorphism, and still be exceptional. See Example 8.6 below.

Proof. Let P_0, P_1, P_2 be the corresponding polygons and $P = P_0 + P_1 + P_2$ their Minkowski sum. Every edge e of P is the sum of edges from one or more of the P_i and vertices from the others. Label an edge by a subset of $\{0, 1, 2\}$ consisting of those polygons for which the summand of e is an edge. Now consider consecutive edges of F. Proceed until we have a sequence containing all three labels $0, 1, 2$. Take the smallest subsequence with this property. We then have the following cases:

(1) The sequence has length 1, so there is a single edge labeled $\boxed{0\,1\,2}$. This will be the locally unmixed case.

(2) The sequence has length 2. Up to relabeling and change of direction the sequence will be either:
   (a) $\boxed{0\,1}$, $\boxed{2}$ or
   (b) $\boxed{0\,1}$, $\boxed{1\,2}$.

   Such sequences will be called *partially unmixed*.

(3) The sequence has length 3 or more. All such sequences can be represented as $\boxed{0\,1}, \boxed{1}, \boxed{1}, \ldots, \boxed{1}, \boxed{1\,2}$.

   The numbers in gray may or may not occur. That is to say, the first term must contain label 0 but may or may not contain 1. Similarly for the final term must contain label 2 and possibly also label 1. There is at least one term labeled just 1 in the middle, but there may be others. Altogether, sequences of this type will be called *generically mixed*.

Case 1: The three degrees share an edge, hence share a complete flag consisting of this edge and either of the two vertices. So the polygons are locally unmixed in the sense of the previous section. Therefore, we know we can always find a partition yielding a residue 1 matrix. We illustrate the partition via the diagram in Figure 8.1.

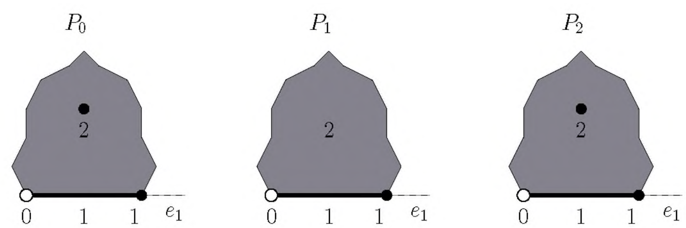

FIGURE 8.1. Case 1: Locally Unmixed Partition

Each of the three figures represents one of the three polygons. The edge e_1 on the bottom is shared by all three polygons. The white vertex in each P_i is in partition set M_{i0} hence marked as "0" in the diagram, the rest of the lattice points on e_1 are in set M_{i1}, and finally any lattice points off of e_1 are in set M_{i2}. Hence the partition matrix M_{ij} is exactly the one constructed in the previous section. Since the polygons are essential, at most one of them is 1-dimensional, thus two of them, say P_0 and P_2 as shown, have at least one point off the edge e_1 marked "2".

Case 2a: The polygons are partitioned according to Figure 8.2.

There are two distinguished edges e_1 and e_2. Polygons P_0 and P_1 share a complete flag along edge e_1 and are partitioned accordingly into three sets M_{i0}, M_{i1}, and M_{i2} for $i = 0, 1$ as shown. But, this time the third polygon P_2 has only one point on e_1 represented by the dotted line. This point is put into set M_{20} and all other points of P_2 are put into set M_{22}. Notice that in the third polygon $M_{21} = \emptyset$.

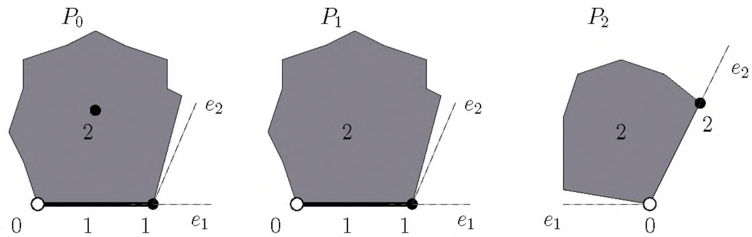

FIGURE 8.2. Case 2a

Also note that P_0 and P_1 each have only one point (marked "1") on the dotted lines representing edges parallel to e_2.

To see that this is a partition matrix we must show that the sum of lattice points in M_{i0}, M_{j1}, and M_{k2} with $\{i, j, k\} = \{0, 1, 2\}$ is in the interior of P. From the diagram this corresponds to taking three points marked "0", "1" and "2" respectively from three different polygons and showing they cannot all lie on parallel edges with the same inner normal. If "0" and "1" come from the first two polygons then their sum is in the interior of the edge e_1 of the two-dimensional (by essentiality) sum of P_0 and P_1. By definition however a point marked "2" from P_2 is not on this edge.

If instead "0" comes from the P_2 and "1" comes from P_0 or P_1 the Minkowski sum of these two points is either the vertex lying only on edges e_1 and e_2 or in the interior of e_1. However, the points marked "2" from the third polygon P_0 or P_1 are not on either of these two edges.

For the combinatorial degree we apply Lemma 6.8. This shows that any face of P (facet of $\tilde{P}$) maximally canonically colored by C_2 must only be colored by C_2. Such a face must have its coloring matrix with 1's only in the first two columns. It is easy to see from the picture that this can only happen for the bottom edge e_1. It is also easy to see that the two vertices of this edge are colored $\{1, 2\}$ and $\{0, 2\}$ respectively. In particular there is a unique complete flag colored $(\{2\}, \{1, 2\})$, as desired.

Case 2b: The difference between Case 2a and Case 2b is that P_1 now also contains an edge parallel to e_2. One attempt to account for this would be to use the same partition as in Case 2a above except all the lattice points in P_1 along the edge e_2 are placed in M_{11}.

If this were a partition matrix, edge e_1 would remain the only edge colored just $\{2\}$ and its vertices would still be colored $\{1, 2\}$ and $\{0, 2\}$.

To check if this is a partition matrix, most of the arguments from the previous case go through. If we take points marked "0" from P_2 and "1" from P_0 or P_1, the sum is either the vertex between edge e_1 and e_2 or in the interior of one of these edges. Taking "0" from P_1 and "1" from P_0 again yields a point in the interior of edge e_1. However, if we take "0" from P_0 and "1" from P_1 there is a problem if there is some edge other than e_1 passing through both these points. But this can only happen if the edge e_3 of P_0 directly before e_1 passes through the endpoint of e_2 in P_1, marked "1" as in Figure 8.3.

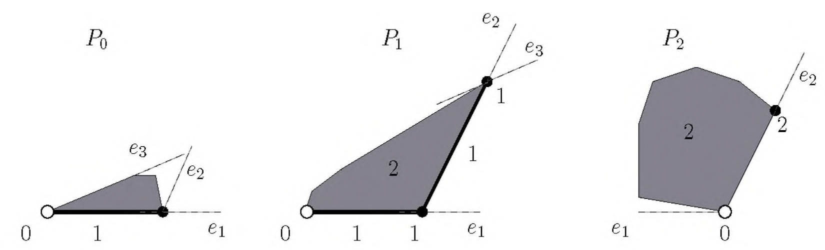

FIGURE 8.3. Case 2b: Failed partition

In this case the partition fails, so we try to partition in a different way, reversing the roles of e_1 and e_2 as in Figure 8.4.

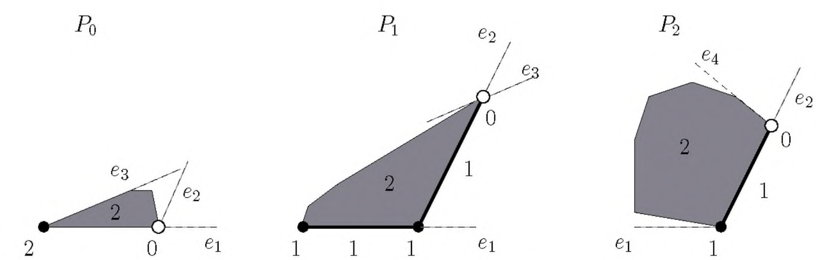

FIGURE 8.4. Case 2b: Switched partition

Just as before this partition works as long as the next edge e_4 of P_2 after e_2 does not pass through the left endpoint of e_1 in P_1. If both of the above attempts fail, we have that the edge e_3 of P_0 before e_1 passes through the end point of e_2 in P_1, and the edge e_4 of P_2 after e_2 passes through the left endpoint of e_1 in P_1.

In this final case, we show that we can find a partition matrix unless we are in the exceptional case. First, assume that one of P_0 or P_2 is actually two-dimensional. Assume without loss of generality it is P_0. Take the partition in Figure 8.5.

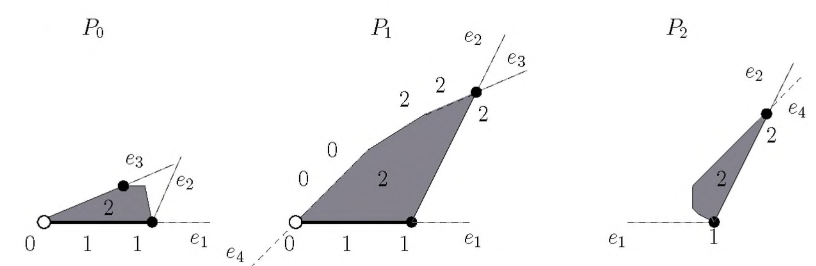

FIGURE 8.5. Case 2b: Non-exceptional

Once again, a sum of points marked “0” from P_0 and “1” from P_1 must lie on the interior of e_1 which contains no points marked “2” from P_2. If we take “0” from P_1 then all the edges on which it lies are between e_1 and e_3. But since P_0 was assumed two-dimensional, the only such edge that could pass through a point marked "1" is e_1 itself.

If we take "1" from P_2 we must lie on an edge of P_2 on or before e_2 but on or after e_4. The edge e_2 passes only through points marked "1" or "2" in P_0 and P_1. The next edge before e_2 is e_1 which passes through only "0" and "1" in P_0 and P_1. Furthermore all edges before e_1 and on or after e_4 pass only through "0" in P_1 and, since e_3 is before e_4, only through "0" in P_0. In every case we do not get three points from different partition sets from three different polygons all lying on the same edge.

A similar argument with an analogous partition applies if P_2 is two-dimensional. Finally, if both P_0 and P_2 are one-dimensional, the only way the partition above can fail is if the edge e_3, parallel to e_1, which is known to go through a point marked "2" in P_1, also goes through a point marked "0". But this can only happen if this is the only other point of P_1 and moreover the edge connecting this point to the endpoint of e_1 is parallel to e_4 which is also parallel to e_2. In other words, we must have P_1 have the same normal fan as the Minkowski sum of the two non-parallel segments that are P_0 and P_2. That is to say we are in the exceptional case.

Case 3: The mixed case is somewhat easier and is partitioned as in Figure 8.6.

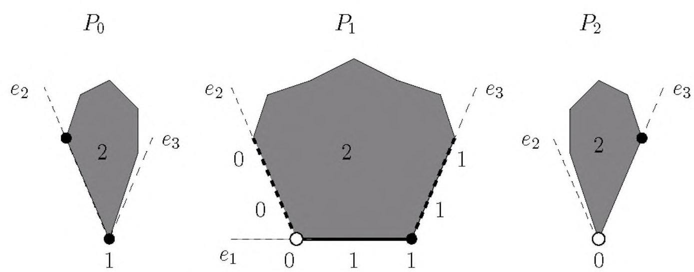

FIGURE 8.6. Case 3: Mixed partition

There are now three edges e_1, e_2, and e_3. There may actually be several edges of P_1 between e_2 and e_3 in which case e_1 is the left most such edge. The edge e_2 intersects P_1 in either a point or a whole edge, marked as a dashed line in the diagram. All of the points on this edge are placed in set M_{10}. The edge e_2 together with e_1 and all other edges before e_3 intersects P_2 in a single point, represented by the dotted lines, placed in partition set M_{20} hence also marked "0" in the diagram. The edge e_3 together with e_1 and all other edges before e_3 intersects P_0 in a single point placed in set M_{01} Finally e_3 intersects P_1 in either a single point or a whole edge marked by a dashed line and all of the points are in set M_{11}. All the points in P_1 on the edges between e_2 and e_3 are also in set M_{11}. All other lattice points in each P_i is in partition set M_{i2} thus marked as "2".

To show that this is a partition matrix we must take a point marked "1" from P_0 or 0 from P_2 (otherwise we would have to take "2" from both). First if we took both "0" from P_2 and "1" from P_0, then their Minkowski sum is on both e_2 and e_3. However any point marked "2" from P_1 is on neither edge nor any edge in between.

If we took "0" from P_2 and "1" from P_1, the Minkowski sum is a point lying only on edges on or between e_1 and e_3. However, any point marked "2" from P_0 is not on any of these edges. The case of "1" from P_1 is similar.

For combinatorial degree we note that the only edge colored just $\{2\}$ is the edge e_1. Every other edge either intersects a point marked "2" or, if it is one of the other edges missing "2", can be colored $\{0,2\}$. The two vertices of this edge are colored $\{0,2\}$, and $\{1,2\}$ respectively, completing the combinatorial degree computation.

□

To finish we show that an exceptional set of degrees *never* has a compatible vertex partition resulting in an element of residue one.

Proposition 8.3. *Let $\alpha_0, \alpha_1, \alpha_2$ be an exceptional family of degrees on a two-dimensional toric variety X. Any compatible vertex partition yields a residue matrix with determinant zero.*

Proof. We can assume that the corresponding polygons P_0 and P_1 are one-dimensional and P_2 has the same normal fan as P_0+P_1. Therefore P_0 and P_1 each have two vertices which we denote by u_0, u_1 and v_0, v_1, and P_2 has four vertices w_0, w_1, w_2, w_3. For each pair u_i, v_j there is a unique w_k such that $u_i+v_j+w_k \in \operatorname{int}(P_0+P_1+P_2)$. Now suppose we have a compatible vertex partition with associated partition matrix M. If u_0 and u_1 are in the same partition set, by the above we know there do not exist v_j, w_k such that both $u_0+v_j+w_k$ and $u_1+v_j+w_k$ are in $\operatorname{int}(P_0+P_1+P_2)$. Hence, there are no non-zero terms in the expansion of the determinant of the induced residue matrix. If u_0, u_1, v_0, v_1 all lie in two columns of the partition matrix, then again there is no possible compatible choice of w_k in the complementary entry and the induced residue matrix will have determinant zero. Therefore, up to relabeling we are left with only one choice of partition matrix of the form:

$$\begin{bmatrix} u_0 & u_1 & \emptyset \\ v_0 & \emptyset & v_1 \\ P_2^0 & P_2^1 & P_2^2 \end{bmatrix}.$$

For compatibility each of P_2^i can only contain a unique vertex w_k. But this implies there is some vertex w_k which cannot lie in any of the P_2^i, a contradiction. Hence there are no non-trivial compatible partition matrices. In particular there is no partition yielding an element of residue one. □

This last result shows that while we know for essential degrees there must always exist a polynomial of residue one by Theorem 3.2, it cannot always be obtained as the determinant of a matrix. On the flip side the method does work for all but one quite degenerate situation. We illustrate this by constructing residue matrices for some examples.

Example 8.4. Consider the polynomials:

$$\begin{aligned} f_0 &= a_0x + a_1xy + a_2y^2 \\ f_1 &= b_0 + b_1x + b_2x^2 + b_3xy \\ f_2 &= c_0 + c_1y + c_2xy^2. \end{aligned}$$

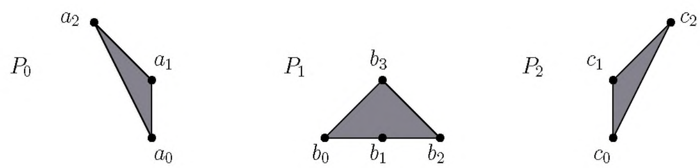

FIGURE 8.7.

The Newton polygons are shown in Figure 8.7 with the lattice points labeled by their corresponding coefficients. This falls under Case 3 of the previous theorem so applying the partition as in Figure 5.1 yields the following residue matrix:

$$\begin{bmatrix} 0 & a_0x & a_1xy + a_2y^2 \\ b_0 & b_1x + b_2x^2 & b_3xy \\ c_0 & 0 & c_1y + c_2xy^2. \end{bmatrix}$$

The determinant is

$$a_0b_3c_0xy - a_0b_0c_1xy - a_0b_0c_2xy^2 - a_1b_1c_0x^2y - a_1b_2c_0x^3y - a_2b_1c_0xy^2 - a_2b_2c_0x^2y^2.$$

This is a polynomial supported on the interior of the Minkowski sum $P_0 + P_1 + P_2$. The homogenization up to critical degree has toric residue equal to 1.

Example 8.5. Consider the polynomials:

$$\begin{aligned} f_0 &= a_0 + a_1x \\ f_1 &= b_0 + b_1x + b_2y \\ f_2 &= c_0 + c_1xy. \end{aligned}$$

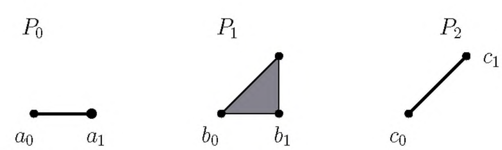

FIGURE 8.8.

The corresponding Newton polygons are shown in Figure 8.8. These polygons can be classified under Case 2a of the above theorem. As such we get the following residue matrix:

$$\begin{bmatrix} a_0 & a_1x & 0 \\ b_0 & b_1x & b_2y \\ c_0 & 0 & c_1xy \end{bmatrix}.$$

The determinant is

$$a_1b_2c_0xy + (a_0b_1c_1 - a_1b_0c_1)x^2y$$

which is supported in the interior of $P_0 + P_1 + P_2$ (consisting of two points). Once again the homogenization up to critical degree yields the desired element of residue 1.

Example 8.6. Let us now consider an exceptional system:

$$f_0 = a_0 + a_1 x$$
$$f_1 = b_0 + b_1 x + b_2 y + b_3 xy$$
$$f_2 = c_0 + c_1 y.$$

The Newton polygons consists of two line segments and their Minkowski sum (a square). Since we are in the exceptional case the theorem does not apply.

However, there is a unique interior point of the Minkowski sum, so the critical degree is trivial. Thus, there is a unique element of residue 1, namely the resultant itself which in this case is:

$$a_1 b_0 c_1 - a_0 b_1 c_1 - a_1 b_2 c_0 + b_3 a_0 c_0.$$

By Proposition 8.3 this polynomial is not expressible as the determinant of a residue matrix.

9. Further Work and Conclusions

Given a collection of $n+1$ semi-ample divisors on a toric variety X which do not have a common zero, there exists a toric residue map which is not identically zero if and only if the degrees of the divisors are essential. The goal of this work was to explicitly construct an element of residue one.

We have shown how compatible partitions of the Newton polytopes lead to matrices whose determinant is an element of critical degree with toric residue equal to a certain integer constant, namely the combinatorial degree of a canonical induced coloring. In the case the polytopes share a complete flag of faces and in almost every case in dimension 2 we have shown how to choose this partition to yield an element of residue exactly one.

The most obvious open question is to find compatible partitions yielding elements of residue one in higher dimensions when the polytopes do not necessarily share a complete flag. We have computed a large number of examples in dimension three where the four polytopes are simplices. In every case we have found working partitions. Of course there will be exceptional families, as in dimension two, where no such partitions exist. However, it is hoped that these will be relatively rare and perhaps nonexistent in the most important case when the polytopes are all full dimensional.

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