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On The Construction of Mixed Orthogonal Arrays of Strength Two

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1. Introduction

An orthogonal array of strength two, $L_N(s_1 \cdots s_k)$, is an $N \times k$ matrix with symbols in the ith column from a finite set of s_i symbols $(1 \leqslant i \leqslant k)$, such that in every $N \times 2$ submatrix, all possible combinations of symbols occur equally often as a row. If among $s_1, \ldots, s_k$, there are n_i that equal μ_i $(1 \leqslant i \leqslant u)$ where $\{n_i\}$ and $\{\mu_i\}$ are positive integers, $\mu_i \geqslant 2$, $n_1 + \cdots + n_u = k$, then we will write $L_N(\mu_1^{n_1} \cdots \mu_u^{n_u})$ for $L_N(s_1 \cdots s_k)$. When $s_1 = \cdots = s_k$ the orthogonal array is called *symmetric*; otherwise it is called asymmetric or *mixed*. An orthogonal array $L_N(s_1 \cdots s_k)$ is *tight* if $\sum_{i=1}^{k} s_i - k = N - 1$. Orthogonal arrays have extensive applications in statistical design of experiments, computer science, and cryptography, and large orthogonal arrays, sometimes with hundreds of runs, are becoming increasingly popular among researchers modelling consumer choice (Kuhfeld, 2004). Methods for constructing mixed orthogonal arrays of strength two have been developed recently by Wang and Wu (1991), Dey and Midha (1996, 2001), Wang (1996a, b), Zhang et al. (1999), Xu (2002), and many other authors. For an excellent description of the methods of construction of orthogonal arrays, see Hedayat et al. (1999).For extensive construction methods, see Kuhfeld (2004).

Wang and Wu (1991) used the Kronecker sum of orthogonal arrays and difference schemes to construct several families of mixed orthogonal arrays. Dey and Midha (1996, 2001) extended the method of Wang and Wu (1991) to construct more families of mixed orthogonal arrays. In this paper, we modify this method to allow more flexibility. As a consequence, some new families of mixed orthogonal arrays are obtained.

2. Basic concepts and notations

Let G be an additive group with p elements, $0, 1, \ldots, p-1$. An $rp \times k$ matrix with entries from G is called a *difference scheme* $D_{rp,k,p}$, if among the differences of the corresponding elements of any two columns, each element of G appears r times.

Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times r$ and $m \times s$ matrices respectively with entries from an additive group G of p elements. The *Kronecker sum* of A and B, denoted by $A * B$, is defined to be an $nm \times rs$ matrix $(B^{a_{ij}})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant r}$, where B^a is an $m \times s$ matrix $(b_{ij} + a)_{1 \leqslant i \leqslant m, 1 \leqslant j \leqslant s}$.

Throughout, we let 0_n be the $n \times 1$ vector of zeros and let τ_n be the $n \times 1$ vector $(0, 1, \ldots, n-1)'$. We now list some useful properties of difference schemes.

1. We can assume, without loss of generality, that the first column of a difference scheme $D_{rp,k,p}$ is 0_{rp}. Then every element of G appears exactly r times in all other columns.
2. The Kronecker sum of a difference scheme $D_{rp,k,p}$ and an orthogonal array $L_N(p^s)$ is an orthogonal array $L_{rpN}(p^{ks})$. The Kronecker sum of two difference schemes $D_{r_1 p,k_1,p}$ and $D_{r_2 p,k_2,p}$ is also a difference scheme $D_{r_1 r_2 p^2, k_1 k_2, p}$.
3. If $D_{rp,k,p}$ exists then $k \leqslant rp$. $D_{rp,rp,p}$ is called a generalized Hadamard matrix. $D_{h,h,2}$ is a Hadamard matrix of order h. If a Hadamard matrix of order h exists, h is called a Hadamard number. It is conjectured that h is a Hadamard number if $h = 1, 2$, or a multiple of 4.
4. If p is a prime or a prime power then $D_{rp,rp,p}$ exists in each of the following cases: (a) $r = 2$ or 4; (b) r and p are powers of the same prime; (c) $r = q^m(q+1)/p$ for all $m \geqslant 0$ if q is a prime power and $D_{q+1,q+1,p}$ exists.

Suppose an $L_N(s_1^{n_1} \cdots s_u^{n_u})$ and difference schemes $D_{M,k_1,s_1}, \ldots, D_{M,k_u,s_u}$ exist. Partition the $L_N(s_1^{n_1} \cdots s_u^{n_u})$ as $[L_N(s_1^{n_1}), \ldots, L_N(s_u^{n_u})]$. By using Kronecker sum, Wang and Wu (1991) constructed the following mixed orthogonal array $L_{MN}(s_1^{k_1 n_1} \cdots s_u^{k_u n_u} M^1)$,

$$[D_{M,k_1,s_1} * L_N(s_1^{n_1}), \ldots, D_{M,k_u,s_u} * L_N(s_u^{n_u}), \tau_M * 0_N].$$

If $L_M(t_1^{m_1}\cdots t_v^{m_v})$ exists, then we can replace the M-symbol column of the above array by $\sum_{1=1}^{v} m_i$ columns of symbols $t_1, \ldots, t_v$ respectively and obtain an $L_{MN}(s_1^{k_1 n_1} \cdots s_u^{k_u n_u} t_1^{m_1} \cdots t_v^{m_v})$. Furthermore, if a_1, a_2, a_3 are three 2-symbol columns such that $a_1 + a_2 = a_3$, then we can replace these three 2-symbol columns by a 4-symbol column. By using this procedure, Wang and Wu (1991) constructed several families of mixed orthogonal arrays.

Dey and Midha (1996) modified the construction of Wang and Wu (1991) and obtained the following result. If there exist an orthogonal array $L_N(w^1 s_1^{n_1} \cdots s_u^{n_u})$ and difference schemes $D_{M,k_1,s_1}, \ldots, D_{M,k_u,s_u}$, then an $L_{MN}(s_1^{k_1 n_1} \cdots s_u^{k_u n_u}(Mw)^1)$ can be constructed as follows. Arrange the rows of the $L_N(w^1 s_1^{n_1} \cdots s_u^{n_u})$ such that the first column is $\tau_w * 0_{N/w}$. Partition $L_N(w^1 s_1^{n_1} \cdots s_u^{n_u})$ as $[\tau_w * 0_{N/w}, L_N(s_1^{n_1}), \ldots, L_N(s_u^{n_u})]$. Then an $L_{MN}(s_1^{k_1 n_1} \cdots s_u^{k_u n_u}(Mw)^1)$ can be constructed as

$$[D_{M,k_1,s_1} * L_N(s_1^{n_1}), \ldots, D_{M,k_u,s_u} * L_N(s_u^{n_u}), \tau_{Mw} * 0_{N/w}].$$

3. Main results

We first modify the result of Wang and Wu (1991) to obtain an N-symbol column by sacrificing several columns in the construction.

Theorem 1. *If there exists an orthogonal array $L_N(s_1^{n_1} \cdots s_u^{n_u})$ and difference schemes $D_{M,k_1,s_1}, \ldots, D_{M,k_u,s_u}$, then we can construct an orthogonal array $L_{MN}(s_1^{(k_1-1)n_1} \cdots s_u^{(k_u-1)n_u} M^1 N^1)$.*

Proof. For $i=1, \ldots, u$, let $D_{M,k_i,s_i} = [0_M, D_{M,k_i-1,s_i}]$. Then each of the s_i symbols appears M/s_i times in every column of D_{M,k_i-1,s_i}. We can verify that

$$[D_{M,k_1-1,s_1} * L_N(s_1^{n_1}), \ldots, D_{M,k_u-1,s_u} * L_N(s_u^{n_u}), \tau_M * 0_N, 0_M * \tau_N]$$

is an $L_{MN}(s_1^{(k_1-1)n_1} \cdots s_u^{(k_u-1)n_u} M^1 N^1)$. $\square$

For examples, we obtain an $L_{216}(18^1 12^1 6^5 3^{66})$ by using $L_{18}(6^1 3^6)$, $D_{12,6,6}$, and $D_{12,12,3}$; obtain an $L_{216}(18^1 12^1 3^{77} 2^{11})$ by using $L_{18}(3^7 2^1)$, $D_{12,12,3}$, and $D_{12,12,2}$; and obtain an $L_{144}(12^2 3^{11} 2^{44})$ by using $L_{12}(3^1 2^4)$, $D_{12,12,3}$, and $D_{12,12,2}$ in Theorem 1.

The result in Theorem 1 was, in a slightly different formulation, also obtained by Dey and Midha (2001) by a slightly different method. Note that $L_{MN}(s_1^{(k_1-1)n_1} \cdots s_u^{(k_u-1)n_u} M^1 N^1)$ in Theorem 1 is *tight* if $L_N(s_1^{n_1} \cdots s_u^{n_u})$ is tight and $D_{M,k_1,s_1}, \ldots, D_{M,k_u,s_u}$ are generalized Hadamard matrices. Several families of tight orthogonal arrays are constructed in the following by using Theorem 1.

Corollary 1.1. *If p is a prime power and D_{rp^2,rp^2,p^2} exists, then we can construct a tight orthogonal array $L_{rp^5}((rp^2)^1 (p^3)^1 (p^2)^{rp^2-1} p^{(rp^2-1)p^2})$.*

Proof. Orthogonal arrays can be constructed by using $L_{p^3}((p^2)^1 p^{p^2})$, D_{rp^2,rp^2,p^2}, and $D_{rp^2,rp^2,p}$ in Theorem 1. The existence of $D_{rp^2,rp^2,p}$ is implied by the existence of D_{rp^2,rp^2,p^2}. $\square$

For $p=2$ and $r=3$ in Corollary 1.1, we have a new array $L_{96}(12^1 8^1 4^{11} 2^{44})$. For $p=3$ and $r=2$ we obtain $L_{486}(27^1 18^1 9^{17} 3^{153})$.

Corollary 1.2. *If p is a prime power and $D_{rp,rp,p}$ exists, then we can construct a tight orthogonal array $L_{rp^n}((rp)^1 (p^{n-1})^1 p^{(rp-1)(p^{n-1}-1)/(p-1)})$ for all $n \geqslant 3$.*

Proof. The orthogonal array can be constructed by using $L_{p^{n-1}}(p^{(p^{n-1}-1)/(p-1)})$ and $D_{rp,rp,p}$ in Theorem 1. $\square$

For $r=2$, $n=3$, and $p=4,5$ in Corollary 1.2, we have new arrays $L_{128}(16^1 8^1 4^{35})$ and $L_{250}(25^1 10^1 5^{54})$. In particular, we obtain the following orthogonal arrays by Corollaries 1.1 and 1.2, since $D_{2p,2p,p}$, $D_{4p,4p,p}$, and $D_{4r,4r,2}$ exist.

Corollary 1.3. *If p is a prime power, $r \geqslant 1$, and $n \geqslant 3$, we can construct tight orthogonal arrays* (a) $L_{2p^5}((2p^2)^1 (p^3)^1 (p^2)^{2p^2-1} p^{(2p^2-1)p^2})$; (b) $L_{4p^5}((4p^2)^1 (p^3)^1 (p^2)^{4p^2-1} p^{(4p^2-1)p^2})$; (c) $L_{2p^n}((2p)^1 (p^{n-1})^1 p^{(2p-1)(p^{n-1}-1)/(p-1)})$; (d) $L_{4p^n}((4p)^1 (p^{n-1})^1 p^{(4p-1)(p^{n-1}-1)/(p-1)})$; *and* (e) $L_{r2^{n+1}}((4r)^1 (2^{n-1})^1 2^{(4r-1)(2^{n-1}-1)})$.

For $p=3$ and $n=3,4$ in Corollary 1.3 (c) and (d), we obtain tight arrays $L_{54}(9^1 6^1 3^{20})$, $L_{162}(27^1 6^1 3^{65})$, $L_{108}(12^1 9^1 3^{44})$, $L_{324}(27^1 12^1 3^{143})$. $L_{54}(9^1 6^1 3^{20})$ was also constructed by Wang and Wu (1991), the other three arrays are believed to be new.

We next modify the construction of Dey and Midha (1996) to obtain the following orthogonal array.

Theorem 2. *If there exist orthogonal arrays $L_N(w^1 s_1^{n_1} \cdots s_u^{n_u})$ and $L_N(w^1 t_1^{m_1} \cdots t_v^{m_v})$ and difference schemes $D_{M,k_1,s_1}, \ldots, D_{M,k_u,s_u}$, then we can construct an orthogonal array $L_{MN}(s_1^{(k_1-1)n_1} \cdots s_u^{(k_u-1)n_u} t_1^{m_1} \cdots t_v^{m_v} (Mw)^1)$.*

Proof. Partition the orthogonal arrays as $L_N(w^1 s_1^{n_1} \cdots s_u^{n_u}) = [\tau_w * 0_{N/w}, L_N(s_1^{n_1}), \cdots, L_N(s_u^{n_u})]$ and $L_N(w^1 t_1^{m_1} \cdots t_v^{m_v}) = [\tau_w * 0_{N/w}, L_N(t_1^{m_1} \cdots t_v^{m_v})]$. For $i=1,\ldots,u$, let $D_{M,k_i,s_i} = [0_M, D_{M,k_i-1,s_i}]$. Then we can verify that

$$[D_{M,k_1-1,s_1} * L_N(s_1^{n_1}), \ldots, D_{M,k_u-1,s_u} * L_N(s_u^{n_u}), 0_M * L_N(t_1^{m_1} \cdots t_v^{m_v}), \tau_{Mw} * 0_{N/w}]$$

is an $L_{MN}(s_1^{(k_1-1)n_1} \cdots s_u^{(k_u-1)n_u} t_1^{m_1} \cdots t_v^{m_v} (Mw)^1)$. $\square$

Example 1. We use Theorem 2 to construct many new 72-run orthogonal arrays in the following by combining two 36-run orthogonal arrays.

(a) By using $L_{36}(3^1 2^{27})$, assorted L_{36}, and $D_{2,2,2}$ in Theorem 2, we obtain L_{72}:

$L_{36}(3^1 12^1 3^{11})$	$\rightarrow$	$L_{72}(12^1 6^1 3^{11} 2^{27})$	$L_{36}(3^1 6^3 3^6)$	$\rightarrow$	$L_{72}(6^4 3^6 2^{27})$
$L_{36}(3^1 6^3 3^2 2^1)$	$\rightarrow$	$L_{72}(6^4 3^2 2^{28})$	$L_{36}(3^1 6^3 3^1 2^3)$	$\rightarrow$	$L_{72}(6^4 3^1 2^{30})$
$L_{36}(3^1 6^3 2^4)$	$\rightarrow$	$L_{72}(6^4 2^{31})$			

(b) By using $L_{36}(2^1 2^{34})$, assorted L_{36}, and $D_{2,2,2}$ in Theorem 2, we obtain L_{72}.

$L_{36}(2^1 18^1 2^1)$	$\rightarrow$	$L_{72}(18^1 4^1 2^{35})$	$L_{36}(2^1 9^1 2^{12})$	$\rightarrow$	$L_{72}(9^1 4^1 2^{46})$
$L_{36}(2^1 6^3 3^3)$	$\rightarrow$	$L_{72}(6^3 4^1 3^3 2^{34})$	$L_{36}(2^1 6^3 3^2 2^2)$	$\rightarrow$	$L_{72}(6^3 4^1 3^2 2^{36})$
$L_{36}(2^1 6^3 3^1 2^3)$	$\rightarrow$	$L_{72}(6^3 4^1 3^1 2^{37})$	$L_{36}(2^1 6^3 2^7)$	$\rightarrow$	$L_{72}(6^3 4^1 2^{41})$
$L_{36}(2^1 6^2 3^8)$	$\rightarrow$	$L_{72}(6^2 4^1 3^8 2^{34})$	$L_{36}(2^1 6^2 3^5 2^1)$	$\rightarrow$	$L_{72}(6^2 4^1 3^5 2^{35})$
$L_{36}(2^1 6^2 3^4 2^8)$	$\rightarrow$	$L_{72}(6^2 4^1 3^4 2^{42})$	$L_{36}(2^1 6^2 3^1 2^9)$	$\rightarrow$	$L_{72}(6^2 4^1 3^1 2^{43})$
$L_{36}(2^1 6^1 3^9 2^2)$	$\rightarrow$	$L_{72}(6^1 4^1 3^9 2^{36})$	$L_{36}(2^1 6^1 3^8 2^9)$	$\rightarrow$	$L_{72}(6^1 4^1 3^8 2^{43})$
$L_{36}(2^1 6^1 3^1 2^{17})$	$\rightarrow$	$L_{72}(6^1 4^1 3^1 2^{51})$	$L_{36}(2^1 3^2 2^{19})$	$\rightarrow$	$L_{72}(4^1 3^2 2^{53})$
$L_{36}(2^1 3^1 2^{26})$	$\rightarrow$	$L_{72}(4^1 3^1 2^{60})$			

We now construct two families of orthogonal arrays by using Theorem 2. Let $r(\geqslant 3)$ be an odd number. It is known that $L_{4r}((2r)^1 2^2)$ exists, and it is not possible to have more than two 2-symbol columns. Let ϕ_r denote the largest possible m in an $L_{4r}(r^1 2^m)$. It is known that $\phi_3 = 4$ and $\phi_5 = 8$. For $r \geqslant 7$ we do not know the exact value of ϕ_r except that $\phi_7 \geqslant 12$, $\phi_9 \geqslant 13$, $\phi_{11} \geqslant 12$, $\phi_{13} \geqslant 12$, and $\phi_r \geqslant 13$ for $r \geqslant 15$.

Corollary 2.1. *If $r \geqslant 3$ is an odd number and h is an Hadamard number, then we can construct* (a) $L_{4rh}((2h)^1(2r)^1 2^{(4r-2)(h-1)+1})$; *and* (b) $L_{4rh}((2h)^1 r^1 2^{(4r-2)(h-1)+\phi_r-1})$, *where ϕ_r is the maximum number m such that $L_{4r}(r^1 2^m)$ exists.*

Proof. $L_{4rh}((2h)^1(2r)^1 2^{(4r-2)(h-1)+1})$ is obtained by using $L_{4r}(2^1 2^{4r-2})$, $L_{4r}(2^1(2r)^1 2^1)$, and $D_{h,h,2}$ in Theorem 2. $L_{4rh}((2h)^1 r^1 2^{(4r-2)(h-1)+\phi_r-1})$ is obtained by using L_{4r} $(2^1 2^{4r-2})$, $L_{4r}(2^1 r^1 2^{\phi_r-1})$, and $D_{h,h,2}$ in Theorem 2. $\square$

For $h=2$ in Corollary 2.1(a), we obtain $L_{8r}((2r)^1 4^1 2^{4r-1})$ which was also constructed by Agrawal and Dey (1982). For $h=2$ and $r=3$, 5, 7, 9, and 11 in Corollary 2.1(b), we obtain $L_{24}(4^1 3^1 2^{13})$, $L_{40}(5^1 4^1 2^{25})$, $L_{56}(7^1 4^1 2^{37})$, $L_{72}(9^1 4^1 2^{46})$, and $L_{88}(11^1 4^1 2^{53})$ respectively. The first two arrays were also obtained by Wang and Wu (1991), and the last three arrays are believed to be new. Also for $h=4$, 8, 12 and $r=3$, 5, 7, 9 in Corollary 2.1, we obtain new arrays $L_{112}(14^1 8^1 2^{79})$, $L_{112}(8^1 7^1 2^{89})$, $L_{144}(24^1 6^1 2^{111})$, $L_{144}(24^1 3^1 2^{113})$, $L_{144}(18^1 8^1 2^{103})$, $L_{144}(9^1 8^1 2^{114})$, $L_{160}(16^1 10^1 2^{127})$, $L_{160}(16^1 5^1 2^{133})$, $L_{224}(16^1 14^1 2^{183})$, $L_{224}(16^1 7^1 2^{193})$, $L_{240}(24^1 10^1 2^{199})$, $L_{240}(24^1 5^1 2^{205})$, $L_{288}(18^1 16^1 2^{239})$,$L_{288}(16^1 9^1 2^{250})$, $L_{336}(24^1 14^1 2^{287})$, $L_{336}(24^1 7^1 2^{297})$, $L_{432}(24^1 18^1 2^{375})$, and $L_{432}(24^1 9^1 2^{386})$.

In the following example, we obtain two new 96-run orthogonal arrays by using Dey and Midha's (1996) construction, and replacing several 2-symbol columns by 4-symbol columns.

Example 2. Partition the $L_8(4^1 2^4)$ as $[\tau_2 * 0_4,\ L_8(4^1),\ L_8(2^3)]$. By sacrificing a 2-symbol column in $L_8(4^1 2^4)$, Dey and Midha (1996) obtained $L_{96}(24^1 4^{12} 2^{36})$ as

$$[D_{12,12,4} * L_8(4^1),\ D_{12,12,2} * L_8(2^3),\ \tau_{24} * 0_4].$$

More arrays can be obtained by replacing τ_{24} with any 24-run array L_{24}. Let $L_8(2^3) = [a_1, a_2, a_3]$ and $D_{12,12,2} = [0_{12}, b_1, b_2, B]$. If the 24-run array has at least two 2-symbol columns, we can permute the rows of L_{24} such that $L_{24} = [b_1 * 0_2, b_2 * 0_2, L]$. Since for $i = 1, 2$ we have

$$0_{12} * a_i + b_i * a_i = b_i * 0_8 = b_i * 0_2 * 0_4,$$

the three 2-symbol columns $0_{12} * a_i$, $b_i * a_i$, $b_i * 0_2 * 0_4$ can be replaced by a 4-symbol column. For example, if we choose L_{24} to be $L_{24}(12^1 2^{12})$ and $L_{24}(6^1 4^1 2^{11})$, we obtain two new 96-run arrays $L_{96}(12^1 4^{14} 2^{42})$ and $L_{96}(6^1 4^{15} 2^{41})$, respectively.

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