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BEZOUT INEQUALITY FOR MIXED VOLUMES

IVAN SOPRUNOV AND ARTEM ZVAVITCH

ABSTRACT. In this paper we consider the following analog of Bezout inequality for mixed volumes:

$$V(P_1, \dots, P_r, \Delta^{n-r})V_n(\Delta)^{r-1} \leq \prod_{i=1}^r V(P_i, \Delta^{n-1}) \quad \text{for } 2 \leq r \leq n.$$

We show that the above inequality is true when Δ is an n -dimensional simplex and P_1, \dots, P_r are convex bodies in \mathbb{R}^n . We conjecture that if the above inequality is true for all convex bodies P_1, \dots, P_r , then Δ must be an n -dimensional simplex. We prove that if the above inequality is true for all convex bodies P_1, \dots, P_r , then Δ must be indecomposable (i.e. cannot be written as the Minkowski sum of two convex bodies which are not homothetic to Δ), which confirms the conjecture when Δ is a simple polytope and in the 2-dimensional case. Finally, we connect the inequality to an inequality on the volume of orthogonal projections of convex bodies as well as prove an isomorphic version of the inequality.

INTRODUCTION

The classical Bezout inequality in algebraic geometry relates the degrees of hypersurfaces to the degree of their intersection. More precisely, let X be a closed algebraic set in an affine space \mathbb{A}^n over the field of complex numbers \mathbb{C} (or an algebraically closed field). Its degree $\deg X$ is defined as the number of intersection points of X with a generic affine subspace of complementary dimension. Given $1 \leq r \leq n$ hypersurfaces X_1, \dots, X_r in \mathbb{A}^n whose intersection has pure codimension r , the Bezout inequality says

$$\deg(X_1 \cap \dots \cap X_r) \leq \deg X_1 \dots \deg X_r.$$

(see, for instance, Proposition 8.4 of [F] and examples therein). The theory of Newton polytopes provides a beautiful interconnection between algebraic geometry and convex geometry. One of its central results is the Bernstein–Kushnirenko–Khovanskii theorem which expresses the number of intersection points of n hypersurfaces in $(\mathbb{C} \setminus \{0\})^n$ with fixed Newton polytopes and generic coefficients as the normalized mixed volume of the Newton polytopes, see [Ber, Ku, Kho]. In particular, if X is a generic hypersurface with Newton polytope P and H is a line given by $n-1$ generic linear forms, the degree of X equals

$$\deg X = \#(X \cap H) = n!V(P, \Delta_n, \dots, \Delta_n),$$

where $\Delta_n = \text{conv}\{0, e_1, \dots, e_n\}$ is the standard n -simplex (the Newton polytope of a generic linear form), and $V(A_1, \dots, A_n)$ denotes the mixed volume of convex bodies A_1, \dots, A_n .

Similarly, the degree of the intersection $X_1 \cap \dots \cap X_r$ is given by

$$\deg(X_1 \cap \dots \cap X_r) = n!V(P_1, \dots, P_r, \Delta_n, \dots, \Delta_n),$$

where P_i is the Newton polytope of X_i . In the mixed volume above, the simplex Δ_n is repeated $n - r$ times. In what follows we abbreviate this as $V(P_1, \dots, P_r, \Delta_n^{n-r})$, using standard notation from the theory of mixed volumes.

Therefore, the Bezout inequality becomes the inequality for mixed volumes:

$$n!V(P_1, \dots, P_r, \Delta_n^{n-r}) \leq \prod_{i=1}^r n!V(P_i, \Delta_n^{n-1}).$$

As the volume of Δ_n is $1/n!$ we can rewrite this as

$$(1.1) \quad V(P_1, \dots, P_r, \Delta_n^{n-r})V_n(\Delta_n)^{r-1} \leq \prod_{i=1}^r V(P_i, \Delta_n^{n-1}).$$

We call this inequality *the Bezout Inequality for mixed volumes*, and the purpose of this paper is to study this inequality and its offspring and applications.

In Section 2 we start by giving a geometric proof of the inequality which works for arbitrary convex bodies P_1, \dots, P_r and arbitrary simplex Δ_n . In addition, we give a necessary and sufficient condition when the inequality becomes equality when the P_i are convex polytopes. It is thus natural to ask if the class of simplices is a unique class of convex bodies for which the above inequality is true when $r \geq 2$. More precisely, fix a convex body D and assume

$$(1.2) \quad V(K_1, \dots, K_r, D^{n-r})V_n(D)^{r-1} \leq \prod_{i=1}^r V(K_i, D^{n-1})$$

holds for all convex bodies K_1, \dots, K_r in \mathbb{R}^n . What can be said about D ?

In Section 3 we prove that D must be indecomposable. In particular, if D is a simple polytope then D must be an n -simplex. Based on this observation, as well as other results in the paper, we propose the following conjecture.

Conjecture 1.1. *Fix integers $2 \leq r \leq n$ and let D be an n -dimensional convex body which satisfies the Bezout inequality (1.2) for all convex bodies K_1, \dots, K_r in \mathbb{R}^n . Then D is an n -simplex.*

The indecomposability result of Theorem 3.3 gives us that the above conjecture is true in the case of $n = 2$ (see Section 3). We also show in Example 3.6, that there are indecomposable bodies D in dimension 3 and higher for which (1.2) is false. In Section 4 we connect inequality (1.2) to the inequalities related to the volume of orthogonal projections of convex bodies. This connection helps us to provide more examples in support of Conjecture 1.1, including that the body D cannot be rotation invariant. In Section 5 we provide an isomorphic version of (1.2), that is we show that

there exists a constant $c_{n,r}$, depending on n and r only, such that

$$(1.3) \quad V(K_1, \dots, K_r, D^{n-r})V_n(D)^{r-1} \leq c_{n,r} \prod_{i=1}^r V(K_i, D^{n-1})$$

is true for all convex bodies K_1, \dots, K_r and D in \mathbb{R}^n . Using the results of Fradelizi, Giannopoulos, Hartzoulaki, Meyer, and Paouris in [FGM, GHP] we show that $c_{n,r} = r^r/r!$ if K_1, \dots, K_r are zonoids and $c_{n,r} \leq n^r r^r/r!$ in the case of general convex bodies. Finally, in Section 6 we give a direct geometric proof of inequality (1.3) in the two dimensional case for the class of general convex bodies with the optimal constant $c_{2,2} = 2$. It was pointed out to us by Christos Saroglou that this 2-dimensional result was also proved by S. Artstein-Avidan, D. Florentin, and Y. Ostrover in [AFO, Proposition 5.1].

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BEZOUT INEQUALITY FOR MIXED VOLUMES

We start by giving basic definitions and setting up notation. As a general reference on the theory of convex sets and mixed volumes we use R. Schneider's book "Convex bodies: the Brunn-Minkowski theory" [Sch1].

For $x, y \in \mathbb{R}^n$, $x \cdot y$ denotes the inner product of x and y . If X and Y are sets in \mathbb{R}^n then $X + Y = \{x + y \mid x \in X, y \in Y\}$ is the *Minkowski sum* of X and Y . A *convex body* is a non-empty convex compact set. For a convex body K the function $h_K(u) = \max\{x \cdot u \mid x \in K\}$ is the *support function* of K . A (*convex*) *polytope* is the convex hull of a finite set of points. An n -dimensional polytope is called *simple* if every its vertex is adjacent to exactly n edges. A polytope which is the Minkowski sum of finitely many line segments is called a *zonotope*. Limits of zonotopes in the Hausdorff metric are called *zonoids*, see [Sch1, Section 3.2] for details. Let $V(K_1, \dots, K_n)$ denote the n -dimensional mixed volume of n convex bodies K_1, \dots, K_n in \mathbb{R}^n . We will also denote by $V_n(K)$ the n -dimensional (Euclidean) volume of K .

In this section we give a geometric proof of inequality (1.1) and describe when the inequality becomes equality. We need the following definition.

Definition 2.1. A collection of m convex sets in \mathbb{R}^n is called *essential* if the Minkowski sum of any k of them has dimension at least k , for $1 \leq k \leq m$.

It is well known that n convex bodies in \mathbb{R}^n form an essential collection if and only if their mixed volume is non-zero. This result goes back to Minkowski's work [Min] (see also [Sch1, Theorem 5.1.7]).

In the next statement and in what follows we use A^u to denote the face of a convex body A defined by the support hyperplane to A with normal vector u .

Proposition 2.2. *Let A_1, \dots, A_n be convex polytopes in \mathbb{R}^n and $K \subseteq A_1$, a convex body. Then $V(K, A_2, \dots, A_n) = V(A_1, A_2, \dots, A_n)$ if and only if K intersects every face A_1^u for u in the set*

$$E = \{u \in \mathbb{S}^{n-1} \mid (A_2^u, \dots, A_n^u) \text{ is essential}\}.$$

Proof. We use the inductive formula for the mixed volume [Sch1, Theorem 5.1.6]. We have

$$V(K, A_2, \dots, A_n) = \frac{1}{n} \int_{u \in \mathbb{S}^{n-1}} h_K(u) dS(A_2, \dots, A_n, u),$$

where $h_K(u)$ is the support function of K and $S(A_2, \dots, A_n, u)$ is the mixed area measure on the unit sphere \mathbb{S}^{n-1} .

Note that the set E is finite. Indeed, the Minkowski sum $A_2^u + \dots + A_n^u$ is $(n-1)$ -dimensional, hence, must be a facet of $A' = A_2 + \dots + A_n$. But each such facet corresponds to a unique $u \in \mathbb{S}^{n-1}$. By the above, we can replace the right hand side with a finite sum

$$V(K, A_2, \dots, A_n) = \frac{1}{n} \sum_{u \in E} h_K(u) V(A_2^u, \dots, A_n^u).$$

Here $V(A_2^u, \dots, A_n^u)$ is the $(n-1)$ -dimensional mixed volume of A_2^u, \dots, A_n^u translated to the orthogonal subspace u^\perp .

Clearly, $h_K(u) \leq h_{A_1}(u)$ for any $u \in \mathbb{S}^{n-1}$, as $K \subseteq A_1$. Since $V(A_2^u, \dots, A_n^u) > 0$ for $u \in E$, the equality $V(K, A_2, \dots, A_n) = V(A_1, A_2, \dots, A_n)$ holds if and only if $h_K(u) = h_{A_1}(u)$ for all $u \in E$. The latter means that $K^u \cap A_1^u$ is non-empty. \square

Corollary 2.3. *Let P be a convex polytope and $K \subseteq P$ a convex body. Then $V(K, P^{n-1}) = V_n(P)$ if and only if K intersects every facet of P .*

Proof. Indeed, the set $E = \{u \in \mathbb{S}^{n-1} \mid (P^u, \dots, P^u) \text{ is essential}\}$ consists of exactly the normal vectors to the facets of P . \square

Remark 2.4. It is a natural question to ask what necessary and sufficient conditions are required for $V(K, A_2, \dots, A_n) = V(A_1, A_2, \dots, A_n)$ to hold when $K \subset A_1$ and A_2, \dots, A_n are arbitrary convex bodies. This is an open question in general. See a detailed discussion in [Sch1, page 277], as well as Conjecture 6.6.13, and Theorem 6.6.16 therein.

Let K be a convex body and Δ an n -simplex in \mathbb{R}^n . We say that K is in *convenient position with respect to Δ* if $K \subset \Delta$ and K has a non-empty intersection with every facet of Δ . In general let $\lambda > 0$ be the largest number such that $\lambda K + v$ is contained in Δ for some $v \in \mathbb{R}^n$. Then it is easy to see that $\lambda K + v$ is in convenient position with respect to Δ .

The next theorem is a generalization of (1.1) to arbitrary convex bodies and an arbitrary simplex.

Theorem 2.5. *Let K_1, \dots, K_r be convex bodies in \mathbb{R}^n , for $1 \leq r \leq n$, and Δ an n -simplex. Then the following inequality holds:*

$$(2.1) \quad V(K_1, \dots, K_r, \Delta^{n-r}) V_n(\Delta)^{r-1} \leq \prod_{i=1}^r V(K_i, \Delta^{n-1}).$$

Proof. The inequality (2.1) is homogeneous in the K_i and is independent of translations of the K_i in \mathbb{R}^n . Therefore, after a possible translation and dilation we may assume that all K_i are contained in Δ and intersect each facet of Δ , i.e. each K_i

is in convenient position with respect to Δ . In this case, by Corollary 2.3, we have $V(K_i, \Delta^{n-1}) = V_n(\Delta)$. Now (2.1) is equivalent to

$$V(K_1, \dots, K_r, \Delta^{n-r}) \leq V_n(\Delta),$$

which is true by the monotonicity of the mixed volume. \square

Next we will give a description of the K_i for which the Bezout inequality (2.1) becomes equality in the case of polytopes.

Theorem 2.6. *Let K_1, \dots, K_r for $2 \leq r \leq n$ be convex polytopes in \mathbb{R}^n which are in convenient position with respect to a simplex Δ . Then equality in the Bezout inequality (2.1) is attained at K_1, \dots, K_r if and only if for any subset $\{K_{i_1}, \dots, K_{i_s}\} \subseteq \{K_1, \dots, K_r\}$ the union $K_{i_1} \cup \dots \cup K_{i_s}$ has a non-empty intersection with every $(n-s)$ -dimensional face of Δ .*

Proof. First, suppose we have equality in (2.1). As in the proof of Theorem 2.5 this is equivalent to

$$(2.2) \quad V(K_1, \dots, K_r, \Delta^{n-r}) = V_n(\Delta).$$

By monotonicity of the mixed volume, it follows that for any subset $\{K_{i_1}, \dots, K_{i_s}\}$ we also have

$$(2.3) \quad V(K_{i_1}, \dots, K_{i_s}, \Delta^{n-s}) = V_n(\Delta).$$

We will show that in this case the union $K_{i_1} \cup \dots \cup K_{i_s}$ intersects every $(n-s)$ -dimensional face of Δ . Suppose not, so there exists a $(n-s)$ -dimensional face Γ of Δ with $(K_{i_1} \cup \dots \cup K_{i_s}) \cap \Gamma = \emptyset$. Let K be the convex hull of $K_{i_1} \cup \dots \cup K_{i_s}$ which is contained in Δ . Since Γ is a face of Δ , it follows that $K \cap \Gamma = \emptyset$, as well. Choose $v \in \mathbb{S}^{n-1}$ such that the hyperplane H orthogonal to v strictly separates K and Γ and $\Gamma = \Delta^v$. Furthermore, let $\bar{\Delta}$ be the ‘‘truncated’’ simplex, i.e. $\bar{\Delta} = \Delta \cap H^+$, where H^+ is the half-space containing K . Then we have

$$(2.4) \quad V(K_{i_1}, \dots, K_{i_s}, \Delta^{n-s}) \leq V(\bar{\Delta}^s, \Delta^{n-s}),$$

by monotonicity. Furthermore, we have $h_{\bar{\Delta}}(u) \leq h_{\Delta}(u)$ for all $u \in \mathbb{S}^{n-1}$ and $h_{\bar{\Delta}}(v) < h_{\Delta}(v)$ by construction. In addition,

$$V((\bar{\Delta}^v)^{s-1}, (\Delta^v)^{n-s}) > 0,$$

since $\dim \bar{\Delta}^v = n-1$ and $\dim \Delta^v = n-s$. Therefore,

$$V(\bar{\Delta}^s, \Delta^{n-s}) = \frac{1}{n} \sum_{u \in \mathbb{S}^{n-1}} h_{\bar{\Delta}}(u) V((\bar{\Delta}^u)^{s-1}, (\Delta^u)^{n-s}) <$$

$$\frac{1}{n} \sum_{u \in \mathbb{S}^{n-1}} h_{\Delta}(u) V((\bar{\Delta}^u)^{s-1}, (\Delta^u)^{n-s}) = V(\bar{\Delta}^{s-1}, \Delta^{n-s+1}) \leq V_n(\Delta),$$

which together with (2.4) contradicts (2.3).

For the other implication, suppose for any subset $\{K_{i_1}, \dots, K_{i_s}\} \subseteq \{K_1, \dots, K_r\}$ the union $K_{i_1} \cup \dots \cup K_{i_s}$ has a non-empty intersection with every $(n-s)$ -dimensional face of Δ . We claim that this implies (2.2). Clearly, this is true for $r=1$ since (2.1) is

trivial for $r = 1$. We use induction on r . By the inductive hypothesis, the equalities (2.3) hold whenever $s < r$. In particular, we have

$$V(K_1, \dots, K_{r-1}, \Delta^{n-r+1}) = V_n(\Delta).$$

Thus, it is enough to show that

$$(2.5) \quad V(K_1, \dots, K_r, \Delta^{n-r}) = V(K_1, \dots, K_{r-1}, \Delta^{n-r+1}),$$

or, equivalently,

$$\sum_{u \in \mathbb{S}^{n-1}} h_{K_r}(u) V(K_1^u, \dots, K_{r-1}^u, (\Delta^u)^{n-r}) = \sum_{u \in \mathbb{S}^{n-1}} h_{\Delta}(u) V(K_1^u, \dots, K_{r-1}^u, (\Delta^u)^{n-r}).$$

Consider $u \in \mathbb{S}^{n-1}$. If $K_r \cap \Delta^u \neq \emptyset$ then $h_{K_r}(u) = h_{\Delta}(u)$. Otherwise, we claim that the mixed volume $V(K_1^u, \dots, K_{r-1}^u, (\Delta^u)^{n-r})$ is zero and the above equality follows. Indeed, if $\dim \Delta^u < n - r$ then the collection $(K_1^u, \dots, K_{r-1}^u, (\Delta^u)^{n-r})$ is not essential. Assume $\dim \Delta^u = n - s$ for some $2 \leq s \leq r$. By the condition of the theorem and since $K_r \cap \Delta^u = \emptyset$, the set $\{K_1, \dots, K_{r-1}\}$ can contain at most $s - 2$ of the K_i that are disjoint from Δ^u . Therefore, there exist $r - s + 1$ of the K_i among $\{K_1, \dots, K_{r-1}\}$ which intersect Δ^u . We may assume they are K_1, \dots, K_{r-s+1} . But then the sum of $n - s + 1$ bodies $K_1^u + \dots + K_{r-s+1}^u + (n - r)\Delta^u$ has dimension $n - s$, which means that the collection $(K_1^u, \dots, K_{r-1}^u, (\Delta^u)^{n-r})$ is not essential. \square

THE CASE OF OTHER POLYTOPES

In this section we study Conjecture 1.1. We start with two classical statements describing the properties of log-concave sequences (see [DP], [W] or [H]).

Lemma 3.1. *Let a_0, \dots, a_n be a sequence of non-negative real numbers. Then*

$$a_i^2 \geq a_{i-1}a_{i+1} \text{ for all } 1 \leq i \leq n - 1$$

if and only if

$$a_i a_j \geq a_{i-1} a_{j+1} \text{ for all } 1 \leq i \leq j \leq n - 1.$$

Proof. The “if” statement is obvious. The “only if” follows by induction on $j - i$. \square

Lemma 3.2. *Let a_0, \dots, a_n be a sequence of non-negative real numbers. Fix $0 \leq m \leq n$ and let*

$$C_i = \sum_{j=0}^m \binom{m}{j} a_{i+j}, \quad 0 \leq i \leq n - m.$$

Then

$$(3.1) \quad a_i^2 \geq a_{i-1}a_{i+1} \text{ for all } 1 \leq i \leq n - 1$$

implies

$$(3.2) \quad C_i^2 \geq C_{i-1}C_{i+1} \text{ for all } 1 \leq i \leq n - m - 1.$$

If, in addition, all the inequalities in (3.2) are equalities then so are (3.1).

Proof. The proof is by induction on m . For $m = 0$ the lines (3.1) and (3.2) are the same. Let $C'_i = C_i + C_{i+1}$ for $0 \leq i \leq n - m - 1$. It is easy to see that

$$C'_i = \sum_{j=0}^{m+1} \binom{m+1}{j} a_{i+j}.$$

For every $1 \leq i \leq n - m - 2$, by the inductive hypothesis and Lemma 3.1, we have

$$(3.3) \quad C_i^2 \geq C_{i-1}C_{i+1}, \quad C_iC_{i+1} \geq C_{i-1}C_{i+2}, \quad C_{i+1}^2 \geq C_iC_{i+2}.$$

This implies

$$(3.4) \quad (C_i + C_{i+1})^2 \geq (C_{i-1} + C_i)(C_{i+1} + C_{i+2}), \text{ i.e. } C_i^2 \geq C'_{i-1}C'_{i+1}$$

and the first statement follows. If the inequality (3.4) is, in fact, equality then so are (3.3) and we obtain the second statement of the lemma. \square

We recall that the convex body $D \subset \mathbb{R}^n$ is called *indecomposable* if a representation $D = A + B$, where A and B are convex bodies, is only possible when A and B are homothetic to D (see [Sch1, Section 3.2]).

Theorem 3.3. *Let D be an n -dimensional convex body. Suppose (1.2) holds for all convex bodies K_1, \dots, K_r in \mathbb{R}^n where $2 \leq r \leq n$. Then D is indecomposable.*

Proof. Suppose $D = A + B$ for some convex bodies A and B . We will show that A is homothetic to B .

First, we set $K_1 = A$, $K_2 = B$, and $K_i = D$ for $3 \leq i \leq r$. Then the inequality (1.2) simplifies to

$$(3.5) \quad V(A, B, D^{n-2})V_n(D) \leq V(A, D^{n-1})V(B, D^{n-1}).$$

Since $D = A + B$ we have

$$V(A, B, D^{n-2}) = \sum_{j=0}^{n-2} \binom{n-2}{j} V(A^{n-1-j}, B^{j+1}).$$

Similarly,

$$V(A, D^{n-1}) = \sum_{j=0}^{n-1} \binom{n-1}{j} V(A^{n-j}, B^j), \quad V(B, D^{n-1}) = \sum_{j=0}^{n-1} \binom{n-1}{i} V(A^{n-1-j}, B^{j+1}),$$

and

$$V_n(D) = \sum_{j=0}^n \binom{n}{j} V(A^{n-j}, B^j).$$

Denoting $a_j = V(A^{n-j}, B^j)$, the inequality (3.5) becomes

$$\left(\sum_{j=0}^{n-2} \binom{n-2}{j} a_{j+1} \right) \left(\sum_{j=0}^n \binom{n}{j} a_j \right) \leq \left(\sum_{j=0}^{n-1} \binom{n-1}{j} a_j \right) \left(\sum_{j=0}^{n-1} \binom{n-1}{j} a_{j+1} \right).$$

Let

$$C_i = \sum_{j=0}^{n-2} \binom{n-2}{j} a_{i+j}, \quad 0 \leq i \leq 2.$$

Then the above inequality states

$$(3.6) \quad C_1(C_0 + 2C_1 + C_2) \leq (C_0 + C_1)(C_1 + C_2).$$

This is equivalent to $C_1^2 \leq C_0C_2$. Since the sequence a_i satisfies the Alexandrov–Fenchel inequalities $a_i^2 \geq a_{i-1}a_{i+1}$, by Lemma 3.2 we must have $C_1^2 \geq C_0C_2$. Therefore, $C_1^2 = C_0C_2$ which implies $a_i^2 = a_{i-1}a_{i+1}$, again by Lemma 3.2.

First, assume that A and B are n -dimensional. Notice that the equations $a_i^2 = a_{i-1}a_{i+1}$ are homogeneous, so after possible rescaling we may assume that $B \subset A$. From the proof of [Sch1, Theorem 6.6.18] it follows that $a_0 = a_1$. Indeed, $a_1^2 = a_0a_2$ is the case of equality in (6.6.52) from [Sch1, page 396] for $K = A$ and $L = B$ which gives $a_0 = a_1$, see the end of the proof on [Sch1, page 372]. Together with $a_i^2 = a_{i-1}a_{i+1}$ this implies that $a_0 = \dots = a_n$. In particular, $a_0 = a_n$, i.e. A and B have the same volume. Therefore, A is homothetic to B .

Now assume $\dim A = k < n$. Then $a_0 = 0$ and we obtain $a_0 = \dots = a_n = 0$. In particular, $\dim B < n$. Let $l = \dim B$. Since D is n -dimensional we have $k + l \geq \dim(A + B) = n$. But then the collection (A^{n-l}, B^l) is essential (the Minkowski sum of any m of them is at least m -dimensional for any $1 \leq m \leq n$), which contradicts $a_l = V(A^{n-l}, B^l) = 0$. \square

Corollary 3.4. *Suppose D is a simple n -dimensional polytope satisfying (1.2) for all bodies K_1, \dots, K_r where $2 \leq r \leq n$. Then D is an n -simplex.*

Proof. This follows from the fact that any simple n -dimensional polytope is decomposable unless it is an n -simplex, see [Gru2, Theorem 15.1.4]. \square

Corollary 3.5. *Suppose D is a 2-dimensional convex body in \mathbb{R}^2 satisfying*

$$V(K_1, K_2)V_2(D) \leq V(K_1, D)V(K_2, D)$$

for all convex bodies $K_1, K_2 \subset \mathbb{R}^2$. Then D is a 2-simplex.

Proof. It is well known that any convex body D in \mathbb{R}^2 is decomposable, unless D is a segment or a 2-simplex, see for example [Sch1, Theorem 3.2.11]. \square

The next example shows the existence of indecomposable polytopes for which (1.2) fails.

Example 3.6. Let $O = \{x \in \mathbb{R}^3 \mid \sum_{i=1}^3 |x_i| \leq 1\}$ be an octahedron in \mathbb{R}^3 . Then O is indecomposable since all its 2-dimensional faces are triangles (see [Sch1, Corollary 3.2.13]). However, one can find K_1, K_2 such that (1.2) fails with $r = 2$ and $D = O$. Indeed, let K_1 be a segment connecting two opposite (non-intersecting) faces of O and K_2 one of those faces. Let O' (resp. K_2') be the projection of O (resp. K_2) onto the plane orthogonal to K_1 . Then by (5.3.23) of [Sch1, page 294] we obtain

$$V(K_1, K_2, O) = \frac{V_1(K_1)}{3}V(K_2', O') \quad \text{and} \quad V(K_1, O, O) = \frac{V_1(K_1)}{3}V_2(O').$$

Since K_2' intersects all facets (sides) of O' we have $V(K_2', O') = V_2(O')$, by Corollary 2.3. On the other hand, $V(K_2, O, O) < V_3(O)$ since K_2 does not intersect all facets of O , again by Corollary 2.3. Therefore, we obtain

$$V(K_1, K_2, O)V_3(O) > V(K_1, O, O)V(K_2, O, O).$$

CONNECTIONS TO PROJECTIONS

In this section we relate Conjecture 1.1 to inequalities involving orthogonal projections. In particular, we see how (1.2) may fail when we choose K_1 and K_2 to be segments and $K_i = D$ for $3 \leq i \leq r$.

Let K_1 and K_2 be unit segments and $\xi, \nu \in \mathbb{S}^{n-1}$ the corresponding direction vectors (chosen up to a sign). Then, by (5.3.23) of [Sch1, page 294],

$$V(K_1, D^{n-1}) = \frac{1}{n} V_{n-1}(D|\xi^\perp) \text{ and } V(K_2, D^{n-1}) = \frac{1}{n} V_{n-1}(D|\nu^\perp),$$

where $D|\xi^\perp$ denotes the orthogonal projection of D onto the hyperplane orthogonal to ξ . Let $U = K_1 + K_2$. Similarly, for the orthogonal projection $D|(\xi, \nu)^\perp$ of D onto the $(n-2)$ -dimensional subspace $(\xi, \nu)^\perp$ we have

$$V_{n-2}(D|(\xi, \nu)^\perp) V_2(U) = \binom{n}{2} V(U, U, D^{n-2}).$$

By the linearity of the mixed volume we get $V(U, U, D^{n-2}) = 2V(K_1, K_2, D^{n-2})$, and

$$V_{n-2}(D|(\xi, \nu)^\perp) V_2(U) = n(n-1) V(K_1, K_2, D^{n-2}).$$

Substituting the above calculations in (1.2) we obtain

$$(4.1) \quad \frac{n}{n-1} V_{n-2}(D|(\xi, \nu)^\perp) V_2(U) V_n(D) \leq V_{n-1}(D|\xi^\perp) V_{n-1}(D|\nu^\perp).$$

Equation (4.1) turns out to be quite useful to check the particular cases of Conjecture 1.1. Direct calculations and the fact that the inequality (1.2) is invariant under linear transformations shows that (4.1) (and thus (1.2)) are false when D is an n -dimensional ellipsoid or parallelepiped (also both of those cases follow from Theorem 3.3 as both are decomposable).

We may also generalize Example 3.6, using (4.1). Indeed, consider the n -dimensional octahedron $O_n = \{x \in \mathbb{R}^n \mid \sum_{i=1}^n |x_i| \leq 1\}$. Then $V_n(O_n) = 2^n/n!$. Choose

$$\xi = (1/\sqrt{2}, 1/\sqrt{2}, 0, \dots, 0) \text{ and } \nu = (1/\sqrt{2}, -1/\sqrt{2}, 0, \dots, 0).$$

Then

$$V_{n-1}(O_n|\xi^\perp) = V_{n-1}(O_n|\nu^\perp) = \frac{1}{\sqrt{2}} V_{n-1}(O_{n-1}) = \frac{1}{\sqrt{2}} \frac{2^{n-1}}{(n-1)!}.$$

Also notice that $V_{n-2}(O_n|(\xi, \nu)^\perp) = V_{n-2}(O_{n-2}) = 2^{n-2}/(n-2)!$. Thus inequality (4.1) is false when $D = O_n$.

We next show that (4.1) is false if D is a rotation invariant body, which gives another evidence of Conjecture 1.1.

Proposition 4.1. *Consider a concave function $f : [a, b] \rightarrow \mathbb{R}^+$ and define convex body $K_f = \{x \in \mathbb{R}^n \mid \sum_{i=2}^n x_i^2 \leq f^2(x_1)\}$. Choose $\xi = (0, 1, 0, \dots, 0)$ and $\nu = (0, 0, 1, 0, \dots, 0)$. Then*

$$\frac{n}{n-1} V_{n-2}(K_f|(\xi, \nu)^\perp) V_n(K_f) > V_{n-1}(K_f|\xi^\perp) V_{n-1}(K_f|\nu^\perp).$$

Proof. Using the Fubini theorem, we get

$$V_n(K_f) = \int_a^b f(t)^{n-1} V_{n-1}(B^{n-1}) dt = \kappa_{n-1} \int_a^b f(t)^{n-1} dt,$$

where $B^n = \{x \in \mathbb{R}^n \mid \sum x_i^2 \leq 1\}$ is the Euclidean unit ball and $\kappa_n = V_n(B^n)$. Moreover,

$$V_{n-1}(K_f|\xi^\perp) = V_{n-1}(K_f|\nu^\perp) = \kappa_{n-2} \int_a^b f(t)^{n-2} dt,$$

and

$$V_{n-2}(K_f|(\nu, \xi)^\perp) = \kappa_{n-3} \int_a^b f(t)^{n-3} dt.$$

Thus to prove the proposition we need to show that

$$\left(\kappa_{n-2} \int_a^b f(t)^{n-2} dt \right)^2 < \frac{n}{n-1} \left(\kappa_{n-3} \int_a^b f(t)^{n-3} dt \right) \left(\kappa_{n-1} \int_a^b f(t)^{n-1} dt \right).$$

First, by Hölder's inequality, we get

$$\left(\int_a^b f(t)^{n-2} dt \right)^2 \leq \left(\int_a^b f(t)^{n-3} dt \right) \left(\int_a^b f(t)^{n-1} dt \right).$$

Next, we will show that

$$\kappa_{n-2}^2 < \frac{n}{n-1} \kappa_{n-3} \kappa_{n-1}.$$

Indeed, using $\kappa_k = \pi^{k/2} / \Gamma(k/2 + 1)$ (see [MS], [Sch1]) we get that the above inequality is equivalent to

$$\Gamma^2(n/2) > \frac{n-1}{n} \Gamma(n/2 - 1/2) \Gamma(n/2 + 1/2),$$

which follows immediately from the classical inequality for the Gamma function:

$$\sqrt{x + \frac{1}{2}} > \frac{\Gamma(x+1)}{\Gamma(x+1/2)}.$$

□

ISOMORPHIC VERSION

As we have already seen, (1.2) may not hold for a general convex body D . However, we can relax the inequality by introducing a constant $c_{n,r} > 0$ depending on the dimension n and the number r only:

$$(5.1) \quad V(K_1, \dots, K_r, D^{n-r}) V_n(D)^{r-1} \leq c_{n,r} \prod_{i=1}^r V(K_i, D^{n-1}).$$

It is an interesting question to find the minimal such constant $c_{n,r}$. In this section we give an upper bound for $c_{n,r}$ in the general and in the symmetric cases (see Theorem 5.7). In addition, we prove that $c_{n,r} = r^r / r!$ in the special case when $K_1, \dots, K_r \subset \mathbb{R}^n$ are zonoids and $D \subset \mathbb{R}^n$ is an arbitrary convex body (see Theorem 5.6).

We start with a rough estimate for the constant $c_{n,r}$.

Proposition 5.1. *There exists a constant $c_{n,r} < (n+2)^{n-r} n^{n(r-1)}$ such that*

$$V(K_1, \dots, K_r, D^{n-r}) V_n(D)^{r-1} \leq c_{n,r} \prod_{i=1}^r V(K_i, D^{n-1}).$$

is true for all convex bodies K_1, \dots, K_r , and D in \mathbb{R}^n .

Proof. It is enough to consider the case when D is n -dimensional. We use a standard idea of approximating D by a simplex of maximal volume contained in D (see [Gru1], [L1], [L2] and [Sch2]). Let Δ be a simplex of maximal volume inscribed in D . Using the invariance of (5.1) under affine transformation we may assume that Δ is a regular simplex with the barycenter at the origin. For every vertex $v \in \Delta$ choose a hyperplane H such that $v \in H$ and H is parallel to the facet of Δ opposite to v . Then H supports D , otherwise we would get a contradiction with the maximality of Δ . This shows that $D \subset -n\Delta$. (Note that if D is a symmetric convex body then we immediately get $D \subset n\Delta$, which would slightly improve our estimate on the constant $c_{n,r}$.) Also by a result of Lassak [L1] $D \subseteq (n+2)\Delta$. Therefore we obtain

$$\Delta \subset D \subseteq ((n+2)\Delta) \cap (-n\Delta).$$

From Theorem 2.5 we get that

$$V(K_1, \dots, K_r, \Delta^{n-r}) V_n(\Delta)^{r-1} \leq \prod_{i=1}^r V(K_i, \Delta^{n-1}),$$

and thus

$$(n+2)^{-(n-r)} V(K_1, \dots, K_r, D^{n-r}) n^{n(1-r)} V_n(D)^{r-1} \leq \prod_{i=1}^r V(K_i, D^{n-1}).$$

□

One can improve the above estimate for $c_{n,r}$ by computing the volume of $((n+2)\Delta) \cap (-n\Delta)$. This would show $c_{n,r} < (cn)^{r(n-1)}$ for some absolute constant $c < 1$, which we still feel is not the optimal bound. Instead, we are going to first prove a better estimate in the case of K_i being zonoids and then apply John's Theorem ([J], see also [MS]).

First recall the inequality for projections in [GHP, Lemma 4.1] (see also [FGM]):

$$\frac{n}{2(n-1)} V_{n-2}(D|(\xi, \nu)^\perp) V_n(D) \leq V_{n-1}(D|\xi^\perp) V_{n-1}(D|\nu^\perp)$$

for any convex body $D \subset \mathbb{R}^n$. Thus using (5.3.23) of [Sch1, page 294] we see that

$$(5.2) \quad V(K_1, K_2, D^{n-2}) V_n(D) \leq 2V(K_1, D^{n-1}) V(K_2, D^{n-1})$$

is true for any convex body D and two orthogonal segments K_1, K_2 . Second, the fact that the above inequality is invariant under linear transformations implies that the orthogonality assumption is not necessary. Finally, by the additivity of the mixed volume with respect to the Minkowski sum and continuity with respect to the Hausdorff metric, (5.2) generalizes to arbitrary zonoids K_1, K_2 . We have obtained the following lemma.

Lemma 5.2. For any zonoids $K_1, K_2 \subset \mathbb{R}^n$ and any convex body $D \subset \mathbb{R}^n$ we have

$$V(K_1, K_2, D^{n-2})V_n(D) \leq 2V(K_1, D^{n-1})V(K_2, D^{n-1}).$$

It is well known that any symmetric convex body D in \mathbb{R}^2 is a zonoid (see [Sch1, Theorem 3.2.11]). This implies the following.

Corollary 5.3. Suppose K_1, K_2 and D are convex bodies in \mathbb{R}^2 and K_1, K_2 are symmetric. Then

$$V(K_1, K_2)V_2(D) \leq 2V(K_1, D)V(K_2, D).$$

The next fact is an analog of [GHP, Lemma 4.1].

Lemma 5.4. Consider a convex body $D \subset \mathbb{R}^n$. Let $P_i(D)$ be the orthogonal projection of D onto e_i^\perp and $P_{[r]}(D)$ be the orthogonal projection of D onto $(e_1, e_2, \dots, e_r)^\perp$. Then

$$\left(\frac{n}{r}\right)^r \binom{n}{r}^{-1} V_{n-r}(P_{[r]}(D))V_n(D)^{r-1} \leq \prod_{i=1}^r V_{n-1}(P_i(D)).$$

Proof. Consider $x \in \mathbb{R}^n$ and write $x = (x_1, \dots, x_r, y)$ where $y \in \mathbb{R}^{n-r}$. For every $y \in P_{[r]}(D)$ define

$$D_i(y) = \{(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r) \mid (x_1, \dots, x_{i-1}, 0, x_{i+1}, \dots, x_r, y) \in P_i(D)\}$$

and

$$D_{[r]}(y) = \{(x_1, \dots, x_r) \in \mathbb{R}^r \mid (x_1, \dots, x_r, y) \in D\}.$$

Note that $D_i(y)$ is the orthogonal projection of $D_{[r]}(y)$ onto the coordinate plane e_i^\perp in \mathbb{R}^r and thus by the Loomis-Whitney inequality [LW]

$$V_r(D_{[r]}(y))^{r-1} \leq \prod_{i=1}^r V_{r-1}(D_i(y)).$$

Next

$$\begin{aligned} \prod_{i=1}^r V_{n-1}(P_i(D)) &= \prod_{i=1}^r \int_{P_{[r]}(D)} V_{r-1}(D_i(y)) dy \geq \\ &\left[\int_{P_{[r]}(D)} \prod_{i=1}^r V_{r-1}^{\frac{1}{r}}(D_i(y)) dy \right]^r \geq \left[\int_{P_{[r]}(D)} V_r(D_{[r]}(y))^{\frac{r-1}{r}} dy \right]^r. \end{aligned}$$

By the Brunn-Minkowski inequality the function $\phi(y) = V_r(D_{[r]}(y))^{\frac{1}{r}}$ is concave on $P_{[r]}(D)$. We apply Berwald's Lemma ([B], see also [GHP, Lemma 4.2]) with

$$\phi(y) = V_r(D_{[r]}(y))^{\frac{1}{r}},$$

$$A = P_{[r]}(D); \quad p = r - 1, \quad q = r, \quad \text{and} \quad k = n - r.$$

We get

$$\left[\binom{n-1}{n-r} \frac{1}{V_{n-r}(P_{[r]}(D))} \int_{P_{[r]}(D)} |\phi(y)|^{r-1} \right]^{\frac{1}{r-1}} \geq \left[\binom{n}{n-r} \frac{1}{V_{n-r}(P_{[r]}(D))} \int_{P_{[r]}(D)} |\phi(y)|^r \right]^{\frac{1}{r}}$$

which is equivalent to

$$\left[\int_{P_{[r]}(D)} |\phi(y)|^{r-1} \right]^r \geq V_{n-r}(P_{[r]}(D)) \frac{\binom{n}{n-r}^{r-1}}{\binom{n-1}{n-r}^r} \left[\int_{P_{[r]}(D)} |\phi(y)|^r \right]^{r-1},$$

i.e.

$$\left[\int_{P_{[r]}(D)} V_r(D_{[r]}(y))^{\frac{r-1}{r}} \right]^r \geq V_{n-r}(P_{[r]}(D)) \frac{\binom{n}{n-r}^{r-1}}{\binom{n-1}{n-r}^r} \left[\int_{P_{[r]}(D)} V_r(D_{[r]}(y)) \right]^{r-1},$$

and the statement of the lemma follows. \square

Remark 5.5. We note that the estimate in Lemma 5.4 is the best possible. Indeed, let O^k be the octahedron as in Section 4 and $Q^k = \{x \in \mathbb{R}^k \mid |x_i| \leq 1\}$ a cube in \mathbb{R}^k . We define the convex body D as the convex hull of the union $(Q^r \times \{0\}) \cup (\{0\} \times O^{n-r}) \subset \mathbb{R}^r \times \mathbb{R}^{n-r} = \mathbb{R}^n$. Then

$$V_n(D) = 2^n \frac{r!}{n!}.$$

Furthermore, $P_{[r]}(D) = O^{n-r}$ and if $1 \leq i \leq r$ then $P_i(D)$ is the convex hull of the union of an $(r-1)$ -dimensional cube and the octahedron $\{0\} \times O^{n-r}$. Thus

$$V_{n-1}(P_i(D)) = 2^{n-1} \frac{(r-1)!}{(n-1)!} \quad \text{and} \quad V_{n-r}(P_{[r]}(D)) = \frac{2^{n-r}}{(n-r)!}.$$

Therefore D gives equality in the statement of Lemma 5.4.

Now let us restate Lemma 5.4 in the language of mixed volumes. Let K_1, \dots, K_r be pairwise orthogonal unit segments. Then again by (5.3.23) in [Sch1, page 294],

$$V(K_i, D^{(n-1)}) = \frac{1}{n} V_{n-1}(D|K_i^\perp).$$

Moreover, if $U = \sum_{i=1}^r K_i$ then

$$\binom{n}{r} V(D^{n-r}, U^r) = V_r(U) V_{n-r}(D|(K_1, \dots, K_r)^\perp).$$

Thus

$$V_{n-r}(D|(K_1, \dots, K_r)^\perp) = \binom{n}{r} r! V(K_1, \dots, K_r, D^{n-r}),$$

and Lemma 5.4 implies

$$\frac{r!}{r^r} V(K_1, \dots, K_r, D^{n-r}) V_n(D)^{r-1} \leq \prod_{i=1}^r V(K_i, D^{n-1}).$$

Since the above formula is invariant under a linear transformation the orthogonally assumption on K_1, \dots, K_r is not necessary. Again, by the additivity and continuity of the mixed volume, and taking into account the example in Remark 5.5 we obtain the following result.

Theorem 5.6. *Suppose D is a convex body in \mathbb{R}^n and K_1, \dots, K_r are zonoids then*

$$V(K_1, \dots, K_r, D^{n-r})V_n(D)^{r-1} \leq \frac{r^r}{r!} \prod_{i=1}^r V(K_i, D^{n-1}),$$

and the inequality is sharp.

Now we may apply John's theorem to improve the bound for $c_{n,r}$ in Proposition 5.1. Indeed, for any convex body K_i there exists an ellipsoid E_i such that $E_i \subset K_i \subset nE_i$. Moreover, $E_i \subset K_i \subset \sqrt{n}E_i$ when K_i is symmetric. It is also well known that an ellipsoid is a particular case of a zonoid. This implies the following theorem.

Theorem 5.7. *There exists a constant $c_{n,r} \leq n^r r^r / r!$ such that*

$$V(K_1, \dots, K_r, D^{n-r})V_n(D)^{r-1} \leq c_{n,r} \prod_{i=1}^r V(K_i, D^{n-1})$$

holds for all convex bodies K_1, \dots, K_r and D in \mathbb{R}^n . Moreover $c_{n,r} \leq n^{r/2} r^r / r!$ when K_1, \dots, K_r are symmetric with respect to the origin.

ISOMORPHIC BEZOUT INEQUALITY IN DIMENSION 2

Applying Theorem 5.7 in the case of $n = r = 2$ and K_1, K_2 being symmetric we obtain $c_{2,2} \leq 2$ (see also Corollary 5.3). It turns out that $c_{2,2} = 2$ always, even in the non-symmetric case.

Theorem 6.1. *Let K, L and D be any convex bodies in \mathbb{R}^2 . Then*

$$(6.1) \quad V(K, L)V_2(D) \leq 2V(K, D)V(L, D).$$

Moreover, the equality is attained when K and L are line segments and $D = \lambda_1 K + \lambda_2 L$ for some $\lambda_i \geq 0$.

Proof. The second statement follows immediately since $V_2(D) = 2\lambda_1\lambda_2V(K, L)$ and $V(K, D) = \lambda_1V(K, L)$, $V(L, D) = \lambda_2V(K, L)$, by the additivity of the mixed volume.

To show the inequality (6.1), first note that it is enough to consider the case of convex polygons K, L as they approximate convex 2-dimensional bodies. Recall that any convex polygon is the Minkowski sum of triangles and line segments (see [YB, p. 177]). Let

$$K = \sum_{i=1}^k K_i \quad L = \sum_{i=1}^l L_i$$

be the corresponding Minkowski decompositions into triangles and line segments. Then, by additivity, (6.1) can be written as

$$\sum_{i=1}^k \sum_{j=1}^l V(K_i, L_j)V_2(D) \leq \sum_{i=1}^k \sum_{j=1}^l 2V(K_i, D)V(L_j, D).$$

This reduces the proof to the case of two triangles K and L . (The case when one or both of K, L is a line segment follows by contracting one of the sides of a triangle to a point.) Furthermore, (6.1) is invariant under translations and dilations of each K, L , and D . Thus, after a possible dilation and translation of K and L we may

assume that D is inscribed in both K and L , in which case $V(K, D) = V_2(K)$ and $V(L, D) = V_2(L)$ by Corollary 2.3. Therefore, we may assume that $D = K \cap L$. The statement now follows from Theorem 6.2 below. \square

In the statement and the proof of the next theorem we use $|K|$ to denote the area (2-dimensional volume) of a triangle K and $|\overline{AB}|$ the length of a line segment \overline{AB} .

Theorem 6.2. *Let K, L be two triangles in the plane whose intersection is inscribed in each of them. Then*

$$(6.2) \quad V(K, L)|K \cap L| \leq 2|K||L|.$$

Proof. Let $c > 0$ be the smallest number such that K is contained in $L' = cL + v$ for some $v \in \mathbb{R}^2$. We distinguish two cases: either the normals to the sides of K and L are distinct and alternate (as points on \mathbb{S}^1) or not. Notice that in the latter case K and L' have a common vertex, and in the former the three vertices of K lie on three different sides of L' . Note that (6.2) is invariant under affine transformations applied to both K and L . Therefore we may assume that L' equals the standard 2-simplex Δ .

First, suppose K and L' share a vertex, which we may assume to be $(1, 0)$. Then $K = \triangle ABC$ with $C = (1, 0)$, and $L = \triangle A'B'C'$ is a translate of $\mu\Delta$ where $\mu = 1/c > 0$ (see Figure 1).

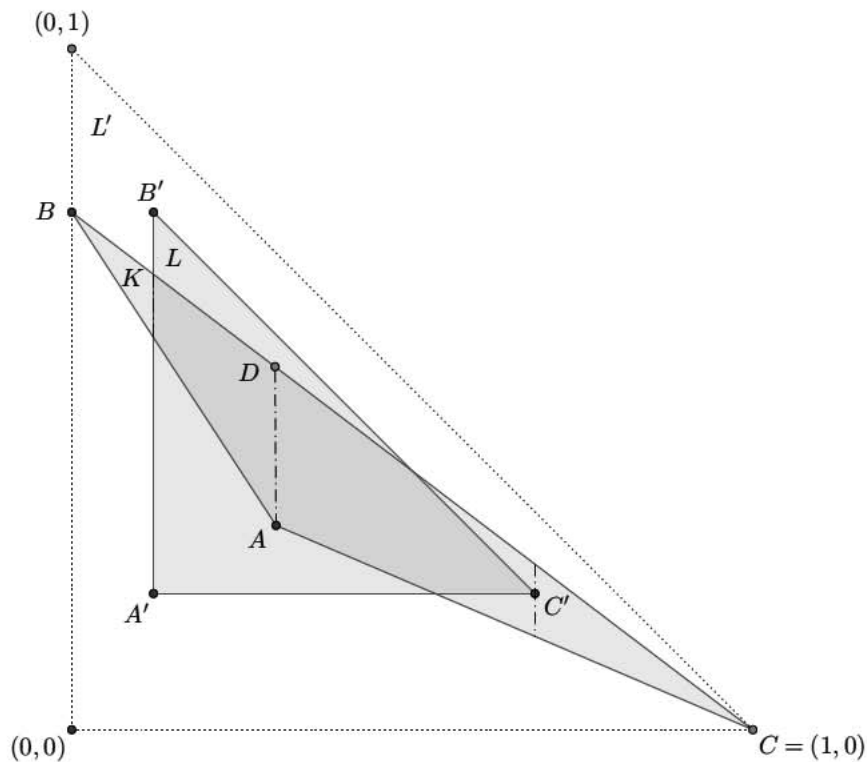


FIGURE 1. Case 1.

Note that K is inscribed in Δ , so by Corollary 2.3,

$$V(K, L) = V(K, \mu\Delta) = \mu V(K, \Delta) = \mu|\Delta| = \mu/2.$$

Since $|L| = \mu^2/2$ the inequality (6.2) becomes

$$|K \cap L| \leq 2\mu|K|.$$

Consider the vertical segment \overline{AD} subdividing K into two triangles as in Figure 1. Note that its length $|\overline{AD}|$ equals $2|K|$ divided by the horizontal width of K , which by our choice of coordinates equals 1, i.e. $|\overline{AD}| = 2|K|$. Next, notice that $K \cap L$ is contained in the vertical strip of width μ and its longest vertical section is at most $|\overline{AD}|$. Therefore,

$$|K \cap L| \leq \mu|\overline{AD}| = 2\mu|K|,$$

and this case is proved.

For the other case, K has vertices $A = (\alpha, 0)$, $B = (0, \beta)$, $C = (\gamma, 1 - \gamma)$ for some $0 < \alpha, \beta, \gamma < 1$, and $L = \Delta A'B'C'$ is a translate of $\mu\Delta$ (see Figure 2). As in the

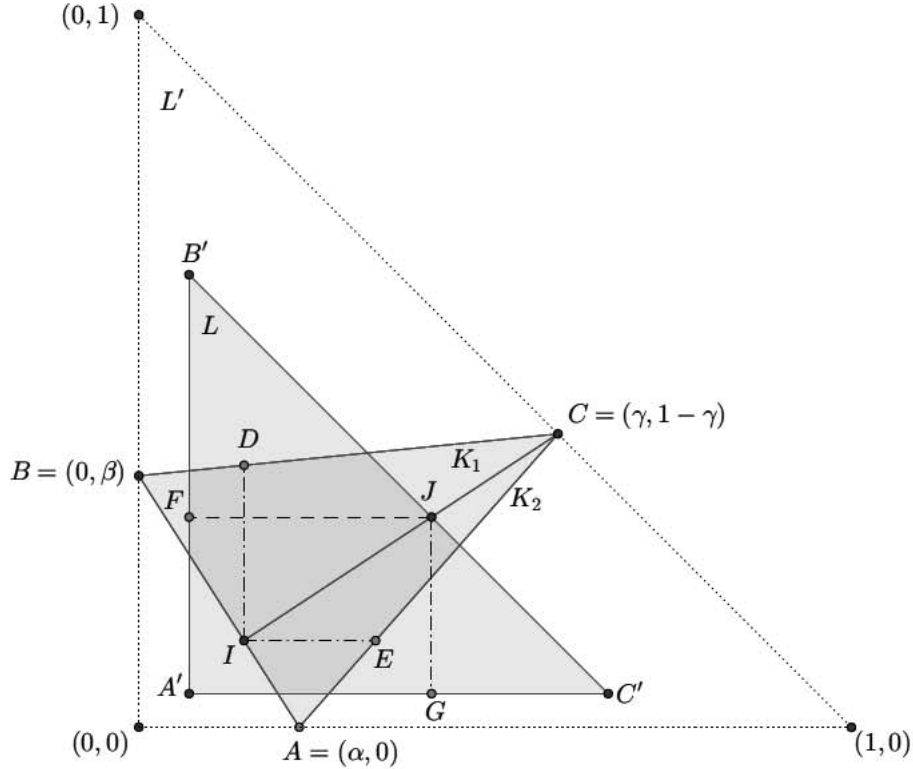


FIGURE 2. Case 2.

previous case, K is inscribed in Δ , so we need to show

$$|K \cap L| \leq 2\mu|K|.$$

We subdivide K into two triangles $K_1 = \triangle BIC$ and $K_2 = \triangle AIC$ where I is a point on the side \overline{AB} chosen in such a way that $|K_1|/|K| = \gamma$. In coordinates

$$I = (t\alpha, (1 - \gamma)\beta).$$

Next, let \overline{ID} and \overline{IE} be the vertical and the horizontal segments subdividing the triangles K_1 and K_2 , respectively. Using the same argument as above we get

$$\frac{2|K_1|}{\gamma} = |\overline{ID}|, \quad \text{and} \quad \frac{2|K_2|}{1 - \gamma} = |\overline{IE}|.$$

Furthermore, let J be the intersection of \overline{CI} and $\overline{B'C'}$, and let \overline{JF} and \overline{JG} be the horizontal and the vertical segments as in Figure 2. Then, similar to the first case, we have

$$|K_1 \cap L| \leq |\overline{ID}||\overline{JF}| \quad \text{and} \quad |K_2 \cap L| \leq |\overline{IE}||\overline{JG}|.$$

We denote $\mu_1 = |\overline{JF}|$, and so $\mu - \mu_1 = |\overline{JG}|$. Therefore we can write

$$|K \cap L| \leq \frac{2|K_1|}{\gamma} \mu_1 + \frac{2|K_2|}{1 - \gamma} (\mu - \mu_1).$$

It remains to show that

$$\frac{|K_1|}{\gamma} \mu_1 + \frac{|K_2|}{1 - \gamma} (\mu - \mu_1) \leq \mu|K|,$$

which is immediate once we note that $|K_1|/|K| = \gamma$ and $|K_2|/|K| = 1 - \gamma$. \square

As a corollary we obtain the following interesting result in elementary plane geometry.

Corollary 6.3. *Let $H = AA'BB'CC'$ be a convex hexagon and $K = \triangle ABC$, $L = \triangle A'B'C'$. Then*

$$|H||K \cap L| \leq 2|K||L|.$$

Proof. This follows from Theorem 6.2 and the fact that $V(K, L) = |H|$. Indeed, $|H| = V(H, L)$, since L is inscribed in H (see Corollary 2.3). On the other hand,

$$V(K, L) = \frac{1}{2} \sum_u h_K(u) |L^u|,$$

where the sum is over the unit normals to L . Clearly, $h_H(u) = h_K(u)$ for all such u and so $V(K, L) = V(H, L)$. \square

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