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EXTENSIONS OF THE MORSE-HEDLUND THEOREM

by

Eben Blaisdell

A Thesis

Presented to the Faculty of

Bucknell University

in Partial Fulfillment of the Requirements for the Degree of

Bachelor of Science with Honors in Mathematics

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Approved: ____________________________

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Abstract

Bi-infinite words are sequences of characters that are infinite forwards and backwards; for example "...*ababababab*...". The Morse-Hedlund theorem says that a bi-infinite word f repeats itself, in at most n letters, if and only if the number of distinct subwords of length n is at most n. Using the example, "...*ababababab*...", there are 2 subwords of length 3, namely "*aba*" and "*bab*". Since 2 is less than 3, we must have that "...*ababababab*..." repeats itself after at most 3 letters. In fact it does repeat itself every two letters. Interestingly, there are many extensions of this theorem to multiple dimensions and beyond. We prove a few results in two-dimensions, including a specific partial result of a question known as the Nivat conjecture. We also consider a novel extension to the more general setting of 'group actions', and we prove an optimal analogue of the Morse-Hedlund theorem in this setting.

Chapter 1

Introduction

Any English word can be seen as a mapping from a set $\{0, 1, ..., n-1\}$ to the alphabet $\{``a", ``b", ``c", ..., ``x", ``y", ``z"\}$. Consider the word "*math*"; this can be seen as the mapping $0 \mapsto$ "*m*", $1 \mapsto$ "*a*", $2 \mapsto$ "*t*", and $3 \mapsto$ "*h*". We can generalize this idea to define bi-infinite words. A bi-infinite word is a mapping from the integers, $\mathbb{Z}$, to a finite alphabet. Standard words are finite strings of characters, and infinite words are any sequence of characters that goes forever both forwards and backwards. Bi-infinite words are useful in many areas of mathematics, natural science, and engineering. They have been used as abstract tools for symbolic dynamics, information theory, and data storage.

One interesting property that bi-infinite words can possess, that finite words cannot, is periodicity. A function, such as $\sin x$, is called periodic (with period 2π) since shifting the function by 2π does not change its value. Formally, we know $\sin(x + 2\pi) = \sin x$. We can intuitively see this type of repetition in words as well. Consider the standard finite word "*murmur*". It appears to repeat itself, but no shift truly yields the same word. For example, shifting by three yields "___*murmur*", which is poorly defined and is definitely not equal to "*murmur*". Bi-infinite words are a way around this problem. Consider the bi-infinite word "...*murmurmurmur*...".

Shifting this word by three yields "...*murmurmurmur*...", so we can reasonably say that this word is periodic with period three. Similarly "...*abababab*..." has period 2 and "...*abacabacabac*..." has period 4. Note that not all bi-infinite words are periodic; for example the word "...*aaaaabaaaaa*..." is not periodic with any period.

The Morse-Hedlund theorem is a central tool for assessing whether a word is periodic or not. To state this theorem, we must first explore the measurement of complexity.

Definition 1.1. *The scale n block complexity of a bi-infinite word f is defined as the number of distinct length n subwords of f; this is denoted as $p_f(n)$.*

For example, the scale 5 block complexity of $f :=$ "...*murmurmurmur*..." is 3, since there are 3 distinct length 5 subwords of f, namely "*murmu*", "*urmur*", and "*rmurm*"; thus we say $p_f(5) = 3$. Using this, the Morse-Hedlund theorem is as follows.

Theorem 1.1 (Morse-Hedlund Theorem [11])**.** *Let f be a bi-infinite word, then f is periodic, with a period of length at most n, if and only if $p_f(n) \leq n$.*

In order to get a deeper understanding of bi-infinite words and the Morse-Hedlund theorem, we prove a related result. Note that a 26 character alphabet does not apply to all human languages, and certainly not to all practical uses of bi-infinite words; therefore, we consider bi-infinite words as having characters in an arbitrary finite alphabet.

Theorem 1.2 (Morse-Hedlund Theorem with Alphabetic Strengthening)**.** *Let f be a bi-infinite word with alphabet Σ, where all letters in Σ appear in f. If there is an n such that $p_f(n) \leq n + |\Sigma| - 2$ then f is periodic, with period at most $p_f(n)$.*

Proof. We will proceed by induction on n.

Consider first if $n = 1$. Notice that $p_f(1) = |\Sigma|$, since the number of length 1 subwords is exactly the number of letters. Therefore, $p_f(1) > |\Sigma| - 1$, and so the result is vacuously true.

Now pick an arbitrary $n \geq 2$, and assume the claim for $n-1$. Assume $p_f(n) \leq n + |\Sigma| - 2$, we will show that f is periodic. To do this, we consider two cases, $p_f(n-1) < n + |\Sigma| - 2$ and $p_f(n-1) \geq n + |\Sigma| - 2$.

Say that $p_f(n-1) < n + |\Sigma| - 2$. Notice that $p_f(n-1) \leq p_f(n)$, since every length n subword of f contains a length $n-1$ subword, starting at the same index. Therefore, $p_f(n-1) \leq p_f(n) \leq (n-1) + |\Sigma| - 2$. Thus, by the induction hypothesis, f is periodic with period at most $p_f(n-1)$. Since every distinct subword of length n contains a subword of length $n-1$ subword. Therefore, $p_f(n-1) \leq p_f(n)$, meaning f is periodic with period at most $p_f(n)$.

Now say that $p_f(n-1) \geq n + |\Sigma| - 2$. Since $p_f(n) \leq n + |\Sigma| - 2$, we have $p_f(n-1) \leq n + |\Sigma| - 2$. Therefore, $p_f(n-1) = p_f(n) = n + |\Sigma| - 2$. Notice that since every length n subword contains a length $n-1$ subword starting at the same index, and $p_f(n-1) = p_f(n)$, every length $n-1$ subword appearing anywhere in the word is contained in exactly one length n subword. In other words, when a specific length $n-1$ subword appears in the word, the same letter must appear after it. Notice that this means that the presence of a length $n-1$ subword at a position implies that a specific length $n-1$ subword must appear in the next position. There are finitely many distinct subwords of length $n-1$, since $p_f(n-1) \leq |\Sigma|^{n-1}$. Notice that since every subword appears in the word, it appears to the right of some other subword, thus all length $n-1$ subwords are implied as the next subword, by some other length $n-1$ subword. This means that the mapping representing what subword is implied by another subword is actually a permutation of the set of length $n-1$ subwords. Since this mapping is a permutation on a set of size $p_f(n-1)$, we can say that there is some $v \leq p_f(n-1)$ such that applying this permutation v times yields the identity permutation: the permutation that maps everything to itself. This means that the subword after the subword after the subword after the subword and so on v times must always be the original subword. This implies that v places after a length $n-1$ subword, the same subword appears. This specifically implies, by looking at the first letter in each length $n-1$ subword, that the letter appearing v spaces after any letter must be the same letter. By definition, this means that f is periodic with period $v \leq p_f(n-1) = p_f(n) = n + |\Sigma| - 2$.

In all cases, we have that f is periodic with period at most n, proving the inductive step. Therefore, if there is an n such that $p_f(n) \leq n + |\Sigma| - 2$, then f is periodic with period at most $n + |\Sigma| - 2$. $\square$

The Morse-Hedlund theorem compares the 'global' property of periodicity of bi-infinite words to the local problem of complexity of bi-infinite words. It quickly becomes a question whether this idea can extend to more general domains. A bi-infinite word f is a mapping from $\mathbb{Z}$ to the alphabet Σ, that is, $f : \mathbb{Z} \to \Sigma$. The integers can be seen on the number line as discrete points; we call this the one-dimensional integer lattice. Since there are finitely many letters in the alphabet, we can think of this as finitely many colors in a color palette. In this way, bi-infinite words can be conceptualized as colorings of the one-dimensional integer lattice; throughout this thesis, we use the language of letters and colors interchangeably. A natural question that arises is whether something like the Morse-Hedlund theorem holds in the two-dimensional integer lattice, $\mathbb{Z}^2 := \{(x, y) : x, y \in \mathbb{Z}\}$. We say that a two-dimensional word $f : \mathbb{Z}^2 \to \Sigma$ is periodic with a nonzero period vector $\mathbf{v} = \langle v_x, v_y \rangle$ if at every point $\mathbf{x}$, we have $f(\mathbf{x}) = f(\mathbf{x} + \mathbf{v})$. This intuitively means that shifting the word f by $\mathbf{v}$ in the plane does not change the colors at any point. There is a more general definition for complexity as well.

Definition 1.2. *The $n \times k$ rectangle complexity of a two-dimensional word $f : \mathbb{Z}^2 \to \Sigma$, written $p_f(n, k)$, is the number of ways to color an $n \times k$ rectangle in f. More precisely, by translating the $n \times k$ rectangle and placing the colors in f at the points in the $n \times k$ rectangle yields a coloring of that rectangle, and the number of distinct colorings is $p_f(n, k)$.*

The extension of the Morse-Hedlund theorem to two-dimensions is a widely researched topic in symbolic dynamics known as Nivat's conjecture [3, 6, 7, 8, 9, 10].

Conjecture 1.1 (Nivat's Conjecture)**.** *A two-dimensional word $f : \mathbb{Z}^2 \to \Sigma$ with finite alphabet Σ that satisfies $p_f(n,k) \leq nk$, for some n and k, must be periodic.*

To unpack this, we must explore some properties of two-dimensional words. There is no nice analogy between two-dimensional words and standard finite words like "*math*"; the name "word" is mainly a borrowing from the one-dimensional case. Two-dimensional words are most usefully though of as colorings of the two-dimensional integer lattice. This is like pixels on an infinite screen; every point is assigned some color. Nivat's conjecture posits a relationship between the complexity and periodicity of words in two-dimensions.

Many partial results of the Nivat conjecture have been proven. For example, it is known that if $p_f(n,k) \leq nk$ for infinitely many n and k then f is periodic [10]. It is also known that the Nivat conjecture holds if $k \leq 3$ [6]. Interestingly, because of a connection to algebraic geometry discovered recently, it is also known that if $p_f(n,k) \leq nk$ for some n and k then f is the sum of finitely many periodic complex functions [10].

This thesis provides another partial result of the Nivat conjecture, as well as proving a few other useful results about two-dimensional words. We also generalize the idea of words to have groups (a mathematical structure) as domains.

In Chapter 2, we describe relevant disciplines in mathematics, as well as their utility in and relevance to this problem. In Chapter 3, we describe novel results about two-dimensional words that are not directly subresults of the Nivat conjecture. In Chapter 4, we present a specific subresult of the Nivat conjecture utilizing the machinery accumulated by Cyr and Kra [6]. Chapter 5 describes the generalization of the problem using groups, including an optimal analogue of the Morse-Hedlund theorem in this abstract case. Chapter 6 poses questions on the frontier the abstract case, as well as cataloguing what is yet to be known about the Nivat conjecture.

Chapter 2

Background

This work lies at the intersection of many areas of mathematics. Most significantly, it uses concepts from dynamical systems, combinatorics, abstract algebra, theoretical computer science, topology, and discrete geometry.

Dynamical systems define the state of a geometric space dependent on time. Dynamics has developed many tools to efficiently explore the evolution of these systems over time. Consider a system of Newtonian celestial bodies, or a volume of gas containing many molecules. Both of these exist in the three dimensional space $\mathbb{R}^3$, and they change state over time. Both of these can be modelled with differential equations. This consideration implicitly considers that time acts like $\mathbb{R}$.

Some interesting properties arise if we consider time to act more like $\mathbb{Z}$. This is the idea behind the subfield of dynamics known as symbolic dynamics. Consider once more the Newtonian celestial bodies. To simplify, consider one planet orbiting one star. We will now consider the state of this system at 'discrete' instances of time. Partition the space into finitely many sections. The center of the planet is in only one of these sections at a time. Knowing this we can consider a finite alphabet where the letters represent the section of space that the planet is in. Using symbolic dynamical tools such as the Morse-Hedlund theorem, we can relate local properties (occurring over short time periods) of the system to global properties (occurring over long time periods). The bi-infinite word with letters from the alphabet we constructed represent the evolution of that system throughout all of time. Bi-infinite words are functions from $\mathbb{Z}$ to an alphabet, and we are considering time as estimated by $\mathbb{Z}$; therefore, this is a function representing the position of the planet sampled at specific times. The complexity of this word is the number of ways for the state to evolve over some length of time. It turns out that in specific instances, for a large enough length of time, the complexity is eclipsed by the length of time considered. By an interesting application of the Morse-Hedlund theorem, we know that the system is periodic. That is, after some period of time, the state of the system repeats itself. This matches up with our intuition of orbiting being a highly repetitive process.

Combinatorics is the study of counting. Given some finite set of objects, combinatorics asks how many of those objects there are. For example, if you have 5 shirts, 3 pairs of pants, and 2 pairs of shoes, how many outfits can you make? The Rule of Product from elementary combinatorics says that there are 30 complete outfits. Combinatorial questions can quickly get more complicated. The number of abstract trees with n nodes and ℓ leaves is one combinatorial problem that is very simple to state, but incredibly difficult to calculate. Combinatorics appears naturally when counting the number of subwords of a certain size in an infinite word. Therefore, in this paper, combinatorial methods are used implicitly and explicitly to help find the complexity of words.

Algebra is the general study of operations acting on abstract items. Algebraic objects known as groups, which we will study in-depth later, are a fundamental abstraction of actions on a set. Thinking algebraically can allow for the proof of much more general results in mathematics. When considering some bi-infinite word $f : \mathbb{Z} \to \Sigma$, the properties of the numbers in $\mathbb{Z}$ are central. For example, we say f is periodic if there is some integer where $f(v + x)$ is always equal to $f(x)$. The Nivat conjecture seeks to 'loosen' this structure of words; it considers words over $\mathbb{Z}^2$ rather than $\mathbb{Z}$. In this paper, we will discuss what can be said if almost all specific structure is removed; we will consider words $f : S \to \Sigma$, where S is any set that we can be 'moved around' by some group.

Theoretical computer science is generally the study of any computational system. The foundations of theoretical computer science lie in mathematical models of computers that operate with input/output given as words over a finite alphabet. For this and other reasons, various computational models are relevant to the study of infinite words. The converse is even more straightforward, the study of infinite words has the potential to yield many results in theoretical computer science.

Topology is very roughly geometry without distance. To formalize this for a space, topology calls some subsets of that space "open sets". The open sets define what it means to be 'close' in the space. This conceptualization has many of useful consequences. One in particular is a reimagining of limits. Classically, we say that a sequence $(x_n) = x_0, x_1, x_2, ...$ has a limit point L if for arbitrarily small $\epsilon > 0$ there is some index N such that for all $n \geq N$, the point x_n is within distance ϵ of the point L. Topologically, we can define proximity to L by open sets rather than distance. Thus, topologically, we say that L is a limit point of a sequence $(x_n) = x_0, x_1, x_2, ...$ if for every open set U containing L, there is some index N such that for all $n \geq N$, the point x_n is also in U. Though there is little geometric intuition, work on the extensions of the Morse-Hedlund Theorem often considers words to be points in a topological space; the significance of this stems almost entirely from the ability to consider limit of a sequence of words.

Discrete geometry is the study of geometric objects over spaces that don't have many of the continuous properties of spaces like $\mathbb{R}^2$. In this paper, we will specifically explore much of the geometry of $\mathbb{Z}^2$. In discrete geometry many common sense facts don't necessarily hold. For example, lines that are not parallel do not necessarily intersect.

By utilizing concepts from these fields, much can be said about the Nivat conjecture, and even more general extensions of the Morse-Hedlund theorem.

Chapter 3

General Results in $\mathbb{Z}^2$

The Morse-Hedlund can be seen as completely characterizing the relationship between complexity and periodicity of colorings of $\mathbb{Z}$, the one dimensional integer lattice. The relationship between complexity and periodicity on $\mathbb{Z}^2$, the two-dimensional integer lattice, is still not completely understood. This lack of complete understanding is evident in the diverse body of partial results [6, 7, 8, 9, 10].

The Morse-Hedlund Theorem claims that there exists an n such that $p_f(n) \leq n$ if and only if f is periodic, with period at most n. Analogously, the Nivat Conjecture posits that if there exists an n and k such that $p_f(n,k) \leq nk$, then f is periodic. The Nivat conjecture makes no claim about the magnitude of the period. No optimal general result bounding this magnitude is known, but this bound is necessarily large: significantly larger than the linear bound implicit to the Morse Hedlund theorem. To see this, we start with some notation.

Definition 3.1. *Let X be a boolean valued expression. We say that that $(X) = 1$ if X is true, and $(X) = 0$ if X is false. For example, we have that $(0 < 4) = 1$, $(3 = 4) = 0$, and $(X) = (Y)$ if and only if $X \iff Y$.*

The following example demonstrates how large the period of a word satisfying

$p_f(n,k) \leq nk$ can be.

Example 3.1 (Low Complexity Word with Large Period). *Let $\rho_0, \rho_1, \rho_2, \ldots$ be the prime numbers. Define a family of $\{0,1\}$-valued words $\ell_m : \mathbb{Z}^2 \to \{0,1\}$ by $\ell_m(x,y) := (\rho_y | x)$ for $0 \leq y < m$, and $\ell_m(x,y) := 0$ otherwise. Here $\rho_y | x$ means that ρ_y divides into x.*

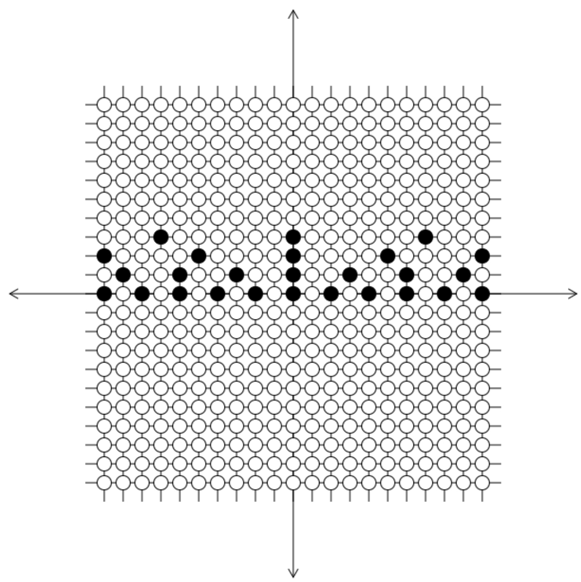

Figure 3.1: A visual representation of ℓ_4 where 0 is white and 1 is black.

Theorem 3.1. *The complexity of ℓ_m satisfies $p_{\ell_m}(n,1) \leq \sum_{i=0}^{m-1} \rho_i + 1$ for all n.*

Proof. First consider the $n \times 1$, that is, hieght 1, rectangles originating from (x_0, y) with $0 \leq y < m$. Fix some such y. We will count the number of ways to color an $n \times 1$ rectangle on some horizontal slice of ℓ_m. This is equivalent to the one dimensional complexity of some word s_y, i.e. $p_{s_y}(n)$ where $s_y(x) := \ell_m(x,y) = (\rho_y | x)$. Notice that $s_y(x) := (\rho_y | x)$ has period ρ_y. Certainly $s_y(x + \rho_y) = (\rho_y | x + \rho_y)$ and $s_y(x) = (\rho_y | x)$. Since $\rho_y | \rho_y$, we know $(\rho_y | x + \rho_y)$ if and only if $(\rho_y | x)$. Therefore, $s_y(x + \rho_y) = s_y(x)$ so ρ_y is a period of s_y. Thus, by the Morse-Hedlund theorem, $p_{s_y}(n) \leq \rho_y$. Thus, for rectangles originating from (x,y) with $0 \leq y < m$ contribute at most ρ_y colorings of $n \times 1$ rectangles that appear in ℓ_m.

Now consider $n \times 1$ rectangles originating from (x_0, y) with $y < 0$ or $y \geq m$. In these regions, $\ell_m(x, y) := 0$. Thus, these regions contribute at most the one coloring of the $n \times 1$ rectangle: the constant 0 coloring.

Therefore, over all of ℓ_m, there are at most $\sum_{i=0}^{m-1} \rho_i + 1$ colorings of the $n \times 1$ rectangle. Equivalently, $p_{\ell_m}(n, 1) \leq \sum_{i=0}^{m-1} \rho_i + 1$. $\square$

The important thing to note is that the complexity bound $\sum_{i=0}^{m-1} \rho_i + 1$ does not depend on n. This means that arbitrarily large $n \times 1$ rectangles have complexity bounded by a constant.

Now that we have examined the complexity of ℓ_m, consider the periods of ℓ_m.

Theorem 3.2. *The word ℓ_m has period vector $\langle \prod_{i=0}^{m-1} \rho_i, 0 \rangle$, and this is the shortest period of ℓ_m.*

Proof. Fix m. We will first show that $\langle \prod_{i=0}^{m-1} \rho_i, 0 \rangle$ is a period of ℓ_m. Pick an arbitrary (x, y). Consider two cases: $0 \leq y < m$ and otherwise.

We begin with the first case. By definition, $\ell_m(x, y) = (\rho_y | x)$. Since $\rho_y | \prod_{i=0}^{m-1} \rho_i$, we know $(\rho_y | x + \prod_{i=0}^{m-1} \rho_i)$ if and only if $(\rho_y | x)$. Therefore, $\ell_m(x, y) = \ell_m(x + \prod_{i=0}^{m-1} \rho_i, y)$, so the result holds in this case.

Now consider the case where $y < 0$ or $y \geq m$. Therefore, by the definition of ℓ_m, we know $\ell_m(x, y) = \ell_m(x + \prod_{i=0}^{m-1} \rho_i, y) = 0$. Therefore the result holds in this case as well.

Since in all cases $\ell_m(x, y) = \ell_m(x + \prod_{i=0}^{m-1} \rho_i, y)$, ℓ_m is periodic with period vector $\langle \prod_{i=0}^{m-1} \rho_i, 0 \rangle$.

We will now show that $\langle \prod_{i=0}^{m-1} \rho_i, 0 \rangle$ is of minimal magnitude among all possible period vectors. Assume for contradiction that ℓ_m has some period $\langle x, y \rangle$ with $y \neq 0$. Since $y \neq 0$, there is some integer κ such that $\kappa y < 0$. We know $\ell_m(0, 0) = 1$

since $\rho_0 | 0$. Since ℓ_m is periodic with period $\langle x, y \rangle$, it is periodic with period $\langle \kappa x, \kappa y \rangle$, since κ is an integer. This implies $\ell_m(\kappa x, \kappa y) = 1$, which contradicts the fact that $\ell_m(\kappa x, \kappa y) = 0$ since $\kappa y < 0$. Therefore, all periods of ℓ_m have y component 0.

Now let $\langle x, 0 \rangle$ be some period of ℓ_m. Note that for all $0 \leq j < m$, we know $\ell_m(0, j) = 1$ because $\rho_j | 0$. Since $\langle x, 0 \rangle$ is a period, $\ell_m(x, 0) = 1$. Thus, $\rho_j | x$. Since ρ_j are all distinct primes, they are clearly pairwise coprime. This means that since for all $0 \leq j < m, \rho_j | x$ we know $\prod_{i=0}^{m-1} \rho_i | x$. Since periods are by definition nonzero, we know $x \neq 0$. Therefore, $|x| \geq \prod_{i=0}^{m-1} \rho_i$. Thus, $||\langle x, 0 \rangle|| = |x| \geq \prod_{i=0}^{m-1} \rho_i$, meaning $\langle \prod_{i=0}^{m-1} \rho_i, 0 \rangle$ is of minimal magnitude. □

The important thing to notice here is the very large size of the minimal period $\langle \prod_{i=0}^{m-1} \rho_i, 0 \rangle$.

To describe this asymptotically, we need some definitions.

Definition 3.2 (Asymptotics)**.** *Let $f, g : \mathbb{N} \to \mathbb{R}$ be real-valued functions of natural numbers.*

We say that $f = O(g)$ if and only if $\limsup_{x \to \infty} \frac{f(x)}{g(x)} < \infty$*. Pronounced Big-O, this intuitively means that f is asymptotically no larger than g.*

We say that $f = \Omega(g)$ if and only if $\liminf_{x \to \infty} \frac{f(x)}{g(x)} > 0$*. Pronounced Big-Omega, this intuitively means that f is asymptotically no smaller than g.*

We say that $f = \Theta(g)$ if and only if both $f = O(g)$ and $f = \Omega(g)$. Pronounced Big-Theta, this intuitively means that f is asymptotically equivalent to g.

We say that $f = o(g)$ if and only if $\lim_{c \to \infty} \frac{f(x)}{g(x)} = 0$*. Pronounced Little-o, this intuitively means that f is asymptotically strictly smaller than g.*

Theorem 3.3. *The hypothesis of the Nivat conjecture alone cannot imply that a word has a period vector asymptotically smaller than $\sqrt{nk} e^{\sqrt{nk}}$.*

Proof. Let ℓ_m be as before. On a word ℓ_m, consider $\left(\sum_{i=0}^{m-1}\rho_i + 1\right) \times 1$ rectangles. Define $n_m = \sum_{i=0}^{m-1}\rho_i + 1$ and $k_m = 1$. We know that $p_{\ell_m}(\sum_{i=0}^{m-1}\rho_i + 1, 1) \leq \sum_{i=0}^{m-1}\rho_i + 1 = n_m k_m$, so ℓ_m satisfies the Nivat hypothesis.

Notice $n_m k_m = \sum_{i=0}^{m-1}\rho_i + 1 = \Theta(\frac{m^2}{\log m}) = O(m^2)$ [1]. Therefore, $m^2 = \Omega(n_m k_m)$ and $m = \Omega(\sqrt{n_m k_m})$. Now suppose $\mathbf{v}_m = \langle \prod_{i=0}^{m-1}\rho_i, 0\rangle$ is the minimal period of ℓ_m. Therefore, $||\mathbf{v}_m|| = \prod_{i=0}^{m-1}\rho_i = e^{(1+o(1))m\log m} = \Omega(me^m)$ [13]. This means that $||\mathbf{v}_m|| = \Omega(me^m) = \Omega(\sqrt{n_m k_m} e^{\sqrt{n_m k_m}})$.

Assume for contradiction there is some bound $b(nk)$ on the magnitude of the minimal period of a word f satisfying $p_f(n,k) \leq nk$ that is asymptotically stricter than $\sqrt{nk}e^{\sqrt{nk}}$, that is, $b(nk) = o(\sqrt{nk}e^{\sqrt{nk}})$. Remember that all ℓ_m satisfy $p_{\ell_m}(n_m k_m) \leq n_m k_m$. Therefore, $||\mathbf{v}_m|| \leq b(n_m k_m)$, meaning $||\mathbf{v}_m|| = o(\sqrt{n_m k_m} e^{\sqrt{n_m k_m}})$. Contradiction, $||\mathbf{v}_m|| = \Omega(\sqrt{n_m k_m} e^{\sqrt{n_m k_m}})$. Thus, any bound on the period of words satisfying the Nivat hypothesis cannot be asymptotically stricter than $\sqrt{nk}e^{\sqrt{nk}}$. □

The Morse-Hedlund theorem imposes a linear bound on the magnitude of the period of the word. This theorem showed that the Nivat conjecture cannot impose a bound on the magnitude of the period that is better than this nearly exponential $\sqrt{nk}e^{\sqrt{nk}}$.

While there is no magnitude bound on the period in $\mathbb{Z}^2$ analogous to the Morse-Hedlund theorem, stricter statements can be made about the direction of the period implied by the Nivat hypothesis (that there exist n, k such that $p_f(n,k) \leq nk$). The construction of the η-generating set in [6] implicitly shows that the Nivat hypothesis implies that if there is a period, it is in the direction of the difference of points in the $n \times k$ rectangle $R_{n\times k}$. Notationally, the period is in $\mathbb{Q}(R_{n\times k} - R_{n\times k}) := \{q(r_1 - r_2) : q \in \mathbb{Q}, r_1 \neq r_2 \in R_{n\times k}\}$.

By imposing a stronger bound on complexity, one can say something significantly stronger about the period of a word.

Theorem 3.4. *Let $f : \mathbb{Z}^2 \to \Sigma$ be a word satisfying $p_f(n,k) \leq n + k + |\Sigma| - 3$, then f*

is periodic with a period that is a multiple of a standard basis vector (i.e., the period is $\langle x, 0\rangle$ or $\langle 0, y\rangle$).

Proof. We will consider two cumulatively exhaustive cases: when there is some n such that $p_f(n,1) \le n + |\Sigma| - 2$ and when $p_f(n,1) \ge n + |\Sigma| - 1$ for all n.

Consider the case when there is some n such that $p_f(n,1) \le n + |\Sigma| - 2$. Fix such an n. Use $f \in \Sigma^{\mathbb{Z}^2}$ to construct an $f' \in (\Sigma^{\mathbb{Z}})^{\mathbb{Z}}$. Specifically, define $f'(y)(x) = f(x,y)$.

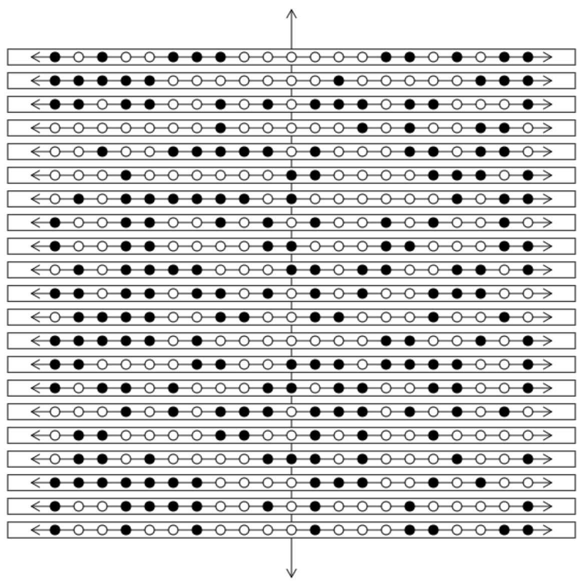

Figure 3.2: A representation of a possible $f' : \mathbb{Z} \to \Sigma^{\mathbb{Z}}$. It is visually similar to its corresponding $f : \mathbb{Z}^2 \to \Sigma$, which is just a coloring of the two dimensional integer lattice.

For any y, notice we can say that $p_{f'(y)}(n) \le n + |\Sigma| - 2$. By the alphabetically strengthened Morse-Hedlund theorem, all $f'(y)$ are periodic with some period $0 < v_y \le p_{f'(y)}(n)$. Since all v_y are at most n, we can say $v_y | p_{f'(y)}(n)!$. Thus all $f'(y)$ are periodic with period $p_{f'(y)}(n)!$. This means that for all x and y, $f'(y)(x + p_{f'(y)}(n)!) = f'(y)(x)$ and therefore, $f(x + p_{f'(y)}(n)!, y) = f(x,y)$. Thus, f is periodic with period $\langle p_{f'(y)}(n)!, 0\rangle$, which is a multiple of a standard basis vector.

Now consider the case when $p_f(n,1) \ge n + |\Sigma| - 1$. Define the set $N := \{0, 1, ..., k-$

$1\}$. Now use $f \in \Sigma^{\mathbb{Z}^2}$ to construct a new $f' \in \left((\Sigma^N)^{\mathbb{Z}}\right)^{\mathbb{Z}}$. Define $f'(x)(y)(h) := f(x+h, y)$.

Pick an arbitrary x. Notice that $f'(x)$ maps from $\mathbb{Z}$ to Σ^N. Thus, $f'(x)$ is a one-dimensional word whose alphabet is the finite set of functions Σ^N. Note $f'(x)$ need not be surjective; in fact, it achieves exactly the colorings of $n \times 1$ rectangles that actually appear in f. Call this set of colorings $P_f(n,1)$, and note that $|P_f(n,1)| = p_f(n,1)$. Using this, we can consider $f'(x) : \mathbb{Z} \to P_f(n,1)$ to be a surjective function. Thus, by the alphabetically strengthened Morse-Hedlund theorem, if $p_{f'(x)}(k) \leq k + |P_f(n,1)| - 2 = k + p_f(n,1) - 2$, then $f'(x)$ is periodic with a period at most $p_{f'(x)}(k)$.

Notice that $p_{f'(x)}(k) \leq p_f(n,k)$; every coloring in $P_{f'(x)}(k)$, which is a k long segment colored in with colored n long segments, is equivalent to some coloring in $P_f(n,k)$, which is a coloring of a $n \times k$ rectangles in f. We know $p_f(n,k) \leq n + k + |\Sigma| - 3$ by assumption. Since $p_f(n,1) \geq n + |\Sigma| - 1$, we know $p_f(n,1) \leq k + p_f(n,1) - 2$. Therefore,

$$p_{f'(x)}(k) \leq p_f(n,k) \leq k + p_f(n,1) - 2,$$

implying $f'(x)$ is periodic with some period of length at most k.

Since all x have $f'(x)$ periodic with a period $0 < v \leq p_{f'(x)}(k)$, and $v | p_{f'(x)}(k)!$, it is also periodic with period $p_{f'(x)}(k)!$. Therefore, $f'(x)(y + p_{f'(y)}(k)!) = f'(x)(y)$. Specifically, we have, $f'(x)(y + p_{f'(y)}(k)!)(0) = f'(y)(x)(0)$, implying $f(x, y + p_{f'(y)}(k)!) = f(x,y)$. Thus, f is periodic with period $\langle 0, p_{f'(y)}(k)! \rangle$, which is a multiple of a standard basis vector.

Thus, in all cases, the statement that there exist n, k such that $p_f(n,k) \leq n + k + |\Sigma| - 3$ implies f is periodic with a period that is a multiple of a standard basis vector, proving the claim. $\square$

Interestingly, this result is optimal. To observe this, examine the following example.

Example 3.2 (Low Complexity Word with Poorly Behaved Period)**.** *Define the clamp* *function $clamp_{a,b}(x)$ by $clamp_{a,b}(x) := a$ if $x \leq a$, $clamp_{a,b}(x) := x$ if $a \leq x \leq b$, and $clamp_{a,b}(x) := b$ if $x \geq b$.*

Now define a family of words $c_\sigma : \mathbb{Z}^2 \to \{0, 1, ..., \sigma-1\}$ by $c_\sigma(x, y) := clamp_{0,\sigma-1}(x+y)$.

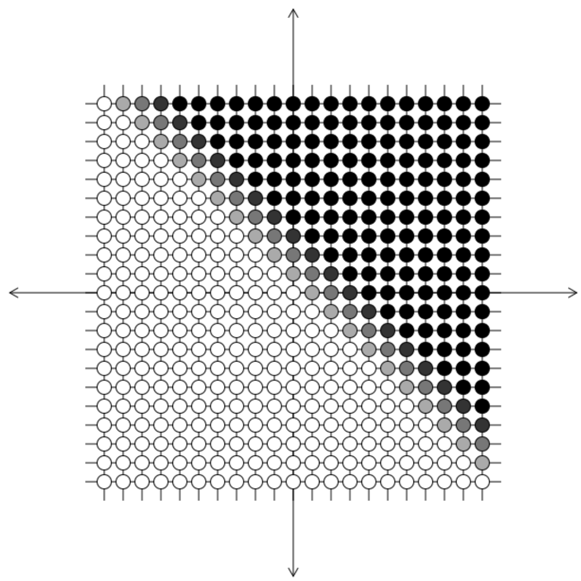

Figure 3.3: A visual representation of c_5. Here, 0 is white, 4 is black, and $1, 2$, and 3 are shades of gray.

Lemma 3.5. *The word c_σ is not periodic with a period that is a multiple of a standard basis vector.*

Proof. Fix σ, and assume for contradiction that c_σ is periodic with a multiple of a standard basis vector. Thus, it has a period $\langle x, y\rangle$ with $x = 0$ or $y = 0$, but not both. Therefore, $x + y \neq 0$. Thus, there is some integer κ such that $\kappa(x + y) \geq \sigma$.

Calculate that $c_\sigma(0, 0) = \text{clamp}_{0,\sigma-1}(0) = 0$. Since c_σ is periodic with $\langle x, y\rangle$, it is periodic with $\kappa\langle x, y\rangle$. Thus, $c_\sigma(\kappa x, \kappa y) = c_\sigma(0, 0) = 0$. This is a contradiction; we can calculate that $c_\sigma(\kappa x, \kappa y) = \text{clamp}_{0,\sigma-1}(\kappa(x + y)) = \sigma - 1 \neq 0$. Thus, c_σ is not periodic with a multiple of any standard basis vector. $\square$

Since c_σ is not periodic with a multiple of a standard basis vector, we necessarily have that $p_{c_\sigma}(n,k) > n+k+\sigma-3$. Surprisingly, c_σ misses this bound by exactly one.

Lemma 3.6. *For all c_σ, we have $p_{c_\sigma}(n,k) = n+k+\sigma-2$.*

Proof. Notice that the value of $c_\sigma(x,y) := \text{clamp}_{0,\sigma-1}(x+y)$ is only dependent on the value of $x+y$. Consider the values of $x+y$ that are achieved in an $n \times k$ rectangle originating from $\langle x_0, y_0 \rangle$. It achieves x_0+y_0, and $x_0+n-1+y_0+k-1 = x_0+y_0+n+k-2$. Since the rectangle is contiguous (one can construct a path of points distance one from each other from the bottom left to the top right) the rectangle achieves all values between x_0+y_0 and $x_0+y_0+n+k-2$. Thought of in a different way, it achieves the values in the contiguous segment of length $n+k-1$ originating from x_0+y_0.

Since x_0+y_0 can take on any integral value, colorings of $n \times k$ rectangles on c_σ over $\mathbb{Z}^2$ are analogous to colorings of $n+k-1$ long segments on $\text{clamp}_{0,\sigma-1}$ over $\mathbb{Z}$. More specifically, $p_{c_\sigma}(n,k) = p_{\text{clamp}_{0,\sigma-1}}(n+k-1)$. In general, $p_{\text{clamp}_{a,b}}(c) = c+b-a$. Thus, $p_{\text{clamp}_{0,\sigma-1}}(n+k-1) = n+k+\sigma-2$, implying $p_{c_\sigma}(n,k) = n+k+\sigma-2$, proving the claim. $\square$

It clearly follows from this complexity that the bound to imply a period that is a multiple of a standard basis vector is optimal.

Theorem 3.7. *The existence of $p_f(n,k) \leq n+k+|\Sigma|-3$ is the weakest statement that can imply a period that is a multiple of a standard basis vector for all words.*

Proof. Assume for contradiction that there is some bound $b(n,k) > n+k+|\Sigma|-3$ such that $p_f(n,k) \leq b(n,k)$ implies f is periodic with a period that is a multiple of a standard basis vector. Since $b(n,k) > n+k+|\Sigma|-3$, we know $b(n,k) \leq n+k+|\Sigma|-2$. Therefore, $p_f(n,k) \geq n+k+|\Sigma|-2$ would also imply this variety of periodicity.

Notice that the c_σ has an alphabet of size $|\Sigma| = \sigma$. We know $p_{c_\sigma}(n,k) = n+k+\sigma-2$, so c_σ is periodic with a period that is a multiple of a standard basis vector.

Contradiction, it is not. Thus, no such b can exist, so $n + k + |\Sigma| - 3$ is optimal. $\square$

There is much to be said about words with very strict complexity bounds. The hypothesis of the Nivat conjecture, however, is the most widely studied complexity bound [9, 12, 6, 7, 10]. Kari and Szabados showed that a finitary integral word (a word whose alphabet is a finite subset of $\mathbb{Z}$) satisfying the Nivat hypothesis can be written as a sum of finitely many periodic complex functions [10]. Unfortunately we can not directly say that every such word can be written as a sum of finitely many periodic words, since the periodic complex functions need not be finitary. Interestingly, we can say that a word satisfying the Nivat hypothesis actually is a finite modular sum of periodic finitary words. In order to see this, some background is required.

First, we will explore the ideas behind the field extension $\mathbb{C} : \mathbb{Q}$; that is considering $\mathbb{C}$ as a vector space over $\mathbb{Q}$. Remembering that a field is a set where addition, subtraction, multiplication, and division are defined, recall the formal definition of a vector space.

Definition 3.3. *A field F of scalars, and a set V of vectors, equipped with two operations, vector addition $+ : V \times V \to V$ and scalar multiplication $\cdot : F \times V \to V$, is a vector space if and only if it satisfies the following conditions.*

1. *For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$; vector addition is associative.*
2. *For all $\boldsymbol{u}, \boldsymbol{v} \in V$, $\boldsymbol{u} + \boldsymbol{v} = \boldsymbol{v} + \boldsymbol{u}$; vector addition is commutative.*
3. *There is a $\boldsymbol{0} \in V$ such that for all $\boldsymbol{v} \in V$, $\boldsymbol{v} + \boldsymbol{0} = \boldsymbol{v}$; there is a zero vector.*
4. *For all $\boldsymbol{v} \in V$, there is some $-\boldsymbol{v} \in V$ such that $\boldsymbol{v} + (-\boldsymbol{v}) = \boldsymbol{0}$; there are additive inverses.*
5. *For all $a, b \in F$ and $\boldsymbol{v} \in V$, $a(b\boldsymbol{v}) = (ab)\boldsymbol{v}$; multiplication is compatible.*
6. *Let $1 \in F$ be the identity. Then for all $\boldsymbol{v} \in V$, $1\boldsymbol{v} = \boldsymbol{v}$.*
7. *For all $a \in F$ and $\boldsymbol{u}, \boldsymbol{v} \in V$, $a(\boldsymbol{u} + \boldsymbol{v}) = a\boldsymbol{u} + a\boldsymbol{v}$; vector addition distributes.*

8. *For all $a, b \in F$ and $\boldsymbol{v} \in V$, $(a+b)\boldsymbol{v} = a\boldsymbol{v} + b\boldsymbol{v}$; field addition distributes.*

There are many different examples of vector spaces over many different fields. The set $\mathbb{R}^3$ is a vector space over the field $\mathbb{R}$; it satisfies all of the above properties. In $\mathbb{R}^3$, we can construct what is known as a basis. The standard basis is the set $\{\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle\}$. We call this a basis because every element $\langle x,y,z\rangle \in \mathbb{R}^3$ can be written uniquely as a finite sum of scalar multiples of elements in the basis, e.g. $\langle x,y,z\rangle = x\langle 1,0,0\rangle + y\langle 0,1,0\rangle + z\langle 0,0,1\rangle$. Interestingly, by assuming the axiom of choice, we can say that every single vector space has some basis, and that given a vector, there is some basis containing that vector [5].

Now consider $\mathbb{C}$ as a vector space with scalars in the field $\mathbb{Q}$. We know addition in $\mathbb{C}$ is associative and commutative. Notice $0 \in \mathbb{C}$ has that for all $z \in \mathbb{C}$, $z + 0 = z$. Every element in $\mathbb{C}$ has an additive inverse. See that since $\mathbb{Q} \subseteq \mathbb{C}$, we have that multiplication is compatible, that $1 \in \mathbb{Q}$ is the identity, and that addition distributes. Therefore, we can say that $\mathbb{C}$ is a vector space with scalars in $\mathbb{Q}$. This means that there is some basis for $\mathbb{C}$ containing the vector $1 \in \mathbb{C}$. Intriguingly, this means that there is some $B \subseteq \mathbb{C}$ with $1 \in B$ such every element $z \in \mathbb{C}$ can be written uniquely as a *finite* sum $\sum_{i=0}^{n} q_i b_i$ with $q_i \in \mathbb{Q}$ and $b_i \in B$. Since $\mathbb{C}$ with scalars in $\mathbb{Q}$ is an infinite vector space, this is known as a Hamel basis. This will eventually help show that every word satisfying the Nivat conjecture can actually be written as a sum of periodic rational valued functions.

Some sums of finitely many periodic functions can be conceptualized in an interesting way. Specifically, we can consider points in the sum as sums of lines over a strip.

Theorem 3.8. *Let $c : \mathbb{Z}^2 \to \mathbb{C}$ be a word such that there are periodic functions $c_i : \mathbb{Z}^2 \to \mathbb{C}$ that have $c = \sum_{i=0}^{m} c_i$. Say that all c_i are periodic with distinct periods $\langle 1, v_i\rangle$. Then, there is some 'vertical strip' function $s : \mathbb{Z}^2 \to \mathbb{C}$ satisfying $c(m,b) = \sum_{y=mx+b} s(x,y)$ and having some bound $B \in \mathbb{Z}^+$ where $s(x,y) = 0$ if $|x| > B$.*

Proof. We will construct s in the following way. For all $0 \leq i < m$, define $s(-v_i, y) :=$

$c_i(0, y)$. If $x \neq -v_i$ for any $0 \leq i < m$, define $s(x, y) := 0$.

There are finitely many v_i, so $B := \max_{0 \leq i < m} |v_i|$ is defined. Notice therefore that for all $|x| > B$, $s(x, y) = 0$.

We will now calculate $\sum_{y=mx+b} s(x, y)$. For all x not equal to any $-v_i$, we have $s(x, y) = 0$ by construction, so we need only consider x values equal to some $-v_i$. That is, $\sum_{y=mx+b} s(x, y) = \sum_{i=0}^{m-1} s(-v_i, -mv_i + b)$. By the definition of s, $\sum_{i=0}^{m-1} s(-v_i, -mv_i + b) = \sum_{i=0}^{m-1} c_i(0, -mv_i + b)$. Since c_i is periodic with period $\langle 1, v_i \rangle$, it is periodic with period $\langle m, mv_i \rangle$. Therefore, $c_i(0, -mv_i+b) = c_i(m, b)$. Thus, $\sum_{i=0}^{m-1} c_i(0, -mv_i + b) = \sum_{i=0}^{m-1} c_i(m, b)$. Since $c = \sum_{i=0}^{m-1} c_i$, $\sum_{i=0}^{m-1} c_i(m, b) = c(m, b)$. Thus, $\sum_{y=mx+b} s(x, y) = c(m, b)$, proving the claim. $\square$

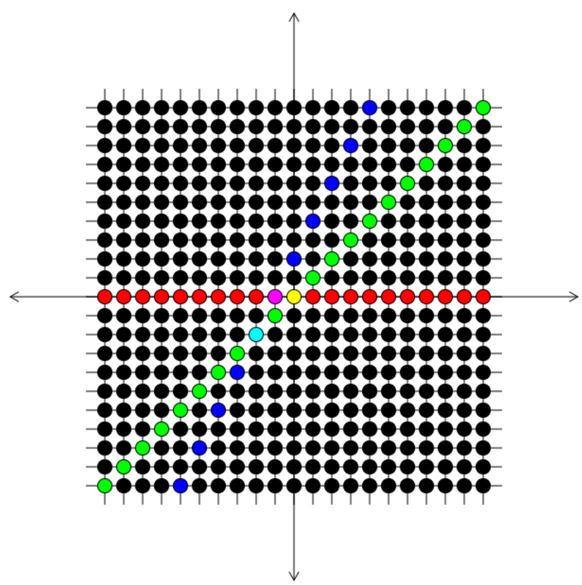

Figure 3.4: An example of a word that is a sum of finitely many words. Specificially 3 in this case.

By utilizing these concepts, we can determine that any integral finitary word is actually a finite modular sum of periodic finitary rational words.

Theorem 3.9. *Let $c : \mathbb{Z}^2 \to \mathbb{Z}$ be a finitary word satisfying the Nivat hypothesis, $p_c(n, k) \leq nk$. There are some $M, D \in \mathbb{Z}^+$ such that there are finitely many singly*

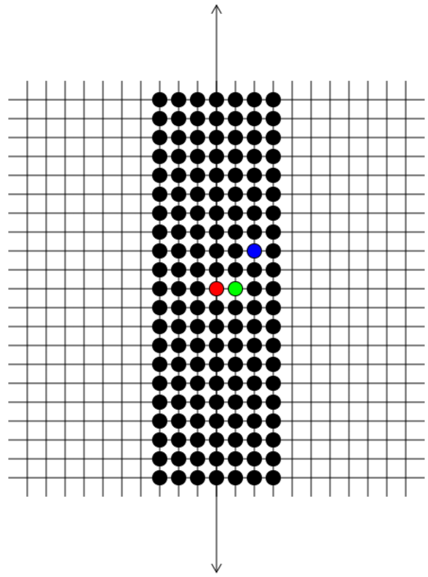

Figure 3.5: The strip coloring that generates the previous figure. Draw a line $y = mx + b$ on this strip, and add up the colors. Notice that this is always the color at the point (m, b) in the previous figure.

periodic $\hat{q}_i : \mathbb{Z} \to \frac{1}{D}\mathbb{Z}/\langle M \rangle$ such that $c \equiv \sum_{i=0}^{n'-1} \hat{q}_i (mod\ M)$, where $\frac{1}{D}\mathbb{Z}/\langle M \rangle$ is the set of all fractions $\frac{x}{D}$ with $x \in \mathbb{Z}$ considered modulo M.

Proof. Since c satisfies the Nivat hypothesis, by Kari-Szabados [10], we know there are finitely many periodic $c_i : \mathbb{Z}^2 \to \mathbb{C}$ such that $c = \sum_{i=0}^{n-1} c_i$.

Remember that there is some Hamel basis $B \subseteq \mathbb{C}$ such that $1 \in B$ and every element of $\mathbb{C}$ is a unique sum of rational multiples of elements in B. We will define a function $\mathbb{1} : \mathbb{C} \to \mathbb{Q}$. Let $z \in \mathbb{C}$; we know that we can uniquely write $z = \sum_{i=0}^{\mu_z} q_{z,i} b_{z,i}$ for $q_i \in \mathbb{Q} \setminus \{0\}$ and $b \in B$. By examining the coefficient of $1 \in B$ (which may be zero) specifically, we can write $z = q_{z,\mathbf{1}} \cdot 1 + \sum_{i=0}^{\mu'_z} q_{z,i} b_{z,i}$ for $q_{z,\mathbf{1}} \in \mathbb{Q}$ and $b \in B \setminus \{1\}$. Using this, define $\mathbb{1}(z) := q_{z,\mathbf{1}}$; that is, $\mathbb{1}(z)$ is the coefficient on $1 \in B$ in the finite sum of multiples of basis elements representing z. We know this function is well defined since the sum representation of z is unique. Let $z_1 = q_{z_1,\mathbf{1}} \cdot 1 + \sum_{i=0}^{\mu'_{z_1}} q_{z_1,i} b_{z_1,i}$ and $z_2 = q_{z_2,\mathbf{1}} \cdot 1 + \sum_{i=0}^{\mu'_{z_2}} q_{z_2,i} b_{z_2,i}$. Therefore, $z_1 + z_2 = (q_{z_1,\mathbf{1}} + q_{z_2,\mathbf{1}}) \cdot 1 + \sum_{i=0}^{\mu'_{z_1}} q_{z_1,i} b_{z_1,i} + \sum_{i=0}^{\mu'_{z_2}} q_{z_2,i} b_{z_2,i}$. See that $\mathbb{1}(z_1) = q_{z_1,\mathbf{1}}$, $\mathbb{1}(z_2) = q_{z_2,\mathbf{1}}$, and $\mathbb{1}(z_1 + z_2) = q_{z_1,\mathbf{1}} + q_{z_2,\mathbf{1}}$.

Thus, for any $z_1, z_2 \in \mathbb{C}$ we know $\mathbb{1}(z_1 + z_2) = \mathbb{1}(z_1) + \mathbb{1}(z_2)$. Now pick $q \in \mathbb{Q} \subseteq \mathbb{C}$. Since $q = q \cdot 1$, we can say $\mathbb{1}(q) = q$.

Use $\mathbb{1}$ to define new functions q_i. Specifically, define $q_i := \mathbb{1} \circ c_i$. Since $c : \mathbb{Z}^2 \to \mathbb{Z}$ and $\mathbb{Z} \subseteq \mathbb{Q}$, we know $\mathbb{1} \circ c = c$. Since $\mathbb{1}(z_1 + z_2) = \mathbb{1}(z_1) + \mathbb{1}(z_2)$ for any $z_1, z_2 \in \mathbb{C}$, $\mathbb{1}$ is a homomorphism for addition. This means that $\mathbb{1} \circ \sum_{i=0}^{n-1} c_i = \sum_{i=0}^{n-1} \mathbb{1} \circ c_i$. We can therefore calculate that $c = \mathbb{1} \circ c = \mathbb{1} \circ \sum_{i=0}^{n-1} c_i = \sum_{i=0}^{n-1} \mathbb{1} \circ c_i = \sum_{i=0}^{n-1} q_i$, or more succinctly $c = \sum_{i=0}^{n-1} q_i$. Since $\mathbb{1}$ is a well defined function and all c_i are periodic, all $q_i = \mathbb{1} \circ c_i$ are periodic. Therefore c is a sum of periodic rational functions.

Say that all q_i are periodic with $\langle x_i, y_i \rangle$. Since there are finitely many q_i, we know $\mu := \max_{0 \le i < n} |x_i| + 1$ exists. Use μ to construct a 'sheared' q_i' defined by $q_i'(x, y) = q_i(x - \mu y, y)$ and c' by $c'(x, y) = c(x - \mu y, y)$. Notice that we analogously have $c' = \sum_{i=0}^{n-1} q'$. Since $q_i'(x + x_i + \mu y_i, y + y_i) = q_i(x + x_i + \mu(-y), y + y_i) = q_i(x + \mu(-y), y) = q_i'(x, y)$, we can say all q_i' is periodic with $\langle x_i + \mu y_i, y_i \rangle$. Notationally say $\langle x_i', y_i' \rangle := \langle x_i + \mu y_i, y_i \rangle$. Intuitively, all periods are 'sheared' as well.

If $y_i = 0$, then $x_i' = x_i + \mu y_i = x_i \neq 0$. If $y_i \neq 0$, then $|y_i| \geq 1$. Thus, since $\mu := \max_{0 \le i < n}\{|x_i|\} + 1$, we have $|\mu y_i| = \mu |y_i| \geq \mu > |x_i|$ for all i. Thus, $x_i' = x_i + \mu y_i \neq 0$. In all cases, $x_i' = x_i + \mu y_i \neq 0$. Thus, no period of any q_i' is vertical.

Define $\gamma := \gcd_{0 \le i < n} x_i'$ since no x_i' is zero. Define $\kappa_i \in \mathbb{Z}$ by $\kappa_i := \gamma / x_i'$. Notice q_i' is periodic with $\kappa_i \langle x_i', y_i' \rangle = \langle \gamma, \kappa_i y_i' \rangle$. Define $v_i := \kappa_i y_i'$. For $0 \le j < \gamma$ define $c_{*,j}'(x, y) := c'(\gamma x + j, y)$ and $q_{i,j}'(x, y) := q_i'(\gamma x + j, y)$. Since they are defined in a similar way, $c_{*,j}' = \sum_{i=0}^{n-1} q_{i,j}'$ for all j.

Notice that $q_{i,j}'(x + 1, y + v_i) := q_i'(\gamma(x + 1) + j, y + v_i) = q_i'(\gamma x + j, y) = q_{i,j}'(x, y)$. Therefore, $q_{i,j}'$ has a period $\langle 1, v_i \rangle$. This means that we can construct γ many s_j by $s_j(-v_i, y) := q_{i,j}'(0, y)$ and $\sum_{y = mx + b} s_j(x, y) = c_{*,j}'(m, b)$ as in Theorem 3.8. Additionally, there is some B such that for $|x| > B$, $s_j(x, y) = 0$.

We will now consider $s_j (\text{mod } 1)$. Notice that $c_{*,j}'(\mathbb{Z}^2) \subseteq c'(\mathbb{Z}^2) = c(\mathbb{Z}^2) \subseteq \mathbb{Z}$. Thus, $c_{*,j}' \equiv 0 (\text{mod } 1)$. Thus, $\sum_{i=0}^{n-1} q_{i,j} \equiv 0 (\text{mod } 1)$ for all j. We will show that there are finitely many elements of $s_j(\mathbb{Z}^2)$ such that all elements of $s_j(\mathbb{Z}^2) \pmod 1$ are the sums and negations of those elements modulo 1. For $|x| > B$, $s_j(x,y) = 0$, which is the value of the empty sum. Now construct $S_j := \{s_j(x,y) : |x| \leq B, |y| \leq x^2 + 4B^2 + 1\}$. Notice that S_j is contained in the rectangle $\{s_j(x,y) : |x| \leq B, |y| \leq 5B^2 + 1\}$, so S_j is finite.

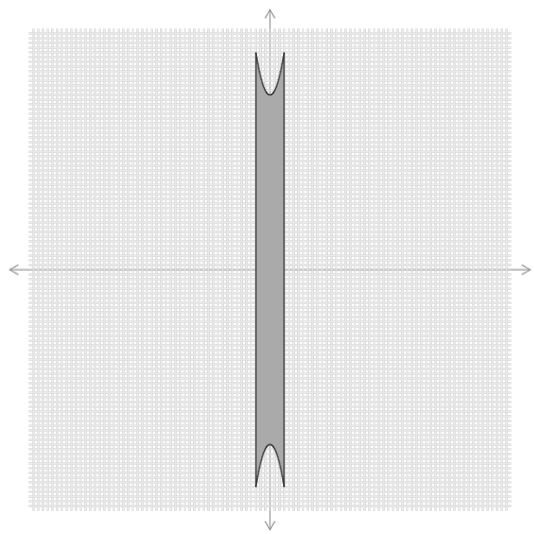

Figure 3.6: A visual representation of the lattice points represented in S_j for strip width $B = 3$.

We will show specifically that all values of $s_j(x,y)$ with $|x| \leq B$ are sums or negations of elements of S_j; that is $s_j(x,y) = \sum_{s \in \mathcal{S}} (-1)^{c_s} s$ for some powers c_s where $\mathcal{S} \subseteq S_j$. To show this we will show that all elements of $\{s_j(x,y) : |x| \leq B, |y| \leq x^2 + 4B^2 + y_0\}$ for all $y_0 \geq 1$ are sums and negations of elements of S_j. Proceed by induction on y_0. For y_0 the claim is clear because $\{s_j(x,y) : |x| \leq B, |y| \leq x^2 + 4B^2 + 1\} = S_j$.

Now assume the claim for $y_0 - 1$ to prove it for y_0. Pick an arbitrary element of $\{s_j(x,y) : |x| \leq B, |y| \leq x^2 + 4B^2 + y_0\} \setminus \{s_j(x,y) : |x| \leq B, |y| \leq x^2 + 4B^2 + y_0 - 1\}$. This element necessarily is $s_j(x_0, y')$ for $|x_0| \leq B$ and $y' = \pm(x_0^2 + 4B^2 + y_0)$. By symmetry, without loss of generality, say the element is $s_j(x_0, x_0^2 + 4B^2 + y_0)$. Now consider the line $y = (2x_0)x + (-x_0^2 + 4B^2 + y_0)$. Also notice that since $|x - x_0| \leq |x| + |x_0| \leq 2B$, we know $(x - x_0)^2 \leq 4B^2$. Rewrite $y = x^2 - (x - x_0)^2 + 4B^2 + y_0$.

Notice $x^2-(x-x_0)^2+4B^2+y_0 \geq x^2-4B^2+4B^2+y_0 \geq x^2+y_0 \geq 0$, so $y \geq 0$. Say $x \neq x_0$, so $(x-x_0)^2 \geq 1$. Thus, $x^2-(x-x_0)^2+4B^2+y_0 \leq x^2+4B^2+y_0-1$ and therefore $y \leq x^2+4B^2+y_0-1$. Thus, for $x \neq x_0$, we have $0 \leq y \leq x^2+4B^2+y_0-1$ and $|y| \leq x^2+4B^2+y_0-1$. This means that elements on the line $y=(2x_0)x+(-x_0^2+4B^2+y_0)$ with $x \neq x_0$ are inside $\{\langle x,y\rangle : |x| \leq B, |y| \leq x^2+4B^2+y_0-1\}$. We know that $\sum_{(2x_0)x+(-x_0^2+4B^2+y_0)} s_j(x,y) = c'_{*,j}(2x_0, -x_0^2+4B^2+y_0) \equiv 0 \pmod 1$. Therefore, $s_j(x_0, x_0^2+4B^2+y_0) = s_j(x_0, (2x_0)x_0+(-x_0^2+4B^2+y_0)) \equiv -\sum_{(2x_0)x+(-x_0^2+4B^2+y_0), x\neq x_0} s_j(x,y) \pmod 1$. All elements in that sum are elements of $\{s_j(x,y) : |x| \leq B, |y| \leq x^2+4B^2+y_0-1\}$, which by the induction hypothesis are sums and negations of elements of S_j mod 1. Thus, $s_j(x_0, x_0^2+4B^2+y_0)$ is a sum of negations of elements of S_j mod 1. Since $s_j(x_0, x_0^2+4B^2+y_0)$ was chosen arbitrarily, all elements of $\{s_j(x,y) : |x| \leq B, |y| \leq x^2+4B^2+y_0\} \setminus \{s_j(x,y) : |x| \leq B, |y| \leq x^2+4B^2+y_0-1\}$ are sums and negations of elements in S_j, proving the inductive step.

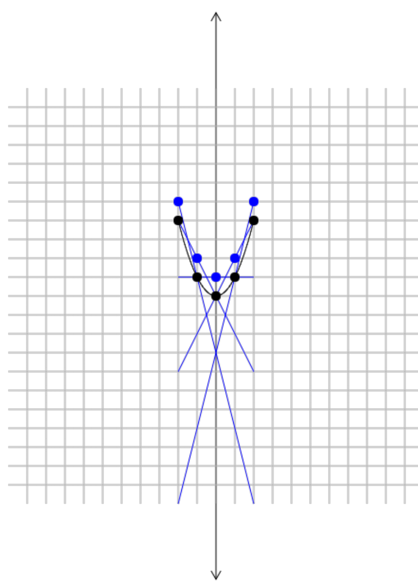

Figure 3.7: A representation of the 'frontier' $\{s_j(x,y) : |x| \leq B, |y| \leq x^2+4B^2+y_0\} \setminus \{s_j(x,y) : |x| \leq B, |y| \leq x^2+4B^2+y_0-1\}$. The inductive step calculates the values of the blue points modulo 1 based on the values of the points on the blue lines. Notice that each blue line goes through exactly one blue point and the rest of its integer points are on or below the parabola.

Thus, all elements in $\{s_j(x,y) : |x| \leq B, |y| \leq x^2+4B^2+y_0\}$ for any y_0 are sums and negations of elements in S_j. Pick an arbitrary $s_j(x', y')$ with $|x'| \leq B$ and notice that this is in $\{s_j(x, y) : |x| \leq B, |y| \leq x^2 + 4B^2 + y'\}$. Therefore, every $s_j(x, y)$ with $|x| \leq B$ are sums and negations of the finitely many elements of S_j. The set S_j contains finitely many rational numbers, so there is some common denominator D_j such that $S_j \subseteq \frac{1}{D_j}\mathbb{Z} := \{\frac{x}{D_j} : x \in \mathbb{Z}\}$. Since $\frac{1}{D_j}\mathbb{Z}$ is closed under addition and negation, and all elements of $s_j(\mathbb{Z})$ are sums and negations of elements of S_j modulo 1, we know $s_j(\mathbb{Z}^2) \subseteq \frac{1}{D_j}\mathbb{Z}$.

Pick an arbitrary $q'_{i,j}(x, y)$. Remember $q'_{i,j}(x, y) = s_j(-v_i, y)$, so $q'_{i,j}(x, y) \in \frac{1}{D_j}\mathbb{Z}$. Therefore, $q'_{i,j}(\mathbb{Z}^2) \subseteq \frac{1}{D_j}\mathbb{Z}$. Define $D := \gcd_{0 \leq j < \gamma} D_j$. Notice that $\frac{1}{D_j}\mathbb{Z} \subseteq \frac{1}{D}\mathbb{Z}$. Thus, for all j, we have $q'_{i,j}(\mathbb{Z}^2) \subseteq \frac{1}{D}\mathbb{Z}$

Pick an arbitrary $q'_i(x, y)$. We know $x = x'\gamma + j$ with $0 \leq r < \gamma$. Remember, $q'_i(x, y) = q_{i,j}(x', y)$. Thus, $q'_i(x, y) \in \frac{1}{D}\mathbb{Z}$, implying $q'_i(\mathbb{Z}^2) \subseteq \frac{1}{D}\mathbb{Z}$. Since the 'shearing' transformation was a bijection, $q_i(\mathbb{Z}^2) = q'_i(\mathbb{Z}^2)$. Therefore, $q_i(\mathbb{Z}^2) \subseteq \frac{1}{D}\mathbb{Z}$.

Since c is finitary, we can define $M := \max c(\mathbb{Z}^2) - \min c(\mathbb{Z}^2) + 1$. Let $\frac{1}{D}\mathbb{Z}/\langle M \rangle$ denote $\frac{1}{D}\mathbb{Z}$ modulo M. From $q_i : \mathbb{Z}^2 \to \mathbb{Q}$ which can be seen as $q_i : \mathbb{Z}^2 \to \frac{1}{D}\mathbb{Z}$, directly construct $\hat{q}_i : \mathbb{Z}^2 \to \frac{1}{D}\mathbb{Z}/\langle M \rangle$. If $q_i(x + x_i, y + y_i) = q_i(x, y)$, then $q_i(x + x_i, y + y_i) \equiv q_i(x, y) (\text{mod } M)$, so $\hat{q}_i(x + x_i, y + y_i) = \hat{q}_i(x, y)$. Therefore, since q_i is periodic, $\hat{q}_i$ is periodic the the same direction.

Since $M := \max c(\mathbb{Z}^2) - \min c(\mathbb{Z}^2) + 1$, for any $a, b \in c(\mathbb{Z}^2)$, $a = b$ if and only if $a \equiv b (\text{mod } M)$, so intuitively, no information is lost considering c modulo M. Since $c = \sum_{i=0}^{n-1} q_i$, we know $c \equiv \sum_{i=0}^{n-1} \hat{q}_i$.

Now consider the $\hat{q}_i$ that are doubly periodic. Summing these gives one doubly periodic word. Summing this to some singly periodic $\hat{q}_i$ yields one singly periodic word. Considering these sums in the context of $c \equiv \sum_{i=0}^{n-1} \hat{q}_i$, we see that c is the sum of finitely many singly periodic words, proving the claim. □

This means that any word satisfying the Nivat hypothesis can be written as a finite sum of words with finite alphabet, where the "sum" takes place in $\frac{1}{D}\mathbb{Z}/\langle M \rangle$.

The advantage of a finite alphabet is that many dynamical properties hold that do not necessarily hold for infinite alphabets. Expansiveness is one such property. When considering expansiveness with respect to this decomposition, useful results follow.

We now prove a lemma that is very similar to a result of Kari-Szabados [10].

Lemma 3.10. *Let $c : \mathbb{Z}^2 \to \mathbb{Z}$ be finitary integral word and suppose that we have a decomposition $c = \sum_{i=0}^{n-1} \hat{q}_i$ where all of the $\hat{q}_i : \mathbb{Z}^2 \to \mathbb{Q}$ are periodic with periods in the set $V := \{\mathbf{v}_i : 0 \leq i < n\}$. Then $\sum_{V' \subseteq V} (-1)^{|V'|} T^{\Sigma V'} c = 0$. Here, $T^{\mathbf{v}}$ represents a shift: $T^{\mathbf{v}} c(\mathbf{x}) := c(\mathbf{x} - \mathbf{v})$, and $\Sigma V'$ is shorthand for $\sum_{\mathbf{v} \in V'} \mathbf{v}$.*

Proof. To prove this claim for all $c = \sum_{i=0}^{n-1} \hat{q}_i$, we proceed by induction on n.

Consider the case when $n = 0$. Thus, $c = 0$ is the empty sum. Similarly, $\sum_{V' \subseteq \emptyset} (-1)^{|V'|} T^{\Sigma V'} c = 0$ is also the empty sum; this proves the claim for $n = 0$.

Assume the claim is true for $n - 1$ to prove the claim for n. Pick some function $c = \sum_{i=0}^{n-1} \hat{q}_i$. Let $\mathbf{v}_{n-1}$ be a period of $\hat{q}_{n-1}$. Define $V^* := V \setminus \{\mathbf{v}_{n-1}\}$. Notice that

$$\sum_{V' \subseteq V} (-1)^{|V'|} T^{\Sigma V'} c = \sum_{V' \subseteq V*} (-1)^{|V'|} T^{\Sigma V'} c + \sum_{V' \subseteq V*} (-1)^{|V' \cup \{\mathbf{v}_{n-1}\}|} T^{\Sigma (V' \cup \{\mathbf{v}_{n-1}\})} c;$$

essentially, this splits the sum into subsets that don't contain $\mathbf{v}_{n-1}$ and those that do. We can calculate that

$$(-1)^{|V' \cup \{\mathbf{v}_{n-1}\}|} T^{\Sigma (V' \cup \{\mathbf{v}_{n-1}\})} c = -(-1)^{|V'|} T^{\Sigma V' + \mathbf{v}_{n-1}} c = -(-1)^{|V'|} T^{\mathbf{v}_{n-1}} T^{\Sigma V'} c.$$

This means that

$$\begin{aligned}
\sum_{V' \subseteq V*} (-1)^{|V'|} T^{\Sigma V'} c + \sum_{V' \subseteq V*} (-1)^{|V' \cup \{\mathbf{v}_{n-1}\}|} T^{\Sigma (V' \cup \{\mathbf{v}_{n-1}\})} c \\
&= \sum_{V' \subseteq V*} (-1)^{|V'|} T^{\Sigma V'} c - \sum_{V' \subseteq V*} (-1)^{|V'|} T^{\mathbf{v}_{n-1}} T^{\Sigma V'} c \\
&= \sum_{V' \subseteq V*} (-1)^{|V'|} \left(T^{\Sigma V'} c - T^{\mathbf{v}_{n-1}} T^{\Sigma V'} c \right).
\end{aligned}$$

Since the translation of sums of words is the sum of translations of words, from $c = \sum_{i=0}^{n-1} \hat{q}_i$ we can deduce that

$$\begin{aligned}
&\sum_{V' \subseteq V*} (-1)^{|V'|} \left(T^{\Sigma V'} c - T^{\mathbf{v}_{n-1}} T^{\Sigma V'} c \right) \\
&\quad = \sum_{V' \subseteq V*} (-1)^{|V'|} \left(T^{\Sigma V'} \sum_{i=0}^{n-1} \hat{q}_i - T^{\mathbf{v}_{n-1}} T^{\Sigma V'} \sum_{i=0}^{n-1} \hat{q}_i \right) \\
&\quad = \sum_{V' \subseteq V*} (-1)^{|V'|} \left(T^{\Sigma V'} \sum_{i=0}^{n-2} \hat{q}_i - T^{\mathbf{v}_{n-1}} T^{\Sigma V'} \sum_{i=0}^{n-2} \hat{q}_i + T^{\Sigma V'} \hat{q}_{n-1} - T^{\mathbf{v}_{n-1}} T^{\Sigma V'} \hat{q}_{n-1} \right)
\end{aligned}$$

Since $\mathbf{v}_{n-1}$ is a period of $\hat{q}_{n-1}$, it is also a period of $T^{\Sigma V'} \hat{q}_{n-1}$. Thus, $T^{\mathbf{v}_{n-1}} T^{\Sigma V'} \hat{q}_{n-1} = T^{\Sigma V'} \hat{q}_{n-1}$, meaning that $\hat{q}_{n-1} - T^{\mathbf{v}_{n-1}} T^{\Sigma V'} \hat{q}_{n-1} = 0$. Therefore, the above sum is equal to

$$\sum_{V' \subseteq V*} (-1)^{|V'|} \left(T^{\Sigma V'} \sum_{i=0}^{n-2} \hat{q}_i - T^{\mathbf{v}_{n-1}} T^{\Sigma V'} \sum_{i=0}^{n-2} \hat{q}_i \right).$$

Notice that this is equal to $\sum_{V' \subseteq V*} (-1)^{|V'|} T^{\Sigma V'} \sum_{i=0}^{n-2} (\hat{q}_i - T^{\mathbf{v}_{n-1}} \hat{q}_i)$. Since every $\mathbf{v}_i$ is a period of $\hat{q}_i$, it is also a period of $T^{\mathbf{v}_{n-1}} \hat{q}_i$, and therefore a period of $\hat{q}_i - T^{\mathbf{v}_{n-1}} \hat{q}_i$. Define $c^* := \sum_{i=0}^{n-2} (\hat{q}_i - T^{\mathbf{v}_{n-1}} \hat{q}_i)$. We can now write the sum as $\sum_{V' \subseteq V*} (-1)^{|V'|} T^{\Sigma V'} c^*$. Notice that c^* is the sum of $n-1$ many period words whose periods are the elements of V^*. Thus, by the inductive hypothesis, $\sum_{V' \subseteq V*} (-1)^{|V'|} T^{\Sigma V'} c^* = 0$. By the transitive property of equality, this means $\sum_{V' \subseteq V} (-1)^{|V'|} T^{\Sigma V'} c = 0$, proving the inductive step.

Therefore, for any $\sum_{V' \subseteq V} (-1)^{|V'|} T^{\Sigma V'} c = 0$ for any $c = \sum_{i=0}^{n-1} \hat{q}_i$ where $\hat{q}_i$ have periods V. $\square$

These results are useful in a statement about the expansiveness of directions, a tool used extensively in the study of the Nivat conjecture [2, 6, 7].

Definition 3.4. *Let $c : \mathbb{Z}^2 \to \Sigma$ be a word. We say that a direction is expansive if and only if the half-plane defined by that direction (the half-plane to one side of the line in that direction), call it H, has the following property. For any $c_1, c_2 \in \overline{\mathcal{O}(f)}$, if*

$c_1|_H = c_2|_H$*, then* $c_1 = c_2$*. Here* $\overline{\mathcal{O}(f)}$ *is the orbit closure of* f*; that is,* $\overline{\mathcal{O}(f)}$ *contains all translations and limits of translations of* f*. Additionally,* $c_1|H : H \to \Sigma$ *is the restriction of* c_1 *to* H*, which has the same value as* c_1 *over* H*, but is undefined outside* H*.*

The previous results allow us to say that nonexpansive directions come in pairs.

Theorem 3.11. *Let* $c : \mathbb{Z}^2 \to \mathbb{Z}$ *be a finitary word satisfying the hypothesis of Nivat's conjecture. Then,* c *is nonexpansive in finitely many paired antiparallel directions.*

Proof. Since $c : \mathbb{Z}^2 \to \mathbb{Z}$ satisfies the Nivat hypothesis, there are $D, M \in \mathbb{Z}^+$ and $\hat{q}_i : \mathbb{Z}^2 \to \frac{1}{D}\mathbb{Z}/\langle M \rangle$ such that $c = \sum_{i=0}^{n-1} \hat{q}_i$ and all $\hat{q}_i$ are singly periodic with some $\mathbf{v}_i$. Without loss of generality, we may say that these periods are pairwise linearly independent.

Let H be a half-plane in $\mathbb{Z}^2$ in a direction $\mathbf{v}$ not parallel to $\mathbf{v}_i$ for any $0 \leq i < n$. Assume for contradiction that this direction is nonexpansive. From the definition of expansiveness, there are $c_1 \neq c_2 \in \overline{\mathcal{O}(c)}$ such that $c_1|_H = c_2|_H$. Since $c_1 \neq c_2$, there is some $\mathbf{x}'$ such that $c_1(\mathbf{x}') \neq c_2(\mathbf{x}')$.

Let $\mathbf{n}$ be the normal vector of H. We know there is some b such that $H = \{\mathbf{x} : \mathbf{n} \cdot \mathbf{x} \geq b\}$. Since $\mathbf{v}$ is not parallel to any $\mathbf{v}_i$, we have $\mathbf{n} \cdot \mathbf{v}_i \neq 0$. By the Archimedean property, for all i there is some k_i such that $k_i(\mathbf{n} \cdot \mathbf{v}_i) \geq b - \mathbf{n} \cdot \mathbf{x}'$. Notice that $b - \mathbf{n} \cdot \mathbf{x}' > 0$ since $\mathbf{x}' \notin H$. Thus, $k_i(\mathbf{n} \cdot \mathbf{v}_i) \geq 0$. Thus, for any subset $I \neq \emptyset \subseteq [0, n)$, we have $\sum_{i \in I} k_i(\mathbf{n} \cdot \mathbf{v}_i) \geq b - \mathbf{n} \cdot \mathbf{x}'$. By the properties of the dot product, this is equivalent to $\mathbf{n} \cdot \mathbf{x}' + \sum_{i \in I} \mathbf{n} \cdot (k_i \mathbf{v}_i) \geq b$ and also $\mathbf{n} \cdot (\mathbf{x}' + \sum_{i \in I} k_i \mathbf{v}_i) \geq b$. Notice that is is the same as saying that $\mathbf{x}' + \sum_{i \in I} k_i \mathbf{v}_i \in H$ for any $I \neq \emptyset \subseteq [0, n)$.

Now notice that $W := \{\sum_{i \in I} k_i \mathbf{v}_i : I \subseteq [0, n)\} \subseteq \mathbb{Z}^2$ is a finite set of size at most 2^n. Since c_1, c_2 are limit points of c, (that is $c_1, c_2 \in \overline{\mathcal{O}(c)}$) we know that the colorings $c_1|_W$ and $c_2|_W$ are the same as the coloring of some shift of W in c.

Now consider an arbitrary shift $\mathbf{x} + W$ of W. Notice that a coloring of this

$\mathbf{x}+W$ on c is equivalent to a coloring of $\{\sum_{i\in I} k_i\mathbf{v}_i : I \subseteq [0,n)\}$ over $T^{-\mathbf{x}}c$. Also, since $c = \sum_{i=0}^{n-1} \hat{q}_i$, we have $T^{-\mathbf{x}}c = \sum_{i=0}^{n-1} T^{-\mathbf{x}}\hat{q}_i$. Since $\hat{q}_i$ is periodic with period $\mathbf{v}_i$, it is also periodic with period $k_i\mathbf{v}_i$. Thus, $T^{-\mathbf{x}}\hat{q}_i$ is periodic with $-k_i\mathbf{v}_i$ as well. By applying the last lemma we know that $\sum_{I\subseteq[0,n)}(-1)^{|I|}T^{\sum_{i\in I} -k_i\mathbf{v}_i}T^{-\mathbf{x}}c = 0$. By composition of shifts, we can say that $\sum_{I\subseteq[0,n)}(-1)^{|I|}T^{-\mathbf{x}-\sum_{i\in I} k_i\mathbf{v}_i}c = 0$.

Let $\mathbf{x}_1 + W$ and $\mathbf{x}_2 + W$ be specific shifts of W that are colored in c the same way that W is colored in c_1 and c_2 respectively. Use the previous result for arbitrary shifts to realize that $\sum_{I\subseteq[0,n)}(-1)^{|I|}T^{-\mathbf{x}_1-\sum_{i\in I} k_i\mathbf{v}_i}c = 0$ and $\sum_{I\subseteq[0,n)}(-1)^{|I|}T^{-\mathbf{x}_2-\sum_{i\in I} k_i\mathbf{v}_i}c = 0$. By the choice of $\mathbf{x}_1$ and $\mathbf{x}_2$, we know that $\sum_{I\subseteq[0,n)}(-1)^{|I|}T^{-\sum_{i\in I} k_i\mathbf{v}_i}c_1 = 0$ and that $\sum_{I\subseteq[0,n)}(-1)^{|I|}T^{-\sum_{i\in I} k_i\mathbf{v}_i}c_2 = 0$. By examining the point $\mathbf{x}'$, we see that $\sum_{I\subseteq[0,n)}(-1)^{|I|}c_1(\mathbf{x}' + \sum_{i\in I} k_i\mathbf{v}_i) = 0$ and $\sum_{I\subseteq[0,n)}(-1)^{|I|}c_2(\mathbf{x}' + \sum_{i\in I} k_i\mathbf{v}_i) = 0$. We can notationally separate the $I = \emptyset$ case to see that $c_1(\mathbf{x}') = -\sum_{I\neq\emptyset\subseteq[0,n)}(-1)^{|I|}c_1(\mathbf{x}'+ \sum_{i\in I} k_i\mathbf{v}_i)$ and that $c_2(\mathbf{x}') = -\sum_{I\neq\emptyset\subseteq[0,n)}(-1)^{|I|}c_2(\mathbf{x}' + \sum_{i\in I} k_i\mathbf{v}_i)$. Remember that for all $I \neq \emptyset \subseteq [0,n)$, we have that $\mathbf{x}' + \sum_{i\in I} k_i\mathbf{v}_i \in H$. Since $c_1|_H = c_2|_H$ this means that $c_1(\mathbf{x}' + \sum_{i\in I} k_i\mathbf{v}_i) = c_2(\mathbf{x}' + \sum_{i\in I} k_i\mathbf{v}_i)$. Therefore, $-\sum_{I\neq\emptyset\subseteq[0,n)}(-1)^{|I|}c_1(\mathbf{x}' + \sum_{i\in I} k_i\mathbf{v}_i) = -\sum_{I\neq\emptyset\subseteq[0,n)}(-1)^{|I|}c_2(\mathbf{x}' + \sum_{i\in I} k_i\mathbf{v}_i)$. Thus, $c_1(\mathbf{x}') = c_2(\mathbf{x}')$. This is a contradiction, as $c_1(\mathbf{x}') \neq c_2(\mathbf{x}')$. Thus, H must be expansive.

Now we know that all directions that are not parallel to any $\mathbf{v}_i$ are expansive. We will now show that any direction parallel to some $\mathbf{v}_i$ is nonexpansive. Fix some specific $0 \leq i' < n$. We can say that no two $\mathbf{v}_i$ are parallel to each other; therefore, $\mathbf{v}_{i'}$ is not parallel to any $\mathbf{v}_i$ for $i \neq i'$. Consider $c' := \sum_{i\neq i'\in[0,n)} \hat{q}_i$. By the previous argument, c' is expansive in the directions parallel to $\mathbf{v}_{i'}$, since they would not be parallel to any other $\mathbf{v}_i$. Since $\hat{q}_{i'}$ is singly periodic with period $\mathbf{v}_i$, the word $\hat{q}_{i'}$ is nonexpansive in all directions parallel to $\mathbf{v}_i$. If one word is expansive in a direction, and another word in nonexpansive in that same direction, the sum is nonexpansive in that direction. Thus, $c' + \hat{q}_{i'} = c$ is nonexpansive in all directions parallel to an arbitrary $\mathbf{v}_{i'}$.

Since c is nonexpansive in all directions parallel to some $\mathbf{v}_i$, and expansive in all other directions, the nonexpansive directions of c are exactly the finitely many paired antiparallel directed lines defined by $\pm\mathbf{v}_i$. Thus, the claim is proven. $\square$

Chapter 4

A Case in Height 4

Cyr and Kra [6] accumulate a large quantity of machinery for handling short rectangles. Specifically, the application of this machinery in their paper proves Nivat's conjecture for rectangles of height at most 3 [6]. Much of this machinery can be applied to rectangles height 4. Thus, in order to efficiently prove a result in height 4, we start by summarizing the relevant results of Cyr and Kra.

It is first important to define "extending uniquely".

Definition 4.1 (Unique Extension [6]). *For some $S_1 \subseteq S_2 \subseteq \mathbb{Z}^2$, we say that S_1 extends uniquely to S_2 if and only if for all $\eta_1, \eta_2 \in \overline{\mathcal{O}(\eta)}$, having $\eta_1|_{S_1} = \eta_2|_{S_1}$ implies that $\eta_2|_{S_1} = \eta_2|_{S_2}$.*

Notice that given this definition, a direction with half-plane H is expansive if and only if H extends uniquely to $\mathbb{Z}^2$. Using unique extension, we can define η-generating sets.

Definition 4.2 (Generating Sets [6]). *Let S be a finite convex subset of $\mathbb{Z}^2$. We say that S is an η-generating set if and only if for all $\boldsymbol{x} \in S$, the subset $S \setminus \{\boldsymbol{x}\}$ extends uniquely to S.*

The existence of η-generating sets is implied by the hypothesis of the Nivat conjecture.

Proposition 4.1. *Let $\eta : \mathbb{Z}^2 \to \Sigma$ be a word satisfying $p_f(n, 4) \leq 4n$, then there is some η-generating set S that is a subset of the $n \times 4$ rectangle. Additionally, if ℓ is a nonexpansive direction ℓ, there is some edge of S that is parallel to ℓ.*

Proof. The argument is a straightforward extension of Proposition 2.7 in [6]. □

Balanced sets are a tool used extensively in [6], which are defined as follows.

Definition 4.3 (Balanced Sets [6])**.** *Let S be a finite, convex, subset of $\mathbb{Z}^2$, and ℓ a line. We say that S is ℓ-balanced if an only if all of the following hold.*

1. *S has an edge w parallel to ℓ.*
2. *The endpoints of w, call them x_0, x_1 have that $S \setminus \{x_0\}$ and $S \setminus \{x_1\}$ both extend uniquely to S.*
3. *The number of ways to color $S \setminus w$ is less than $|w \cap S|$ fewer than the total number of ways to color S.*
4. *Every line parallel to ℓ that intersects S intersects it in at least $|w \cap S| - 1$ many points.*

This machinery leads naturally to a specific subcase of the Nivat conjecture.

Theorem 4.2. *There is no word $\eta' : \mathbb{Z}^2 \to \Sigma$ with 4 nonexpansive lines satisfying $p_{\eta'}(n, 4) \leq 4n$ for some n (in the case of incident simultaneous witnesses to nonexpansiveness).*

Proof. Assume for contradiction that such an η' exists. Since $p_{\eta'}(n, 4) \leq 4n$, let us construct an η'-generating set S' that is a subset of the $n \times 4$ rectangle. Notice that this implies that S' has height at most 4.

Notice that since there are 4 nonexpansive lines, there are 8 nonexpansive directions, which come in antiparallel pairs. Remember that every nonexpansive direction is parallel to some edge of any generating set, specifically S'. Therefore, S' has at least 8 edges. The set S' is convex and has height 4, geometrically implying that it can have at most 8 edges, opposite edges are antiparallel, and the nonhorizontal edges contain exactly 2 points in $\mathbb{Z}^2$ and have y-component 1. Therefore, S' has exactly eight edges, and the nonhorizontal edges have exactly 2 integer points and have y-component 1.

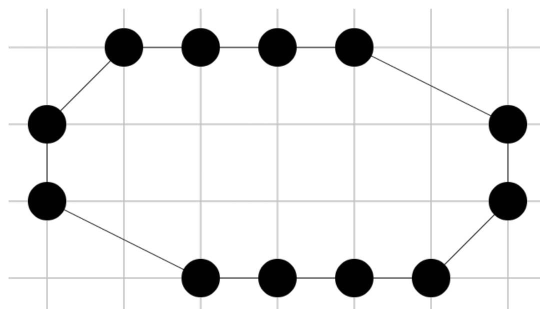

Figure 4.1: The edges of a possible η'-generating set S'.

Notice that since the center left edge of S' has y-component 1, there is a shear transformation of $\mathbb{Z}^2$ such that the horizontal direction stays horizontal, but the center left edge of S', and the corresponding nonexpansive direction become vertical. Apply this transformation to S' and η' to yield S and η respectively. Notice that we still have that S is an η-generating set, that S has 8 edges that are antiparallel to the opposite edge, and that all nonhorizontal edges contain exactly 2 integer points.

By [6], we can construct what we will call 'simultaneous witnesses to nonexpansiveness'. By assumption, we can construct 'incident witnesses to nonexpansiveness'. This means that there is a ray u_0 parallel to w_0 and ray u_1 that shares an endpoint with u_0 and is parallel to some edge w_1 of S_1 that is incident to w_0, such that the sector enclosed by u_0 and u_1 does not extend uniquely to either of the rays directly outside of u_0 and u_1; call these rays b_0 and b_1 respectively. Call the enclosed sector L.

Say that S has width n'. Notice that this means $S \setminus w_0$ has width $n' - 1$. Let ℓ_0 and ℓ_1 be lines in parallel or antiparallel to u_0 and u_1 respectively, such that they are oriented to contain the sector between u_0 and u_1. Notice that S is both ℓ_0- and ℓ_1- balanced; the four conditions all follow from S being an η-generating set with the edges parallel to ℓ_0 and ℓ_2 having two integer points (since neither can be horizontal, as it must touch the center left edge) and having the opposite edges being exactly antiparallel. By the argument of Proposition 2.16 of [6], we can say that all vertical strips of width $n' - 1$ that extend non-uniquely to the left are vertically constant and strips of width $n' - 1$ that extend uniquely to the left are vertically periodic with period at most 2. Also by this argument, we know that b_0 is vertically periodic with period at most 2. Additionally, since S is ℓ_1 balanced, we can use the same argument to say that the analogous strip along u_1, which geometrically has height at least 3, is periodic with period $\langle v_x, 1\rangle$, which is the vector between the integer points in w_0 which defines the direction of u_1.

Notice that since the vertical strips that do not extend uniquely are vertically constant, they contain only one coloring of $S \setminus w_0$. Notice that this must not extend uniquely to S, because this would imply the values of all translations of w_0 to the left of $S \setminus w_0$, meaning that it extends uniquely when it does not. Therefore, all width $n' - 1$ strips that don't extend uniquely to the left contain only the coloring of $S \setminus w_0$ that does not extend uniquely to S.

Since all vertical lines have period at most two, we can consider two cases, the case when b_0 is vertically constant, and when b_0 is vertically genuinely two periodic.

Consider if b_0 is vertically constant. Without loss of generality, say that b_0 is colored with all 0s. Notice the width $n' - 1$ strip that has left edge u_0 does not extend uniquely to the left onto b_0 because the entire sector does not extend uniquely to b_0. Thus, since b_0 is all $0's$, there are all 0s to the left of a strip that does not extend uniquely. Since this strip only contains the coloring of $S \setminus w_0$ that does not extend uniquely, this means that some extension thereof colors w_0 as $0, 0$; notice that coloring w_0 as a, b, means color one point of w_0 with a and the other with b. Remember that the coloring of $S \setminus w_0$ that does not extend uniquely to S has exactly two extensions to S, which are differentiated only by there coloring of w_0. Assume for contradiction that the extension to w_0 that is not $0,0$ has two different colors, a, b. We know that S colored with this extension must appear is some width $n' - 1$ strip that does not extend uniquely to the left. If it did not then all such strips would only extend to the word of all zeros since all coloring of $S \setminus w_0$ in that strip don't extend uniquely, and the only extension of this to w_0 other than a, b is $0, 0$, and the only coloring completely covered by $0, 0$ is the constant zero words. If all strips that do not extend uniquely always extend to the constant word, then they do extend uniquely, which is a contradiction. Thus, the coloring with w_0 colored as a, b is in some width $n' - 1$ strip that does not extend uniquely to the left. As before, the line to the left of this must be covered by the colorings $0, 0$ and a, b. No word using with subword a, b with $a \neq b$ can be covered by $0, 0$ and a, b, which is a contradiction. Therefore, the other extension to w_0 must have the same color on both points. Without loss of generality, say that this extension has w_0 colored as $1, 1$.

Since all extensions of $S \setminus w_0$ keep w_0 constant, any vertical line to the left of a non-uniquely extending strip is vertically constant. notice that this actually implies that all lines are vertically constant, and this vertical constancy extends to u_1. Thus, L is vertically constant.

Remember the diagonal strip with edge u_1 that has height at least three and is periodic with period $\langle v_x, 1\rangle$ as defined above. Consider translating S so that it places w_1 on u_1. Now consider the coloring in that position of $S \setminus w_1$. Translate this by $\langle v_x, 1\rangle$, knowing that it is still contained in the diagonal strip. By moving this period length, we must have witnessed an non-uniquely extending vertical strip, and thus S is colored in two different ways, but there can only be one coloring of $S \setminus w_1$ that does not extend uniquely. Thus, one of these does extend uniquely, contradicting the fact that strips that do not extend uniquely, such as that with edge u_1, contain only words that extend non-uniquely.

Now consider the case when b_0 is genuinely vertically two periodic. Without loss of generality, say that b_0 is colored with alternating 0s and 1s. Remember that there is exactly one coloring of $S \setminus w_0$ that does not extend uniquely to S, and this coloring extends in exactly two ways to S. Also remember that any width $n' - 1$ strip that does not extend uniquely to the left must contain only this coloring of $S \setminus w_0$. Notice that since the width $n' - 1$ strip directly to the right of b_0 does not extend uniquely to its left because the whole sector does not extend uniquely to b_0. Therefore, the coloring of $S \setminus w_0$ that does not extend uniquely is a subcoloring of the colorings that put w_0 directly over b_0. Notice that this would color w_0 as $0, 1$ and $1, 0$. Since $S \setminus w_0$ extends to S in exactly two ways and we can say that two of them are $0, 1$ and $1, 0$, these colorings of w_0 represent all possible extensions to S of the coloring of $S \setminus w_0$ that do not extend uniquely.

Notice that there are infinitely many width $n' - 1$ strip to the right of u_0. Assume for contradiction that all of these strips extend uniquely to their left. Notice that all of these strips are vertically periodic with (a not necessarily minimal) period 2. Therefore, knowing $2(n' - 1)$ colors in a strip uniquely determines a width $n' - 1$ strip. Thus there are at most $|\Sigma|^{2(n'-1)}$ colorings of such strips; importantly, this is finite. Since there are finitely many distinct colorings of infinitely many strips, at least two strips have the same coloring; say these same colors appear d_0 and d_1 units to the right of the strip whose rightmost edge is u_0. If a coloring of a strip implies a coloring of a strip, which implies a coloring of a strip and so on, by our assumption that all colorings of width $n' - 1$ strips to the right of b_0 extend uniquely, we know that the coloring of a strip implies the coloring of the strip d_0 units to the left for any d_0. This means that since the strips distance d_0 and d_1 from u_0 have the same coloring, the strips distances $d_0 - d_0 = 0$ and $d_1 - d_0 > 0$ to the right of u_0 have the same coloring. That is the coloring of the strip whose leftmost edge is u_0 is the same as the coloring of the strip $d_1 - d_0$ units to the right of that. By our assumption, the coloring of the strip $d_1 - d_0$ units to the right extends uniquely to the left. Therefore, the same coloring on the strip whose left edge is u_0 also extends uniquely to the left. This contradicts the fact that this strip does not extend unique as discussed previously. Therefore, not all strips to the right of u_0 extend uniquely to the left.

Pick a width $n' - 1$ strip to the right of u_0 whose coloring does not extend uniquely to the left. Notice that this implies that this strip is vertically constant and all colorings of $S \setminus w_0$ on this strip are the coloring that does not extend uniquely to S. Notice that the two possible colorings of w_0 to the left of this particular coloring of $S \setminus w_0$ are $0, 1$ and $1, 0$. Thus, the vertical line to the left of this strip is completely covered with colors $0, 1$ and $1, 0$. The only such coloring is the word ...01010101.... Therefore, the line to the left of this non-uniquely extending strip has genuine period 2.

Notice that this line continues to b_1. Notice geometrically that this must have at two adjacent points in the height 3 strip whose edge is u_1, strictly because of this height; call these points $\mathbf{x}_1$ and $\mathbf{x}_2$, where $\mathbf{x}_1$ is directly above $\mathbf{x}_2$. Notice that since the line is genuinely two periodic, $\mathbf{x}_1$ and $\mathbf{x}_2$ have different colors. Remember that this extends non-uniquely and is thus periodic with period $\langle v_x, 1\rangle$, where this vector represents the difference of the two integer points on w_1. Notice trivially that $n' > v_x$, since the set S contains w_1. Thus, $n' - 1 \leq v_x$. Say that the points $\mathbf{x}_1$ and $\mathbf{x}_2$ are both horizontal distance d_1 from u_0 (since they lie on the same vertical line). By the division algorithm, there are some integers q, r such that $d' = v_x q + r$ where $0 \leq r < n' - 1$. Equivalently, $d' - v_x q = r$. Since the diagonal strip adjacent to b_1 is period with period $\langle v_x, 1\rangle$, it is also periodic with period $\langle -q v_x, -q\rangle$. Translate both $\mathbf{x}_1$ and $\mathbf{x}_2$ by this vector to yield $\mathbf{x}_1'$ and $\mathbf{x}_2'$. See that $\mathbf{x}_1$ and $\mathbf{x}_1'$ must have the same color and $\mathbf{x}_2$ and $\mathbf{x}_2'$ must have the same color, by the period $\langle -q v_x, -q\rangle$. Note that this implies that $\mathbf{x}_1'$ is a point above directly above $\mathbf{x}_2'$ that has a different color. Since these points now have distance $d' - v_x q = r$ from u_0 with $0 \leq r < n' - 1$ from u_0, we know that these points are both in the vertical strip of width $n' - 1$ whose left edge is u_0. Since this strip is vertically periodic, $\mathbf{x}_1'$ and $\mathbf{x}_2'$ have the same color, which is a contradiction.

Since both cases yield a contradiction, so it is impossible for a word η to have 4 nonexpansive lines satisfying $p_{\eta'}(n, 4) \leq 4n$ for some n, in the case of incident simultaneous witnesses to nonexpansiveness.

$\square$

Chapter 5

Other Results

The Morse-Hedlund theorem relates the global dynamical property of periodicity with the local combinatorial property of complexity for colorings on $\mathbb{Z}$. The Nivat conjecture seeks a similar relationship for the two-dimensional integer lattice $\mathbb{Z}^2$. The Nivat conjecture appears to be a natural generalization of the Morse-Hedlund theorem, posing the question of the possibility of further generalization. Morse-Hedlund analogues have been lightly explored over the general case in $\mathbb{Z}^d$ [8, 3, 4]. This section seeks to begin exploration of more general analogues of the Morse-Hedlund theorem over arbitrary groups. Groups are a mathematical object underlying the modern theory of abstract algebra, defined as follows.

Definition 5.1. *A set G equipped with an operation $* : G \times G \to G$ is called a group if and only if*

1. *For all $a, b, c \in G$, $(a * b) * c = a * (b * c)$. That is, $*$ is associative.*
2. *There is some $e \in G$, called the identity, such that for all $g \in G$, $e * g = g * e = g$.*
3. *Every element $g \in G$ has some inverse $g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.*

We often write $a * b$ simply as ab, because the operation is usually clear from *context.*

For the remainder of this paper, when discussing groups, let $e \in G$ denote the identity of G. We will see G as a set of 'actions' that translate elements of some other set. We can formalize this as follows.

Definition 5.2. *We say a function $\phi : G \times S \to S$ is a group action if it satisfies the following properties.*

1. *The identity maps everything to itself; that is, for all $s \in S$, $\phi(e, s) = s$.*
2. *Composing element actions results in the product action; that is, for all $a, b \in G$ and $s \in S$, $\phi(a, \phi(b, s)) = \phi(ab, s)$.*

Since ϕ is often clear from context, we write $\phi(g, s)$ as gs.

Intuitively, G defines the way that elements in S move around, or permute.

In this chapter, we discuss a very general analogue of the Morse-Hedlund theorem in the setting of a group G acting on a set S. Given some alphabet Σ, words are functions $f : S \to \Sigma$. Intuitively, the elements of G can be seen as translations of S.

Definition 5.3 (Word Translation)**.** *Let a group G act on a set S. The translation of a word $f : S \to \Sigma$ by some $g \in G$, denoted gf, is defined by $gf(s) := f(g^{-1}s)$.*

Lemma 5.1 (Associativity of Translation)**.** *Let $g_1, g_2 \in G$, then $g_1(g_2 f) = (g_1 g_2)f$.*

Proof. We can compute that for all $s \in S$,

$$g_1(g_2 f)(s) = g_2 f(g_1^{-1} s) = f(g_2^{-1} g_1^{-1} s) = f((g_1 g_2)^{-1} s) = (g_1 g_2) f(s).$$

Therefore, $g_1(g_2 f) = (g_1 g_2)f$, proving the claim. $\square$

This means that we don't need to write parentheses for translations.

Definition 5.4 (Period Subgroup)**.** *Let a group G act on a set S. The period subgroup of f, denoted V_f, is defined as the elements of G that leave f unaltered under translation. Formally $V_f := \{g \in G : gf = f\}$. The group G and set S are implied by the context. If $V_f \neq \{e\}$, f is periodic; otherwise, f is called aperiodic. For $v \neq e \in V_f$, f is said to be periodic with period v.*

Lemma 5.2 (Period Subgroup is a Group)**.** *The period subgroup is a subgroup of G.*

Proof. By definition, $V_f \subseteq G$. Notice $ef = f$, so $e \in V_f$. Let $v_1, v_2 \in V_f$, then $v_1 v_2 f = v_1 f = f$, so $v_1 v_2 \in V_f$. Let $v \in V_f$, $vf = f$, so $f = v^{-1} f$, meaning $v^{-1} \in V_f$. Thus, V_f is a subgroup of G. □

While this structure is clearly motivated by the Morse-Hedlund theorem, this setup describes a much more general situation. When considering the symmetries of some object, one is often constrained to the symmetries of some overarching space. For example, imagine a symmetric vase. The intuitive notion of the symmetries of that vase is formally the rigid motions of space that leave the vase unaltered. In this context, the symmetries of the vase are the period subgroup of the vase when considering the space $\mathbb{R}^3$ acted upon by $E(3)$, the rigid motions of three-dimensional space. In this sense, this section seeks to provide a set of tools for examining arbitrary symmetries.

Definition 5.5 (Colorings and Complexity)**.** *Let a group G act on a set S. Given some 'pattern' $N \subseteq G$, the colorings of that pattern, denoted $P_f(N)$, are defined as the set $P_f(N) := \{w_s \in \Sigma^N : s \in S, w_s(n) := f(n^{-1}s)\}$. The 'complexity' of f with pattern N, denoted $p_f(N)$, is simply the number of colorings. That is,*

$$p_f(N) := |P_f(N)|.$$

Using this definition of complexity, there is a weak analogue to the Morse-Hedlund theorem that holds for an arbitrary group.

Theorem 5.3 (Periodicity and Complexity of Arbitrary Groups.)**.** *Let a group G act on a set S, $f : S \to \Sigma$ be a word and $N \subseteq G$ a pattern. If $p_f(N) < \log_{|\Sigma|} |N|$ (or $|\Sigma|^{p_f(N)} < |N|$), then f is periodic with some period in $N^{-1}N \setminus \{e\} = \{n^{-1}n' \neq e : n, n' \in N\}$.*

Proof. Assume $p_f(N) < \log_{|\Sigma|} |N|$. Therefore, $|\Sigma|^{p_f(N)} < |N|$. Notice $|\Sigma|^{p_f(N)} = |\Sigma|^{|P_f(N)|} = |\Sigma^{P_f(N)}|$; thus, $|\Sigma^{P_f(N)}| < |N|$.

Consider the following function $b : N \to \Sigma^{P_f(N)}$ defined by $b(n)(w) = w(n)$. Notice here that $w \in P_f(N)$ is some coloring of the pattern. Since $|\Sigma^{P_f(N)}| < |N|$, by the pigeonhole principle b cannot be injective. Thus, there are $n_1 \neq n_2 \in N$ such that $b(n_1) = b(n_2)$. Thus, for all $w \in P_f(N)$, we know $b(n_1)(w) = b(n_2)(w)$. By the definition of the evaluation b, we know $w(n_1) = w(n_2)$. By the definition of colorings of a pattern, for all $s \in S$, we have $f(n_1^{-1}s) = f(n_2^{-1}s)$. In terms of translations, for all $s \in S$, $n_1 f(s) = n_2 f(s)$, implying $n_1 f = n_2 f$. Therefore $f = n_1^{-1} n_2 f$, implying $n_1^{-1}n_2 \in V_f$. Since $n_1 \neq n_2$, we know $n_1^{-1}n_2 \neq e$. Thus, f is periodic with period $n_1^{-1}n_2$.

Since $n_1, n_2 \in N$, the set $N^{-1}N := \{n^{-1}n' : n, n' \in N\}$ contains $n_1^{-1}n_2$. Thus, f is periodic with some period in $N^{-1}N \setminus \{e\}$. □

Notice that this puts no restrictions on G or S, and is true in complete generality. The pattern N and alphabet Σ can be infinite if the $|\Sigma|^{p_f(N)} < |N|$ hypothesis is considered.

A natural question is whether the complexity bound can be improved from $\log_{|\Sigma|} |N|$ by imposing conditions on S, G, Σ, N, or f. One such strengthening is obtained by assuming that G is torsion-free (that is, there are no $g \neq e \in G$ and $n \in \mathbb{Z}^+$ such that $g^n = e$), that Σ is finite, and that f is surjective; these assumptions improve the bound to $\log_{|\Sigma|} |N| + \log_{|\Sigma|}(|\Sigma|!)$. Note that if N is infinite, this bound is equivalent to the previously proven bound; thus, we need only consider if N is finite.

Theorem 5.4 (Strengthened Periodicity and Complexity of Arbitrary Groups.)**.** *Let* *a group G act on a set S. Let $f : S \to \Sigma$ be a surjective word with Σ finite and $N \subseteq G$ be a finite pattern. If $p_f(N) < \log_{|\Sigma|}|N| + \log_{|\Sigma|}(|\Sigma|!)$, then f is periodic.*

Proof. Assume $p_f(N) < \log_{|\Sigma|}|N| + \log_{|\Sigma|}(|\Sigma|!)$. Thus, $p_f(N) < \log_{|\Sigma|}(|N||\Sigma|!)$. Therefore, $|\Sigma|^{p_f(N)} < |N||\Sigma|!$. Note that $|\Sigma|^{p_f(N)} = |\Sigma|^{|P_f(N)|} = |\Sigma^{P_f(N)}|$. Since Σ is finite, $|\Sigma|! = |S_\Sigma|$, and therefore $|N||\Sigma|! = |N \times S_\Sigma|$. Overall, we conclude $|\Sigma^{P_f(N)}| < |N \times S_\Sigma|$.

Now consider the function $b : N \times S_\Sigma \to \Sigma^{P_f(N)}$ defined by $b(n, \pi)(w) = \pi(w(n))$. Since $|\Sigma^{P_f(N)}| < |N \times S_\Sigma|$, by the pigeonhole principle, b cannot be injective. Thus, there exist $n_1, n_2 \in N$ and $\pi_1, \pi_2 \in S_\Sigma$ such that $b(n_1, \pi_1) = b(n_2, \pi_2)$ and $(n_1, \pi_1) \neq (n_2, \pi_2)$. This implies that for all $w \in P_f(N)$ the function b has that $b(n_1, \pi_1)(w) = b(n_2, \pi_2)(w)$. Thus, by the definition of b, we know $\pi_1(w(n_1)) = \pi_2(w(n_2))$, and therefore $w(n_1) = \pi_1^{-1}\pi_2(w(n_2))$. Since $w \in P_f(N)$, the definition of colorings implies that for all $s \in S$, we have $f(n_1^{-1}s) = \pi_1^{-1}\pi_2(f(n_2^{-1}s))$. Define s' by $s = n_2 s'$. Since $n_2 \in N \subseteq G$, $n_2 : S \to S$ is a permutation. Thus, we can requantify that for all $s' \in S$, $f(n_1^{-1}n_2 s') = \pi_1^{-1}\pi_2(f(s))$. In terms of translations, $n_2^{-1}n_1 f(s') = \pi_1^{-1}\pi_2(f(s'))$.

Since $\pi_1^{-1}\pi_2 \in S_\Sigma$, we have $(\pi_1^{-1}\pi_2)^{|\Sigma|!} = e$. Thus, for all $s' \in S$, we have the formula $(n_2^{-1}n_1)^{|\Sigma|!} f(s') = (\pi_1^{-1}\pi_2)^{|\Sigma|!}(f(s')) = f(s')$. Therefore, $(n_2^{-1}n_1)^{|\Sigma|!} f = f$ and $(n_2^{-1}n_1)^{|\Sigma|!} \in V_f$.

Assume for contradiction that $n_1 = n_2$. Thus, $n_2^{-1}n_1 = e$. Therefore, for all $s' \in S$, $f(s') = n_2^{-1}n_1 f(s') = \pi_1^{-1}\pi_2(f(s'))$. Pick some 'color' $\sigma \in \Sigma$. Since f is surjective, there is some $s_\sigma \in S$ such that $f(s_\sigma) = \sigma$. Notice $f(s_\sigma) = \pi_1^{-1}\pi_2(f(s_\sigma))$ implies $\sigma = \pi_1^{-1}\pi_2(\sigma)$. Since the color was arbitrary, for all $\sigma \in \Sigma$, $\pi_1^{-1}\pi_2(\sigma) = \sigma$. Thus, necessarily, $\pi_1^{-1}\pi_2 = e$ and thus, $\pi_1 = \pi_2$. Therefore, $(n_1, \pi_1) = (n_2, \pi_2)$, contradicting that $(n_1, \pi_1) \neq (n_2, \pi_2)$. Therefore, $n_1 \neq n_2$.

Since $n_1 \neq n_2$, $n_2^{-1}n_1 \neq e$. Since G is torsion-free, $(n_2^{-1}n_1)^{|\Sigma|!} \neq e$. Thus, f is periodic with period $(n_2^{-1}n_1)^{|\Sigma|!}$. $\square$

Adding a few reasonable constraints to the system slightly improves the complexity bound for an arbitrary group. This naturally poses questions of the optimality of this bound and the effect of stronger assumptions. The following example displays the asymptotic optimality of this bound and the inefficacy of the many seemingly useful assumptions.

Example 5.1 (Aperiodic Word of Low Complexity)**.** *Consider $\mathbb{Z}^{\mathbb{N}}$, all infinite sequences $(x_i) = x_0, x_1, x_2, ...$ with elements in $\mathbb{Z}$, acting on itself with addition; that is, for $(a_i) = a_0, a_1, a_2, ...$ and $(y_i) = b_0, b_1, b_2, ...$, the sum is $(a_i) + (b_i) = a_0 + b_0, a_1 + b_1, a_2 + b_2, ...$ where all a_i and b_i are integers. Let $\Sigma := \{0, 1\}$. Define the word $a : \mathbb{Z}^{\mathbb{N}} \to \Sigma$ by $a(x_0, x_1, x_2, ...) = (x_{|x_0|+1} \geq 0)$. Here, as before, $(x_{|x_0|+1} \geq 0) = 1$ if $x_{|x_0|+1} \geq 0$, and it is zero otherwise. Also define a family of finite hyperrectangles $R_{k,d}$ by $R_{k,d} := \{0\} \times \{0, ..., k\}^d \times \{0\}^{\aleph_0}$.*

Theorem 5.5 (a is aperiodic)**.** *The word a is aperiodic.*

Proof. Assume for contradiction a is periodic with some translation $(x_i) \neq (0) \in \mathbb{Z}^{\mathbb{N}}$. We consider two cases: $x_0 = 0$ and $x_0 \neq 0$.

Consider the case when $x_0 = 0$. Since $(x_i) \neq (0)$, there is some $j > 0$ such that $x_j \neq 0$; note $j - 1 \geq 0$, so $|j - 1| = j - 1$. Now consider the point (y_i) defined by $y_0 := j - 1$, $y_j := -x_j$ and $y_i := 0$ otherwise. Since $(x_i) \in V_a$, $a(y_i) = a(y_i + 2x_i)$. Calculate that $a(y_i) = (y_{|y_0|+1} \geq 0) = (y_{|j-1|+1} \geq 0) = (y_j \geq 0) = (-x_j \geq 0)$. Define $z_i := y_i + 2x_i$. Now, calculate

$$\begin{aligned}
a(y_i + 2x_i) &= a(z_i) \\
&= (z_{|z_0|+1} \geq 0) \\
&= (z_{|y_0+2x_0|+1} \geq 0) \\
&= (z_{|j-1+2\cdot 0|+1} \geq 0) \\
&= (z_j \geq 0) = (y_j + 2x_j \geq 0) \\
&= (-x_j + 2x_j \geq 0) \\
&= (x_j \geq 0).
\end{aligned}$$

Since $x_j \neq 0$, we know $(-x_j \geq 0) \neq (x_j \geq 0)$. Thus, $a(y_i) \neq a(y_i + 2x_i)$, a contradiction.

Now consider if $x_0 \neq 0$. Consider the point (y_i) defined by the conditions $y_0 := 0$, $y_1 := -1$, $y_{|x_0|+1} := -x_{|x_0|+1}$, and $y_i := 0$ otherwise. Notice $a(y_i) = (y_{|y_0|+1} = y_1 = -1 \geq 0)$ is false. Also notice

$$a(y_i + x_i) = ((y_i + x_i)_{|y_0+x_0|+1} = y_{|x_0|+1} + x_{|x_0|+1} = 0 \geq 0)$$

is true. Since $(x_i) \in V_a$, $a(y_i) = a(y_i + x_i)$, a contradiction.

All cases lead to contradiction. Thus, a cannot be periodic, proving the claim. □

Proposition 5.6 (Complexity of a)**.** *The complexity of a is $p_a(R_{k,d}) \leq dk + 2$.*

Proof. Consider an arbitrary point (x_i). Consider if $|x_0| + 1 \leq d$. Notice that for all $(r_i) \in R_{k,d}$, $r_0 = 0$, and $(x_i + r_i)_0 = x_0$.

If $|x_0| + 1 > d$, then for any $(r_i) \in R_{k,d}$, $r_{|x_0|+1} = 0$. Therefore, either $a(x_i + r_i)$ or $\neg a(x_i + r_i)$ for all $r_i \in R_{k,d}$. Thus, there are two possible colorings of $R_{k,d}$ for $|x_0| + 1 > d$.

Now consider $|x_0| + 1 \leq d$ fixed. Notice that all colors of R are determined by $x_{|x_0|+1}$. This is because for point $(r_i) \in R$, $a(r_i + x_i) = (r_{|x_0+r_0|+1})$

Summing the d cases of $|x_0| + 1 \leq d$ and the $|x_0| + 1 > d$ case, there are at most $dk + 2$ colorings of $R_{k,d}$ by a. □

Theorem 5.7. *The general bound $p_f(N) < \log_{|\Sigma|} |N|$ is asymptotically optimal for implying periodicity (even in abelian minimally-generated groups). Formally, if there is some bound b such that $p_f(N) < b(|N|)$ implies f is periodic, then $b(|N|) = \mathcal{O}(\log_{|\Sigma|} |N|)$.*

Proof. Clearly, $|R_{k,d}| = k^d$. Thus, $\log_{|\Sigma|} |R_{k,d}| = \log_2 k^d = d \log_2 k$. Thus, for fixed k, we have $dk + 2 = \Theta(d \log_2 k)$. Equivalently, $p_a(R_{k,d}) = \Theta(\log_{|\Sigma|} |R_{k,d}|)$.

Notice that since a is aperiodic and $\mathbb{Z}^{\mathbb{N}}$ is abelian, and minimally generated, a cannot satisfy any complexity bound that implies periodicity for words on abelian or minimally-generated groups. Notice also that $p_a(R_{k,d}) \leq \log_{|\Sigma|} |R_{k,d}|$, so $p_a(R_{k,d})$ is asymptotically equal to the periodicity bound for arbitrary groups. Thus, there can be no bound asymptotically strictly larger than $\log_{|\Sigma|} |N|$, since a would necessarily satisfy it for sufficiently large d, but a is aperiodic. Therefore, $p_f(N) < \log_{|\Sigma|} |N|$ is asymptotically optimal for implying periodicity (even in abelian minimally-generated groups). □

Fundamentally, even given a few strong and seemingly useful assumptions, no bound significantly stronger $p_f(N) < \log_{|\Sigma|} |N|$.

The lack of a larger bound in a sufficiently general case leads to a search for conditions on S, G, Σ, N, or f lead to better bounds. If $S, G = \mathbb{Z}$, then the existence of an N such that $p_f(N) \leq |N| + |\Sigma| - 2$ implies that f is periodic. If $S, G = \mathbb{Z}^2$ then $p_f(\{0, ..., n\} \times \{0, 1, 2\}) \leq 3n$ implies f is periodic. These very strong conditions generate large bounds. We seek similar bounds in more general cases.

One tool that is invaluable in a dynamical consideration of the $\mathbb{Z}^2$ case is that of expansiveness. Examining the expansiveness and nonexpansiveness of the directions in $\mathbb{Z}^2$ yields meaningful bounds on complexity for specific cases [7]. Generalizing this concept to general groups promises to be useful. However, the Cyr-Kra definition of expansiveness relies heavily on half-planes, an inherently geometric object. Even so, there is a generalized definition of half-planes that are satisfiable on an any set acted upon by some group.

Definition 5.6 (Half-Plane)**.** *Let a group G act on a set S. Let $H \subseteq S$ with $V_H \neq G$. We say that H is a half-plane if*

1. $\forall g \in G, gH \subseteq H \vee g^{-1}H \subseteq H$ *and*
2. $\forall s \in S, \forall g \in G \setminus V_H, \exists n \in \mathbb{Z}, s \in H \veebar g^n s \in H$

To understand important aspects of half-planes, we must first further examine the period subgroup.

Definition 5.7 (Set Period Subgroup)**.** *Let G act on S, and $T \subseteq S$. The period subgroup V_T is defined by $V_T := V_{\chi_T}$ where $\chi_T : S \to \{0, 1\}$ is the indicator function of T.*

This definition coincides with the intuition that V_S should be all elements of G that leave S unchanged under translation.

Lemma 5.8. *Let G act on S. Let $f : S \to \Sigma$ be a word and $\pi : \Sigma \to \Sigma$ be a permutation. Then $V_f = V_{\pi f}$.*

Proof. If $g \in V_f$, then $\forall s \in S, f(s) = gf(s) = f(g^{-1}s)$. Thus, $\forall s \in S, \pi f(s) = \pi f(g^{-1}s) = g\pi f(s)$. Thus, $\pi f = g\pi f$ and $g \in V_{\pi f}$.

If $g \in V_{\pi f}$, then $\forall s \in S, \pi f(s) = g\pi f(s) = \pi f(g^{-1}s)$. Since π is injective, $\forall s \in S, f(s) = f(g^{-1}s) = gf(s)$. Thus, $f = fg$ and $g \in V_f$.

Therefore, $V_f = V_{\pi f}$, proving the claim. □

This lemma quickly shows that the complement of a set will have the same period subgroup as the original set.

Corollary 5.9. *Let G act on S, and $T \subseteq S$. Then $V_T = V_{S \setminus T}$.*

Proof. Let (01) be the transposition of 0 and 1. Notice $\chi_T = (01)\chi_{S\setminus T}$. Therefore, $V_T = V_{\chi_T} = V_{(01)\chi_T} = V_{\chi_{S\setminus T}} = V_{S\setminus T}$. □

In $\mathbb{Z}$, and $\mathbb{Z}^2$, as well as any abelian group, translating a word does not change the periods of the word. This is not true in general. Even so, we can make a very strong statement about the periods of a translated version of a word in an any set acted upon by some group.

Lemma 5.10. *Let G act on S and $f : S \to \Sigma$. For $g \in G$, $V_{gf} = gV_f g^{-1}$.*

Proof. Fix $g \in G$. Pick $gvg^{-1} \in gV_f g^{-1}$ with $v \in V_f$. Thus, $f = vf$, $f = vg^{-1}gf$, and $gf = gvg^{-1}gf$. Thus, $gvg^{-1} \in V_{gf}$. Therefore, $gV_f g^{-1} \subseteq V_{gf}$. Next pick $g^{-1}vg \in gV_{gf}g^{-1}$ with $v \in V_{gf}$. Thus, $gf = vgf$ and $f = g^{-1}vgf$. Thus, $g^{-1}vg \in V_f$. Therefore, $g^{-1}V_{gf}g \subseteq V_f$ and $V_{gf} \subseteq gV_f g^{-1}$.

Thus, $V_{gf} = gV_f g^{-1}$, proving the claim. □

Now there is sufficient background to prove some fundamental features of half-planes in arbitrary sets acted upon by groups. Specifically, that the complement or translation of a half-plane is again a half-plane.

Lemma 5.11 (Complement of a half-plane)**.** *Let G act on S. If $H \subseteq S$ be a half-plane, then $S \setminus H$ is a half-plane.*

Proof. Note first that $V_{S\setminus H} = V_H \neq G$.

Now note that $\forall g \in G, gH \subseteq H \vee g^{-1}H \subseteq H$. Equivalently, for all $g \in G$ we have $H \subseteq g^{-1}H \vee H \subseteq gH$ and $g^{-1}(S \setminus H) \subseteq S \setminus H \vee g(S \setminus H) \subseteq S \setminus H$. Therefore, $S \setminus H$ satisfies the first half-plane axiom.

Now see that $\forall s \in S, \forall g \in G \setminus V_H, \exists n \in \mathbb{Z}, s \in H \veebar g^n s \in H$. Equivalently, for all $s \in S$, for all $g \in G \setminus V_H = G \setminus V_{S\setminus H}$, there is some $n \in \mathbb{Z}$ such that $s \notin H \veebar g^n s \notin H$ and also $s \in S \setminus H \veebar g^n s \in \setminus H$. Thus, $S \setminus H$ satisfies the second half-plane axiom.

Therefore, $S \setminus H$ is a half-plane. □

Lemma 5.12 (Translation of a half-plane)**.** *Let G act on S. If $H \subseteq S$ be a half-plane and $g_0 \in G$, then $g_0 H$ is a half-plane.*

Proof. Notice that since $V_H \neq G$, $V_{g_0 H} = g_0 V_H g_0^{-1} \neq G$.

Now note that $\forall g \in G, gH \subseteq H \vee g^{-1}H \subseteq H$. Since $G = g_0^{-1}Gg_0$, we can requantify to see that $\forall g_0^{-1}gg_0 \in G, g_0^{-1}gg_0H \subseteq H \vee (g_0^{-1}gg_0)^{-1}H \subseteq H$ and $\forall g \in G, g_0^{-1}gg_0H \subseteq H \vee g_0^{-1}g^{-1}g_0H \subseteq H$. Therefore, $\forall g \in G, gg_0H \subseteq g_0H \vee g^{-1}g_0H \subseteq g_0H$. This means that g_0H satisfies the first half-plane axiom.

Now see that $\forall s \in S, \forall g \in G \setminus V_H, \exists n \in \mathbb{Z}, s \in H \veebar g^n s \in H$. Since $V_{g_0H} = g_0V_Hg_0^{-1}$, we can requantify, $\forall s \in S, \forall g \in G \setminus V_{g_0H}, \exists n \in \mathbb{Z}, s \in H \veebar (g_0^{-1}gg_0)^n s \in H$ and $s \in H \veebar g_0^{-1}g^ng_0s \in H$. Since $g_0S = S$, we can requantify, $\forall s \in S, \forall g \in G \setminus V_{g_0H}, \exists n \in \mathbb{Z}, g_0^{-1}s \in H \veebar g_0^{-1}g^ng_0g_0^{-1}s \in H$ as well as $g_0^{-1}s \in H \veebar g_0^{-1}g^ns \in H$ and $s \in g_0H \veebar g^ns \in g_0H$. Therefore, g_0H satisfies the second half-plane axiom.

Therefore, g_0H is a half-plane. □

Proposition 5.13. *Let G act on a set S and $H \subseteq S$ be a half-plane. Then V_H is normal.*

Proof. Pick arbitrary $v \in V_H$ and $g \in G \setminus V_H$. We will show $gvg^{-1} \in V_H$, which is sufficient.

Assume for contradiction $gvg^{-1} \notin V_H$. By the first axiom, $gH \subseteq H$ or $g^{-1}H \subseteq H$. Equivalently, $gH \subseteq H$ or $g(S \setminus H) \subseteq S \setminus H$. Thus, $\exists H' \in \{H, S \setminus H\}$ s.t. $gH' \subseteq H'$. Notice H' is a half-plane and that $V_{H'} = V_H = V_{S\setminus H}$.

Pick $s' \in H'$ since $H' \neq \emptyset$. Since $gH' \subseteq H'$, $gs' \in H$. Define $s := gs'$. Thus, $g^{-1}s, s \in H'$.

Since $gvg^{-1} \in G \setminus V_{H'}$, by the second axiom we know $\exists n \in \mathbb{Z}, (gvg^{-1})^n s \notin H'$ since $s \in H'$. Equivalently $gv^ng^{-1}s \notin H'$.

Since $g^{-1}s \in H'$ and $v \in V_{H'}$ we know $v^ng^{-1}s \in H'$. Since $gH' \subseteq H'$, we have that $gv^ng^{-1}s \in H'$, a contradiction. Thus, $gvg^{-1} \in V_H$, proving the claim. □

Definition 5.8 (Intersection Period Subgroup)**.** *Let G act on S and $f : S \to \Sigma$. The intersection period subgroup, denoted $\hat{V}_f$, is defined by $\hat{V}_f := \bigcap_{g\in G} V_{gf}$.*

Proposition 5.14 (Normality of $\hat{V}_f$)**.** *Let G act on S and $f : S \to \Sigma$. Then, $\hat{V}_f$ is normal in G.*

Proof. Pick arbitrary $g_0 \in G$. Consider the conjugate $g_0\hat{V}_f g_0^{-1}$. See that $g_0\hat{V}_f g_0^{-1} = g_0(\bigcap_{g\in G} V_{gf})g_0^{-1} = \bigcap_{g\in G} g_0 V_{gf} g_0^{-1} = \bigcap_{g\in G} V_{g_0 gf}$. Notice that since $G = g_0 G$, we can requantify, meaning $\bigcap_{g\in G} V_{g_0 gf} = \bigcap_{g_0 g\in G} V_{g_0 gf} = \bigcap_{g\in G} V_{gf} = \hat{V}_f$. Therefore, $g_0\hat{V}_f g_0^{-1} = \hat{V}_f$, implying $\hat{V}_f$ is normal, proving the claim. □

One group theoretic concept particularly useful for considering half-planes is the radical of a subgroup.

Definition 5.9. *Let K be a subgroup G. Define the radical of K, denoted $\sqrt{K}$, by $\sqrt{K} := \{g \in G : g^n \in K\}$. That is, the radical of K is all of the elements of G that have some power in K.*

For any half-plane, the radical of its period subgroup is actually just it's period subgroup. Intuitively, this imposes some 'smoothness' on the periodicity of H. It does have periodicity that 'jumps'; if H has some period v, it is periodic with all periods 'between' e and v.

Lemma 5.15. *Let H be a half-plane, then $V_H = \sqrt{V_H}$.*

Proof. Consider for contradiction that $V_H \neq \sqrt{V_H}$. Thus, there are $u \in G$ and $n > 1$ such that $u^n \in V_H$ and $u \notin V_H$. Since $u \notin V_H$, by the first axiom, $uH \subseteq H \veebar u^{-1}H \subseteq H$. Consider these two cases separately.

If $uH \subseteq H$ and $u^{-1}H \nsubseteq H$. Since $u^{-1}H \nsubseteq H$, there is some $s \in H$ such that $u^{-1}s \notin H$. Also since $uH \subseteq H$, $u^{n-1}H \subseteq H$ and $u^{n-1}s \in H$. Since $u^n \in V_H$, $u^{-1}s \in H \iff u^n u^{-1} s = u^{n-1}s \in H$, a contradiction.

If $u^{-1}H \subseteq H$ and $uH \nsubseteq H$. Since $uH \nsubseteq H$, there is some $s \in H$ such that $us \notin H$. Also since $u^{-1}H \subseteq H$, $u^{-(n-1)}H \subseteq H$ and $u^{-(n-1)}s \in H$. Since $u^n \in V_H$, $u^{-(n-1)}s \in H \iff u^n u^{-(n-1)}s = us \in H$, a contradiction.

Therefore, we must have $V_H = \sqrt{V_H}$. □

This proves a few other useful facts.

Corollary 5.16. *Let G act on S and H be a half-plane, then V_H contains all torsions of G, that is, elements $g \in G$, such that $g^n = e$.*

Proof. Since $e \in V_H$, by the defintion of radical all torsions are in $\sqrt{V_H} = V_H$. □

Corollary 5.17. *Let G act on S and H be a half-plane, then $G \setminus V_H$ contains an element of infinite order.*

Proof. Assume $G \setminus V_H$ does not contain an element of infinite order. Note $G \setminus V_H$ can not contain any torsions in G since V_H contains all torsions of G. Thus, since it contains no elements of infinite or finite order, $G \setminus V_H = \emptyset$. Thus, $V_H = G$, contradicting the that $V_H \neq G$ for half-planes. □

These abstract half-planes are defined by two algebraic axioms, rather than any geometric construction, so it is not immediate that half-planes coincide with the standard geometric half-planes. Thankfully, the two definitions of half-planes do coincide in the finite dimensional integer lattice $\mathbb{Z}^d$. The geometric half-planes all satisfy the two half-plane axioms, so they are all abstract half-planes. We will now see why the converse is also true.

Lemma 5.18. *Abstract half-planes are convex in $\mathbb{Z}^d$ acting on itself.*

Proof. Let $H \subseteq \mathbb{Z}^d$. Pick two points $\mathbf{x}_1, \mathbf{x}_2 \in H$. Let $\mathbf{x}_b$ be between those points, meaning there is some vector $\mathbf{v}$, and two positive integers a, b such that $\mathbf{x}_1 + a\mathbf{v} = \mathbf{x}_b$ and $\mathbf{x}_2 - b\mathbf{v} = \mathbf{x}_b$. Since H is a half-plane, either $\mathbf{v} + H \subseteq H$ or $-\mathbf{v} + H \subseteq H$ by the first half-plane axiom. Consider these cases separately.

If $\mathbf{v} + H \subseteq H$, then we also know $a\mathbf{v} + H \subseteq H$. Since $\mathbf{x}_1 \in H$, this means $\mathbf{x}_1 + a\mathbf{v} = \mathbf{x}_b$ is also in H.

If $-\mathbf{v} + H \subseteq H$, then we have $-b\mathbf{v} + H \subseteq H$. Since, $\mathbf{x}_2 \in H$, we know that $\mathbf{x}_2 - b\mathbf{v} = \mathbf{x}_b$ is in H.

In all cases, $\mathbf{x}_b \in H$. Therefore, all points between points in H are also in H, so H is convex. □

Theorem 5.19. *All abstract half-planes on $\mathbb{Z}^d$ are geometric half-planes.*

Proof. Let H be an abstract half-plane. Notice that we know $\mathbb{Z}^d \setminus H$ is also a half-plane. Therefore, both H and $\mathbb{Z}^d \setminus H$ are convex. The only sets satisfying this are geometric half-planes plus part of the boundary.

Consider for contradiction if some (but not all) of the boundary is in H, then there is some period $\mathbf{u} \in \mathbb{Z}^d$ of the geometric half-plane that is not a period of the boundary. This means that $\mathbf{u} \in \mathbb{Z}^d$ is not a period of H, that is $\mathbf{u} \in \mathbb{Z}^d \setminus V_H$. Now pick some point in the geometric half-plane, and call it $\mathbf{x}$. By the second half-plane axiom, there is some $n \in \mathbb{Z}$ such that $\mathbf{x}+n\mathbf{u} \notin H$. This is a contradiction. All $\mathbf{x}+n\mathbf{u}$ are in H, since they are in the geometric half-plane contained in H, because $\mathbf{u}$ is a period of the geometric half-plane.

Thus, H is exactly a geometric half-plane with none of the boundary. □

Using all of the above information about half-planes, we can define an analogue of expansiveness for arbitrary group actions.

Definition 5.10 (Expansiveness)**.** *Let a group G act on a set S.*

The set $\mathcal{O}_G(f)$ is the orbit of f in G, formally $\mathcal{O}_G(f) := \{gf : g \in G\}$; intuitively, $\mathcal{O}_G(f)$ contains all translations of f. The related set $\overline{\mathcal{O}_G(f)}$ is the orbit closure of f over G, that is, the topological closure of the orbit of f in the product topology Σ^S where Σ is endowed with the discrete topology; intuitively, this contains all limits of translations of f.

Let $f : S \to A$ be a word, and let H be a half-plane. We say that H is an expansive half-plane if and only if $\forall f_1, f_2 \in \overline{\mathcal{O}_G(f)}, f_1|_H = f_2|_H \to f_1 = f_2$.

Lemma 5.20. *Let (f_i) converge to f_∞, that is, $f_i \to f_\infty$. Then, $V_{f_\infty} \supseteq \bigcap_{i=0}^{\infty} V_{f_i}$. Note that here convergence means convergence in the product topology Σ^S (where Σ is endowed with the discrete topology).*

Proof. Pick some arbitrary $v \in \bigcap_{i=0}^{\infty} V_{f_i}$. Pick any $s \in S$. We will show $f_\infty(s) = vf_\infty(s)$. Consider the set $\{s, v^{-1}s\}$. Clearly $\{s, v^{-1}s\}$ is finite, so there is some $N \in \mathbb{Z}^+$ such that $\forall n \geq N, \forall s' \in \{s, v^{-1}s\}, f_i(s') = f_\infty(s')$. Therefore, for all $n \geq N$, $f_i(s) = f_\infty(s)$ and $f_i(v^{-1}s) = f_\infty(v^{-1}s)$, or equivalently $f_i(s) = f_\infty(s)$ and $vf_i(s) = vf_\infty(s)$. Since $v \in \bigcap_{i=0}^{\infty} V_{f_i}$, we have $v \in V_i$, implying $f_i = vf_i$. Thus, $f_\infty(s) = f_i(s) = vf_i(s) = vf_\infty(s)$. Therefore, for all $s \in S$, $f_\infty(s) = vf_\infty(s)$, implying $v \in V_{f_\infty}$. Therefore, $\bigcap_{i=0}^{\infty} V_{f_i} \subseteq V_{f_\infty}$. □

One may wonder if when $f_i \to f_\infty$ we have $V_{f_\infty} = \bigcap_{i=0}^{\infty} V_{f_i}$. This is in fact not true. Let $\mathbb{Z}$ act on itself. If f is defined by $f(0) := 1$ and $f(x) = 0$ otherwise. Define $f_i := if$. Therefore, $V_{f_i} = \{0\}$. Notice that $f_i \to f_\infty \equiv 0$ and $V_{f_\infty} = \mathbb{Z}$. Since $\mathbb{Z} \neq \{0\}$, $V_{f_\infty} \neq \bigcap_{i=0}^{\infty} V_{f_i}$.

Corollary 5.21. *The intersection period subgroup of $f : S \to \Sigma$ can be written as $\hat{V}_f = \bigcap_{f' \in \overline{\mathcal{O}_G(f)}} V_{f'}$.*

Proof. We know that for all $g \in G$, $gf \in \overline{\mathcal{O}_G(f)}$. Therefore, $\hat{V}_f = \bigcap_{g \in G} V_{gf} \subseteq \bigcap_{f' \in \overline{\mathcal{O}_G(f)}} V_{f'}$.

Any $f' \in \overline{\mathcal{O}_G(f)}$ is the limit of $(g_i f)$ for $g_i \in G$. Therefore, $V_{f'} \subseteq \bigcap_{i=0}^{\infty} V_{g_i f} \subseteq \bigcap_{g \in G} V_{gf}$. This is true for all $f' \in \overline{\mathcal{O}_G(f)}$, so $\bigcap_{f' \in \overline{\mathcal{O}_G(f)}} V_{f'} \subseteq \bigcap_{g \in G} V_{gf}$.

Therefore, $\hat{V}_f = \bigcap_{f' \in \overline{\mathcal{O}_G(f)}} V_{f'}$, proving the claim. □

Theorem 5.22 (Expansiveness Criterion)**.** *Let G act on S. Let $f : S \to \Sigma$ be a word and $H \subseteq S$ be a half-plane. If $\hat{V}_f \not\subseteq V_H$, then H is expansive.*

Proof. Assume that $\hat{V}_f \not\subseteq V_H$. Therefore, there is some $u \in \hat{V}_f$ with $u \notin V_H$.

Assume for contradiction that H is nonexpansive. Thus, there are $f_1 \neq f_2 \in \overline{\mathcal{O}_G(f)}$ such that $f_1|_H = f_2|_H$. Since $f_1 \neq f_2$, there is some $s \in S \setminus H$ such that $f_1(s) \neq f_2(s)$. By the second half-plane axiom, since $s \notin H$ there is some $n \in \mathbb{Z}$ such that $u^n s \in H$. Because $f_1|_H = f_2|_H$, we know $f_1(u^n s) = f_2(u^n s)$. Since $u \in \hat{V}_f$, $u \in V_{f_1}, V_{f_2}$. Therefore, $u^n \in V_{f_1}, V_{f_2}$. This implies $f_1(s) = f_1(u^n s) = f_2(u^n s) = f_2(s)$, contradicting that $f_1(s) \neq f_2(s)$. Therefore, H must be expansive. $\square$

The utility of expansiveness in the study of $\mathbb{Z}^2$ stems directly from its relationship to periodicity. There is a partial analogue for these results that hold in the general case.

Corollary 5.23 (Periodicity and Nonexpansiveness)**.** *Let $\mathcal{H}$ be the set of all nonexpansive half-planes. Then, $\hat{V}_f \subseteq \bigcap_{H \in \mathcal{H}} V_H$.*

Proof. Assume for contradiction $\hat{V}_f \not\subseteq \bigcap_{H \in \mathcal{H}} V_H$. Thus, there is some nonexpansive H such that $\hat{V}_f \not\subseteq V_H$. This is a contradiction, because if $\hat{V}_f \not\subseteq V_H$ then H is expansive. $\square$

This in some sense bounds $\hat{V}_f$ to $\bigcap_{H \in \mathcal{H}} V_H$. Notice that $\bigcap_{H \in \mathcal{H}} V_H$ is intuitively smaller when there are more half-planes; more specifically, since $\bigcap_{H \in \mathcal{H}} V_H$ is the intersection of half-planes all having $V_H \supseteq \hat{V}_f$, the size and number of half-plane period subgroups V_H that are slightly larger than $\hat{V}_f$ appear to be crucial in the efficacy of $\bigcap_{H \in \mathcal{H}} V_H$ as a bound.

Consider for example $\mathbb{Z}_2^n$, the set of binary n-tuples e.g. $(0, 1, 0, 0, ..., 1, 1)$ acting on itself with modular addition. Any $\mathbf{x} \in \mathbb{Z}_2^n$ has that $2\mathbf{x} = \mathbf{x} + \mathbf{x} = \mathbf{0}$, so all elements are torsions. Half-plane period subgroups contain all torsions, so we must have $V_H = \mathbb{Z}_2^n$, violating the definition of a half-plane. Thus, there are no half-planes in $\mathbb{Z}_2^n$. This implies that $\bigcap_{H \in \mathcal{H}} V_H = \mathbb{Z}_2^n$, by the definition of an empty intersection.

Since $\bigcap_{H\in\mathcal{H}} V_H = \mathbb{Z}_2^n$, regardless of what word is being considered, it is a rather unhelpful bound on $\hat{V}_f$.

Clearly, we would like to attain much more information from the fact that $\hat{V}_f \not\subseteq \bigcap_{H\in\mathcal{H}} V_H$. We now see that the utility of this is dependent on the prevalence of half-planes in S. We will proceed to develop tools to quickly demonstrate the existence of half-planes with specific period subgroups over an arbitrary group action. To begin, we will define an irrational half-plane.

Definition 5.11. *Call a half-plane H irrational if and only if $V_H = \{e\}$; that is H has trivial period subgroup.*

The usefulness of irrational half-planes is predominantly that the existence of irrational half-planes in quotient sets implies the existence of half-planes in the original set. We formalize this as follows.

Lemma 5.24. *Let G act on S. Let $N \triangleleft G$ be a proper normal subgroup. If $\hat{H}$ is an irrational half-plane with G/N acting on $S : N$, then there is some half-plane H with $V_H = N$.*

Proof. Define H in the following way. Say $s \in H$ if and only if $Ns \in \hat{H}$. We will show this is a half-plane.

Pick some $v \in N$ and any $s \in S$. Notice that $Ns = Nvs$, so $Ns \in \hat{H} \iff Nvs \in \hat{H}$. Thus, by the definition of H, we know $s \in H \iff vs \in H$. Thus, $v \in V_H$, so $N \subseteq V_H$.

Pick some $u \in G \setminus N$. Since $\hat{H}$ is irrational, $V_{\hat{H}} = \{Ne\}$. Since $u \notin N$, we know $Nu \neq Ne$. Therefore, $Nu \notin V_{\hat{H}}$. This implies that there is some $s \in S$ such that $Ns \in \hat{H} \veebar Nus \in \hat{H}$. By the definition of H, this means that $s \in H \veebar us \in H$. Thus, $u \notin V_H$. Thus, $V_H \subseteq N$.

Therefore, we know that $V_H = N$. Since N is a proper subgroup, we know that $V_H \neq G$. We will examine the half-plane axioms separately.

Since $\hat{H}$ is a half-plane, for all $Ng \in G/N$, we have $Ng\hat{H} \subseteq \hat{H}$ or $Ng^{-1}\hat{H} \subseteq \hat{H}$. By the definition of H, this straightforwardly shows that for all $g \in G$, we have $gH \subseteq H$ or $g^{-1}H \subseteq H$, so H satisfies the first half-plane axiom.

Also because $\hat{H}$ is a half-plane, for all $Ns \in S$ and $Nu \in G \setminus V_{\hat{H}} = G \setminus \{Ne\}$, there is some $n \in \mathbb{Z}$ such that $Ns \in \hat{H} \veebar Nu^n s \in \hat{H}$. If $u \notin V_H = N$, then $Nu \neq Ne$. Thus, $u \in G \setminus V_H$ implies that $Nu \in G \setminus \{Ne\}$. Thus, by the definition of H, we know that for all $s \in S$ and $u \in G \setminus V_H$, there is some $n \in \mathbb{Z}$ such that $s \in H \veebar u^n s \in H$. Thus, H satisfies the second half-plane axioms.

Since it satisfies both half-plane axioms, H is a half-plane and $V_H = N$, proving the claim. □

This simplifies the problem of finding half-planes with specific normal period groups to the problem of finding irrational half-planes. One specific example is finite dimensional rational spaces.

Lemma 5.25. *Let $\mathbb{Q}^n$ act on $\mathbb{Q}^n$. There is some irrational half-plane $H \subseteq \mathbb{Q}^n$.*

Proof. Note π is transcendental. This means that there is no polynomial $p(t) = \sum_{i=0}^{m-1} q_i t^i$ with rational coefficients such that $p(\pi) = 0$. That is, for any rational q_i for $0 \leq i < m$, we have $\sum_{i=0}^{m-1} q_i \pi^i \neq 0$

Fix $n \in \mathbb{Z}^+$. Define a half-plane $H \subseteq \mathbb{Q}^n$ by $H := \{\langle s_0, s_1, ..., s_{n-1}\rangle : \sum_{i=0}^{n-1} s_i \pi^i \geq 0\}$.

We will show that $V_H = \{\mathbf{0}\}$. Assume for contradiction there is some $\langle u_0, u_1, ..., u_{n-1}\rangle \neq \mathbf{0} \in V_H$. Since π is transcendental, $\sum_{i=0}^{n-1} u_i \pi^i \neq 0$. Thus, by the Archimedian property there is a $k \in \mathbb{Z}$ such that $k \sum_{i=0}^{n-1} u_i \pi^i = \sum_{i=0}^{n-1} k u_i \pi^i < 0$. Since $\langle u_0, u_1, ..., u_{n-1}\rangle \in V_H$, we know $k\langle u_0, u_1, ..., u_{n-1}\rangle = \langle ku_0, ku_1, ..., ku_{n-1}\rangle \in V_H$. Notice $\mathbf{0} \in H$, since $\sum_{i=0}^{n-1} 0\pi^i = 0 \geq 0$. Thus, since $\langle ku_0, ku_1, ..., ku_{n-1}\rangle \in V_H$, we have $\mathbf{0} + \langle ku_0, ku_1, ..., ku_{n-1}\rangle = \langle ku_0, ku_1, ..., ku_{n-1}\rangle \in H$. Contradiction, $\sum_{i=0}^{n-1} u_i \pi^i < 0$. Thus, $V_H = \{\mathbf{0}\}$.

We will now show that this satisfies the half-plane axioms.

Pick an arbitrary $\langle g_0, g_1, ..., g_{n-1}\rangle \in \mathbb{Q}^n$. Thus, $\sum_{i=0}^{n-1} g_i\pi^i \neq 0$. Consider if $\sum_{i=0}^{n-1} g_i\pi^i > 0$. Pick $\langle s_0, s_1, ..., s_{n-1}\rangle \in H$. Thus, $\sum_{i=0}^{n-1} s_i\pi^i > 0$, so $\sum_{i=0}^{n-1} g_i\pi^i + \sum_{i=0}^{n-1} s_i\pi^i = \sum_{i=0}^{n-1}(g_i + s_i)\pi^i > 0$. Thus $\langle s_0, s_1, ..., s_{n-1}\rangle \in \langle g_0, g_1, ..., g_{n-1}\rangle + H$ implying $H \subseteq \langle g_0, g_1, ..., g_{n-1}\rangle + H$ and $-\langle g_0, g_1, ..., g_{n-1}\rangle + H \subseteq H$. Consider if $\sum_{i=0}^{n-1} g_i\pi^i < 0$. Pick $\langle s_0, s_1, ..., s_{n-1}\rangle \in \langle g_0, g_1, ..., g_{n-1}\rangle + H$ Thus, $\sum_{i=0}^{n-1}(g_i + s_i)\pi^i = \sum_{i=0}^{n-1} g_i\pi^i + \sum_{i=0}^{n-1} s_i\pi^i > 0$. Since $\sum_{i=0}^{n-1} g_i\pi^i < 0$, we have $-\sum_{i=0}^{n-1} g_i\pi^i > 0$. Thus, $\sum_{i=0}^{n-1} g_i\pi^i - \sum_{i=0}^{n-1} g_i\pi^i + \sum_{i=0}^{n-1} s_i\pi^i = \sum_{i=0}^{n-1} s_i\pi^i > 0$. Therefore, $\langle g_0, g_1, ..., g_{n-1}\rangle + \langle s_0, s_1, ..., s_{n-1}\rangle \in H$ implying that $\langle g_0, g_1, ..., g_{n-1}\rangle + H \subseteq H$. Therefore, $\langle g_0, g_1, ..., g_{n-1}\rangle + H \subseteq H$ or $-\langle g_0, g_1, ..., g_{n-1}\rangle + H \subseteq H$, so H satisfies the first half-plane axiom.

Pick an arbitrary $\langle s_0, s_1, ..., s_{n-1}\rangle \in \mathbb{Q}^n$ and $\langle u_0, u_1, ..., u_{n-1}\rangle \in \mathbb{Q}^n \setminus V_H = \mathbb{Q} \setminus \{\mathbf{0}\}$. We know that $\sum_{i=0}^{n-1} u_i\pi^i \neq 0$. Thus, by the Archimedean property, there is some $k \in \mathbb{Z}$ such that $\sum_{i=0}^{n-1} s_i\pi^i + k\sum_{i=0}^{n-1} u_i\pi^i \geq 0 \veebar \sum_{i=0}^{n-1} s_i\pi^i \geq 0$. By the definition of H, this means that $\langle s_0, s_1, ..., s_{n-1}\rangle \in H \veebar \langle s_0, s_1, ..., s_{n-1}\rangle + k\langle u_0, u_1, ..., u_{n-1}\rangle \in H$, so H satisfies the second half-plane axiom.

Since it satisfies both of the axioms, H is a half-plane. Thus, $H \subseteq \mathbb{Q}^n$ is a half-plane with $V_H = \{\mathbf{0}\}$ proving the claim. $\square$

We now know that all finite dimensional rational spaces have irrational half-planes. We can use this to make an even deeper statement about half-planes in finite dimensional rational spaces.

Theorem 5.26. *Given a proper subgroup $N \subseteq \mathbb{Q}^n$ such that $\sqrt{N} = N$, there is a half-plane $H \subseteq \mathbb{Q}^n$ such that $V_H = N$.*

Proof. Since $\mathbb{Q}^n$ is abelian, we know that N is normal.

Since $\sqrt{N} = N$ in $\mathbb{Q}^n$, we know that N is isomorphic to $\mathbb{Q}^m$ for some $m < n$. Thus, $\mathbb{Q}^n/N$ acting on $\mathbb{Q}^n : N$ is isomorphic to $\mathbb{Q}^n/\mathbb{Q}^m$ acting on $\mathbb{Q}^n : \mathbb{Q}^m$, which is in turn isomorphic to $\mathbb{Q}^{n-m}$ acting on itself with $n - m \geq 1$.

We know that finite dimensional rational spaces have irrational half-planes, so specifically, $\mathbb{Q}^{n-m}$ acting on itself admits an irrational half-plane. By isomorphism, this means that $\mathbb{Q}^n/N$ acting on $\mathbb{Q}^n : N$ has an irrational half-plane.

Thus, there is a proper normal subgroup $N \subseteq \mathbb{Q}^n$ such that $\mathbb{Q}^n/N$ acting on $\mathbb{Q}^n : N$ has an irrational half-plane. By a previous theorem, this implies that $\mathbb{Q}^n$ acting on itself has some half-plane H such that $V_H = N$, proving the claim. □

Remember that by construction all half-planes H must have that $\sqrt{V_H} = V_H$. Therefore, the period subgroups of half-planes are completely characterized in all $\mathbb{Q}^n$; a subgroup of $\mathbb{Q}^n$ is a period subgroup of some half-plane exactly when $\sqrt{V_H} = V_H$.

This can be extended even further with the following tool.

Theorem 5.27. *Let G acting on S admit some half-plane H. Let $K \not\subseteq V_H$ be a subgroup of G and let $s \in S$. Let $S' \subseteq S$ be such that $kS' = S'$ for all $k \in K$. Note that K acts on S'. We have that $H \cap S'$ is a half-plane on S' and that $V_{H\cap S'} = V_H \cap K$.*

Proof. Since $K \not\subseteq V_H$, there is some $k \in K \setminus V_H$. Pick some $s \in S'$. Since H is a half-plane, by the second half-plane axiom, we have that there is some n such that $k^n s \in H \veebar s \in H$. Notice that since $k \in K$, we have that $k^n \in K$. Therefore, $k^n S' = S'$, implying that $k^n s \in S'$. This means that $k^n s \in H$ and $s \in H$ if and only if $k^n s \in H \cap S'$ and $s \in H \cap S'$ respectively. This implies that $k^n s \in H \cap S' \veebar s \in H \cap S'$. Therefore, $k^n \in K$ is not in $V_{H\cap S'}$, so $V_{H\cap S'} \neq K$. We will now consider both half-plane axioms separately.

Since H is a half-plane we know that for all $g \in G$, we have $gH \subseteq H$ or $g^{-1}H \subseteq H$. Since $K \subseteq G$, this implies that for all $k \in K$, we have $kH \subseteq H$ or $k^{-1}H \subseteq H$. This implies that $(kH) \cap S' \subseteq H \cap S'$ or $(k^{-1}H) \cap S' \subseteq H \cap S'$. Notice that since $k \in K$, we know $kS' = S'$. Thus, $(kH) \cap S' = (kH) \cap kS' = k(H \cap S')$. Similarly, $(k^{-1}H) \cap S' = k^{-1}(H \cap S')$. Therefore, for all $k \in K$, we have that $k(H \cap S') \subseteq H \cap S'$ or $k^{-1}(H \cap S') \subseteq H \cap S'$. Thus, $H \cap S'$ satisfies the first half-plane axiom.

Since H is a half-plane, for all $g \in G$ and $s \in S$, there is some $n \in \mathbb{Z}$ such that $g^n s \in H \veebar s \in H$. More specifically, for all $k \in K$ and $s \in s'$, there is some $n \in \mathbb{Z}$ such that $k^n s \in H \veebar s \in H$. Notice that since $s \in S'$, the element $k^n s$ is also in S'. Thus, $k^n s \in H$ and $s \in H$ if and only if $k^n s \in H \cap S'$ and $s \in H \cap S'$ respectively. Therefore, for all $k \in K$ and $s \in s'$, there is some $n \in \mathbb{Z}$ such that $k^n s \in H \cap S' \veebar s \in H \cap S'$. Thus, $H \cap S'$ satisfies the second half-plane axiom.

Therefore, $H \cap S'$ is a half-plane over K acting on S'. We will now calculate $V_{H \cap S'}$.

Pick $v \in V_H \cap K$. Therefore, for all $s \in S$, $s \in H$ if and only if $vs \in H$. More specifically, for all $s \in S'$, $s \in H$ if and only if $vs \in H$. If $s \in S'$, then $vs \in S'$. Thus $s \in H \cap S'$ and $vs \in H \cap S'$ if and only if $s \in H$ and $vs \in H$. Thus, $s \in H \cap S'$ if and only if $vs \in H \cap S'$. Since $v \in K$ satisfies this property, $v \in V_{H \cap S'}$. Therefore, $V_H \cap K \subseteq V_{H \cap S'}$.

Now pick $u \notin V_H \cap K$. If $u \notin K$, clearly $u \notin V_{H \cap S'}$. Consider $u \notin V_H$ but $u \in K$. Pick some $s \in S'$. Since H is a half-plane, by the second half-plane axiom we have that there is an n such that $u^n s \in H \veebar s \in H$. Notice $u^n \in K$, so $u^n s \in S'$. Thus, we know $u^n s \in H \cap S' \veebar s \in H \cap S'$. This means $u^n \notin V_{H \cap S'}$, which implies that $u \notin V_{H \cap S'}$. This implies that $V_{H \cap S'} \subseteq V_H \cap K$.

Thus, $V_{H \cap S'} = V_H \cap K$, proving the claim. □

A simple application of this extends the results in $\mathbb{Q}^n$ to the integer lattice.

Corollary 5.28. *Let $\mathbb{Z}^d$ act on itself. Let $N \subseteq \mathbb{Z}^d$ be a subgroup with $\sqrt{N} = N$. There is a half-plane $H \subseteq \mathbb{Z}^d$ such that $V_H = N$.*

Proof. To avoid confusion, let $\sqrt[G]{K}$ be the radical subgroup of K over G.

Notice that $\mathbb{Z}^d$ is a subgroup of $\mathbb{Q}^d$. Notice therefore that $\sqrt[\mathbb{Q}^d]{N}$ is a subgroup of $\mathbb{Q}^d$. By the properties of the radical, $\sqrt[\mathbb{Q}^d]{\sqrt[\mathbb{Q}^d]{N}} = \sqrt[\mathbb{Q}^d]{N}$. By the theorem about finite dimensional rational spaces, there is some $H' \subseteq \mathbb{Q}^d$ such that $V_{H'} = \sqrt[\mathbb{Q}^d]{N}$.

By the last the theorem, $H := H' \cap \mathbb{Z}^d$ is a half-plane with $V_H = V_{H'} \cap \mathbb{Z}^d$. Substituting, we know that $V_H = \sqrt[\mathbb{Q}^d]{N} \cap \mathbb{Z}^d$. This is equal to $\sqrt[\mathbb{Z}^d]{N \cap \mathbb{Z}^d}$. Since $N \subseteq \mathbb{Z}^d$, this is equal to $\sqrt[\mathbb{Z}^d]{N} = N$. Therefore, $V_H = N$, implying there is a half-plane in $\mathbb{Z}^d$ with period subgroup N. This proves the claim. $\square$

Similar to the rational case, since we know that $\sqrt{V_H} = V_H$ for all half-planes, this means that the subgroups N of $\mathbb{Z}^d$ with a half-plane H having $V_H = N$ are exactly those having $\sqrt{N} = N$.

Chapter 6

Further Directions

Remember that by a result of this thesis we now know that nonexpansive directions come in finitely many antiparallel pairs. Also remember that given a rectangle satisfying $p_\eta(n, 4) \leq 4n$, there is an η-generating set that is a convex subset of the $n \times 4$ rectangle whose edges correspond to the nonexpansive directions of η. A convex set of height at most four cannot have more than 8 edges, thus η cannot have more than 8 nonexpansive directions, or equivalently 4 nonexpansive lines.

We know that if there are no nonexpansive directions that η is doubly periodic [2]. We also know that if there is exactly one nonexpansive line, then η is singly periodic in a direction on that line [6]. Therefore, if we can show that η does not have 2, 3, or 4 nonexpansive directions, since it cannot have more than 4, we know that η is periodic, proving Nivat's conjecture for height 4.

This thesis proves that some of the instances with 4 nonexpansive lines are impossible. The other instances of 4 nonexpansive lines, as well as the cases of 2 or 3 nonexpansive lines remain to be proven in order to prove Nivat's conjecture for height 4 rectangles.

The utility of half-planes stems from their connection to periodicity. We know in general that $\hat{V}_f \subseteq \bigcap_{H\in\mathcal{H}} V_H$, but this result can be strengthened significantly in specific group actions. Consider the following definitions.

Definition 6.1. *Let G act on S, and $f : S \to \Sigma$ be a word. We call f pseudo-aperiodic if and only if $\hat{V}_f = \{e\}$. That is, for any $v \in G$, there is some shift $g \in G$ such that gf is not periodic with v.*

Definition 6.2 (Discerning Group Action)**.** *Let a group G act on a set S. We say that this is a discerning group action if for all pseudo-aperiodic words $f : S \to \Sigma$, we have $\bigcap_{H\in\mathcal{H}} V_H = \{e\}$.*

Intuitively, a group action being discerning means that it has sufficiently many half-planes that the intersection $\bigcap_{H\in\mathcal{H}} V_H$ is small enough to be trivial. Discerning group actions, whose quotients are also discerning admit a much stronger statement about $\hat{V}_f$.

Theorem 6.1. *Let a group G act on a set S. Let this group action, and all of its quotients, be discerning. Let $f : S \to \Sigma$ be a word, then $\hat{V}_f = \bigcap_{H\in\mathcal{H}} V_H$.*

Proof. In general, remember that we know $\hat{V}_f \subseteq \bigcap_{H\in\mathcal{H}} V_H$.

Remember that $\hat{V}_f$ must be normal. Now consider the quotient action $G/\hat{V}_f$ acting on $S : \hat{V}_f$.

Construct the word $f' : (S : \hat{V}_f) \to \Sigma$ by $f'(\hat{V}_f s) = f(s)$. To show that this is well defined, pick arbitrary $\hat{V}_f s_1 = \hat{V}_f s_2$. Thus, there is some $v \in \hat{V}_f \subseteq V_H$ such that $s_1 = vs_2$. Calculate that $f'(\hat{V}_f s_2) = f(s_2) = f(vs_2) = f(s_1) = f'(\hat{V}_f s_1)$, showing that f' is well defined.

Pick arbitrary $\hat{V}_f g \neq \hat{V}_f e \in G/\hat{V}_f$. Since $\hat{V}_f g \neq \hat{V}_f e$, we have $g \notin \hat{V}_f$. Thus, there is some $a \in G$ such that $g \notin V_{af}$. Therefore, there is some $s \in S$ such that $af(s) \neq af(gs)$ or equivalently $f(a^{-1}s) \neq f(a^{-1}gs)$. Consequently, $f'(\hat{V}_f a^{-1}s) \neq f'(\hat{V}_f a^{-1}gs)$ and $\hat{V}_f af'(\hat{V}_f s) \neq \hat{V}_f af'(\hat{V}_f gs)$. This implies that $\hat{V}_f g \notin \hat{V}_{f'}$. Since $\hat{V}_f g$ was arbitrary, $\hat{V}_{f'} = \{\hat{V}_f e\}$, meaning f' is pseudo-aperiodic.

Since f' is pseudo-aperiodic in a quotient action, by assumption we have that $\bigcap_{H'\in\mathcal{H}'} V_{H'} = \{\hat{V}_f e\}$, where $\mathcal{H}'$ is the set of nonexpansive half-planes of $S : \hat{V}_f$.

Pick an arbitrary $g \in G \setminus \hat{V}_f$. Thus, $\hat{V}_f g \neq \hat{V}_f e$. Therefore, $\hat{V}_f g \notin \bigcap_{H'\in\mathcal{H}'} V_{H'} = \{\hat{V}_f e\}$. Thus, there is some nonexpansive half-plane $H' \subseteq (S : \hat{V}_f)$ such that $\hat{V}_f g \notin V_{H'}$.

Now construct a half-plane $H \subseteq S$ by $H := \{s : \hat{V}_f s \in H'\}$.

Assume for contradiction that $g \in V_H$. Thus, for all $s \in S$, we have $s \in H \iff gs \in H$. By the definition of H', this implies that for all $s \in S$ and thus for all $\hat{V}_f s \in (S : \hat{V}_f)$, $\hat{V}_f s \in H' \iff \hat{V}_f gs \in' H$. Therefore, $\hat{V}_f g \in V_{H'}$, a contradiction. Thus, $g \notin V_H$.

Since H' is nonexpansive, there are $f'_1 \neq f'_2 \in \overline{\mathcal{O}(f')}$ such that $f'_1|_{H'} = f'_2|_{H'}$. Construct the words f_1, f_2 by $f_1(s) := f'_1(\hat{V}_f s)$ and $f_2(s) := f'_2(\hat{V}_f s)$ for all $s \in S$. Notice $f_1, f_2 \in \overline{\mathcal{O}(f)}$. Since $f'_1 \neq f'_2$, there is some $s \in S$ such that $f'_1(\hat{V}_f s) \neq f'_2(\hat{V}_f s)$ and therefore $f_1(s) \neq f_2(s)$. Thus, $f_1 \neq f_2$. Consider arbitrary $h \in H$. Thus, $\hat{V}_f h \in H'$. Therefore, $f'_1(\hat{V}_f h) = f'_2(\hat{V}_f h)$ and consequently, $f_1(h) = f_2(h)$. Since $h \in H$ was arbitrary, $f_1|_H = f_2|_H$. Thus, there are $f_1 \neq f_2 \in \overline{\mathcal{O}(f)}$ such that $f_1|_H = f_2|_H$, implying that H is nonexpansive. Thus, $V_H \supseteq \bigcap_{H\in\mathcal{H}} V_H$.

Since $g \notin V_H$, this implies $g \notin \bigcap_{H\in\mathcal{H}} V_H$. Since $g \in G \setminus \hat{V}_f$ was arbitrary, this means $\bigcap_{H\in\mathcal{H}} V_H \subseteq \hat{V}_f$. Therefore, $\hat{V}_f = \bigcap_{H\in\mathcal{H}} V_H$, proving the claim. $\square$

The ideal statement connecting periodicity and expansiveness is $\hat{V}_f = \bigcap_{H\in\mathcal{H}} V_H$, but having all quotient actions being discerning is a very rare property. Future work into the extension of arbitrary group actions will seek to find weaker preconditions for statements similar to $\hat{V}_f = \bigcap_{H\in\mathcal{H}} V_H$.

This work should also search for connections between complexity and expansiveness, similar to the strong connections present in $\mathbb{Z}^2$.

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