

Bucknell University

Bucknell Digital Commons

Faculty Journal Articles

Faculty Scholarship

2014

On Groups with a Class-Preserving Outer Automorphism

Peter A. Brooksbank

Bucknell University, pbrooksb@bucknell.edu

Follow this and additional works at: https://digitalcommons.bucknell.edu/fac_journal



Part of the [Algebra Commons](#)

Recommended Citation

Brooksbank, Peter A.. "On Groups with a Class-Preserving Outer Automorphism." *Involve* (2014) : 171-179.

This Article is brought to you for free and open access by the Faculty Scholarship at Bucknell Digital Commons. It has been accepted for inclusion in Faculty Journal Articles by an authorized administrator of Bucknell Digital Commons. For more information, please contact dcadmin@bucknell.edu.

involve

a journal of mathematics

On groups with a class-preserving outer automorphism

Peter A. Brooksbank and Matthew S. Mizuhara



On groups with a class-preserving outer automorphism

Peter A. Brooksbank and Matthew S. Mizuhara

(Communicated by Nigel Boston)

Four infinite families of 2-groups are presented, all of whose members possess an outer automorphism that preserves conjugacy classes. The groups in these families are central extensions of their predecessors by a cyclic group of order 2. For each integer $r > 1$, there is precisely one 2-group of nilpotency class r in each of the four families. All other known families of 2-groups possessing a class-preserving outer automorphism consist entirely of groups of nilpotency class 2.

1. Introduction

Let G be a group, $\text{Aut}(G)$ the automorphism group of G , and $\text{Inn}(G)$ the subgroup of inner automorphisms. Then $\text{Aut}(G)$ acts naturally on the set of conjugacy classes of G , and we denote the kernel of this action by $\text{Aut}_c(G)$. We refer to the elements of $\text{Aut}_c(G)$ as *class-preserving automorphisms*. Evidently $\text{Inn}(G) \trianglelefteq \text{Aut}_c(G)$, and the elements of $\text{Out}_c(G) = \text{Aut}_c(G)/\text{Inn}(G)$ will be referred to as *class-preserving outer automorphisms*.

Over a century ago, William Burnside [1911, Note B, p. 463] asked the question: *Are there groups G such that $\text{Out}_c(G) \neq 1$?* He himself settled the question soon thereafter [Burnside 1913]: *for each prime $p \equiv \pm 3 \pmod{8}$, there is a group G_p of order p^6 and nilpotency class 2 with $\text{Out}_c(G_p) \neq 1$.*

Since Burnside's initial discovery, the problem has been revisited on many occasions, and new families of groups G with $\text{Out}_c(G) \neq 1$ have been found. Until fairly recently, however, most of those families consisted of p -groups of nilpotency class 2. The object of this paper is to prove the following result.

Theorem 1.1. *There are four distinct infinite families $\mathcal{H} = \{H_j\}_{j=1}^{\infty}$, where H_j is a 4-generator 2-group of order 2^{5+j} and nilpotency class $j+1$ such that $\text{Out}_c(H_j) \neq 1$.*

MSC2010: 20D15, 20D45, 20E45.

Keywords: p -groups, class-preserving automorphisms, polycyclic groups.

Project sponsored by the National Security Agency under Grant Number H98230-11-1-0146. The United States Government is authorized to reproduce and distribute reprints notwithstanding any copyright notation herein.

It is evident from the statement of [Theorem 1.1](#) that the nilpotency class of the groups H_j in each family grows in an elementary way as a function of the group orders. This is because H_{j+1} is built as a central extension of H_j by $\mathbb{Z}/2$. Indeed, each \mathcal{H} may be constructed algorithmically using the p -group generation algorithm [[O'Brien 1990](#)]; this is precisely how the families were discovered and studied. Furthermore, the groups in all four families have coclass 4, so we have shown that they are all “mainline groups” in the coclass graph $\mathcal{G}(2, 4)$ (see [[Eick and Leedham-Green 2008](#)]).

Readers interested in the history and applications of Burnside’s problem are referred to the recent comprehensive survey of Yadav [[2011](#)]; we restrict ourselves here to a brief summary of those results pertaining directly to [Theorem 1.1](#).

Wall [[1947](#)] showed that, for each integer m divisible by 8, the general linear group $\text{GL}(1, \mathbb{Z}/m)$ (i.e., the group of linear permutations $x \mapsto \sigma x + \tau$ on integers modulo m with σ, τ integral) has a class-preserving automorphism that is not inner. This family includes the smallest group G such that $\text{Out}_c(G) \neq 1$, namely $\text{GL}(1, \mathbb{Z}/8)$ of order 32 (there, in fact, are two nonisomorphic groups of order 32 having this property). The 2-groups in Wall’s family, namely $\text{GL}(1, \mathbb{Z}/2^k)$, have nilpotency class 2.

Heineken [[1979](#)] constructed, for each odd prime p , an infinite family of p -groups of nilpotency class 2, *all* of whose automorphisms are class-preserving. As far as we are aware, these are the only known infinite families of groups G for which $\text{Aut}_c(G) = \text{Aut}(G)$.

Hertweck [[2001](#)] constructed a family of Frobenius groups as subgroups of affine semilinear groups $A\Gamma(F)$, where F is a finite field, which possess class-preserving automorphisms that are not inner.

Malinowska [[1992](#)] exhibited, for each prime $p > 5$ and each $r > 2$, a p -group G of nilpotency class r such that $\text{Out}_c(G) \neq 1$. Unlike the groups in our families, however, it is not clear how the order of G relates to r .

We remark that the absence of simple groups in the above summary is explained by Feit and Seitz [[1989](#), Section C]: *if G is a finite simple group then $\text{Out}_c(G) = 1$.*

Briefly, the paper is organized as follows. In [Section 2](#) we summarize the necessary background on p -groups. The families \mathcal{H} in [Theorem 1.1](#) are introduced in [Section 3](#); they are naturally parametrized by vectors $\epsilon \in \{0, 1\}^4$, but there only four distinct families. The proof of [Theorem 1.1](#) is given in [Section 4](#).

2. Preliminaries

Our notation and terminology is standard. For elements x, y of a group, we write $x^y = y^{-1}xy$ and $[x, y] = x^{-1}x^y$. For subsets X and Y of a group, we denote by $[X, Y]$ the subgroup generated by all commutators $[x, y]$, where $x \in X$ and $y \in Y$.

The *lower central series* of a group G is the series

$$G = \gamma_1(G) \geq \gamma_2(G) \geq \cdots, \quad (1)$$

where $\gamma_{i+1}(G) = [G, \gamma_i(G)]$. A group G is *nilpotent* if $\gamma_i(G) = 1$ for some $i \geq 1$, in which case the smallest r such that $\gamma_{r+1}(G) = 1$ is called the *nilpotency class* (or simply *class*) of G . A finite group G is a p -group if $|G| = p^n$ for some prime p . All p -groups are nilpotent, and if G has class r , then G has *coclass* $n - r$. A p -group minimally generated by d elements is called a d -generator group.

Each nilpotent group (more generally, each soluble group) possesses a *polycyclic generating sequence* [Holt et al. 2005, Chapter 8]. This in turn gives rise to a *power-conjugate presentation* (or simply *pc-presentation*), an extremely efficient model for computing with soluble groups. We describe these presentations specifically for p -groups.

Fix a p -group G . Let $X = [x_1, \dots, x_n] \subset G$ be such that if $P_i = \langle x_i, \dots, x_n \rangle$ ($i = 1, \dots, n$), then P_i/P_{i+1} has order p , and $G = P_1 > P_2 > \cdots > P_n > 1$ refines the lower central series in (1). If G has nilpotency class r , we define a *weighting*, $w: X \rightarrow \{1, \dots, r\}$, where $w(x_i) = k$ if $x_i \in \gamma_{k-1}(G) \setminus \gamma_k(G)$. Evidently, $w(x_i) \geq w(x_j)$ whenever $i \geq j$. Any such sequence X satisfies the conditions needed to serve as the generating sequence of a *weighted pc-presentation* of G . The relations, R , in such a presentation all have the form

$$x_i^p = \prod_{k=i+1}^n x_k^{b(i,k)}, \quad \text{where } 0 \leq b(i,k) < p, \quad 1 \leq i \leq n,$$

or

$$x_j^{x_i} = x_j \prod_{k=j+1}^n x_k^{b(i,j,k)}, \quad \text{where } 0 \leq b(i,j,k) < p, \quad 1 \leq i < j \leq n.$$

We write $\langle X \mid R \rangle$ to denote the p -group defined by such a presentation. We adopt the usual convention that an omitted relation x_i^p implies that $x_i^p = 1$, and an omitted relation $x_j^{x_i}$ implies that x_i and x_j commute. We will often find it convenient to write a conjugate relation $x_j^{x_i} = x_j w$ as a commutator relation $[x_j, x_i] = w$.

Remark 2.1. In general, one requires that $G = P_1 > \cdots > P_n > 1$ refines a related series called the *exponent p -central series* [Holt et al. 2005, p. 355]. For the families of p -groups we consider here, however, the two series coincide.

A critical feature of a pc-presentation for a p -group is that elements of the group inherit a *normal form* $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$, where $0 \leq a_i < p$. Given $g \in G$ as a word in x_1, \dots, x_n , a normal form may be obtained by repeatedly applying the relations in (2) in a process known as *collection*. If each element of G has a unique normal form, the pc-presentation is said to be *consistent*. Clearly if G has a consistent pc-presentation on $X = [x_1, \dots, x_n]$, then $|G| = p^n$.

We conclude this section with a useful test for consistency. We state it just for 2-groups — since this is all we need — and refer the reader to [Holt et al. 2005, Theorem 9.22] for the more general version.

Proposition 2.2. *A weighted pc-presentation of a d -generator 2-group of class r on $[x_1, \dots, x_n]$ is consistent if the following pairs of words in the generators have the same normal form (the products in parentheses are collected first):*

$$\begin{aligned} (x_k x_j) x_i \text{ and } x_k (x_j x_i), & \quad 1 \leq i < j < k \leq n \text{ and } i \leq d, \quad w(x_i) + w(x_j) + w(x_k) \leq r; \\ (x_j x_j) x_i \text{ and } x_j (x_j x_i), & \quad 1 \leq i < j \leq n \text{ and } i \leq d, \quad w(x_i) + w(x_j) < r; \\ (x_j x_i) x_i \text{ and } x_j (x_i x_i), & \quad 1 \leq i < j \leq n, \quad w(x_i) + w(x_j) < r; \\ (x_i x_i) x_i \text{ and } x_i (x_i x_i), & \quad 1 \leq i \leq n, \quad 2w(x_i) < r. \end{aligned}$$

3. The families \mathcal{H}^ϵ

In this section we introduce four infinite families of 4-generator 2-groups of fixed coclass 4. In the next section we will show that each family consists of groups that have a class-preserving outer automorphism, thus proving [Theorem 1.1](#).

We will define the groups in each family by giving consistent pc-presentations. It is convenient to denote the ordered list of pc-generators of the n -th group in each family by $X_n = \{x_1, x_2, x_3, x_4, z, y_1, \dots, y_n\}$, with the group minimally generated by $\{x_1, x_2, x_3, x_4\}$. The commutator relations for each family are identical, namely

$$\begin{aligned} C_n = \{[x_2, x_1] = [x_3, x_2] = [x_4, x_1] = z, \quad [x_3, x_1] = y_1, \\ [x_1, y_i] = [x_3, y_i] = y_{i+1} \quad (i = 1, \dots, n-1)\}. \end{aligned} \quad (3)$$

For each $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4) \in \{0, 1\}^4$, define

$$\begin{aligned} P_n^\epsilon = \{x_j^2 = z^{\epsilon_j} \quad (j = 1, \dots, 4), \quad z^2 = 1, \\ y_n^2 = 1, \quad y_i^2 = y_{i+1} y_{i+2} \quad (i = 1, \dots, n-2), \quad y_{n-1}^2 = y_n\}. \end{aligned} \quad (4)$$

Let $R_n^\epsilon = C_n \cup P_n^\epsilon$, define $H_n^\epsilon = \langle X_n \mid R_n^\epsilon \rangle$, and put $\mathcal{H}^\epsilon = \{H_n^\epsilon\}_{n=1}^\infty$. Note that the pc-presentations for the n -th group in each family differ only in the power relations of the generators x_i .

Proposition 3.1. *Let n be a positive integer, and $\epsilon \in \{0, 1\}^4$. Then $H_n^\epsilon = \langle X_n \mid R_n^\epsilon \rangle$ has order 2^{n+5} and class $n+1$ (hence coclass 4).*

Proof. To confirm the order of H_n^ϵ , it suffices to check that their defining pc-presentations are consistent, for which we use [Proposition 2.2](#). Although there are $O(n^3)$ computations involved in that test, the lion's share of these may be treated uniformly for the groups H_n^ϵ . The following table lists all of the triples that must be checked, together with their normal forms. Triples involving z are

omitted (since z is central), as are triples involving two or more y_s generators (since $\langle y_s : s = 1, \dots, n \rangle$ is abelian).

Triple (a, b, c)	Conditions	Normal form of $a(bc)$ and $(ab)c$
(x_3, x_2, x_1) (x_4, x_2, x_1) (x_4, x_3, x_1) (x_4, x_3, x_2)		$x_1x_2x_3y_1$ $x_1x_2x_4$ $x_1x_3x_4zy_1$ $x_2x_3x_4z$
(y_s, x_2, x_1) (y_s, x_3, x_1) (y_s, x_4, x_1) (y_s, x_3, x_2) (y_s, x_4, x_2) (y_s, x_4, x_3)	$s \leq n - 2$ $s \leq n - 2$ $s \leq n - 2$ $s \leq n - 2$ $s \leq n - 2$ $s \leq n - 2$	$x_1x_2zy_sy_{s+1}$ $x_1x_3y_1y_s$ $x_1x_4zy_sy_{s+1}$ $x_2x_3zy_sy_{s+1}$ $x_2x_4y_s$ $x_3x_4y_sy_{s+1}$
(x_j, x_j, x_i) (y_s, y_s, x_i) (x_j, x_i, x_i) (y_s, x_i, x_i)	$1 \leq i < j \leq 4$ $s \leq n - 2, i = 1, 3$ $1 \leq i < j \leq 4$ $s \leq n - 2, i \leq 4$	$x_i z^{e_j}$ $x_i y_{s+1}$ $x_j z^{e_i}$ $z^{e_i} y_s$
(x_i, x_i, x_i)	$i \leq 4$	$x_i z^{e_i}$

Routine calculations using the pc-relations are all that is needed to verify the normal forms listed in the table. It remains to compute the lower central series of H_n^ϵ :

$$\begin{aligned}\gamma_1(H_n^\epsilon) &= H_n^\epsilon, \\ \gamma_2(H_n^\epsilon) &= \langle z, y_i : 1 \leq i \leq n \rangle, \\ \gamma_j(H_n^\epsilon) &= \langle y_i : j - 1 \leq i \leq n \rangle \quad \text{for } j = 3, \dots, n + 1, \\ \gamma_{n+2}(H_n^\epsilon) &= 1.\end{aligned}$$

This shows that H_n^ϵ has class $n + 1$, as stated. \square

Proposition 3.1 suggests that there are 16 families \mathcal{H}^ϵ , but the following result shows that there is some duplication.

Proposition 3.2. *For each positive integer n , there are four isomorphism classes among the groups $\{H_n^\epsilon : \epsilon \in \{0, 1\}^4\}$.*

Proof. Each group $H = H_n^\epsilon$ determines a quadratic map $\mathbf{q} = \mathbf{q}^\epsilon$ (independent of n) as follows. Let V denote the largest elementary abelian quotient of H , namely $V = H/A \cong (\mathbb{Z}/2)^4$, where $A = \langle z, y_1, \dots, y_n \rangle$. Let W denote the largest elementary abelian quotient of A , namely $W = A/B \cong (\mathbb{Z}/2)^2$, where $B = \langle y_2, \dots, y_n \rangle$. Define maps $\mathbf{q} : V \rightarrow W$ and $\mathbf{b} : V \times V \rightarrow W$, where $\mathbf{q}(xA) = x^2B$ and $\mathbf{b}(xA, yA) =$

$[x, y]B$ for all $x, y \in H$. Using additive notation in V and W , one easily checks that

$$\mathbf{b}(u, v) = \mathbf{q}(u + v) + \mathbf{q}(u) + \mathbf{q}(v) \text{ for all } u, v \in V, \quad (5)$$

so \mathbf{b} is the symmetric bilinear map associated to \mathbf{q} in the familiar sense.

If H_n^ϵ and H_n^δ are isomorphic groups, and $\alpha: H_n^\epsilon \rightarrow H_n^\delta$ is any isomorphism, then α induces isomorphisms $\beta: V^\epsilon \rightarrow V^\delta$ and $\gamma: W^\epsilon \rightarrow W^\delta$ such that $\mathbf{q}^\delta(v\beta) = \mathbf{q}^\epsilon(v)\gamma$ for all $v \in V^\epsilon$. Thus α induces a *pseudo-isometry* between \mathbf{q}^ϵ and \mathbf{q}^δ .

Fixing a basis $\{v_i\}$ for V , one can represent a quadratic map \mathbf{q} as a 4×4 matrix $\mathbf{Q} = [[q_{ij}]]$ with entries in W , where $q_{ii} = \mathbf{q}(v_i)$, $q_{ij} = \mathbf{b}(v_i, v_j)$ if $i < j$, and $q_{ij} = 0$ if $i > j$. Given $v \in V$, write $v = \sum \lambda_i v_i$ with $\lambda_i \in \mathbb{Z}/2$. Using (5) and a finite induction, we see that $\mathbf{q}(v) = \sum_i \sum_{j \geq i} \lambda_i \lambda_j q_{ij}$. An easy matrix calculation then shows that $\mathbf{q}(v) = v \mathbf{Q} v^{\text{tr}}$ for all $v \in V$.

Using the basis $\{x_i A\}$ for V , and identifying A/B on basis $\{zB, y_1 B\}$ with the additive group of the ring $(\mathbb{Z}/2)[t]/(t^2)$ on the usual basis $\{1, t\}$, the matrix representing $\mathbf{q} = \mathbf{q}^\epsilon$, where $\epsilon = (\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$, is

$$\mathbf{Q} = \begin{bmatrix} \epsilon_1 & 1 & t & 1 \\ 0 & \epsilon_2 & 1 & 0 \\ 0 & 0 & \epsilon_3 & 0 \\ 0 & 0 & 0 & \epsilon_4 \end{bmatrix},$$

and the matrix representing the associated bilinear map \mathbf{b} is $\mathbf{B} = \mathbf{Q} + \mathbf{Q}^{\text{tr}}$.

Given maps \mathbf{q}^ϵ and \mathbf{q}^δ representing groups H^ϵ and H^δ ($\epsilon, \delta \in \{0, 1\}^4$), one can easily test for pseudo-isometry as follows. Let \mathbf{Q}^ϵ and \mathbf{Q}^δ be matrices representing \mathbf{q}^ϵ and \mathbf{q}^δ . If $g \in \text{GL}(4, 2)$ represents an isomorphism $H^\epsilon/A^\epsilon \rightarrow H^\delta/A^\delta$ induced by an isomorphism $H^\epsilon \rightarrow H^\delta$, then the induced isomorphism $A^\epsilon/B^\epsilon \rightarrow A^\delta/B^\delta$ is uniquely determined by g , and its matrix $h \in \text{GL}(2, 2)$ is easily computed. Extend h entry-wise to a map $\mathbb{M}_4(W^\epsilon) \rightarrow \mathbb{M}_4(W^\delta)$, and denote the image of $X \in \mathbb{M}_4(W^\epsilon)$ by X^h . Then \mathbf{q}^ϵ and \mathbf{q}^δ are pseudo-isometric if and only if there exists $g \in \text{GL}(4, 2)$ such that

$$g \mathbf{B}^\delta g^{\text{tr}} = (\mathbf{B}^\epsilon)^h \quad \text{and} \quad v_i (g \mathbf{Q}^\delta g^{\text{tr}}) v_i^{\text{tr}} = v_i (\mathbf{Q}^\epsilon)^h v_i^{\text{tr}},$$

as v_i runs over a basis for $(\mathbb{Z}/2)^4$.

Thus, the determination of the pseudo-isometry classes of the quadratic maps associated to the families \mathcal{H}^ϵ is an elementary matrix calculation in $\text{GL}(4, 2)$, which is easily carried out using a computer algebra system such as MAGMA [Bosma et al. 1997]. Those classes are represented by

$$\mathbf{Q}^\epsilon \quad \text{for } \epsilon \in \{(0, 0, 0, 0), (0, 0, 1, 1), (1, 1, 0, 0), (1, 1, 1, 1)\}.$$

Finally, it is not difficult to verify that any pseudo-isometry $\mathbf{Q}^\epsilon \rightarrow \mathbf{Q}^\delta$ lifts to an

isomorphism $H^\epsilon \rightarrow H^\delta$. Thus, for each n , there are precisely four isomorphism classes of group H_n^ϵ , as claimed. \square

4. Proof of Theorem 1.1

In this section we complete the proof of Theorem 1.1 by exhibiting a class-preserving automorphism of each group H_n^ϵ that is not inner.

Proof of Theorem 1.1. Fix $n \geq 1$, $\epsilon \in \{0, 1\}^4$, and put $H = H_n^\epsilon$. Define $\theta: H \rightarrow H$ on generators, sending

$$x \mapsto \begin{cases} x_4 z & \text{if } x = x_4, \\ x & \text{if } x \in X_n \setminus \{x_4\}. \end{cases} \quad (6)$$

One easily verifies (by replacing x_4 with $x_4 z$ in each pc-relation involving x_4 and evaluating) that $\theta \in \text{Aut}(H)$.

First, suppose that θ is an inner automorphism. Then there exists $h \in H$ commuting with x_1 and x_3 , but not with x_4 . Writing

$$h = \prod_{i=1}^4 x_i^{a_i} \cdot z^b \cdot \prod_{j=1}^n y_j^{c_j} \quad (a_i, b, c_j \in \{0, 1\}) \quad (7)$$

and using the defining commutator relations of H , we see that

$$hx_1 = x_1 h \cdot \left(z^{a_2+a_4} y_1^{a_3} \prod_{j=2}^n y_j^{c_j-1} \right).$$

Hence $h \in C_H(x_1)$ if and only if $a_2 = a_4$ and $0 = a_3 = c_1 = \cdots = c_{n-1}$. Also,

$$x_3 h = x_1^{a_1} x_2^{a_2} x_3^{1+a_3} x_4^{a_4} z^{a_2+b} y_1^{a_1+c_1} \prod_{j=2}^n y_j^{c_j},$$

while

$$hx_3 = x_1^{a_1} x_2^{a_2} x_3^{1+a_3} x_4^{a_4} z^b y_1^{c_1} \prod_{j=2}^n y_j^{c_j} \prod_{j=2}^n y_j^{c_j-1},$$

so that $h \in C_H(x_3)$ if and only if $0 = a_1 = a_2 = c_1 = \cdots = c_{n-1}$. It follows that $C_H(x_1) \cap C_H(x_3) = \langle z, y_n \rangle = Z(H)$. Hence θ is not inner.

We next show that θ is class-preserving. To that end, we must show that, for each $h \in H$, there exists $t = t(h) \in H$ with $h^t = h\theta$. Fix $h \in H$, and write

$$h = \prod_{i=1}^4 x_i^{a_i} \cdot z^b \cdot \prod_{j=1}^n y_j^{c_j},$$

as in (7). If $a_4 = 0$, then $h\theta = h$ and $t(h) = 1$ works. Thus, we may assume that $a_4 = 1$, and hence that $h\theta = hz$.

Claim. *If $h\theta = hz$, then either $h^{x_2} = hz$ or $h^{x_1 x_3} = hz$.*

It is clear from the pc-relations that x_2 commutes with every y_j . This is true also of x_1x_3 . For, if $j < n - 1$, then $y_j^{x_1x_3} = (y_jy_{j+1})^{x_3} = y_jy_{j+1}^2y_{j+2}$. Using the relations (and a finite induction) one sees that $y_{j+1}^2y_{j+2} = y_{n-1}^2y_n = y_n^2 = 1$. It is easy to see that $y_{n-1}^{x_1x_3} = y_{n-1}$ and that $y_n^{x_1x_3} = y_n$.

Next, observe that x_2 commutes with x_4 , while $x_4^{x_1x_3} = (x_4z)^{x_3} = x_4z$. Thus, it suffices to show that, if $h = x_1^{a_1}x_2^{a_2}x_3^{a_3}$ with $(a_1, a_2, a_3) \in \{0, 1\}^3$, then either $h^{x_2} = hz$, or $h^{x_1x_3} = h$. First,

$$h^{x_2} = (x_1^{a_1}x_2^{a_2}x_3^{a_3})^{x_2} = x_1^{a_1}x_2^{a_2}x_3^{a_3}z^{a_1+a_3} = hz^{a_1+a_3}.$$

Hence, if $a_1 \neq a_3$, then $h^{x_2} = hz$, as required. It remains to show that x_1x_3 commutes with h whenever $a_1 = a_3$. If $a_1 = a_3 = 0$, then either $h = 1$ or $h = x_2$; clearly x_1x_3 commutes with 1, and $x_2^{x_1x_3} = x_2z^2 = x_2$. Finally, if $a_1 = a_3 = 1$, then either $h = x_1x_3$ or $h = x_1x_2x_3$; clearly x_1x_3 commutes with itself, and

$$\begin{aligned} (x_1x_2x_3)^{x_1x_3} &= (x_1(x_2z)(x_3y_1))^{x_3} \\ &= (x_1y_1^{-1})(x_2z)z x_3(y_1y_2) \\ &= x_1x_2y_1^{-1}x_3y_1y_2 \\ &= x_1x_2x_3y_2^{-1}y_1^{-1}y_1y_2 = x_1x_2x_3. \end{aligned}$$

This establishes our claim, and completes the proof of [Theorem 1.1](#). □

Acknowledgments

The authors would like to thank R. Quinlan for bringing this problem to their attention, and the anonymous referee for some helpful suggestions.

References

- [Bosma et al. 1997] W. Bosma, J. Cannon, and C. Playoust, “The Magma algebra system, I: The user language”, *J. Symbolic Comput.* **24**:3–4 (1997), 235–265. [MR 1484478](#) [Zbl 0898.68039](#)
- [Burnside 1911] W. Burnside, *Theory of groups of finite order*, 2nd ed., Cambridge Univ. Press, New York, 1911. Reprinted Dover, New York, 1955. [MR 16,1086c](#) [Zbl 0064.25105](#)
- [Burnside 1913] W. Burnside, “On the outer isomorphisms of a group”, *Proc. London Math. Soc.* **S2-11**:1 (1913), 40–42. [MR 1577234](#) [JFM 43.0198.03](#)
- [Eick and Leedham-Green 2008] B. Eick and C. Leedham-Green, “On the classification of prime-power groups by coclass”, *Bull. Lond. Math. Soc.* **40**:2 (2008), 274–288. [MR 2009b:20030](#) [Zbl 1168.20007](#)
- [Feit and Seitz 1989] W. Feit and G. M. Seitz, “On finite rational groups and related topics”, *Illinois J. Math.* **33**:1 (1989), 103–131. [MR 90a:20016](#) [Zbl 0701.20005](#)
- [Heineken 1979] H. Heineken, “Nilpotente Gruppen, deren sämtliche Normalteiler charakteristisch sind”, *Arch. Math. (Basel)* **33**:6 (1979), 497–503. [MR 81h:20023](#) [Zbl 0413.20017](#)
- [Hertweck 2001] M. Hertweck, “Class-preserving automorphisms of finite groups”, *J. Algebra* **241**:1 (2001), 1–26. [MR 2002e:20047](#) [Zbl 0993.20017](#)

- [Holt et al. 2005] D. F. Holt, B. Eick, and E. A. O'Brien, *Handbook of computational group theory*, Chapman & Hall, Boca Raton, FL, 2005. [MR 2006f:20001](#) [Zbl 1091.20001](#)
- [Malinowska 1992] I. Malinowska, "On quasi-inner automorphisms of a finite p -group", *Publ. Math. Debrecen* **41**:1–2 (1992), 73–77. [MR 93g:20069](#) [Zbl 0792.20019](#)
- [O'Brien 1990] E. A. O'Brien, "The p -group generation algorithm", *J. Symbolic Comput.* **9**:5–6 (1990), 677–698. [MR 91j:20050](#) [Zbl 0736.20001](#)
- [Wall 1947] G. E. Wall, "Finite groups with class-preserving outer automorphisms", *J. London Math. Soc.* **22** (1947), 315–320. [MR 10,8g](#) [Zbl 0030.00901](#)
- [Yadav 2011] M. K. Yadav, "Class preserving automorphisms of finite p -groups: a survey", pp. 569–579 in *Groups St Andrews 2009* (Bath, 2009), vol. II, edited by C. M. Campbell et al., London Math. Soc. Lecture Note Ser. **388**, Cambridge Univ. Press, 2011. [MR 2012j:20061](#) [Zbl 1231.20024](#)

Received: 2012-08-04

Revised: 2012-11-07

Accepted: 2012-11-17

pbrooks@bucknell.edu

*Department of Mathematics, Bucknell University,
380 Olin Science Building, Lewisburg, PA 17837, United States*

msm030@bucknell.edu

*Department of Mathematics, Bucknell University,
380 Olin Science Building, Lewisburg, PA 17837, United States*

EDITORS

MANAGING EDITOR

Kenneth S. Berenhaut, Wake Forest University, USA, berenhks@wfu.edu

BOARD OF EDITORS

Colin Adams	Williams College, USA colin.c.adams@williams.edu	David Larson	Texas A&M University, USA larson@math.tamu.edu
John V. Baxley	Wake Forest University, NC, USA baxley@wfu.edu	Suzanne Lenhart	University of Tennessee, USA lenhart@math.utk.edu
Arthur T. Benjamin	Harvey Mudd College, USA benjamin@hmc.edu	Chi-Kwong Li	College of William and Mary, USA ckli@math.wm.edu
Martin Bohner	Missouri U of Science and Technology, USA bohner@mst.edu	Robert B. Lund	Clemson University, USA lund@clemson.edu
Nigel Boston	University of Wisconsin, USA boston@math.wisc.edu	Gaven J. Martin	Massey University, New Zealand g.j.martin@massey.ac.nz
Amarjit S. Budhiraja	U of North Carolina, Chapel Hill, USA budhiraj@email.unc.edu	Mary Meyer	Colorado State University, USA meyer@stat.colostate.edu
Pietro Cerone	Victoria University, Australia pietro.cerone@vu.edu.au	Emil Minchev	Ruse, Bulgaria eminchev@hotmail.com
Scott Chapman	Sam Houston State University, USA scott.chapman@shsu.edu	Frank Morgan	Williams College, USA frank.morgan@williams.edu
Joshua N. Cooper	University of South Carolina, USA cooper@math.sc.edu	Mohammad Sal Moslehian	Ferdowsi University of Mashhad, Iran moslehian@ferdowsi.um.ac.ir
Jem N. Corcoran	University of Colorado, USA corcoran@colorado.edu	Zuhair Nashed	University of Central Florida, USA znashed@mail.ucf.edu
Toka Diagana	Howard University, USA tdiagana@howard.edu	Ken Ono	Emory University, USA ono@mathcs.emory.edu
Michael Dorff	Brigham Young University, USA mdorff@math.byu.edu	Timothy E. O'Brien	Loyola University Chicago, USA tbriell@luc.edu
Sever S. Dragomir	Victoria University, Australia sever@matilda.vu.edu.au	Joseph O'Rourke	Smith College, USA orourke@cs.smith.edu
Behrouz Emamizadeh	The Petroleum Institute, UAE bemamizadeh@pi.ac.ae	Yuval Peres	Microsoft Research, USA peres@microsoft.com
Joel Foisy	SUNY Potsdam foisys@potsdam.edu	Y.-F. S. Pétermann	Université de Genève, Switzerland petermann@math.unige.ch
Errin W. Fulp	Wake Forest University, USA fulp@wfu.edu	Robert J. Plemmons	Wake Forest University, USA rplemmons@wfu.edu
Joseph Gallian	University of Minnesota Duluth, USA kgallian@d.umn.edu	Carl B. Pomerance	Dartmouth College, USA carl.pomerance@dartmouth.edu
Stephan R. Garcia	Pomona College, USA stephan.garcia@pomona.edu	Vadim Ponomarenko	San Diego State University, USA vadim@sciences.sdsu.edu
Anant Godbole	East Tennessee State University, USA godbole@etsu.edu	Bjorn Poonen	UC Berkeley, USA poonen@math.berkeley.edu
Ron Gould	Emory University, USA rg@mathcs.emory.edu	James Propp	U Mass Lowell, USA jpropp@cs.uml.edu
Andrew Granville	Université Montréal, Canada andrew@dms.umontreal.ca	József H. Przytycki	George Washington University, USA przytyck@gwu.edu
Jerrold Griggs	University of South Carolina, USA griggs@math.sc.edu	Richard Rebarber	University of Nebraska, USA rrebarbe@math.unl.edu
Sat Gupta	U of North Carolina, Greensboro, USA sgupta@uncg.edu	Robert W. Robinson	University of Georgia, USA rwr@cs.uga.edu
Jim Haglund	University of Pennsylvania, USA jhaglund@math.upenn.edu	Filip Saidak	U of North Carolina, Greensboro, USA f_saidak@uncg.edu
Johnny Henderson	Baylor University, USA johnny_henderson@baylor.edu	James A. Sellers	Penn State University, USA sellersj@math.psu.edu
Jim Hoste	Pitzer College jhoste@pitzer.edu	Andrew J. Sterge	Honorary Editor andy@ajsterge.com
Natalia Hritonenko	Prairie View A&M University, USA nhritonenko@pvamu.edu	Ann Trenk	Wellesley College, USA atrenk@wellesley.edu
Glenn H. Hurlbert	Arizona State University, USA hurlbert@asu.edu	Ravi Vakil	Stanford University, USA vakil@math.stanford.edu
Charles R. Johnson	College of William and Mary, USA crjohnso@math.wm.edu	Antonia Vecchio	Consiglio Nazionale delle Ricerche, Italy antonia.vecchio@cnr.it
K. B. Kulasekera	Clemson University, USA kk@ces.clemson.edu	Ram U. Verma	University of Toledo, USA verma99@msn.com
Gerry Ladas	University of Rhode Island, USA gladas@math.uri.edu	John C. Wierman	Johns Hopkins University, USA wierman@jhu.edu
		Michael E. Zieve	University of Michigan, USA zieve@umich.edu

PRODUCTION


Silvio Levy, Scientific Editor

See inside back cover or msp.org/involve for submission instructions. The subscription price for 2014 is US \$120/year for the electronic version, and \$165/year (+\$35, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Involve (ISSN 1944-4184 electronic, 1944-4176 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

Involve peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2014 Mathematical Sciences Publishers

involve

2014

vol. 7

no. 2

An interesting proof of the nonexistence of a continuous bijection between \mathbb{R}^n and \mathbb{R}^2 for $n \neq 2$	125
HAMID REZA DANESHPAJOUH, HAMED DANESHPAJOUH AND FERESHTE MALEK	
Analysing territorial models on graphs	129
MARIE BRUNI, MARK BROOM AND JAN RYCHTÁŘ	
Binary frames, graphs and erasures	151
BERNHARD G. BODMANN, BIJAN CAMP AND DAX MAHONEY	
On groups with a class-preserving outer automorphism	171
PETER A. BROOKSBANK AND MATTHEW S. MIZUHARA	
The sharp log-Sobolev inequality on a compact interval	181
WHAN GHANG, ZANE MARTIN AND STEVEN WARUHIU	
Analysis of a Sudoku variation using partially ordered sets and equivalence relations	187
ANA BURGERS, SHELLY SMITH AND KATHERINE VARGA	
Spanning tree congestion of planar graphs	205
HIU FAI LAW, SIU LAM LEUNG AND MIKHAIL I. OSTROVSKII	
Convex and subharmonic functions on graphs	227
MATTHEW J. BURKE AND TONY L. PERKINS	
New results on an anti-Waring problem	239
CHRIS FULLER, DAVID R. PRIER AND KARISSA A. VASCONI	