2-1-2006

A C-Symplectic Free S1-Manifold with Contractible Orbits and Cat = 1/2 Dim

Christopher Allday
University of Hawaii

John Oprea
Cleveland State University, J.OPREA@csuohio.edu

Publisher's Statement

First published in Proceedings of The American Mathematical Society in 2006, published by the American Mathematical Society.

A C-SYMPLECTIC FREE S^1-MANIFOLD WITH CONTRACTIBLE ORBITS AND CAT $= \frac{1}{2}$ DIM

CHRISTOPHER ALLDAY AND JOHN OPREA

(Communicated by Ronald A. Fintushel)

ABSTRACT. An interesting question in symplectic geometry concerns whether or not a closed symplectic manifold can have a free symplectic circle action with orbits contractible in the manifold. Here we present a c-symplectic example, thus showing that the problem is truly geometric as opposed to topological. Furthermore, we see that our example is the only known example of a c-symplectic manifold having non-trivial fundamental group and Lusternik-Schnirelmann category precisely half its dimension.

1. INTRODUCTION

A question that has arisen in symplectic geometry (see [MS, p. 156]) is whether a closed symplectic manifold can support a free symplectic action of the circle such that each orbit is contractible in the manifold. In this paper, we will present an example showing that this problem is essentially geometric, not topological. Specifically, we shall give a cohomologically symplectic manifold with a free circle action having orbits contractible in the manifold. Furthermore, we shall show that this manifold provides an example (perhaps the first known) of a c-symplectic manifold with non-trivial fundamental group having Lusternik-Schnirelmann category exactly half the dimension of the manifold.

2. BASICS ON CATEGORY AND C-SYMPLECTICNESS

A space X has *category* n, denoted $\mathrm{cat}(X) = n$, if and only if n is the least integer such that there is an open covering of X, $U_1, \ldots, U_{n+1}$, with each U_i contractible to a point *in the space* X. A $2n$-manifold M^{2n} is *c-symplectic* if it has a cohomology class $\omega \in H^2(M;\mathbb{Q})$ such that $\omega^n \neq 0$. Of course, the point of the latter definition is to extract relevant homotopy properties from the definition of a symplectic manifold and, in this way, delineate the boundary between geometry and topology. This is then a theme of the paper.

Properties 2.1. The basic properties which we shall use are the following (see [CLOT]).

(1) Category is a homotopy-type invariant.

(2) The *cup length* of a space X is the largest integer k such that there exists a product $x_1 \cdots x_k \neq 0$, with $x_i \in \widetilde{H}^*(X;A)$. Here the coefficient ring A may

vary, and the cup length may be considered for any coefficients. The fundamental relation between cup length and category is $\operatorname{cup}(X) \leq \operatorname{cat}(X)$.

(3) An upper bound for category is given by $\operatorname{cat}(X) \leq \dim(X)$ (where, for spaces more general than manifolds, $\dim(X)$ denotes the covering dimension of X). If X is $(r-1)$-connected, then $\operatorname{cat}(X) \leq \dim(X)/r$. For instance, by (2) and the fact that $\mathbb{C}\mathrm{P}^k$ is simply connected, we see that $\operatorname{cat}(\mathbb{C}\mathrm{P}^k) = k$.

(4) We have the following characterization of category for 3-manifolds.

Theorem 2.2 (See [GoGo] or [OR]). *Let M^3 be a closed 3-dimensional manifold. Then*

$$\operatorname{cat}(M) = \begin{cases} 1 & \textit{if} \quad \pi_1(M) = \{1\}, \\ 2 & \textit{if} \quad \pi_1(M) \textit{ is free and non-trivial}, \\ 3 & \textit{otherwise}. \end{cases}$$

(5) We have the following inequality when the inclusion of a fibre is null.

Theorem 2.3 (See [CLOT] for instance). *Suppose $F \xrightarrow{i} E \xrightarrow{p} B$ is a fibration and $i \simeq *$. Then $\operatorname{cat}(E) \leq \operatorname{cat}(B)$.*

(6) If a c-symplectic manifold is simply connected, then we have $\operatorname{cat}(M) \geq \operatorname{cup}(M) \geq n$ and $\operatorname{cat}(M) \leq \dim(M)/2 = n$. Hence, $\operatorname{cat}(M) = n = \dim(M)/2$. Non-simply connected c-symplectic manifolds of dimension $2n$ may have category between n and $2n$. For instance,

$$\operatorname{cat}\left(\left(\prod_1^k S^2\right) \times T^{2n-2k}\right) = 2n - k.$$

We say that M is *c-symplectically aspherical* if $\omega \in H^2(M;\mathbb{Q}) \cong \operatorname{Hom}(H_2(M),\mathbb{Q})$ vanishes on the image of the Hurewicz map $\pi_2(M) \to H_2(M)$. A c-symplectically aspherical manifold (M,ω) has $\operatorname{cat}(M) = \dim(M)$. (See [RO].)

3. The Contractible Orbit Problem

Problem 3.1 (Contractible Orbit Problem [MS]). *Is it possible to have a free symplectic circle action on a closed symplectic manifold with orbits contractible in the manifold?*

Problem 3.2 (The Cohomological Contractible Orbit Problem). *Is it possible to have a free circle action on a closed c-symplectic manifold with orbits contractible in the manifold?*

Recall that an action is *symplectic* if each element of the group, thought of as a diffeomorphism of the manifold, preserves the symplectic form. (Also recall that *Hamiltonian* circle actions have fixed points, so they can never be free.) Note that the Cohomological Contractible Orbit Problem removes the requirement that the action be somehow compatible with a type of geometry. Indeed, since S^1 is connected, every element preserves the c-symplectic cohomology class. Also note that a positive solution for the Contractible Orbit Problem automatically provides a positive solution for the Cohomological Contractible Orbit Problem. Of course, it is also true that if the Cohomological Contractible Orbit Problem has a negative solution for a class of symplectic manifolds, then the Contractible Orbit Problem also has a negative solution for this class. This paper only concerns the more general Cohomological Contractible Orbit Problem.

Now, a free S^1-action with contractible orbits would give a principal bundle to which Theorem 2.3 may be applied. (See [OW] for more general results on actions with contractible orbits.) Thus we obtain

Proposition 3.3. *The cohomological contractible orbit problem has a negative solution for (M, ω) a c-symplectically aspherical manifold.*

Proof. Let $S^1 \xrightarrow{i} M \xrightarrow{p} B$ be a principal bundle corresponding to a free circle action on M. Because the orbits are contractible in M, we have $i \simeq *$. By Theorem 2.3, $\mathrm{cat}(M) \leq \mathrm{cat}(B)$. But this contradicts $\mathrm{cat}(M) = \dim(M) > \dim(B) \geq \mathrm{cat}(B)$. Hence, no such action exists. □

A c-symplectic manifold (M, ω) is said to have the weak Lefschetz property if the cup product with ω^{n-1} gives an isomorphism $H^1(M;\mathbb{Q}) \to H^{2n-1}(M;\mathbb{Q})$. From results of [A] and [LO], we have

Proposition 3.4. *The cohomological contractible orbit problem has a negative solution for a c-symplectic manifold that has the weak Lefschetz property.*

Proof. As before, let $S^1 \xrightarrow{i} M \xrightarrow{p} B$ be the principal bundle. Because the action is free, p is classified by a map $B \to BS^1$. Since $\omega^n \neq 0$, but $\dim(B) = 2n-1$, $\omega \notin \mathrm{Im}(p^*)$. Hence, in the Leray-Serre spectral sequence for $B \to BS^1$, $d_2(\omega) \neq 0$. Let $d_2(\omega) = t \otimes u$, where $t \in H^2(BS^1;\mathbb{Q})$ is the polynomial generator, and $u \in H^1(M;\mathbb{Q})$.

By Poincaré duality and the weak Lefschetz property, there is $v \in H^1(M;\mathbb{Q})$ such that $\omega^{n-1}uv = \omega^n$. Now, by considering $d_2(\omega^n v)$, which must be zero for degree reasons, it follows that $d_2(v) \neq 0$. In particular, $p^* \colon H^1(B;\mathbb{Q}) \to H^1(M;\mathbb{Q})$ is not surjective. So, in the homotopy long exact sequence for p, it follows that $\pi_1(S^1) \to \pi_1(M)$ is not trivial. □

Now let us look at the c-symplectic example which is our main focus.

4. The example

Let $X = S^1 \times S^3$ with S^1 acting trivially on the first factor and by left multiplication on the second. By the slice theorem, there is an equivariant tube around each orbit; let C be one such tube in S^3 and define $B = [\theta, \mu] \times C \subset X$, where $[\theta, \mu]$ is a closed interval in S^1.

Now let $A = X - \mathrm{Int}(B)$ and note that $\partial A = \partial B$. Let $M = A \cup_{\partial A} A$ be the double of A along ∂A. M is then a closed 4-manifold with free circle action. (A similar construction in [Ko] leads to 4-manifolds with zero minimal volume.)

Consider $\partial A = \partial B = \{\theta\} \times C \cup \{\mu\} \times C \cup [\theta, \mu] \times T$, where T is the boundary torus of the tube C. So ∂A consists of two solid tori whose boundaries are joined by line segments between corresponding points. In particular, restricted to corresponding meridians in the upper and lower solid tori, we have a cylinder (with top and bottom). To see the homotopy type of this boundary, crush the meridians of the upper and lower solid tori. This creates two-spheres from the cylinders and gives $\partial A \simeq S^1 \times S^2$. We also see from this that $\mathbb{Z} \cong \pi_1(\partial A) \cong \pi_1(\partial B) \xrightarrow{\cong} \pi_1(B)$. Finally note that, since C is contained in the simply-connected S^3-factor of X

and $\pi_1(C) \cong \pi_1(B)$ via the inclusion, we have $\pi_1(B) \xrightarrow{0} \pi_1(X)$. Now, since $X = A \cup_{\partial A} B$, Van Kampen's theorem gives a pushout diagram:

$$\begin{tikzcd} \mathbb{Z} = \pi_1(\partial A) \arrow[r, "0"] \arrow[d, "\cong"'] & \pi_1(A) \arrow[d] \\ \mathbb{Z} = \pi_1(B) \arrow[r, "0"] & \pi_1(X) \end{tikzcd}$$

The top horizontal homomorphism is zero by the definition of the amalgamated product and the isomorphism $\pi_1(X) \cong \pi_1(A) *_{\pi_1(\partial A)} \pi_1(B)$. Specifically, any element in $\pi_1(A) *_{\pi_1(\partial A)} \pi_1(B)$ can be written as ${a_1}^{\epsilon_1}{b_1}^{\tau_1} \ldots {a_k}^{\epsilon_k}{b_k}^{\tau_k}$, where $a_j \in \pi_1(A)$, $b_j \in \pi_1(B)$ and all exponents are ± 1. Because $\pi_1(\partial A) \to \pi_1(B)$ is an isomorphism, the b's are all identified with elements of $\pi_1(A)$ via the homomorphism $\pi_1(\partial A) \to \pi_1(A)$. But $\pi_1(B) \to \pi_1(X)$ is the zero homomorphism, so all the b's are the identity. This means that $\pi_1(\partial A) \to \pi_1(A)$ is also the zero homomorphism. But then we have

$$\pi_1(A) \xrightarrow{\cong} \pi_1(X) \cong \mathbb{Z}.$$

We also have

$$\pi_1(\text{orbit}) \cong \pi_1(C) \cong \pi_1(B) \cong \pi_1(\partial A) \xrightarrow{0} \pi_1(A).$$

Since any orbit of the action on M lies in one of the two A's, we see that $\pi_1(\text{orbit}) \to \pi_1(M)$ is zero as well. Since any orbit is a circle, we see that orbits are contractible in M.

We can also compute the fundamental group of $M = A \cup_{\partial A} A$ by Van Kampen's theorem:

$$\begin{tikzcd} \pi_1(\partial A) \arrow[r, "0"] \arrow[d, "0"'] & \pi_1(A) \arrow[d] \\ \pi_1(A) \arrow[r] & \pi_1(M) \end{tikzcd}$$

Because $\pi_1(\partial A) \xrightarrow{0} \pi_1(A)$, we obtain $\pi_1(M) \cong \pi_1(A) * \pi_1(A) \cong \mathbb{Z} * \mathbb{Z}$, the free group on two generators.

This immediately implies that $H_1(M;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} = H^1(M;\mathbb{Z})$. By Poincaré duality, we have $H^3(M;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$. Of course $H^4(M;\mathbb{Z}) = \mathbb{Z}$ since M is orientable. To determine $H^2(M;\mathbb{Z})$, note that the existence of a free circle action on M implies that the Euler characteristic vanishes: $\chi(M) = 0$. Hence,

$$0 = 1 + 2 - b_2 - 2 + 1 = b_2 - 2$$

which implies that $b_2 = 2$ and $H_2(M;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$ (since $H^3(M;\mathbb{Z})$ is torsion free). Therefore, $H^2(M;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$.

Finally, we claim that M is also c-symplectic. Let x and y denote two basis elements of $H^2(M;\mathbb{Q}) = \mathbb{Q} \oplus \mathbb{Q}$. If $x^2 \neq 0$ or $y^2 \neq 0$, then we are done, so suppose $x^2 = 0 = y^2$. Then $(x+y)^2 = x^2 + 2xy + y^2 = 2xy$. If $xy = 0$, then all products $H^2 \otimes H^2 \to H^4$ are zero, and this violates Poincaré duality. Thus, $xy \neq 0$ and $(x+y)^2 \neq 0$, providing a c-symplectic class. We summarize the preceding discussion in the following theorem.

Theorem 4.1. *There exists a 4-dimensional closed c-symplectic manifold and a free circle action on it with orbits contractible in the manifold. Thus, the cohomological contractible orbit problem has an affirmative solution.*

5. L. S. CATEGORY OF THE EXAMPLE

From Properties 2.1, we see that symplectic or c-symplectic manifolds split into two types depending on whether they have a trivial or non-trivial fundamental group. If M^{2n} is c-symplectic and $\pi_1(M) = 0$, then $\mathrm{cat}(M) = n$. If $\pi_1(M) \neq 0$, then $n \leq \mathrm{cat}(M) \leq 2n = \dim(M)$. In fact, however, for the non-simply-connected case, it appears that not one single example of the lower bound being attained was known before the following

Theorem 5.1. *The manifold M^4 of Theorem 4.1 has $\mathrm{cat}(M) = 2$.*

Proof. The free circle action gives a principal S^1-bundle

$$S^1 \to M \to M/S^1,$$

where M/S^1 is a closed 3-manifold. Because orbits of the S^1-action are contractible in M, the fibre inclusion $S^1 \to M$ is null-homotopic. By Properties 2.1, we know $\mathrm{cat}(M) \leq \mathrm{cat}(M/S^1)$. But, by the long exact sequence in homotopy, we also know that $\pi_1(M) = \mathbb{Z} * \mathbb{Z} = \pi_1(M/S^1)$ and, again by Properties 2.1, we see that $\mathrm{cat}(M/S^1) = 2$. Since we know that $\mathrm{cup}(M) \geq 2$, we obtain

$$2 \leq \mathrm{cup}(M) \leq \mathrm{cat}(M) \leq \mathrm{cat}(M/S^1) = 2.$$

Hence $\mathrm{cat}(M) = 2$. □

So we see that M is the first-known example of a c-symplectic manifold with non-trivial fundamental group whose category is that of a simply-connected c-symplectic manifold of the same dimension.

6. AFTERTHOUGHTS

The manifold $X = S^1 \times S^3$ is complex, so we can ask how much of this structure is inherited by M. We know that, even though M has the homology of $S^2 \times T^2$, it *cannot* be Kähler because no compact Kähler manifold can have $\pi_1 = \mathbb{Z} * \mathbb{Z}$ (see [ABCKT]). Moreover, by [ABCKT, Chapter 1, section 3], this also means that M cannot have a complex structure.

Can M have an almost-complex structure then? Wu's criterion guarantees the existence of an almost-complex structure on a 4-manifold M if there is an integral class $c \in H^2(M;\mathbb{Z})$ which reduces mod 2 to the second Stiefel-Whitney class w_2 and which satisfies $c^2([M]) = 3\sigma + 2\chi$, where σ is the signature of M and χ is the Euler characteristic. Now, it can be seen that $H_2(M;\mathbb{Z})$ is generated by a pair of embedded 2-spheres each of which has self-intersection zero. (One S^2 comes from the $S^1 \times S^2$ where the gluing takes place, and a transverse 2-sphere is constructed from disks bounded by the Hopf circles.) This corresponds to the situation where $H^2(M;\mathbb{Z})$ is generated by x and y with $x^2 = 0 = y^2$. This immediately implies that $\sigma = 0$. Of course, as we noted above, $\chi = 0$ because M possesses a free circle action. Now (since M is orientable), w_2 is characterized by the equation $w_2 \cup b = b \cup b$ for all $b \in H^2(M;\mathbb{Z}/2)$. But then we have $w_2 \cup \bar{x} = 0 = w_2 \cup \bar{y}$ (where $\bar{x}$ and $\bar{y}$ are the mod 2 reductions of the generators x and y), and we thus see that $w_2 = 0$. Therefore, if we choose $c = 0$, Wu's criterion provides an almost-complex structure on M.

Finally, we can ask if M is symplectic. Using Seiberg-Witten invariants, it can be shown that symplectic 4-manifolds with $b_2^+ > 1$ cannot contain symplectic spheres of non-negative square. In particular, they cannot contain symplectic spheres of square zero. (More specifically, in [Ta], Taubes showed that symplectic manifolds have non-zero Seiberg-Witten invariants while in [FS] and [Ko2] it was shown that these vanish in the presence of 2-spheres of non-negative square.) Although M itself has $b_2^+ = 1$, because $\pi_1(M) = \mathbb{Z} * \mathbb{Z}$, there are finite covers of M with arbitrarily large b_2^+ for which the embedded 2-spheres mentioned above lift. Therefore, these finite covers cannot be symplectic, and this implies that M is not symplectic as well.

All of these questions (and their answers!) point out the real difficulties in moving from the world of topology to that of geometry in proving or constructing counterexamples to Problem 3.1 and in proving a symplectic analogue of Theorem 5.1.

Acknowledgements

We thank Ron Fintushel and Dieter Kotschick for providing the answers to the questions in §6. We also thank the Mathematisches Forschungsinstitut Oberwolfach, where the writing of this paper began, and the Université des Sciences et Technologies de Lille, where the writing was completed.

References

[A] C. Allday, *Lie group actions on cohomology Kähler manifolds*, unpublished manuscript (1978).

[ABCKT] J. Amorós, M. Burger, K. Corlette, D. Kotschick and D. Toledo, *Fundamental groups of compact Kähler manifolds*, Amer. Math. Soc. Surveys and Mono. 44 (1996). MR1379330 (97d:32037)

[CLOT] O. Cornea and G. Lupton and J. Oprea and D. Tanré, *Lusternik-Schnirelmann Category*, Amer. Math. Soc. Surveys and Mono. 103 (2003). MR1990857 (2004e:55001)

[FS] R. Fintushel and R. Stern, *Immersed spheres in 4-manifolds and the immersed Thom conjecture*, Turkish J. Math. 19 (1995) 145-157. MR1349567 (96j:57036)

[GoGo] J.C. Gomez-Larranaga and F. Gonzalez-Acuna, *Lusternik-Schnirelmann category of 3-manifolds*, Topology **31** (1992) 791-800. MR1191380 (93j:55005)

[Ko] D. Kotschick, *Entropies, volumes and Einstein metrics*, preprint 2004.

[Ko2] D. Kotschick, *Orientations and geometrisations of compact complex surfaces*, Bull. London Math. Soc. 29 2 (1997) 145-149. MR1425990 (97k:32047)

[LO] G. Lupton and J. Oprea, *Cohomologically symplectic spaces: toral actions and the Gottlieb group*, Trans. Amer. Math. Soc. 347 (1995) 261-288. MR1282893 (95f:57056)

[MS] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford University Press, 2nd edition, (1998). MR1698616 (2000g:53098)

[OR] J. Oprea and Y. Rudyak, *Detecting elements and Lusternik-Schnirelmann category of 3-manifolds*, Contemp. Math. 316 (2002) 181-191. MR1962163 (2004c:55003)

[OW] J. Oprea and J. Walsh, *Quotient maps, group actions and Lusternik-Schnirelmann category*, Top. Appls. 117 (2002) 285-305. MR1874091 (2002i:55001)

[RO] Y. Rudyak and J. Oprea, *On the Lusternik-Schnirelmann category of symplectic manifolds and the Arnold conjecture*, Math. Zeit. 230 no. 4 (1999) 673-678. MR1686579 (2000b:53115)

[Ta] C. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Letters 1 (1994) 809-822. MR1306023 (95j:57039)