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Generalized Shifts on Cartesian Products

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GENERALIZED SHIFTS ON CARTESIAN PRODUCTS

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It is proved that if E, F are infinite dimensional strictly convex Banach spaces totally incomparable in a restricted sense, then the Cartesian product $E \times F$ with the sum or sup norm does not admit a forward shift. As a corollary it is deduced that there are no backward or forward shifts on the Cartesian product $\ell_{p_1} \times \ell_{p_2}$, $1 < p_1 \neq p_2 < \infty$, with the supremum norm thus settling a problem left open in Rajagopalan and Sundaresan in *J. Analysis* 7(1999), 75-81 and also a problem stated as unsolved in Rassias and Sundaresan, *J. Math. Anal. Applications* (260)(2001), 36-45.

Key words : Generalized shifts; backward shifts; forward shifts; Cartesian products; sub spaces; Banach spaces; strictly convex; totally incomparable; isometrically incomparable; reflexive Banach spaces

1. INTRODUCTION

In this section the basic definitions are recalled and various notations are established. In this paper all Banach spaces under consideration are infinite dimensional unless otherwise stated and all subspaces are closed. We adhere for the terminology concerning Banach spaces to the book on Normed Linear Spaces by Day, [1]. All isometries in this paper are linear. If E is a Banach space the set of extreme points of the unit ball of E is noted as $Ext\ E$. If E, F are Banach spaces the sum and supremum norm on the Cartesian product $E \times F$ are respectively denoted by $\| \ \|_1$, and $\| \ \|_\infty$. Two

Banach spaces are said to be isometrically incomparable if no infinite dimensional subspace of either of the spaces is isometric with a subspace of the other. Two Banach spaces E, F are said to be Quasi isometrically incomparable if E is not isometric with F, and no subspace of $E(F)$ of codim 1 is isometric with $F(E)$. The terminology is in part motivated by the term "totally incomparable" introduced by Haskell Rosenthal in [10]. As examples of spaces which are isometrically incomparable we mention any two distinct Banach spaces in the set $\{\ell_p | 1 \leq p < \infty\} \cup \{c_0\}$, Pelczynski [6]. On the other hand the spaces ℓ_2 and $C[0,1]$ are not isometrically incomparable but they are quasi isometrically incomparable.

A continuous linear operator on a Banach space $E \longrightarrow E$ is said to be a generalized forward shift as defined in Holub [5] if (1) T is an isometry on E onto a subspace M of E of codim 1, and (2) $\bigcap\limits_{k \geq 1} RangeT^K = \{0\}$. Further adhering to the terminology in [5], $T : E \longrightarrow E$ is said to be a generalized backward shift if (1) $KerT$ is one dimensional and (2) $\bigcup\limits_{k \geq 1} KerT^K$ is dense in E and (3) the canonical extension $\hat{T}$ of T to the factor space $E | KerT$ defined by

$$\hat{T}(x + \mathrm{Ker}T) = T(x) \text{ for all } x \in E$$

is an isometry. The linear transformations T, T' on sequences defined by

$$T(x) = y, y_1 = 0, y_2 = x_1, \ldots y_k = x_{k-1}, k \geq 2$$

and

$$T'(x) = y, y_1 = x_2, y_2 = x_3, \ldots y_k = x_{k+1}, k \geq 1$$
$$\text{where } x = \{x_n\}, \text{and } y = \{y_n\}$$

are respectively generalized forward and backward shifts when restricted to sequence spaces ℓ_p 1 $\leq p < \infty$, and c_0. It is evident from the definitions that there are no generalized forward shifts on finite dimensional Banach spaces while if there is a generalized backward shift on a Banach space it is separable. Thus there is no generalized backward shift on ℓ_∞. The transformation T on sequences defined above is a generalized forward shift on the Banach space ℓ_∞. Further we note that if E is infinite dimensional, a backward shift on E is surjective, [RS$_1$]. In this paper we call generalized forward shifts and generalized backward shifts simply as forward shifts and backward shifts respectively.

If E is a Banach space then $\| \|_E$ denotes the norm on the space E.

Holub in [5] raised the fundamental problem of the existence of forward and backward shifts on various Banach spaces. Some of these problems have been settled, see the recent papers of Gutek *et al*. [3, 4], Ragagopalan and Sundaresan [7] and Themistocles Rassias and Sundaresan [9]. The following problems naturally arise in the context of problems stated as unsolved in [5] and have been stated as unsolved in [8] and [9]. Are there backward or foward shifts on the product spaces

$\ell_{p_1} \times \ell_{p_2}$ equipped with sup norm if $1 < p_1 \neq p_2 < \infty$. We settle the problem completely in this paper. In the process interesting geometric properties of Cartesian products of quasi isometrically incomparable strictly convex spaces are obtained.

The following definitions are useful in the discussion to follow. If E, F are Banach spaces, a subspace M of $E \times F$ is said to be factorable (a rectangle) if there are subspaces M_1 of E and M_2 of F such that $M = M_1 \times M_2$. An isometry T on the product $E \times F$ equipped with a norm is said to be factorable if there are isometries $T_1 : E \longrightarrow E$, and $T_2 : F \longrightarrow F$ such that $T = T_1 \times T_2$ i.e. $T(x, y) = (T_1 x, T_2 y)$.

The section is concluded with the following proposition, stated here for convenience of reference.

Proposition 1 — Let E, F be any two Banach spaces, and $X = E \times F$. Then

(1) If the norm on X is $\| \ \|_1$, then

$$Ext\ X = \{(x, y)|\ x \in Ext\ E \text{ and } y = 0 \text{ or } x = 0 \text{ and } y \in Ext\ F\}.$$

(2) If the norm on X is $\| \ \|_\infty$, then

$$Ext\ X = \{(x, y)|\ x \in Ext\ E, \text{ and } y \in Ext\ F\}.$$

The results follow from the definition of an extreme point of a convex set.

2. Shifts on Cartesian Products

It is proved in this section that there are no forward or backward shifts on the Cartesian product $E \times F$ with either $\| \ \|_1$ or $\| \ \|_\infty$ if E and F are quasi isometrically incomparable strictly convex Banach spaces. From this it is deduced that the product space $\ell_{p_1} \times \ell_{p_2}$, $1 < p_1 \neq p_2 < \infty$ with sup norm does not admit either a forward or a backward shift. The proofs for the cases of $(E \times F, \| \ \|_1)$ and $(E \times F, \| \ \|_\infty)$ are very similar. For this reason the proof for the case $\| \ \|_1$ is presented. In the rest of this paper the norm on $E \times F$ is the sum norm $\| \ \|_1$ unless otherwise specified. The main theorem is established after proving several useful results.

Lemma 2 — If M is a subspace of $E \times F$, and $(x, 0) \in M, (0, y) \in M, x \neq 0 \neq y$, and $\|x\|_E + \|y\|_F = 1$ then $(x, y) \notin ExtM$.

PROOF : Clearly $(x, y) \in M$, and $\|(x, y)\| = 1$. However $(x, y) = \|x\|_E \left(\frac{x}{\|x\|_E}, 0 \right) + \|y\|_F \left(0, \frac{y}{\|y\|_F} \right)$. Hence $(x, y) \notin ExtM$, since $\|x\|_E + \|y\|_F = 1$.

***Theorem* 3** — *Let E, F be two strictly convex Banach spaces and T be an isometry on $E \times F$ into $E \times F$ with range $T = M$, a subspace of Codim 1. Then for all $x \in E$, $(1) T(x, 0) \in (E \times \{0\}) \bigcup (\{0\} \times F)$. A similar inclusion holds for $T(0, y)$ for all $y \in F$.*

PROOF : Let T, M, E, F be as in the theorem. From the property of M, there are linear functionals f, and g, $f \in E^*$, and $g \in F^*$, such that

$$M = \{(a, b) | a \in E, b \in F, \text{ suchthat } f(a) + g(b) = 0\}.$$

It is enough to prove (1) for all $x \in E, \|x\|_E = 1$. With such a choice of x, let $T(x, 0) = (x_1, y_1)$. Since $(x, 0) \in Ext(E \times F), (x_1, y_1) \in Ext\ M$. To complete proof of (1) enough to verify $x_1 = 0$ or $y_1 = 0$. Let $x_1 \neq 0$, $y_1 \neq 0$. Since $(x_1, y_1) \in M, f(x_1) + g(y_1) = 0$. If $f(x_1) = 0$, equivalently $g(y_1) = 0, (x_1, 0) \in M$, and $(0, y_1) \in M$, and it follows from Lemma 2, that $(x_1, y_1) \notin Ext\ M$, a contradiction. Thus $f(x_1) \neq 0$, and $g(y_1) \neq 0$.

Now choose a $y \in F, \|y\|_F = 1$. Let $T(0, y) = (x_2, y_2)$. Then $(x_2, y_2) \in ExtM$, since $(0, y) \in Ext(E \times F)$. Since $\|(x, y)\| = 2$, it follows that

$$\begin{aligned} (I)\ \ 2 = \|T(x, y)\| &= \|(x_1 + x_2, y_1 + y_2)\| = \|x_1 + x_2\|_E + \|y_1 + y_2\|_F \\ &\leq \|x_1\|_E + \|x_2\|_E + \|y_1\|_F + \|y_2\|_F = 2. \end{aligned}$$

Hence

$$\|x_1 + x_2\|_E = \|x_1\|_E + \|x_2\|_E$$

and

$$\|y_1 + y_2\|_F = \|y_1\|_F + \|y_2\|_F.$$

Since $\|\ \|_E, \|\ \|_F$ are strictly convex, assuming $x_2 \neq 0$, and $y_2 \neq 0$, it follows that there are positive numbers t, and s such that $x_2 = tx_1$, and $y_2 = sy_1$. Since $(x_2, y_2) \in M, f(x_2) + g(y_2) = tf(x_1) + sg(y_1) = 0$. Since $f(x_1) + g(y_1) = 0$, and $f(x_1) \neq 0$ it follows that $t = s$. Thus $(x_2, y_2) = t(x_1, y_1)$. Since $\|(x_2, y_2)\| = \|(x_1, y_1)\| = 1$, $t = 1$, and $(x_2, y_2) = (x_1, y_1)$, a contradiction since $(x, 0) \neq (0, y)$.

In case $x_2 = 0, (0, y_2) \in ExtM$, and $g(y_2) = 0$. However the inequality I and strict convexity of F imply $y_2 = sy_1$, for some $s > 0$. Thus $g(y_1) = 0$, contradicting that $g(y_1) \neq 0$. Similarly the case $y_2 = 0$, leads to a contradiction. Thus for all $x \in E$, (1) holds. A similar argument leads to the inclusion

$$T(0, y) \in (E \times \{0\}) \bigcup (\{0\} \times F)$$

for all $y \in F$, as desired.

Corollary 4 — If T, M, E, and F are as in Theorem 3 then $ExtM \subset Ext(E \times F)$.

PROOF : If $(x_1, y_1) \in Ext\ M$, it follows from proposition 1, that $(x_1, y_1) = T(x, 0)$ or $(x_1, y_1) = T(0, y)$. Hence from theorem 3, $x_1 = 0$ or $y_1 = 0$. Hence $(x_1, y_1) \in Ext(E \times F)$.

The theorem 6 below reveals the structure of M, if T, M, E, F are as in theorem 3. Before proceeding to the theorem we state a useful lemma.

Lemma 5 — If E, F are strictly convex Banach spaces and M is a subspace of $E \times F$ of codim 1, then if $x \in E$, and $y \in F, x \neq 0, y \neq 0$, are such that $(x, 0) \notin M, (0, y) \notin M, (x, y) \in M, \|(x, y)\| = 1$, then $(x, y) \in Ext\ M$.

PROOF : Since the proof is similar to the proof of Theorem 3, a proof sketch is provided omitting details.

Choosing the functionals f, g as in the proof of Theorem 3, let if possible

$$(x, y) = \frac{1}{2}\{(x_1, y_1) + (x_2, y_2)\}, \tag{2}$$

where (x_i, y_i) are in $M, i = 1, 2$ and each of unit norm. Assuming $x_i \neq 0, y_i \neq 0, i = 1, 2$ and proceeding as in the proof of Theorem 3, invoking strict convexity of E and F we find that there are positive numbers t, and s such that $x_2 = tx_1$, and $y_2 = sy_1$.

In case $f(x_1) = 0$, it follows that $f(x_2) = 0$ implying $f(x) = 0$ as seen from equation (2). Thus $(x, 0) \in M$, a contradiction on the choice of x. Thus $f(x_1) \neq 0$ and $f(x_2) \neq 0$. Similarly it is seen that $g(y_1) \neq 0$ and $g(y_2) \neq 0$. Since $f(x_1) + g(y_1) = f(x_2) + g(y_2) = 0$ arguing as in the proof of Theorem 3 it follows that $t = s = 1$. Hence $(x_1, y_1) = (x_2, y_2)$ and $(x, y) \in ExtM$.

***Theorem* 6** — *If E, F are strictly convex Banach spaces and $T : E \times F \longrightarrow E \times F$ is an isometry with range M, a subspace of $E \times F$ of Codim 1, then M is a rectangle. More precisely there is a subspace E_0 of E of Codim 1 such that $M = E_0 \times F$ or there is a subspace F_0 of F of Codim 1 such that $M = E \times F_0$.*

PROOF : Since M is a proper subspace of $E \times F$, there is either a $x \in E, x \neq 0$ or a $y \in F, y \neq 0$ such that $(x, 0) \notin M$ or $(0, y) \notin M$. Let $(x, 0) \notin M$ for some $x \in E$. If possible let for some $y \in F, (0, y) \notin M$. Thus $f(x) \neq 0$, and $g(y) \neq 0$ where f, g are functionals in E^* and F* respectively such that

$$M = \{(a, b) | f(a) + g(b) = 0\}.$$

Since $f(x) \neq 0$, and $g(y) \neq 0$, there are nonzero numbers t_1, t_2 such that $(t_1 x, t_2 y) \in M$. Now setting $\|(t_1 x, t_2 y)\| = A$, since $(t_1 x, 0) \notin M, (0, t_2 y) \notin M$, and $\|(\frac{t_1 x}{A}, \frac{t_2 y}{A})\| = 1$, it follows from lemma 5, that $(\frac{t_1 x}{A}, \frac{t_2 y}{A}) \in ExtM$. Thus either $x = 0$ or $y = 0$ as a consequence of corollary 4, and proposition 1, a contradiction. Hence if $(x, 0) \notin M$, then for every $y \in F, (0, y) \in M$.

Let $E_0 = \{x^1 | \, x^1 \in E, (x^1, 0) \in M\}$. E_0 is a subspace of E, and x chosen in the previous paragraph, is not in E_0. Let us note that if $(x_1, y_1) \in M$, then since $(0, y_1) \in M$, it follows that

$(x_1, 0) \in M$. From this it is readily verified that $M = E_0 \times F$. Further if x, and x_1 are not in E_0 then $f(x) \neq 0$ and $f(x_1) \neq 0$. Thus there is a λ such that $f(x_1) - \lambda f(x) = 0$. Hence $x_1 - \lambda x_0 \in E_0$. Thus E_0 is of codim 1 in E, as desired.

Similarly if for some y in F, $(0, y) \notin M$, it follows that $M = E \times F_0, F_0$ a subspace of F of codim 1. This completes the proof of the theorem.

***Theorem* 7** — *Let E and F be two infinite dimensional quasi isometrically incomparable strictly convex Banach spaces and E_0 a subspace of E of codim 1 (F_0 a subspace of F of codim 1). If T is an isometry on $E \times F$ onto $E_0 \times F$, then T is factorable. A similar result holds if $E_0 \times F$ is replaced by $E \times F_0$.*

PROOF : Let $T : E \times F \longrightarrow E_0 \times F$ be a linear isometry with range T all of $E_0 \times F$. Let $x \in E, \|x\|_E = 1$. Then $T(x, 0)$ is an extreme point of $E_0 \times F$. Thus there is a $x_1 \in E_0$ of unit norm or a $y_1 \in F$ of unit norm such that $T(x, 0)$ is either $(x_1, 0)$ or $(0, y_1)$. * If $T(x, 0) = (0, y_1)$ it is claimed that the range $T| \, E \times \{0\}$ is a subspace of $\{0\} \times F$. If not there are x^1, x_1^1 in E of unit norm such that $T(x^1, 0) = (x_1^1, 0)$. From our choice of x and y_1 it follows that

$$\|x + x^1\|_E = \|(x_1^1, y_1)\| = 2.$$

Hence strict convexity of E implies $x = x^1$. Thus $(0, y_1) = (x_1^1, 0)$ a contradiction since $x_1^1 \neq 0$. ** Thus the range of $T| \, E \times \{0\}$ is a subspace of $\{0\} \times F$. In fact it is all of $\{0\} \times F$. Otherwise since $\{0\} \times F$ is a subspace of the range of T, there is a $y^1 \in F$ of unit norm such that for some $y \in F$ of unit norm, $T(0, y) = (0, y^1)$. This observation together with (*) again imply $y^1 = y_1$, which is proved by using strict convexity of F and proceeding as in the proof of (**). Thus $(x, 0) = (0, y)$ a contradiction. Hence T maps $E \times \{0\}$ onto $\{0\} \times F$. Thus F is isometric with E contradicting E and F are quasi isometrically incomparable.

The above observations prove that the assumption (*) is false i.e. Range $T| \, E \times \{0\}$ is a subspace of $E \times \{0\}$. Hence the hypothesis on the range of T implies range $T| \, E \times \{0\} = E_0 \times \{0\}$.

Further since T is a linear isometry on $E \times F$ onto $E_0 \times F$, and the range $T| \, E \times \{0\} = E_0 \times \{0\}$, it is deduced using Proposition 1, that Range $T| \, \{0\} \times F = \{0\} \times F$.

To complete the proof that T is factorable, let $T_1 : E \longrightarrow E, T_2 : F \longrightarrow F$ be defined by $T_1(x) = x^1$, if $T(x, 0) = (x^1, 0)$, and $T_2(y) = y^1$ if $T(0, y) = (0, y^1)$. Then T_1, T_2 are linear isometries respectively on E onto E_0, and on F onto F. The remarks on the ranges of the restrictions of T above and the definitions of T_1, T_2 imply that $T = T_1 \times T_2$ i.e. $T(x, y) = (T_1 x, T_2 y)$ completing the proof that T is factorable.

If the range of T is $E \times F_0$, arguing as in the preceding case, it is proved that T is factorable.

Theorem 8 — *If E and F are quasi isometrically incomparable strictly convex Banach spaces, then there is no forward shift on $(E \times F, \| \|)$.*

PROOF : If T is a forward shift on $E \times F$, it follows from the definition of a forward shift and Theorem 6, that either there is a subspace E_0 of E of codim 1 such that the range of T is $E_0 \times F$ or else there is a subspace F_0 of F of codim 1, such that the range of T is $E \times F_0$. Assuming the first alternative and applying theorem 7, and adopting notation introduced in the theorem, it follows that the linear isometry T_1 is a linear isometry on E onto E_0, and $T_2 : F \longrightarrow F$ is a surjective linear isometry. Since the range $T_2^n = F$ for all $n \geq 1$, and $T^n = T_1^n \times T_2^n$ it follows that $\bigcap\limits_{n\geq 1} RangeT^n \supset \{0\} \times F$ contradicting that T is a forward shift.

In case range $T = E \times F_0$, a repetition of the above argument leads again to a contradiction, completing the proof of the Theorem.

In passing we note that as stated in the introduction all the results in this section concerning $E \times F$ equipped with the sum norm $\| \|_1$ have verbatim analogues when $\| \|_1$ is replaced by $\| \|_\infty$. In particular Theorem 8 holds if $\| \|_1$ is replaced by $\| \|_\infty$. Thus we have the following theorem stated for convenience of reference.

Theorem 9 — *If E, F are quasi isometrically incomparable strictly convex spaces then the product space $E \times F$ with norm $\| \|_1(\| \|_\infty)$ does not admit a forward shift.*

We conclude the paper answering the problem of existence of backward or forward shifts on the product space $\ell_{p_1} \times \ell_{p_2}, 1 < p_1 \neq p_2 < \infty$ with the sup norm. We note that the spaces $\ell_{p_1}, \ell_{p_2}, 1 < p_1 \neq p_2 < \infty$ are quasi isometrically incomparable since they are isometrically incomparable as noted in section 1. Further ℓ_p spaces are strictly convex if $1 < p < \infty$.

Before proceeding to the final theorem we note that a separable reflexive Banach space E admits a forward shift if and only if the dual space E^* admits a backward shift, [TRS].

Theorem 10 — *If $1 < p_1 \neq p_2 < \infty$ then the Banach spaces $\ell_{p_1} \times \ell_{p_2}$ admit neither a forward nor a backward shift when equipped with $\| \|_1(\| \|_\infty)$.*

PROOF : If follows at once that the space $\ell_{p_1} \times \ell_{p_2}, 1 < p_1 \neq p_2 < \infty$ with $\| \|_1(\| \|_\infty)$ does not admit a forward shift from theorem 9 since these spaces ℓ_{p_1}, ℓ_{p_2} are strictly convex and quasi isometrically incomparable. To prove that the spaces $\ell_{p_1} \times \ell_{p_2}, 1 < p_1 \neq p_2 < \infty$ with $\| \|_1(\| \|_\infty)$ do not admit a backward shift we simply note that these spaces are separable reflexive Banach spaces such that their duals do not admit a forward shift and apply the duality result stated earlier here. This completes the proof.

It is natural to inquire the existence of shifts on $\ell_p \times \ell_p$. We note that if T is a forward (backward) shift on a Banach space E and if T_1 is the operator on $E \times E \longrightarrow E \times E$, defined by $T_1(x, y) =$

(y, Tx), then T_1 is a forward (backward) shift on $E \times E$ with $\| \|_1$ or $\| \|_\infty$. In fact it follows from the above observation that if E and F are isometric Banach spaces, E admitting a forward (backward) shift, then the product space $E \times F$ admits a forward (backward) shift, with $\| \|_1$ or $\| \|_\infty$ as the norm on the space $E \times F$, by simply noting the easily verified fact that isometries preserve the property that a Banach space has a forward (backward) shift. In particular it follows that $\ell_p \times \ell_p$ with $\| \|_1$ or $\| \|_\infty$ admits forward (backward) shifts if $1 \leq p < \infty$.

In conclusion it is noted that similar results are obtained for the products $E \times F$, where $E = \ell_p$, $1 \leq p \leq \infty$ and F is either c_0 or ℓ_1. Since the proofs for these cases are significantly different from the proofs of the results in this paper and further vary from case to case these are discussed in a separate paper.

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