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Construction of m^4 Run Linear Graphs by Finite Geometries

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Introduction

In a fractional factorial design where the main effect of each factor and some two-factor interactions are to be estimated, a graph, which consists of vertices and edges, is usually used to represent the model. Each vertex in the graph denotes the main effect of a factor, and the interaction of two factors is denoted by the edge joining the two factors. Taguchi (1959, 1960) introduced linear graphs, which associate graphs with orthogonal arrays of strength two, for planning this type of experiments. Linear graphs have since been used extensively in the design of industrial experiments. *Designs obtained from linear graphs, which were shown by Dey and Suen (2002) to satisfy the combinatorial conditions given by Dey and*

Mukerjee (1999), *are universally optimal* (*as defined by Kiefer, 1975*) *for estimating the specified main effects and two factor interactions among all designs involving the same number of runs*.

In this paper, we consider symmetrical factorial designs where each factor has m levels. An orthogonal array $\mathrm{OA}(N, k, m, t)$, having N rows, k columns, m symbols, and strength t, is an $N \times k$ array with elements from a set of m symbols in which all the m^t possible combinations of symbols occur equally often as rows in any $N \times t$ subarray. Here we shall use orthogonal arrays of strength two only. A linear graph for an orthogonal array is an assignment of the columns of the array to the vertices and edges of a graph so that different vertices or edges correspond to different columns. Any column of the orthogonal array can be assigned to a vertex of the graph, but the $m - 1$ columns assigned to an edge must be carefully chosen so that the edge represents the interaction of the two vertices on the edge.

Collections of linear graphs can be found in the appendices of Taguchi (1987) and Wu and Chen (1992). Sun and Wu (1994) also gave interaction graphs for three-level designs, but some of their designs allow only partial estimation of certain two-factor interactions. We now briefly describe Taguchi's method for designing this type of experiments. Given the main effects of a set of factors and some two-factor interactions to be estimated, the investigator first draws a graph according to the model. The investigator then searches through the list of linear graphs to see if any graph contains the model as a subgraph. If a linear graph in the list is found to contain the drawn graph then it can be used in the experiment by dropping the unused vertices and edges. A disadvantage of this method, as Wu and Chen (1992) pointed out, is that the total number of linear graphs is usually too large to be included in the collection. Only six types of 16-run linear graphs are given, out of more than 800 types of graphs, in Taguchi's (1987) table. Wu and Chen (1991) had 190 16-run linear graphs in their collection. While it is true that the total numbers of linear graphs are usually too large for large-run plans, we believe that among those "more than 800 types of graphs" for 16-run designs, most of them are subgraphs of others. An attempt to list all nonisomorphic maximal linear graphs for 16-run designs is given in the last section.

In Section 2, the geometric approach to construct linear graphs is described. The concept of *maximal* linear graphs is introduced to reduce the number of nonisomorphic graphs. In Section 3, several series of m^4-run linear graphs are constructed using the geometric properties of $\mathrm{PG}(3, m)$. 81-run linear graphs are given as examples. However, we are not ready to give a complete list of 81-run linear graphs because we think there are too many of them. In Section 4, we give a list of 27 nonisomorphic 16-run maximal linear graphs. We believe that this list is complete although we cannot prove it.

Linear graphs and finite geometries

Since many orthogonal arrays are constructed by using finite fields and finite geometries, we shall use a geometric approach to demonstrate linear graphs. Let m be a prime or a power of a prime, and let $\mathrm{PG}(r-1, m)$ denote the $(r-1)$-dimensional finite projective geometry over $\mathrm{GF}(m)$, the finite field of m elements. A point in $\mathrm{PG}(r-1, m)$ is represented by an r-tuple $(x_0, x_1, \ldots, x_{r-1})$, where $x_0, x_1, \ldots, x_{r-1}$ are elements of $\mathrm{GF}(m)$ which cannot be all 0's. Two r-tuples represent the same point if one is a multiple of the other. Therefore there are $(m^r-1)/(m-1)$ points in $\mathrm{PG}(r-1, m)$. A t-flat in $\mathrm{PG}(r-1, m)$ is a set of points which are linear combinations of $t+1$ independent points. There are $(m^{t+1}-1)/(m-1)$ points in a t-flat, and there are $\frac{(m^r-1)(m^{r-1}-1)\cdots(m^{r-t}-1)}{(m^{t+1}-1)(m^t-1)\cdots(m-1)}$ t-flats in $\mathrm{PG}(r-1, m)$. 0-flats, 1-flats, and 2-flats are also called points, lines, and planes, respectively. Given integers $s, t,\ s \leqslant t$, there are $\frac{(m^{r-s-1}-1)(m^{r-s-2}-1)\cdots(m^{t-s+1}-1)}{(m^{r-t-1}-1)(m^{r-t-2}-1)\cdots(m-1)}$ t-flats passing through an s-flat in $\mathrm{PG}(r-1, m)$. For more properties about finite projective geometries, we refer to Hirschfeld (1979).

Now let A be an $r \times (m^r-1)/(m-1)$ matrix with the $(m^r-1)/(m-1)$ $r \times 1$ column vectors corresponding to all the points of $\mathrm{PG}(r-1, m)$. Then the $m^r \times (m^r-1)/(m-1)$ array, whose row vectors are elements of the row space of A, forms an $\mathrm{OA}(m^r, (m^r-1)/(m-1), m, 2)$. Since the matrix A generates the orthogonal array, the $(m^r-1)/(m-1)$ points of $\mathrm{PG}(r-1, m)$ can be used to represent the columns of this array. Let L be a graph with n vertices and k edges, then L is a linear graph for the orthogonal array $\mathrm{OA}(m^r, (m^r-1)/(m-1), m, 2)$ if the following assignment is possible: (a) assign each vertex a point in $\mathrm{PG}(r-1, m)$ and assign each edge $m-1$ other points on the line joining the two vertices; and (b) the $n+k(m-1)$ points corresponding to the n vertices and k edges are distinct. We also call L an m^r-run linear graph. Clearly, a necessary condition for L to be an m^r-run linear graph is $n+k(m-1) \leqslant (m^r-1)/(m-1)$. L is said to be *saturated* if $n+k(m-1)=(m^r-1)/(m-1)$. If L is not saturated, we can add $(m^r-1)/(m-1)-n-k(m-1)$ isolated vertices to it and assign them to the remaining unassigned points in $\mathrm{PG}(r-1, m)$ to make it saturated.

Since isolated vertices can be added to a linear graph easily, we ignore all isolated vertices when isomorphisms of linear graphs are considered. Let $e(L)$ denote the set of all edges of a linear graph L, and let L_1 and L_2 be two m^r-run linear graphs. L_1 is said to be a subgraph of L_2 ($L_1 \subseteq L_2$) if it is possible to relabel the vertices of L_1 such that $e(L_1) \subseteq e(L_2)$. If L_1 is a subgraph of L_2 but L_2 is not a subgraph of L_1, then L_1 is said to be a *proper* subgraph of L_2. L_1 and L_2 are said to be *isomorphic* if $L_1 \subseteq L_2$ and $L_2 \subseteq L_1$. Clearly, if L_1 is a subgraph of L_2, then L_1 can be obtained from L_2 by deleting some edges.

Definition 2.1. An m^r-run linear graph L is said to be *maximal* if L is not a proper subgraph of any other m^r-run linear graph.

It is only necessary to list nonisomorphic maximal linear graphs for each orthogonal array since all other linear graphs can be obtained from them by deleting some edges. An m^r-run linear graph with n vertices and k edges is maximal if it has no isolated vertices and $(m^r-1)/(m-1)-n-k(m-1) < m-1$. In particular, a saturated linear graph with no isolated vertices is maximal.

Since a model is completely determined by its graph, we are interested in the isomorphism of the graphs instead of the designs in this paper. The approach used by Wu and Chen (1992) and Sun and Wu (1994) is different. They started with nonisormorphic designs, and listed nonisomorphic graphs for each design. As a result, some of their graphs from nonisomorphic designs are isomorphic.

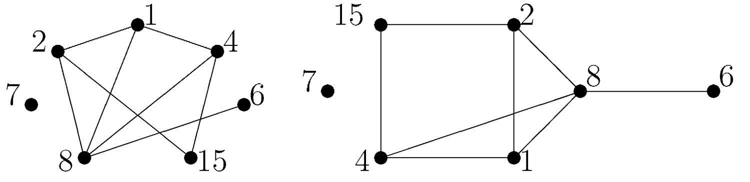
For convenience, we shall use the following notations for graphs.

1. $(F_1, F_2; F_3, F_4; \ldots; F_{2u-1}, F_{2u})_1$: a graph consisting of $2u$ vertices $F_1, \ldots, F_{2u}$ and u edges $F_1F_2, F_3F_4, \ldots, F_{2u-1}F_{2u}$.

2. $(F_1 \ldots, F_u; F_{u+1} \ldots, F_{u+v})_2$: a graph consisting of $u + v$ vertices $F_1, \ldots, F_{u+v}$ and uv edges $F_i F_j$ $(1 \leqslant i \leqslant u, u+1 \leqslant j \leqslant u+v)$.
3. $(F_1, \ldots, F_u)_3$: a graph consisting of u vertices $F_1, \ldots, F_u$ and u edges F_1F_2, F_2F_3, …, $F_{u-1}F_u$, F_uF_1.

A graph may consist of one or more components of the above as subgraphs. There may be several different ways to represent a graph using the above notations. If a graph consists of more than one component, we shall choose the components such that any two components may share the same vertex but not the same edge. In general, we like any two components to share the least number of vertices as possible.

Sometimes it is not easy to see if a graph is a subgraph of another because people draw graphs differently. For example, the left of the following two graphs is a 16-run linear graph taken from Wu and Chen (1992). We cannot easily tell if it is a subgraph of any of the 27 16-run maximal linear graphs in Table 1. But if we draw the graph as the one on the right, it is easily seen to be a subgraph of the graph 8.



A suggestion to help determine the isomorphism of two graphs, as indicated in the above example, is that we should always show the largest polygon when drawing graphs. Most linear graphs in this paper show the largest polygons except when it is difficult to draw that way such as Example 6(a).

m^4-run linear graphs

Dey and Suen (2002) used finite geometries to construct several families of universally optimal designs for estimating certain main effects and two-factor interactions. Their designs are actually Taguchi's linear graphs. However, their constructions are more general and cover only a few specific cases. In this section, we focus on the construction of m^4-run linear graphs only and try to cover as many graphs as possible. Hence some of our linear graphs are special cases of Dey and Suen (2002).

We first consider the m^3-run linear graphs which are associated with the $\mathrm{OA}(m^3, m^2 + m + 1, m, 2)$. As discussed in the previous section, the orthogonal array is represented by the finite projective plane $\mathrm{PG}(2, m)$. The following are the only two nonisomorphic m^3-run maximal linear graphs.

$L_{3.1} = \{(F_1, F_2, F_3)_3\}$, where F_1, F_2, F_3 are three noncollinear points in $\mathrm{PG}(2, m)$.

$L_{3.2} = \{(F_0; F_1, \ldots, F_{m+1})_2\}$, where $F_1, \ldots, F_{m+1}$ are $m + 1$ points on a line and F_0 is a point not on this line.

We next consider the m^4-run linear graphs which are associated with the $\mathrm{OA}(m^4, m^3 + m^2 + m + 1, m, 2)$. In addition to the properties mentioned in Section 2, it is helpful to know that the $m^3 + m^2 + m + 1$ points of $\mathrm{PG}(3, m)$ can be partitioned into $m^2 + 1$ disjoint lines.

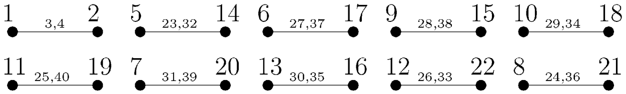
The following maximal linear graphs are constructed by using the geometry of PG$(3, m)$. For each of the graph, we illustrate it with an example of 81-run linear graph, i.e. $m = 3$. Let 0, 1, 2 be the three elements of GF(3). We use the numbers $1, \ldots, 40$ to represent the 40 points of PG(3, 3) as below.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1
0	0	0	0	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2
1	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0

21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
0	0	1	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2	2
2	2	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	2	2	2
1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2

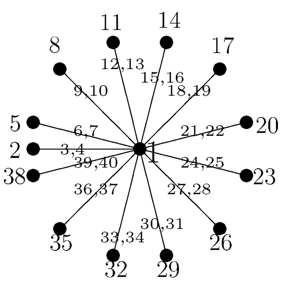
(1) $L_{4.1} = \{(F_1, F_2; F_3, F_4; \ldots; F_{2m^2+1}, F_{2m^2+2})_1\}$. Let $L_1, \ldots, L_{m^2+1}$ be $m^2 + 1$ lines which partition PG$(3, m)$. For $i = 1, \ldots, m^2 + 1$, choose F_{2i-1} and F_{2i} to be two points on the line L_i.

Example 1.



(2) $L_{4.2} = \{(F_0; F_1, \ldots, F_{m^2+m+1})_2\}$, where $F_1, \ldots, F_{m^2+m+1}$ are $m^2 + m + 1$ points on a plane and F_0 is a point not on this plane.

Example 2.



(3) $L_{4.3} = \{(F_0, \ldots, F_m; F'_0, \ldots, F'_m)_2\}$, where $F_0, \ldots, F_m$ are $m + 1$ points on a line L and $F'_0, \ldots, F'_m$ are $m + 1$ points on another line which do not intersect L.

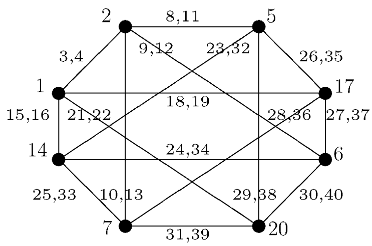
We now construct more graphs from $L_{4.3}$. Let $x^2 - \alpha_1 x - \alpha_0 = 0$ be a primitive polynomial in GF(m^2), and let $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}'_0$, $\mathbf{x}'_1$ be the coordinates of the four independent points F_0, F_1, F'_0, F'_1 in PG$(3, m)$, respectively. For $i \geqslant 2$, define recursively $\mathbf{x}_i \equiv \alpha_0 \mathbf{x}_{i-2} + \alpha_1 \mathbf{x}_{i-1}$ and $\mathbf{x}'_i \equiv \alpha_0 \mathbf{x}'_{i-2} + \alpha_1 \mathbf{x}'_{i-1}$. Let F_i and F'_i be the points with coordinates $\mathbf{x}_i$ and $\mathbf{x}'_i$, respectively. Then it can be shown that $F_i (F'_i)$ and $F_j (F'_j)$ represent the same point if $i \equiv j$(mod

$m+1$). The $(m+1)^2$ two-factor interactions in $L_{4.3}$ can be partitioned into $m+1$ groups $\{F_0F_0', \ldots, F_mF_m'\}$, $\{F_0F_1', F_1F_2', \ldots, F_mF_0'\}, \ldots, \{F_0F_m', F_1F_0', \ldots, F_mF_{m-1}'\}$of $m+1$ two-factor interactions each. Let $\gamma_1, \ldots, \gamma_{m-1}$ be the $m-1$ nonzero elements of $\mathrm{GF}(m)$. For each $j=0, \ldots, m$, the jth group $\{F_0F_j', F_1F_{j+1}', \ldots, F_mF_{m+j}'\}$ are represented by m^2-1 points with coordinates $\mathbf{x}_k + \gamma_i\mathbf{x}_{j+k}'(k=0, \ldots, m; i=1, \ldots, m-1)$. These m^2-1 points can be partitioned into $m-1$ lines $L_{i,j} = \{\mathbf{x}_0+\gamma_i\mathbf{x}_j', \ldots, \mathbf{x}_m+\gamma_i\mathbf{x}_{j+m}'\}$ $(i=1, \ldots, m-1)$. Let $F_{j(m-1)+i,1}$ and $F_{j(m-1)+i,2}$ be any two points on the line $L_{i,j}$, then the jth group of $m+1$ two-factor interactions can be replaced by $2(m-1)$ main effects and $m-1$ two factor interactions $(F_{j(m-1)+1,1}, F_{j(m-1)+1,2}; \ldots; F_{(j+1)(m-1),1}, F_{(j+1)(m-1),2})_1$.

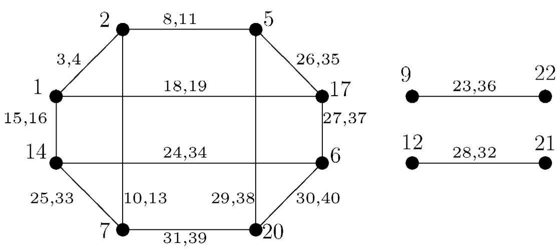
Various linear graphs can be obtained by replacing one or more groups of $m+1$ two-factor interactions by the corresponding $m-1$ pairs of factors. In particular, by replacing the jth group by $m-1$ pairs of factors for each $j=1, \ldots, m-1$, we have the following graph.

$L_{4.3}' = \{(F_0, F_0', \ldots, F_m, F_m')_3, (F_{m,1}, F_{m,2}; \ldots; F_{m^2-m,1}, F_{m^2-m,2})_1\}$.

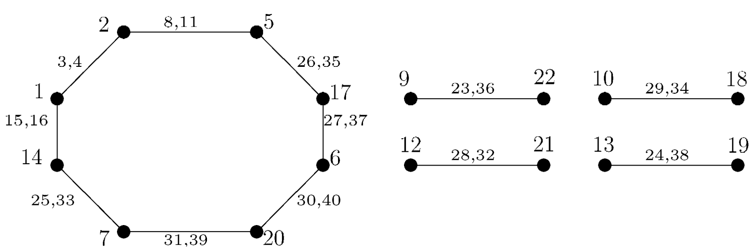
Example 3. (a) By choosing $(F_0, F_1, F_2, F_3, F_0', F_1', F_2', F_3') = (1, 5, 6, 7, 2, 14, 17, 20)$, we have the graph $L_{4.3}$.



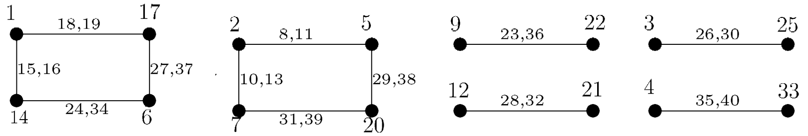
(b) Replacing the two-factor interactions F_0F_2', F_1F_3', F_2F_0', and F_3F_1' in (a) by the graph $(9, 22; 12, 21)_1$, we have the following linear graph.



(c) Replacing the two-factor interactions F_0F_1', F_1F_2', F_2F_3', and F_3F_0' in (b) by the graph $(10, 18; 13, 19)_1$, we have the linear graph $L_{4.3}'$.

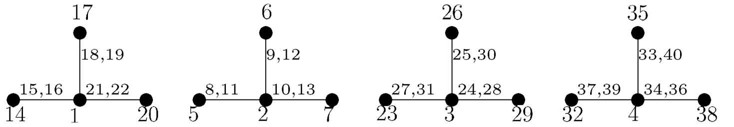


(d) Replacing the two-factor interactions F_0F_0', F_1F_1', F_2F_2', and F_3F_3' in (b) by the graph $(3, 25; 4, 33)_1$, we have the following linear graph.



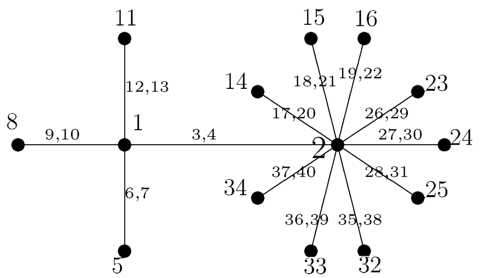
(4) $L_{4.4} = \{(F_{0,1}; F_{1,1}, \ldots, F_{m,1})_2, \ldots, (F_{0,m+1}; F_{1,m+1}, \ldots, F_{m,m+1})_2\}$. Let $F_{0,1}, \ldots, F_{0,m+1}$ be the $m+1$ points on a line L, and let $H_1, \ldots, H_{m+1}$ be the $m+1$ planes through L. For $i = 1, \ldots, m+1$, let L_i be a line on the plane H_i which does not pass through the point $F_{0,i}$. Choose $F_{1,i}, \ldots, F_{m,i}$ to be the m points on the line L_i but not on L.

Example 4. By choosing $(F_{0,1}, F_{0,2}, F_{0,3}, F_{0,4}) = (1, 2, 3, 4)$, we have

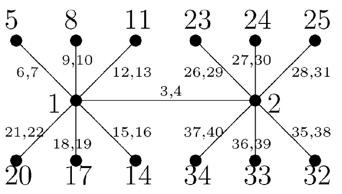


(5) $L_{4.5} = \{(F_{0,1}; F_{0,2}, F_{1,1}, \ldots, F_{um,1})_2, (F_{0,2}; F_{1,2}, \ldots, F_{(m+1-u)m,2})_2\}$, where $1 \leqslant u \leqslant m$. Let $F_{0,1}$ and $F_{0,2}$ be two points on a line L, and let $H_1, \ldots, H_{m+1}$ be the $m+1$ planes through L. For $i = 1, \ldots, u$, let L_i be a line on the plane H_i which does not pass through the point $F_{0,1}$ and choose $F_{(i-1)m+1,1}, \ldots, F_{im,1}$ to be the m points on the line L_i but not on L. For $i = u+1, \ldots, m+1$, let L_i be a line on the plane H_i which does not pass through the point $F_{0,2}$ and choose $F_{(i-u-1)m+1,2}, \ldots, F_{(i-u)m,2}$ to be the m points on the line L_i but not on L.

Example 5. By choosing $(F_{0,1}, F_{0,2}) = (1, 2)$, we have two linear graphs.(a) With $u = 1$ or 3, we have

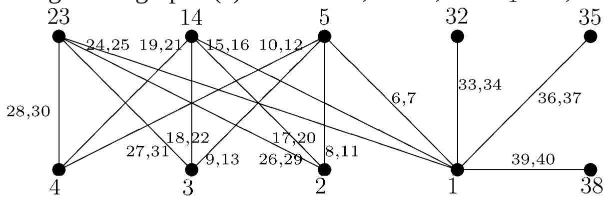


(b) With $u = 2$, we have

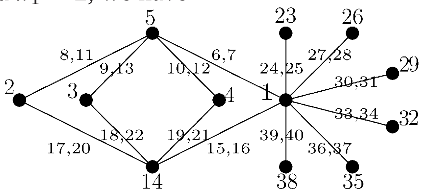


(6) $L_{4.6} = \{(F_1, \ldots, F_t; F_{0,1}, \ldots, F_{0,m+1})_2, (F_{0,1}; F_{1,1}, \ldots, F_{u_1m,1})_2, \ldots, (F_{0,v}; F_{1,v}, \ldots, F_{u_vm,v})_2\}$, where $1 \leqslant t \leqslant m$, $1 \leqslant v \leqslant m+1-t$, $\sum_{i=1}^{v} u_i = m+1-t$, $u_i \geqslant 1$. Let L_1 and L_2 be two disjoint lines, and let $F_{0,1}, \ldots, F_{0,m+1}$ and $F_1, \ldots, F_{m+1}$ be the points on the lines L_1 and L_2, respectively. For $i = 1, \ldots, m+1-t$, let H_i be the plane containing the line L_1 and the point F_{t+i} and let L_i' be a line on the plane H_i which does not pass through the point $F_{0,i}$. For $i = 1, \ldots, v$ and $j = 1, \ldots, u_i$, choose $F_{(j-1)m+1,i}, \ldots, F_{jm,i}$ to be the m points on the line $L'_{u_1+\cdots+u_{i-1}+j}$ but not on L_1.

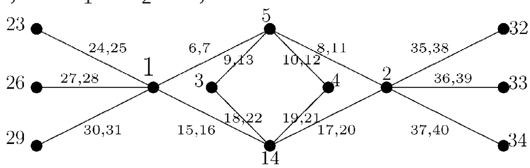
Example 6. By choosing $(F_{0,1}, F_{0,2}, F_{0,3}, F_{0,4}, F_1, F_2, F_3, F_4)$=(1, 2, 3, 4, 5, 14, 23, 32), we have the following linear graphs.(a) With $t = 3$, $v = 1$, and $u_1 = 1$, we have



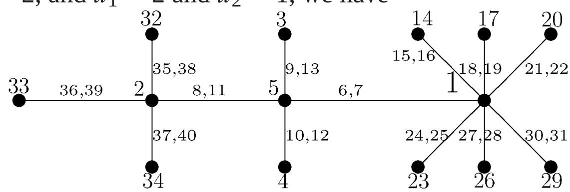
(b) With $t = 2$, $v = 1$, and $u_1 = 2$, we have



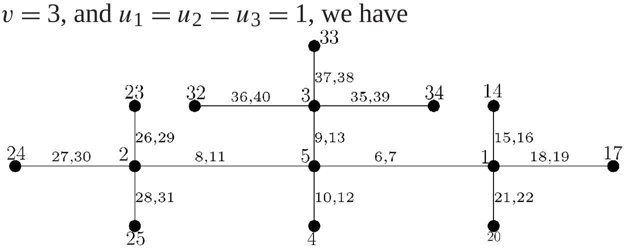
(c) With $t = 2$, $v = 2$, and $u_1 = u_2 = 1$, we have



(d) With $t = 1$, $v = 2$, and $u_1 = 2$ and $u_2 = 1$, we have



(e) With $t = 1$, $v = 3$, and $u_1 = u_2 = u_3 = 1$, we have

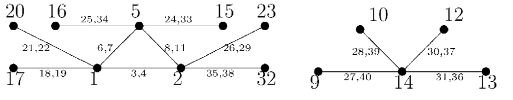


(7) $L_{4.7}=\{(F_{0,0};\, F_{1,0}, \ldots, F_{(m-1)^2,0})_2, (F_{0,1};\, F_{0,2}, F_{1,1}, \ldots, F_{m-1,1})_2, (F_{0,2};\, F_{0,3}, F_{1,2}, \ldots, F_{m-1,2})_2, (F_{0,3};\, F_{0,1}, F_{1,3}, \ldots, F_{m-1,3})_2\}$. Let $F_{0,0}, F_{0,1}, F_{0,2}, F_{0,3}$ be four points which are not on the same plane, and let $L_{i,j}$ be the line through the points $F_{0,i}$ and $F_{0,j}$ for $i, j = 0, 1, 2, 3$, $i \neq j$. Choose $F_{1,0}, \ldots, F_{(m-1)^2,0}$ to be the $(m-1)^2$ points on the plane containing points $F_{0,1}, F_{0,2}, F_{0,3}$ but not on the lines $L_{1,2}, L_{1,3}, L_{2,3}$. Choose $F_{1,1}, \ldots, F_{m-1,1}$ to be the $m-1$ points other than $F_{0,0}$ and $F_{0,2}$ on the line $L_{0,2}$. Choose $F_{1,2}, \ldots, F_{m-1,2}$ to be the $m-1$ points other than $F_{0,0}$ and $F_{0,3}$ on the line $L_{0,3}$. Choose $F_{1,3}, \ldots, F_{m-1,3}$ to be the $m-1$ points other than $F_{0,0}$ and $F_{0,1}$ on the line $L_{0,1}$.

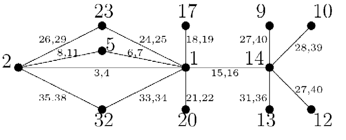
With the same choice of the points in the graph $L_{4.7}$, we also obtain

$$L'_{4.7}=\{(F_{0,0};\, F_{1,0}, \ldots, F_{(m-1)^2,0})_2, (F_{0,1};\, F_{0,0}, F_{0,2}, F_{1,1}, \ldots, F_{m-1,1})_2,$$
$$(F_{0,1}, F_{0,2};\, F_{0,3}, F_{1,2}, \ldots, F_{m-1,2})_2\}.$$

Example 7. By choosing $(F_{0,0}, F_{0,1}, F_{0,2}, F_{0,3}) = (14, 1, 2, 5)$, we have (a) $L_{4.7}$ and (b) $L'_{4.7}$. (a)

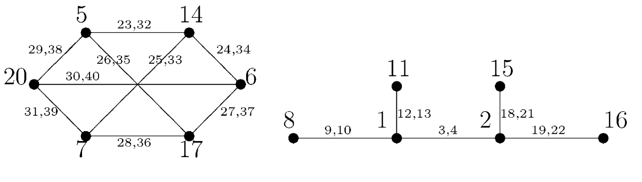


(b)



(8) $L_{4.8} = \{(F_{1,1}, \ldots, F_{m,1};\, F_{1,2}, \ldots, F_{m,2})_2, (F_{m+1,1};\, F_{m+1,2}, F_{1,3}, \ldots, F_{m-1,3})_2, (F_{m+1,2};\, F_{1,4}, \ldots, F_{m-1,4})_2\}$. Let $F_{1,1}, \ldots, F_{m+1,1}$ and $F_{1,2}, \ldots, F_{m+1,2}$ be the points on two disjoint lines L_1 and L_2, respectively. Choose $F_{1,3}, \ldots, F_{m-1,3}$ to be the $m-1$ other points on the line joining the points $F_{1,1}$ and $F_{m+1,2}$, and choose $F_{1,4}, \ldots, F_{m-1,4}$ to be the $m-1$ other points on the line joining the points $F_{1,2}$ and $F_{m+1,1}$.

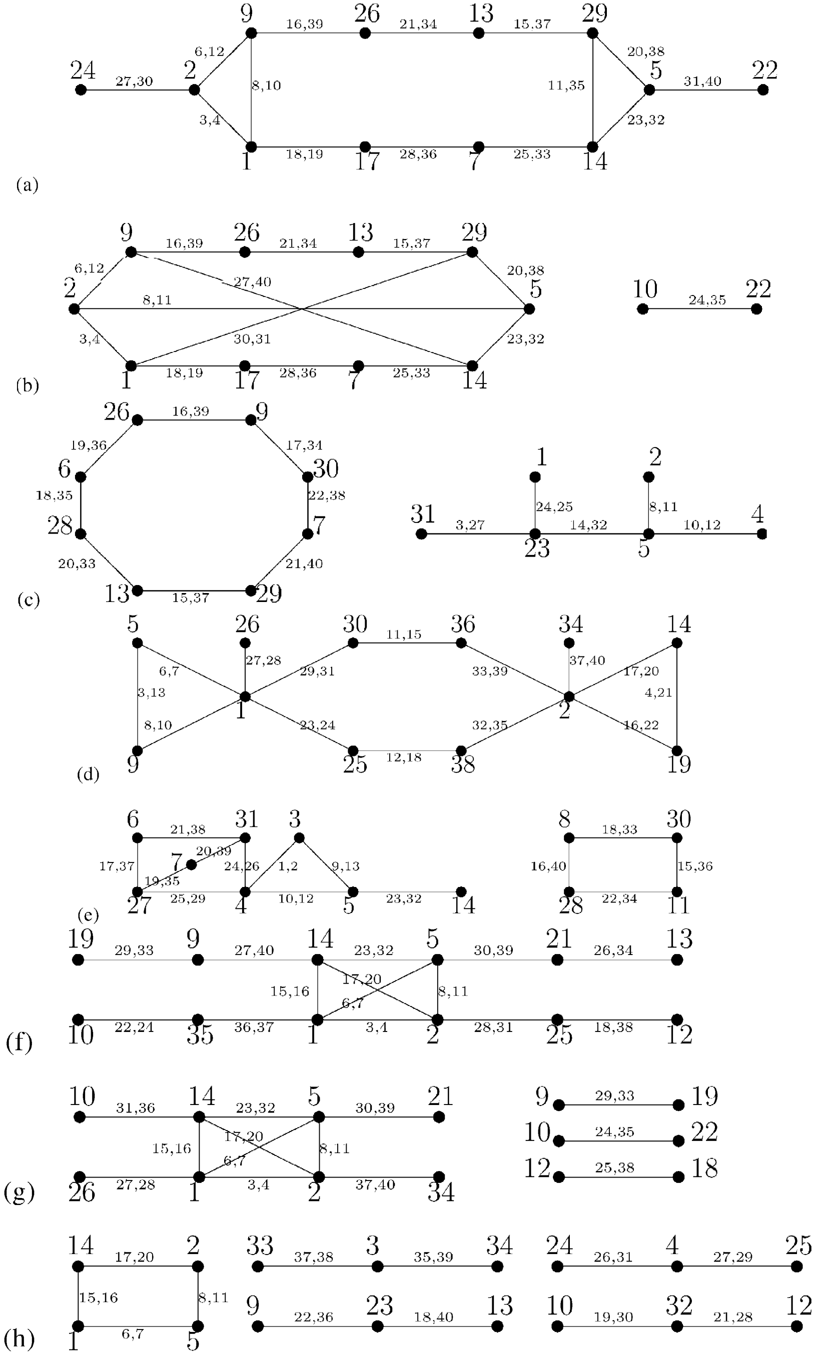
Example 8. By choosing $(F_{1,1}, F_{2,1}, F_{3,1}, F_{4,1})=(5, 6, 7, 1)$ and $(F_{1,2}, F_{2,2}, F_{3,2}, F_{4,2})=(14, 17, 20, 2)$, we have

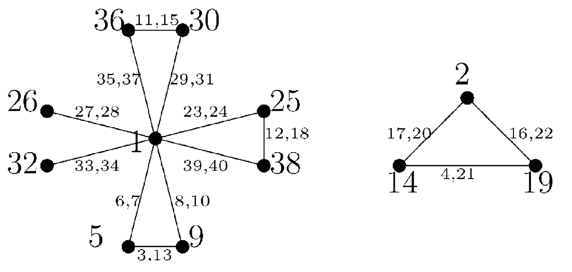


Linear graphs $L_{4.1}$, $L_{4.3}$, $L_{4.4}$, and $L_{4.2}$ are special cases of Theorems 3.3–3.5, and Corollary 3.1 of Dey and Suen (2002), respectively. Graphs derived from $L_{4.3}$ and other series of the above linear graphs are *new*.

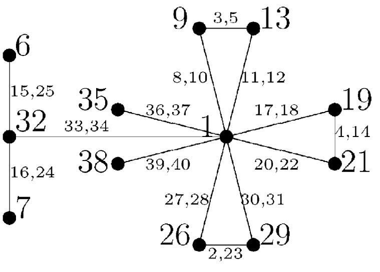
Ten more 81-run maximal linear graphs obtained by trial and error are listed in the following example.

Example 9.


(a)
9 26 13 29
16,39 21,34 15,37
6,12 20,38
24 27,30 2 8,10 11,35 5 31,40 22
3,4 23,32
18,19 28,36 25,33
1 17 7 14
(b)
9 26 13 29
16,39 21,34 15,37
6,12 27,40 20,38
2 8,11 5
3,4 30,31 23,32
18,19 28,36 25,33
1 17 7 14
10 24,35 22
(c)
26 16,39 9
19,36 17,34
6 30
18,35 22,38
28 7
20,33 21,40
13 15,37 29
1 2
24,25 8,11
31 3,27 14,32 10,12 4
23 5
(d)
5 26 30 11,15 36 34 14
27,28 37,40
6,7 29,31 33,39 17,20
3,13 4,21
1 2
8,10 23,24 32,35 16,22
9 25 12,18 38 19
(e)
6 21,38 31 3
7 20,39
17,37 24,26 1,2 9,13
19,35
27 25,29 4 10,12 5 23,32 14
8 18,33 30
16,40 15,36
28 22,34 11
(f)
19 29,33 9 27,40 14 23,32 5 30,39 21 26,34 13
17,20
15,16 8,11
6,7
10 22,24 35 36,37 1 3,4 2 28,31 25 18,38 12
(g)
10 31,36 14 23,32 5 30,39 21
17,20
15,16 8,11
6,7
26 27,28 1 3,4 2 37,40 34
9 29,33 19
10 24,35 22
12 25,38 18
(h)
14 17,20 2
15,16 8,11
1 6,7 5
33 37,38 3 35,39 34
9 22,36 23 18,40 13
24 26,31 4 27,29 25
10 19,30 32 21,28 12



(i)



(j)

Taguchi (1987) listed 14 nonisomorphic 81-run linear graphs, where some of them are not maximal. In fact, his graph 11 is a proper subgraph of his own graph 3. Examples 1, 2, 5(b), and 9(g) are Taguchi's graphs 13, 12, 14, and 3, respectively. Taguchi's graph 8 is a proper subgraph of Example 7. Taguchi's graphs 4 and 7 are proper subgraphs of Example 9(i). Sun and Wu (1994) also listed some 81-run graphs, but their graphs allow partial estimation of two-factor interactions. Most of their graphs are not maximal since their approach is different.

All 81-run maximal linear graphs given in this section are saturated and have no isolated vertices. We believe that there are other such 81-run maximal linear graphs. In addition, we also believe that there are many other 81-run maximal linear graphs with isolated vertices such as Taguchi's graph 1. In our opinion, the number of nonisomorphic 81-run maximal linear graphs is too large to list them all.

16-run linear graphs

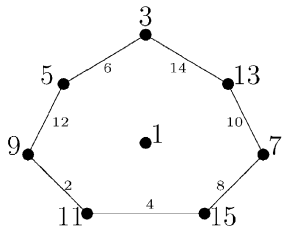
Unlike the 81-run linear graphs, there are not that many nonisomorphic 16-run maximal linear graphs. In this section, we provide a table of 27 nonisomorphic 16-run maximal linear graphs. This table is arranged in the order that the graph with the larger polygon is listed before the one with smaller polygon. The arrangement makes it easier to check if a given graph is a subgraph of one in the table. We believe, though we cannot prove, that there are no more than 27 nonisomorphic 16-run maximal linear graphs. Some graphs in the table are constructed by the methods given in Section 3, others are constructed by trial and error. With $m = 2$ in Section 3, $L_{4.6}$ gives graphs 11 and 22, and $L_{4.1}$, $L_{4.2}$, $L_{4.3}$, $L'_{4.3}$, $L_{4.4}$, $L_{4.5}$, $L_{4.7}$, $L'_{4.7}$, $L_{4.8}$ give graphs 27, 25, 2, 3, 26, 23, 18, 13, 16, respectively. Taguchi's 16-run linear graphs 1, 2, 3, 4, 5, 6 are our graphs

7, 17, 19, 25, 27, 23, respectively. Wu and Chen (1991) listed 190 16-run linear graphs according to their defining relations. We can verify that each one is a subgraph of one of our 27 graphs. Our graphs 3 and 16 are the only two graphs which are not found in Taguchi (1987) or Wu and Chen (1991). Again by letting 0 and 1 be the elements of GF(2), we use the numbers $1, \ldots, 15$ to represent the 15 points in PG(3, 2) as below.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Table 1. 16-run maximal linear graphs

1)



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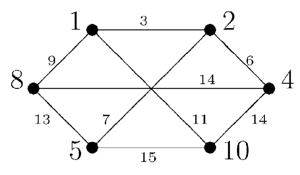
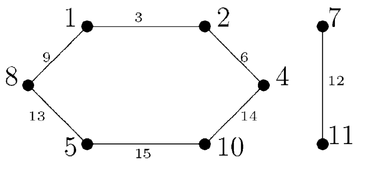


3)



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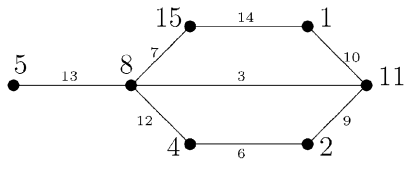
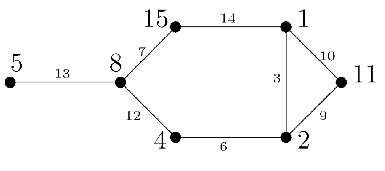
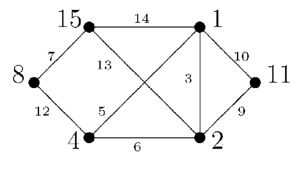


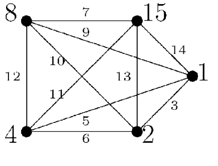
5)



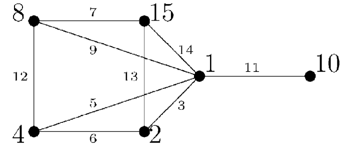
6)



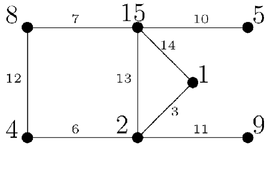
7)



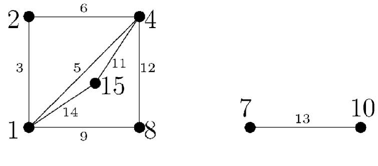
8)



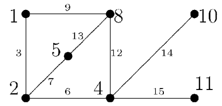
9)



10)



11)



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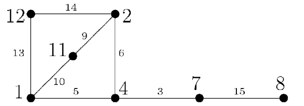
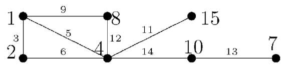


13)



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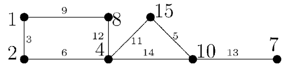

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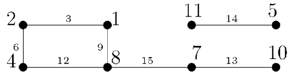
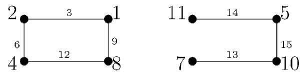
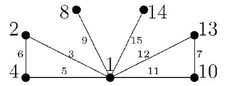


16)



17)



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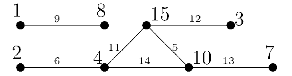

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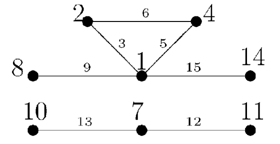

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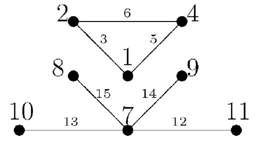

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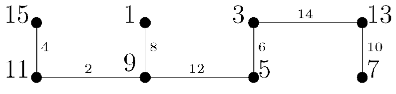

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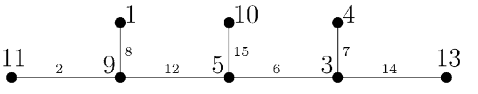

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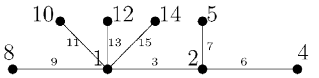

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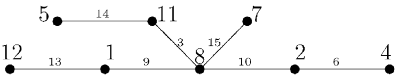

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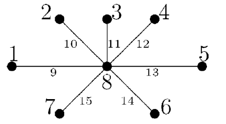

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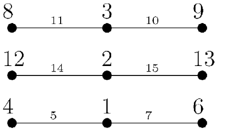

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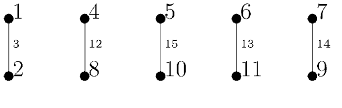

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