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Tropical Determinant on Transportation Polytopes

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Introduction

In this paper we generalize the results of "Tropical determinant of integer doubly stochastic matrices" [3] to the class of all rectangular integer matrices with fixed row and column sums. The discussion in [3] started with cheater's Rubik's cube problem:

When solving Rubik's cube by peeling off and replacing stickers, how many stickers do we need to peel off and replace in the worst case scenario? This problem generalizes to a very natural sorting question: Assume that we have n pails with m balls in each. Each ball is colored in one of n colors and we have m balls of each color. What is the smallest number of balls we need to move from one pail to another in the worst case scenario so that the balls are sorted by color?

This problem turns out to be equivalent to finding the sharp lower bound on the tropical determinant of integer matrices $A = (a_{ij})$ of given size n with given row and column sums m. To see this, let the entry a_{ij} be equal to the number of balls of color i in pail j. We would like to assign each pail a color so that the overall number of balls that we need to move is the smallest possible. That is, we would like to find a transversal of A with the largest possible sum of entries, which is the definition of the tropical determinant $\operatorname{tdet} A$ of A.

The set of all (real) doubly stochastic $n \times n$ matrices forms a convex polytope in $\mathbb{R}^{n^2}$, the Birkhoff polytope $\mathcal{B}_n$ (see [1]). The set of integer $n \times n$ matrices with row and column sums equal m can then be identified with the set of integer points of its m-dilate $m\mathcal{B}_n$. The tropical determinant is a piecewise linear function on $m\mathcal{B}_n$. Therefore, the described problem is equivalent to minimizing this function over the integer points of the polytope, i.e. solving an integer piecewise linear programming problem. This was done in [3].

In the current paper we are working on a natural generalization of this problem, where we replace the Birkhoff polytope with any transportation polytope. A transportation polytope is a convex polytope consisting of non-negative rectangular matrices of given size with fixed row and column sums. The set of integer such matrices is identified with the set of integer points of a transportation polytope. Our goal is to compute the sharp lower bound for the tropical determinant on integer points of a transportation polytope. Surprisingly, this integer piecewise linear programming problem in arbitrary dimension reduces to an integer non-linear (in fact, quadratic) optimization problem in dimension two (see Theorem 3.3).

This problem has a similar combinatorial interpretation. Suppose there are R balls of each of t different colors, totaling tR balls. Suppose they are placed into $s \geq t$ different pails with C balls in each pail (so $sC = tR$). We want to sort the balls by color in some t of the s pails, by replacing balls from one pail to another. What is the smallest number of balls we need to move from one pail to another to achieve this in the worst case scenario? Similar to above, let a_{ij} be the number of balls of color i in pail j. We obtain an $r \times s$ matrix $A = (a_{ij})$ whose row sums are R and column sums are C. The smallest number of moves to sort the balls is then $tR - \operatorname{tdet} A$. Thus, to answer the above question one needs to find the sharp lower bound for the tropical determinant over all such matrices A.

In this paper we build on the methods developed in [3]. We were able to simplify the arguments to the point where the desired generalization became possible. Also the answer in the general setting is more transparent. Our methods are elementary and do not rely on other results except for Hall's marriage theorem.

Following [3] we also consider and solve a version of the problem where in the definition of the tropical determinant the minimum over all the transversals is replaced with the maximum. In this case we are interested in the sharp upper bound over the integer points of the transportation polytope. As in [3], this version of the problem turns out to be significantly easier than the problem we start with.

In 1926 van der Waerden conjectured that the smallest value of the permanent of $n \times n$ doubly stochastic (with row and column sums equal to one) matrices is attained on the matrix all of whose entries are equal to $1/n$, and this minimum is attained only once. This conjecture was proved independently by Egorychev [4] and Falikman [5] in 1979/1980. In [2] Burkard and Butkovich proved a tropical version of the conjecture, where the permanent is replaced with the tropical determinant. Results of this paper and [3] provide an integral tropical version of the van der Waerden conjecture.

1. Definitions

Let $A = (a_{ij})$ be an nk by nl matrix where $\gcd(k, l) = 1$ and a_{ij} are non-negative integers. Let all the row sums in A be equal to a and all the column sums be equal to b. Computing the sum of all the entries in A in two different ways, we get $ka = lb$, which implies $a = ml$, $b = mk$ for some integer m.

Definition 1.1. Let $k \leq l$. Define $\mathcal{D}^{k,l}(m, n)$ to be the set of all $nk \times nl$ matrices with non-negative integer entries whose row sums are ml and columns sums are mk.

Definition 1.2. For an $s \times t$ matrix $A = (a_{ij})$ with $s \leq t$, its *transversal* T is a set $\{a_{1i_1}, \ldots, a_{si_s}\}$ for some subset $\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, t\}$. Furthermore, let $|T| = a_{1i_1} + \cdots + a_{si_s}$ and let $\mathcal{T}(A)$ be the set of all transversals of A. For $t \geq s$ we define transversals of A to be transversals of its transpose A^T.

Definition 1.3. The *tropical determinant* of a matrix $A = (a_{ij})$ is

$$\operatorname{tdet}(A) = \max_{T \in \mathcal{T}(A)} |T|.$$

We will refer to a transversal of A on which this maximum is attained as a *maximal transversal* of A.

Clearly, the set of transversals and, hence, the tropical determinant are invariant under row and column swaps of A.

Let $L^{k,l}(m, n)$ denote the sharp lower bound on the tropical determinant over the set $\mathcal{D}^{k,l}(m, n)$, that is,

$$L^{k,l}(m, n) = \min_{A \in \mathcal{D}^{k,l}(m,n)} \operatorname{tdet}(A).$$

Our main goal in this paper is to compute $L^{k,l}(m, n)$.

Example 1. Let $n = 5$, $k = 1$, $l = 2$, and $m = 6$. Then the matrix

$$A = \begin{pmatrix} 0 & 0 & 2 & 1 & \boxed{2} & 1\,2\,1\,2\,1 \\ 0 & 0 & 1 & \boxed{2} & 1 & 2\,1\,2\,1\,2 \\ 2 & 2 & \boxed{1} & 1 & 1 & 1\,1\,1\,1\,1 \\ 2 & \boxed{2} & 1 & 1 & 1 & 1\,1\,1\,1\,1 \\ \boxed{2} & 2 & 1 & 1 & 1 & 1\,1\,1\,1\,1 \end{pmatrix}$$

lies in $\mathcal{D}^{1,2}(6,5)$. The boxed elements form a maximal transversal of A. Thus tdet $A = 9$. We will later show that $L^{1,2}(6,5) = 9$, that is, the minimum of the tropical determinant on $\mathcal{D}^{1,2}(6,5)$ is attained at this matrix.

One of our tools is Hall's marriage theorem and, following [3], we restate this theorem and its simple corollaries here, making a small adjustment to the case of rectangular matrices. The theorem in our formulation deals with a maximal zero submatrix of A, that is a zero submatrix of A whose sum of dimensions is the largest possible.

Theorem 1.4 *(Philip Hall [6]). Let A be an $s \times s$ 0–1 matrix. Then there is a transversal in A that consists of all 1's if and only if a maximal zero submatrix in A has sum of dimensions less than or equal to s.*

For our future discussion we will need the following two corollaries.

Corollary 1.5. *Let A be an $s \times t$ 0–1 matrix. Then there is a transversal in A that consists of all 1's if and only if a maximal zero submatrix of A has sum of dimensions less than or equal to* $\max(s, t)$.

Proof. Let us assume that $s \leq t$. Extend A to a square 0–1 matrix by appending to A $t - s$ rows consisting of all 1's and apply Hall's marriage theorem to the resulting matrix. □

Let A be an $s \times t$ 0–1 matrix and W be a maximal zero submatrix of A. Then after some row and column swaps, A can be written in the form

$$A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}.$$

Corollary 1.6. *Both Y and Z have a transversal that consists of all 1's.*

Proof. Let W be $d_1 \times d_2$ and a maximal zero submatrix of Y be $s_1 \times s_2$. We can assume that it is in the lower right corner of Y, right on top of W. Then the lower right

$(s_1+d_1)\times s_2$ block of A consists of all zeroes which implies $s_1+d_1+s_2\le d_1+d_2$, and so $s_1+s_2\le d_2$. By Corollary 1.5 there exists a transversal in Y that consists of all 1's. Similarly, such a transversal exists in Z. $\square$

2. Bound on $L^{k,l}(m,n)$

We start with two simple observations concerning the tropical determinant of an arbitrary matrix.

Lemma 2.1. *Let B be an $s\times t$ matrix with $s\ge t$. Then $\operatorname{tdet} B$ is at least the sum of all the entries in B, divided by s. In particular, if all row sums of B are bounded from below by b, then $\operatorname{tdet} B\ge b$.*

Proof. The set of entries of B can be partitioned into s transversals $T_1,\dots,T_s$. Since $|T_i|\le \operatorname{tdet} B$, the sum of all entries of B does not exceed $s\operatorname{tdet} B$. $\square$

Lemma 2.2. *Let Q be an $s\times t$ matrix with $s\le t$. Let a be any element in a maximal transversal of Q. Then*

$$R+C\le \operatorname{tdet} Q+sa,$$

where R and C are the sum of entries in the row and column that contain a.

Proof. After necessary row and column swaps we can assume that $a=a_s$ is in the position (s,s) and that $\operatorname{tdet} Q=a_1+\dots+a_s$, where

$$Q=\begin{pmatrix} a_1 & & & c_1 & & \\ & a_2 & & c_2 & & \\ & & \ddots & \vdots & & \\ b_1 & b_2 & \dots & a_s & \dots & b_t \end{pmatrix},$$

and $C=c_1+\dots+c_{s-1}+a_s$ is the s-th column sum, while $R=b_1+\dots+b_{s-1}+a_s+b_{s+1}+\dots+b_t$ is the s-th row sum.

We have $b_j+c_j\le a_j+a_s$ for $j=1,\dots,s-1$, since otherwise we could switch columns j and s in Q to get a larger transversal. We also have $b_j\le a_s$ for $j=s+1,\dots,t$. Summing these up over $j=1,\dots,t$, we get

$$R+C\le \operatorname{tdet} Q+sa_s. \qquad \square$$

Recall that $A=(a_{ij})\in\mathcal{D}^{k,l}(m,n)$ is an $nk\times nl$ matrix where $k\le l$, $\gcd(k,l)=1$, and a_{ij} are non-negative integers. The row sums of A are equal to ml and the column sums are equal to mk.

Now divide m by n with remainder, $m = qn + r$, where $0 \leq r < n$. Let W be a submatrix of A with entries less than or equal to q with the largest sum of dimensions. Then after some column and row swaps, A can be written in the form

$$A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}.$$

Let X be of size t_1 by t_2.

Lemma 2.3. *We have*

$$qt_1t_2 + r(t_1l + t_2k) \geq klnr.$$

Proof. Let Σ_W and Σ_Y be the sums of all the entries in blocks W and Y. Then $\Sigma_W \leq q(nk - t_1)(nl - t_2)$ since all the entries of W do not exceed q. Hence

$$\Sigma_Y = (nl - t_2)mk - \Sigma_W \geq (nl - t_2)mk - q(nk - t_1)(nl - t_2).$$

On the other hand, $\Sigma_Y \leq t_1ml$. Putting these two inequalities together, we get

$$(nl - t_2)mk - q(nk - t_1)(nl - t_2) \leq t_1ml,$$

which is easily seen to be equivalent to $qt_1t_2 + r(t_1l + t_2k) \geq klnr$ using $m = qn + r$. This argument also shows that $qt_1t_2 + r(t_1l + t_2k) - klnr$ is an upper bound for Σ_X. $\square$

This lemma motivates the following definition.

Definition 2.4. Let x and y be integers satisfying $x \geq rk$, $y \geq rl$, and

$$qxy + r(xl + yk) \geq klnr, \tag{2.1}$$

whose sum $x + y$ is the smallest possible.

Note that while $x + y$ is defined uniquely, this is not necessarily true for x and y. Also, the conditions $x \geq rk$ and $y \geq rl$ will be necessary for the construction in Proposition 3.2.

Recall that

$$A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where W is a maximal submatrix that consists of elements not exceeding q and X is of size t_1 by t_2.

Lemma 2.5. *Let $t_1 + t_2 \le nk$. Then*

$$\operatorname{tdet} A \ge \min\left(nk(q+1), \operatorname{tdet} Y + \operatorname{tdet} Z + (nk - t_1 - t_2)q\right).$$

Proof. Consider the set of all transversals in A which contain a maximal transversal in Y and a maximal transversal in Z. Choose one such transversal T_A with the largest sum $|T_A|$. Let $T_Y \subset T_A$ and $T_Z \subset T_A$ be the corresponding maximal transversals in Y and Z, respectively. Note that since $t_1 + t_2 \le nk$, the transversals in Y and Z have respectively t_1 and t_2 entries. Cross out the rows and columns of A which contain T_Y and T_Z to get an $(nk - t_1 - t_2) \times (nl - t_1 - t_2)$ submatrix Q of W. Then $T_A = T_Y \cup T_Z \cup T_Q$, where the transversal T_Q of Q is also maximal (by construction). Therefore, we obtain

$$\operatorname{tdet} A \ge |T_A| = \operatorname{tdet} Y + \operatorname{tdet} Z + \operatorname{tdet} Q. \tag{2.2}$$

First, assume that Q contains a transversal all of whose elements are equal to q. Then we have $\operatorname{tdet} Q = (nk - t_1 - t_2)q$ and the statement follows from the above inequality.

Next, assume that every maximal transversal of Q has an entry less than or equal to $q-1$. We can rearrange the rows and columns of A as follows

$$A = \left(\begin{array}{cccc|cccccc|cccc}
 & & & & & & & c_1 & & & a_1 & & & \\
 & X & & & & & & c_2 & & & & a_2 & & \\
 & & & & & & & \vdots & & & & & \ddots & \\
 & & & & & & & c_{t_1} & & & & & & a_{t_1} \\
\hline
 & & & & e_1 & & & f_1 & & & & & & \\
 & & & & & e_2 & & f_2 & & & & & & \\
 & & & & & & \ddots & \vdots & & & & & & \\
d_1 & d_2 & \dots & d_{t_2} & g_1 & g_2 & \dots & e_{t_3} & \dots & g_{t_4} & i_1 & i_2 & \dots & i_{t_1} \\
\hline
b_1 & & & & & & & h_1 & & & & & & \\
 & b_2 & & & & & & h_2 & & & & & & \\
 & & \ddots & & & & & \vdots & & & & & & \\
 & & & b_{t_2} & & & & h_{t_2} & & & & & &
\end{array}\right)$$

Here the middle block is Q, $T_Y = \{a_1, \dots, a_{t_1}\}$, $T_Z = \{b_1, \dots, b_{t_2}\}$, and $T_Q = \{e_1, \dots, e_{t_3}\}$. Also, we may assume that $e_{t_3} \le q - 1$.

Applying Lemma 2.2 to the matrix Q, together with $e_{t_3} \le q-1$, we obtain

$$g + f \le \operatorname{tdet} Q + (q-1)t_3, \tag{2.3}$$

where $f = f_1 + \dots + f_{t_3-1} + e_{t_3}$ and $g = g_1 + \dots + g_{t_3-1} + e_{t_3} + g_{t_3+1} + \dots + g_{t_4}$.

Next, note that for every $1 \le j \le t_1$ we have $c_j \le a_j$, by maximality of T_Y. We also know that $i_j \le q$ as it lies in the block W. Assume that we simultaneously have $c_j = a_j$ and $i_j = q$. Then if we swap columns containing c_j and a_j we do not change $|T_Y|$ (as

c_j replaces a_j) but make $|T_Q|$ bigger (since $i_2 = q$ replaces $e_{t_3} \leq q-1$), which contradicts our choice of T_A. Therefore, $c_j + i_j \leq a_j + q - 1$ and, summing these up over $1 \leq j \leq t_1$, we get

$$c + i \leq \operatorname{tdet} Y + (q-1)t_1, \tag{2.4}$$

where $c = c_1 + \cdots + c_{t_1}$ and $i = i_1 + \cdots + i_{t_1}$. Similarly, we have

$$d + h \leq \operatorname{tdet} Z + (q-1)t_2, \tag{2.5}$$

where $d = d_1 + \cdots + d_{t_2}$ and $h = h_1 + \cdots + h_{t_2}$.

Summing up (2.3)–(2.5) and using $c + f + h = mk$, $d + g + i = ml$, and (2.2), we get

$$mk + ml \leq \operatorname{tdet} Y + \operatorname{tdet} Z + \operatorname{tdet} Q + (q-1)(t_1 + t_2 + t_3) \leq \operatorname{tdet} A + (q-1)nk.$$

Finally, this implies

$$\operatorname{tdet} A \geq m(k+l) - (q-1)nk \geq qn(k+l) - (q-1)nk = qnl + nk \geq nk(q+1). \qquad \square$$

Here is our main lower bound on the tropical determinant. In the next section we show that it is sharp.

Theorem 2.6. *Let $m = qn + r$ for $0 \leq r < n$, and x, y as in Definition 2.4. Then*

$$L^{k,l}(m,n) \geq \min(nk(q+1), nkq + x + y).$$

Proof. As before, we can assume that

$$A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

where X is of size $t_1 \times t_2$ and each entry of W is at most q. If $t_1 + t_2 \geq nk$ then sum of dimensions of W is

$$nk - t_1 + nl - t_2 \leq nl,$$

so by Corollary 1.5 there is a transversal in A whose entries are at least $q+1$. Therefore,

$$\operatorname{tdet} A \geq nk(q+1).$$

Now assume that $t_1 + t_2 < nk$. By Corollary 1.6 there exist transversals in Y and Z whose entries are at least $q+1$. Thus, we can write

$$\operatorname{tdet} Y \geq t_1(q+1) \quad \text{and} \quad \operatorname{tdet} Z \geq t_2(q+1).$$

If we also have $x + y \leq t_1 + t_2$, then

$$\operatorname{tdet} Y + \operatorname{tdet} Z + (nk - t_1 - t_2)q \geq nkq + t_1 + t_2 \geq nkq + x + y.$$

The statement now follows from Lemma 2.5.

It remains to consider the case where $t_1 + t_2 < nk$ and $t_1 + t_2 < x + y$. If we had $t_1 \geq rk$ and $t_2 \geq rl$, then Lemma 2.3 and the definition of x and y would imply that $x + y \leq t_1 + t_2$, which is not the case now.

If $t_1 \leq rk$ and $t_2 \leq rl$, then (rk, rl) also satisfies the inequality in Lemma 2.3, so by Definition 2.4 we must have $x = rk$ and $y = rl$. On the other hand, since $t_1 + t_2 < nk$, by Corollary 1.5, every maximal transversal in A contains an entry not exceeding q. Pick a maximal transversal and let e be an entry in that transversal such that $e \leq q$. By Lemma 2.2 we have

$$lm + km \leq \operatorname{tdet} A + lne \leq \operatorname{tdet} A + lnq,$$

which implies

$$\operatorname{tdet} A \geq nkq + kr + lr = nkq + x + y.$$

Finally, assume that $t_1 \geq rk$ and $t_2 \leq rl$. As before, this implies that (t_1, rl) also satisfies the inequality in Lemma 2.3, so by Definition 2.4 we must have $x + y \leq t_1 + rl$. On the other hand, the row sums in Z are bounded below by

$$ml - q(nl - t_2) = rl + qt_2,$$

so by Lemma 2.1, $\operatorname{tdet} Z \geq rl + qt_2$. We have

$$\begin{aligned} \operatorname{tdet} Y + \operatorname{tdet} Z + (nk - t_1 - t_2)q &\geq t_1(q + 1) + rl + qt_2 + (nk - t_1 - t_2)q \\ &= nkq + t_1 + rl \geq nkq + x + y, \end{aligned}$$

and we are done by Lemma 2.5. The case $t_1 \leq rk$ and $t_2 \geq rl$ is similar. □

3. Constructions

To show that the bound in Theorem 2.6 is sharp, we provide two constructions.

Proposition 3.1. *There exists $A \in \mathcal{D}^{k,l}(m, n)$ such that* $\operatorname{tdet} A \leq nk(q + 1)$.

Proof. We describe how to construct such matrix $A \in \mathcal{D}^{k,l}(m, n)$ whose entries equal q or $q + 1$. Each row of A has $q + 1$ repeated rl times and each column has $q + 1$ repeated rk times. To achieve this, in the first row, place $q + 1$ in the first rl positions and fill in the remaining slots with q's. Let then each next row be a circular shift of the previous row by rl slots. In the resulting matrix we will have $rlnk$ entries equal $(q+1)$, so each column will contain rk $q+1$'s since we distributed them evenly among the columns. All the entries of A are less than or equal to $q+1$, so $\operatorname{tdet} A \le nk(q+1)$. Here is an example of this construction with $m=7$, $n=5$, $k=1$, $l=3$, $q=1$, $r=2$:

$$A = \begin{pmatrix} 2&2&2&2&2&2&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&2&2&2&2&2&2&1&1&1 \\ 2&2&2&1&1&1&1&1&1&1&1&1&2&2&2 \\ 1&1&1&2&2&2&2&2&2&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&2&2&2&2&2&2 \end{pmatrix}. \qquad \square$$

Recall that x and y are described in Definition 2.4. Given that $x+y \le nk$ we next explain how to construct a matrix with tropical determinant at most $nkq+x+y$.

Proposition 3.2. *Let $x+y \le nk$. Then there exists $A \in \mathcal{D}^{k,l}(m,n)$ such that $\operatorname{tdet} A \le nkq+x+y$.*

Proof. Let A consist of four blocks X, Y, Z, and W, that is,

$$A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where X is of size $x \times y$. Let W have all of its entries equal to q. The only entries of Z and Y are q's and $(q+1)$'s. We place rk $(q+1)$'s in each column of Y in a pattern similar to that of previous proposition. In the first column of Y we place a string of rk $(q+1)$'s starting at first position (we can do this since $x \ge rk$) and fill in the remaining slots with q's. In the second column we shift down this string by rk positions, circling around, if necessary. We repeat this in every column of Y starting with $(q+1)$'s in the position right after the one where we finished in the previous column. We have distributed $\Sigma_Y = (xq+rk)(nl-y)$ (the sum of all the entries in Y) evenly among the columns and as evenly as possible among the rows of Y. Let $\Sigma_Y = ax+b$ be the result of dividing with remainder of Σ_Y by x, the number of rows in Y. Then the first b rows of Y have $q+1$ in $a+1$ positions and the remaining $x-b$ rows have $q+1$ in a positions. Block Z is constructed in a similar way, but we work with rows instead of columns.

It remains to fill in block X. Its sum of entries is $\Sigma_X = qxy + r(xl+yk) - klnr \ge 0$. We will distribute this sum as evenly as possible among the rows and columns of X. For this, we divide Σ_X by x with a remainder: $\Sigma_X = cx+d$. We want the bottom d row sums of X to be equal to $c+1$ and the remaining row sums to be equal to c. For this we divide $c+1$ by y with remainder $c+1 = ey+f$ and fill the last f slots in the bottom row of X with $e+1$'s and make the remaining slots in the bottom row of X equal to e. Next row upward is a circular leftward shift of this row by f. We continue with these circular shifts until we fill in the bottom d rows of X. We fill the remaining rows of X in a similar fashion: divide c by y with remainder $c = gy+h$, fill the h slots in row $x-d$

with $(g+1)$'s (starting from where we stopped in the row below and going left) and so on. In the resulting block X the sum of entries Σ_X is distributed as evenly as possible between rows and columns of X. Moreover, the bigger row (resp. column) sums at the bottom (resp. rightmost) part of X. We have also evenly distributed row sums in Y and column sums in Z, so that bigger row sums in Y are in the first rows of Y and bigger column sums in Z are in the first columns of Z. Hence first x row sums and y column sums of A are equal to ml and mk, respectively.

Note that $e = g$ unless $c + 1 = ey$ and $f = 0$, so the entries of X differ from each other by at most 1. Hence they are equal to $\lfloor \Sigma_X/xy \rfloor$ or $\lfloor \Sigma_X/xy \rfloor + 1$, where the latter occurs only if xy does not divide Σ_X evenly. We have

$$\frac{\Sigma_X}{xy} = q + \frac{r(xl + yk) - lknr}{xy} \leq q$$

since $xl + yk \leq lkn$ as

$$xl + yk \leq l(x + y) \leq lkn.$$

This implies that each entry in X is at most q. Hence for a maximal transversal of A we can pick at most x $(q+1)$'s in Y and at most y $(q+1)$'s in Z, so

$$\operatorname{tdet} A \leq x(q+1) + y(q+1) + (nk - x - y)q = nkq + x + y,$$

which completes the proof. $\square$

Note that in the above argument we only used the conditions $x + y \leq nk$, $x \geq rk$, $y \geq rl$, and $qxy + r(xl + yk) \geq klnr$, but not the fact that $x+y$ is the smallest possible. We next give an example of the above construction where this last assumption is dropped. This will allow us to have the sum of entries in X not too small, so that the construction of block X can be better illustrated.

Example 2. Let $m = 6$, $n = 5$, $k = 2$, $l = 3$, $q = 1$, $r = 1$, $x = 5$, $y = 4$. Then each column of Y has two 2's and three 1's and the sum of entries in Y is $\Sigma_Y = (xq + rk)(ml - y) = 77 = 5 \cdot 15 + 2$, so we have row sums in first two rows equal to 16 and in the remaining three rows equal to 15. In block Z the overall sum of entries equals

$$\Sigma_Z = (yq + rl)(mk - x) = 35,$$

so first three columns in Z have columns sums equal to 9, and the last column sum is 8. Next,

$$\Sigma_X = qxy + r(xl + yk) - klnr = 13,$$

so last three row sums are 3, and first two are 2. Therefore,

$$A = \left(\begin{array}{cccc|ccccccccccc}
1&0&0&1&2&1&2&1&1&2&1&2&1&1&2\\
0&1&1&0&2&1&1&2&1&2&1&1&2&1&2\\
1&1&0&1&1&2&1&2&1&1&2&1&2&1&1\\
1&0&1&1&1&2&1&1&2&1&2&1&1&2&1\\
0&1&1&1&1&1&2&1&2&1&1&2&1&2&1\\
\hline
2&2&2&1&1&1&1&1&1&1&1&1&1&1&1\\
2&2&1&2&1&1&1&1&1&1&1&1&1&1&1\\
2&1&2&2&1&1&1&1&1&1&1&1&1&1&1\\
1&2&2&2&1&1&1&1&1&1&1&1&1&1&1\\
2&2&2&1&1&1&1&1&1&1&1&1&1&1&1
\end{array}\right)$$

is a matrix with tropical determinant at most $nkq + x + y = 19$.

Theorem 2.6 together with Proposition 3.1 and Proposition 3.2 implies our main result.

Theorem 3.3. *Let $m = qn + r$ for $0 \leq r < n$. Then*

$$L^{k,l}(m,n) = nkq + \min(nk, x + y),$$

where x, y are integers satisfying

$$qxy + r(xl + yk) \geq klnr, \quad x \geq rk, \quad y \geq rl.$$

Example 3. Let $m = 6$, $n = 5$, $k = 1$, $l = 2$, so $q = r = 1$. It is easily seen that $x = y = 2$ satisfy the inequalities

$$xy + 2x + y \geq 10, \quad x \geq 1, \quad y \geq 2,$$

and have the smallest sum $x + y = 4$ (see Fig. 1). Also $x + y < nk = 5$, so $L^{k,l}(m,n) = nkq + x + y = 9$, which is attained at the matrix in Example 1.

4. Corollaries

Corollary 4.1. *If $r = 0$, we have $L^{k,l}(m,n) = nkq$ and this value is attained at the matrix all of whose entries are equal to q.*

Proof. Our conditions on x and y simplify to $qxy \geq 0$, $x \geq 0$, $y \geq 0$, where $x + y$ is the smallest possible, so $x = y = 0$ and the main theorem implies $L^{k,l}(m,n) = nkq$. Also, the matrix all of whose entries are equal to q is in $\mathcal{D}^{k,l}(m,n)$ and its tropical determinant equals nkq. $\square$

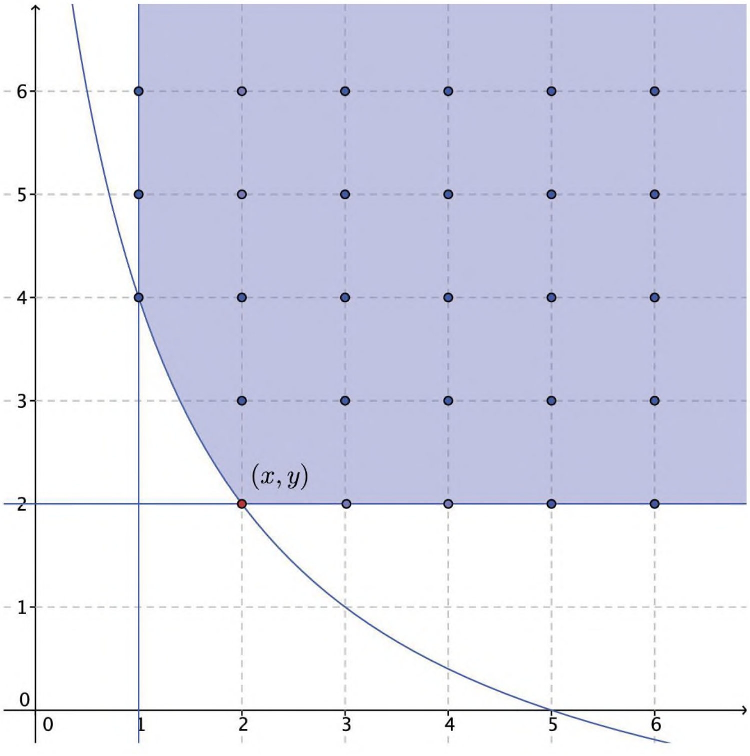

Fig. 1. Region in Example 3 as defined in Definition 2.4.

Corollary 4.2. *If $r \geq \frac{n}{q+2}$, we have*

$$L^{k,l}(m,n) = nkq + \min(nk, r(k+l)).$$

Proof. The condition $r \geq \frac{n}{q+2}$ implies that (rk, rl) satisfies the inequality (2.1). Therefore, $x = rk$ and $y = rl$, and the result follows from the main theorem. □

We now reformulate this corollary.

Corollary 4.3. *If $r \geq \frac{n}{q+2}$ and $r \leq \frac{nk}{k+l}$ then $L^{k,l}(m,n) = nkq + r(k+l)$. If $r \geq \frac{nk}{k+l}$ then $L^{k,l}(m,n) = nk(q+1)$.*

Proof. To prove the second statement, we notice $x + y \geq rk + rl \geq nk$. □

When $k = l = 1$ the above corollary provides Theorems 2.3 and 2.4 from [3]. Next, we show how to recover the result of Theorem 3.3 from [3], which deals with the case $k = l = 1$ and $r < \frac{n}{q+2}$.

First, by definition, $x \geq r$, $y \geq r$ satisfy

$$qxy + r(x+y) - nr \geq 0,$$

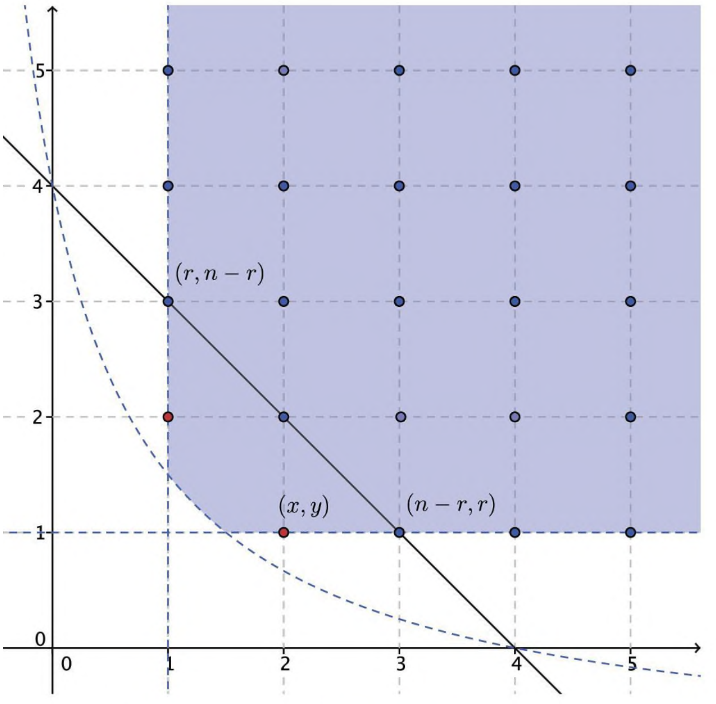

Fig. 2. Region described in Definition 2.4 for $k = l = 1$.

and $x + y$ is smallest possible. The region described by the above inequalities is convex and symmetric with respect to the line $y = x$. It follows that the region contains the segment joining points with coordinates $(r, n-r)$ and $(n-r, r)$ (see Fig. 2). Therefore, any optimal solution (x, y) satisfies $x + y \leq n$. Furthermore, the minimum of $x + y$ is attained either when $y = x$ or $y = x + 1$. Applying Theorem 3.3, we get the statement of [3, Theorem 3.3].

Corollary 4.4. *Let $r < \frac{n}{q+2}$ and x be the smallest positive integer satisfying at least one of the inequalities*

1. $qx^2 + 2rx - nr \geq 0,$
2. $qx^2 + (2r + q)x + r - nr \geq 0.$

Then if x satisfies (1) (and hence (2)), we have $L^{1,1}(m, n) = nq + 2x$. If x satisfies (1) only, we have $L^{1,1}(m, n) = nq + 2x + 1$.

5. Upper bound on the tropical determinant

In this section we consider a version of the problem solved above where the maximum in the definition of the tropical determinant is replaced with the minimum and we are interested in the sharp upper bound of this tropical determinant on the transportation polytope. Recall that $A = (a_{ij})$ is an $nk \times nl$ matrix where $\gcd(k,l) = 1$, $k \leq l$, and a_{ij} are non-negative integers. The row sums of A are equal to ml and the column sums are equal to mk. The set of all such matrices is denoted by $\mathcal{D}^{k,l}(m,n)$. As before, we divide m by n with remainder, so $m = qn + r$, for $0 \leq r < n$.

Definition 5.1. Let $A = (a_{ij})$ be an $s \times t$ matrix with $s \leq t$ and let $\mathcal{T}(A)$ be the set of its transversals. Define the *tropical determinant* of a matrix $A = (a_{ij})$ to be

$$\operatorname{tropdet}(A) = \min_{T \in \mathcal{T}(A)} |T|.$$

Denote its sharp upper bound over the set $\mathcal{D}^{k,l}(m,n)$ by $U^{k,l}(m,n)$,

$$U^{k,l}(m,n) = \max_{A \in \mathcal{D}^{k,l}(m,n)} \operatorname{tropdet}(A).$$

Theorem 5.2. $U^{k,l}(m,n) \leq \max(nkq, nkq + r(k+l) - nl)$.

Proof. Let $A \in \mathcal{D}^{k,l}(m,n)$. Rearrange rows and columns of A so that the tropical determinant is equal to the sum of entries on the main diagonal of A and the entries are non-decreasing along the main diagonal. That is, A is of the form

$$A = \begin{pmatrix} a_{11} & & & a_{1t} & & & \\ & \ddots & & \vdots & & & \\ & & a_{ii} & a_{it} & & & \\ & & & \ddots \; \vdots & & & \\ a_{t1} & \cdots \; a_{ti} & \cdots & a_{tt} & a_{t\,t+1} & \cdots & a_{ts} \end{pmatrix} \tag{5.1}$$

where $t = nk$, $s = nl$, $a_{11} \leq a_{22} \leq \cdots \leq a_{tt}$ and $\operatorname{tropdet} A = a_{11} + \cdots + a_{tt}$. Let us first suppose that $a_{tt} \leq q$. Then

$$\operatorname{tropdet}(A) = a_{11} + \cdots + a_{tt} \leq t\, a_{tt} \leq nkq.$$

Next, let $a_{tt} \geq q + 1$. Observe that

$$a_{ti} + a_{it} \geq a_{ii} + a_{tt} \quad \text{for} \quad i = 1, \ldots, t,$$

since otherwise we could pick a smaller transversal. Also, for the same reason, $a_{tt} \leq a_{ti}$ for $i = t+1, \ldots, s$. Adding up all these inequalities over i, we get

$$a_{t1} + \cdots + a_{t\,t} + a_{t\,t+1} + a_{ts} + a_{1t} + \cdots + a_{t\,t} \geq \operatorname{tropdet} A + s a_{tt},$$

and hence

$$mk + ml \geq \operatorname{tropdet} A + s a_{tt} \geq \operatorname{tropdet} A + nl(q+1),$$

so

$$\operatorname{tropdet} A \leq m(k+l) - nl(q+1) = nkq + r(k+l) - nl. \qquad \square$$

Theorem 5.3. $U^{k,l}(m,n) = \max(nkq, nkq + r(k+l) - nl)$.

Proof. Now it remains to construct matrices that reach the bound of the previous theorem. That is, for $r \leq nl/(k+l)$ we need to construct $A \in \mathcal{D}^{k,l}(m,n)$ such that $\operatorname{tropdet} A \geq nkq$ and for $r \geq nl/(k+l)$ we need to construct $A \in \mathcal{D}^{k,l}(m,n)$ such that $\operatorname{tropdet} A \geq nkq + r(k+l) - nl$. The first task is easy. The entries of A equal q or $q+1$ with rl $q+1$'s in each row, that are evenly distributed among the columns. That is, the first row of A starts with rl $q+1$'s and each next row is a circular shift by rl of the previous row:

$$A = \begin{pmatrix} q+1 & \dots & q+1 & q & \dots & q & q & \dots & q & q \\ q & \dots & q & q+1 & \dots & q+1 & q & \dots & q & q \\ q & \dots & q & q & \dots & q & q+1 & \dots & q+1 & q \\ \dots & q+1 & q & q & \dots & q & q & \dots & q & q+1 \\ \dots & & & & \dots & & & \dots & & \end{pmatrix}.$$

There are $rlnk$ $q+1$'s in this matrix and since they are evenly distributed among the columns, each column contains $rlnk/nl = rk$ of them, so $A \in \mathcal{D}^{k,l}(m,n)$. Since all the entries of A are greater than or equal to q we have $\operatorname{tropdet} A \geq nkq$.

Let us next suppose that $r \geq nl/(k+l)$. Let A consist of four blocks

$$A = \begin{pmatrix} X & Y \\ Z & W \end{pmatrix},$$

where X is an $rk \times rl$ matrix of $q+1$'s, and the entries in Y and Z are all q. We fill in the remaining submatrix W so that $A \in \mathcal{D}^{k,l}(m,n)$. For this, we first make all entries of W equal q. We need to bring up the row sums in W by rl and the column sums by rk. For this, we divide rl by $nl - rl$ with remainder to get $rl = (nl - rl)q' + r'$. We increase the first r' entries in the first row of W by $q'+1$, and the remaining entries in this row by q'. The second row of W is a circular shift by r' of the first row, and so on. Since we distributed $(ml - rlq)(nk - rk)$ as evenly as possible among the columns, the column sums in W are

$$\frac{(ml - rlq)(nk - rk)}{nl - rl} = mk - krq,$$

so $A \in \mathcal{D}^{k,l}(m,n)$. Note that all the entries in W are greater than or equal to q.

We have $nk - rk \leq rl$ and $nl - rl \leq rk$, so for a minimal transversal of A we would need to pick $nk - rk$ entries from Z, $nl - rl$ entries from Y, and the remaining $r(k+l) - nl$ entries from X. Therefore, $\operatorname{tropdet} A = nkq + r(k+l) - nl$. $\square$

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