1997

Diophantine Type Fractional Derivative Representation of Structural Hysteresis, Part I: Formulation

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Publisher's Statement

The final publication is available at Springer via http://dx.doi.org/10.1007/s004660050182

Original Citation

Padovan, J., and Sawicki, J.T. (1997). Diophantine Type Fractional Derivative Representation of Structural Hysteresis, Part I: Formulation. Computational Mechanics, Springer, 19(5), 335-341.

Diophantine type fractional derivative representation of structural hysteresis

Part I: Formulation

J. Padovan, J. T. Sawicki

Abstract Based on a diophantine representation of the operational powers, a fractional derivative modelling scheme is developed to simulate frequency dependent structural damping. The diophantine set of powers is established by employing the curvature properties of the defining empirical data set. These together with a remezed least square scheme are employed to construct a Chebyschev like optimal differintegro simulation. Based on the use of the rational form resulting from the diophantine representation, a composition rule is introduced to reduce the differintegro simulation to first order form. The associated eigenvalue/vector properties are then explored. To verify the robustness-stability accuracy of the overall modelling procedure, correlation studies are also presented. Part I of this series focuses on the diophantine representation, its use in formulating a numerically more workable first order form as well as formal representations of its transient and steady state solutions. This will include investigations of the asymptotic properties of the various formulations. Part II will introduce the model fitting scheme along with a look at eigen properties and fitting effectiveness.

1
Introduction

Under normal circumstances, the actual structural hysteretic response behavior tends to be more complex than traditional viscous, Meirovitch (1967), and proportionally damped, Meirovitch (1967), Fertis (1995), simulations can handle. This is a direct outgrowth of such complicating features as: i) frequency dependent behavior; ii) general multirate/history effects; and iii) multiphasic time shifts, among many. To enable the simulation of such characteristics, typically either the Kelvin-Voigt (*KV*) or more comprehensive Maxwell-Kelvin-Voigt (*MKV*) models are employed, i.e., Mase and Mase (1992), Fung (1965) among many. These generally involve the use of integer differential operators, Mase and Mase (1992).

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Generally *KV* and *MKV* models tend to become increasingly stiffer as the associated operator orders grow. Such behavior is intrinsic to the basis space of integer differential operators. As will be seen, to bypass such difficulties, fractionally based operators can be employed. To date, numerous investigators as Oldham and Spanier (1974), Bagley and Torvik (1983), Padovan et al. (1987), Padovan (1987), Padovan and Guo (1988), Bagley (1989), Bagley and Calico (1991), and Enelund and Josefson (1996), have dealt with various of the properties of fractional operators. Several have explored the use of fractional representation for various analytical or experimentally generated data. These have shown the potential effectiveness of such operators.

This paper will develop a model fitting scheme which enables the development of either *KV* or *MKV* fractional models for arbitrary experimental data. Due to the robustness of the scheme, essentially any level of representational accuracy is possible. This is achieved through the use of diophantinized, Schmidt (1980), fractional operator families. The diophantine approximated, Schmidt (1980), set of fractional power can be established within the appropriate delimiting bounds. These can be obtained by employing the curvature properties of the empirical data set. The fitting coefficients of the *KV* and *MKV* models are then established by a remezed, Remez (1934), Carpenter and Varga (1991), least square formulation, Hamming (1962). This enables an optimal fit for a given diophantine set of powers.

Overall the paper is structured in two parts. The first considers the more or less formal aspects of the development while the second introduces the fitting scheme along with a description of eigen properties and benchmarking. The formal aspects consist of such issues as:

i) introducing diophantine representations of fractional models,
ii) to ease the algorithmic burden of differintegro operators, the diophantine representation is reduced to vector form, and
iii) formal transient and steady solutions are explored.

Since the vector form is based on the formal use of the differintegro operator composition rule, the ramifications of its small time and asymptotic long time properties will be investigated.

In the sections which follow, detailed discussions will be given on: i) fractional and integer based *KV* and *MKV* models and their various forms, ii) the diophantine approximation, iii) the vector form, and iv) formal solutions along with their asymptotic characteristics.

2
Fractional and Integer *KV* and *MKV* models

Prototypically *KV* and *MKV* type simulations are employed to represent system damping. These involve either integer or fractional formulations. This section will develop various of the structural/operational properties of such simulation types.

2.1
Integer *KV* and *MKV* models

Generally *KV* and *MKV* type simulations are cast in terms of integer differential operators, namely:

i) *KV* model, Fung (1965);

$$F_D = \sum_{\ell} \mu_{x\ell} D_\ell(x) \tag{2.1.1}$$

ii) *MKV* model, Fung (1965);

$$\sum_{\ell} \mu_{f\ell} D_\ell(F_D) = \sum_{\ell} \mu_{x\ell} D_\ell(x) \tag{2.1.2}$$

where

$$D_\ell(\cdot) = \frac{\mathrm{d}^\ell}{\mathrm{d}t^\ell}(\cdot) \tag{2.1.3}$$

and F_D, x, and $(\mu_{x\ell}, \mu_{f\ell})$ represent the damping force, kinematic variable, and the coefficients of the *KV*/*MKV* simulations, respectively. Equations (2.1.1 and 2.1.2) can be fit in a variety of ways, i.e., Pade-Laplace, Simhambhatla and Leonov (1993), and so on.

The chief difficulty associated with such simulations stems from the problem of basis completeness and operator admissibility. In particular, to enable fits to more complex response characteristics, higher order operators are required regardless of their physical justification. This naturally causes two forms of model stiffness, namely wherein;

i) the lower and higher order coefficients vary many orders of magnitude thus yielding potential roundoff problems, and;
ii) the higher order operators tend to naturally induce a wide spread in the eigenvalue structure of hysteretic model.

When the small magnitude of the leading coefficient is combined with the order of the operator, the problem stiffness is further exacerbated.

2.2
Fractional models

As noted in the proceeding section, the major difficulty with integer type integro-differential models is the physical admissibility of the higher order operators. This problem can be partially resolved by introducing a fractional differintegro simulation, wherein the physics can be employed to define bounding values for the operator orders. Note, while the underlying problem physics generally produces strong constraint on operator order, the fractional scheme enables us to introduce/develop an essentially complete basis within any interval. This can be achieved by fractionally dissecting the said range of powers.

The fractional differintegro operator adopted here is that by Riemann-Liouville, Oldham and Spanier (1974). It has the form

$$D_q(s) = \frac{1}{\Gamma(-q)} \int_0^t \frac{s(\tau)}{(t-\tau)^{q+1}} \mathrm{d}\tau \tag{2.2.1}$$

where $\Gamma(-q)$ is the gamma function, Abramovitz and Stegun (1965). Employing (2.2.1), the *KV* and *MKV* formulations of the damping/hysteresis simulation take the form:

i) *KV* model;

$$F_D = \sum_{\ell}^{N_x} \mu_{x\ell} D_{p_\ell^x}(x) \tag{2.2.2}$$

ii) *MKV* model

$$\sum_{\ell}^{N_f} \mu_{f\ell} D_{p_\ell^f}(F_D) = \sum_{\ell}^{N_x} \mu_{x\ell} D_{p_\ell^x}(x) \tag{2.2.3}$$

Here the fractional power sets $\underline{p}^f$ and $\underline{p}^x$ are bound by the curvatures of the hysteretic force-kinematic space. Such a relationship can be established either in the Laplace or Fourier transform domains, as established either by transient or steady state behavior.

A detailed discussion is given in the section on fitting in Part II. As will be seen, the fractional formulation has three very important advantages over the integer approach, i.e., i) it is not stiff; ii) the basis space of operators is selected contingent on physical admissibility properties, and iii) a complete basis space can be derived for any interval of powers.

3
Composition rule

A central feature/difficulty of fractional operators is establishing their differintegrable properties. To enable the introduction of any modification in the operator form of the *MKV* formulation, we will require the use of a composition type property, namely

$$D^\alpha\{D^\beta(\cdot)\} = D^{\alpha+\beta}(\cdot) \tag{3.1}$$

For the current purposes, we will assume that all functional representations growing out of the physics will prototypically be differintegrable.

Generally, most special functions of mathematical physics are differintegrable series. These can usually be represented by finite sums of functions which themselves may be defined by series families of the type:

$$g_\theta(t) = (t-\tau)^\theta \sum_{j=0}^{\infty} a_j (t-\tau)^{j/k}; \quad \theta > -1 \tag{3.2}$$

where $a_0 \neq 0$ and k is an arbitrary integer. Such finite sums take the following form

$$G(t) = (t-\tau) \sum_{j_1=0}^{\infty} a_{j_1} (t-\tau)^{j_1} + (t-\tau)^{(n+1)/n} \sum_{j_2=0}^{\infty} a_{j_2} (t-\tau)^{j_2} + \ldots + (t-\tau)^{(n+n-1)/n} \sum_{j_n=0}^{\infty} a_{j_n} (t-\tau)^{j_n} \tag{3.3}$$

In seeking a general composition rule for the differintegration of G, we must determine the relationship between $D\ (D\ (G))$ and $D^{\ +\ }(G)$. Here we must assume that both G and $D\ (G)$ are differintegerable. Noting the form of (3.3), it follows from the linearity-distributive property of fractional operators that, when

$$D\ \left(D\ (g_\theta)\right) \equiv D^{\ +\ }(g_\theta) \tag{3.4}$$

then

$$D\ \left(D\ (G)\right) \equiv D^{\ +\ }(G) \tag{3.5}$$

where here g_θ are the various finite base members of (3.3). If each and every base member satisfies (3.4), then (3.5) follows.

Based on the form of g_θ,

$$\begin{aligned} D\ (g_\theta) &= \mathop{}\limits_{j=0}^{\infty} a_j D\ \left((t-\tau)^{\theta+j}\right) \\ &= \mathop{}\limits_{j=0}^{\infty} a_j \frac{\Gamma(\theta+j+1)(t-\tau)^{\theta+j-\ }}{\Gamma(\theta+j-\ \ +1)} \end{aligned} \tag{3.6}$$

Note, while the property ($g_\theta \equiv 0$ and $D\ (g_\theta) \equiv 0$) automatically satisfies (3.5) for situations in which ($g_\theta \neq 0$ and $D\ (g_\theta) \equiv 0$), then such is not necessarily the case. A necessary and sufficient condition for the nonsingularity of (3.6) requires that $\Gamma(\theta + j - \ \ + 1)$ remains finite for each j for which $a_j \neq 0$. This is satisfied so long as $\ \ < 0$, or for $\ \ < \theta + 1$.

The foregoing conditions are equivalent to requiring that g_θ be regenerated upon application of first $D\ $ and then $D^{-\ }$. This leads to the condition

$$g_\theta - D^{-\ }\left(D\ (g_\theta)\right) \equiv 0 \tag{3.7}$$

For completely general g_θ, it follows from (3.2) that

$$D\ \left(D\ (g_\theta)\right) = D^{\ +\ }(g_\theta) - D^{\ +\ }\left(g_\theta - D^{-\ }\left(D\ (g_\theta)\right)\ \right) \tag{3.8}$$

Due to the linearity of (3.3), (3.8) yields the identity

$$D\ \left(D\ (G)\right) = D^{\ +\ }(G) - D^{\ +\ }\left(G - D^{-\ }\left(D\ (G)\right)\ \right) \tag{3.9}$$

where, if

$$G - D^{-\ }\left(D\ (G)\right) \equiv 0 \tag{3.10}$$

then (3.5) follows. Recalling the earlier discussion, (3.10) is guaranteed for all $\ \ < 0$ and for $\ \ < 1$, wherein G remains bounded at the lower limit of t, i.e., τ.

Based on the foregoing, most special functions of mathematical physics violate (3.1), namely, (3.10) is nonsingular. As an example, the fractional differential equation

$$D_{\frac{1}{q}}(x) - \ \ x = 0 \tag{3.11}$$

has the solution

$$x = cF(\ \ , q, t) \tag{3.12}$$

where c is a constant, and since q is an integer

$$F(\ \ , q, t) = \mathop{}\limits_{k=0}^{q-1} (\ \ )^{q-k-1} H\left(t, -\frac{k}{q}, (\ \ )^q\right) \tag{3.13}$$

such, that

$$H\left(t, -\frac{k}{q}, (\ \ )^q\right) = (t)^{\frac{k}{q}-1} \mathop{}\limits_{\ell=0}^{\infty} \frac{(\ \ )^{q\ell}(t)^{\ell}}{\Gamma(k/q+\ell)} \tag{3.14}$$

Based on (3.13 and 3.14), the decomposed operator yields

$$D_{\frac{1}{q}}\left(D_{\frac{1}{q}}(F(\ \ , q, t))\right) = (\ \ )^2 F(\ \ , q, t) \tag{3.15}$$

For the original operator it follows that

$$D_{\frac{2}{q}}(F(\ \ , q, t)) = (\ \ )^2 F(\ \ , q, t) + \frac{t^{-(1+1/q)}}{\Gamma(-1/q)} \tag{3.16}$$

As can be seen from (3.15 and 3.16), for small times the composition rule is violated. In contrast, for large times, $F(\ \ , q, t)$ asymptotically satisfies (3.1). This is the case for the solutions of all ODE.

Noting the form of (3.14), the structure is reminiscent of a confluent hypergeometric function, Davis (1936). Hence, depending on the choice of k, q, and $\ \ $, a wide variety of the functions of mathematical physics emerge, Davis (1936).

4
Diophantine representation of fractional models

The choice of the set of fractional powers is controlled by the inherent geometric characteristics of the system force-kinematic response behavior. In this context, the range of slopes define upper and lower bounds on potentially admissible powers. This issue will be addressed in the curve fitting section in Part II. Once the bounds are set, several possible choices of fractional powers can be made, namely:

1) As a problem of optimality wherein the choices are determined by the requisite criterion function;
2) By choosing an equally spaced or remezed set, see Remez (1934), Hamming (1962), or Carpenter and Varga (1991), or;
3) By introducing a diophantine representation of the equally spaced set defined under 2.

Given, that the bounds on the *MKV* model are

$$\begin{aligned} &p^{fU} - upper, p^{fL} - lower \\ &p^{xU} - upper, p^{xL} - lower \end{aligned} \tag{4.1}$$

then least upper/lower integer bounds can be selected by respectively rounding up or down to the nearest whole integer. Such a process yields

$$\begin{aligned} &\left(I^{fU}; I^{xU}\right) > \left(p^{fU}; p^{xU}\right) \\ &\left(I^{fL}; I^{xL}\right) < \left(p^{fL}; p^{xL}\right) \end{aligned} \tag{4.2}$$

Based on the limits set by (4.2), the equally spaced diophantine approximation of the $\underline{p}^f$ and $\underline{p}^x$ sets takes the form

$$a_\ell = p_\ell^f = I^{fL} + \frac{\ell}{\ \ _f}\left(I^{fU} - I^{fL}\ \ , \quad \ell \in [0, \ \ _f] \tag{4.3}$$

$$b_\ell = p_\ell^x = I^{xL} + \frac{\ell}{\ \ _x}\ \ I^{xU} - I^{xL}\right), \quad \ell \in [0, \ \ _x] \tag{4.4}$$

where here the common fractional denominators are defined by $_f$ and $_x$. As can be seen from (4.3 and 4), for $(I^{fL}; I^{xL}) \equiv \; 0$ it follows, that

$$a_\ell \sim \ell \frac{I^{fU}}{\;_f} \tag{4.5}$$

$$b_\ell \sim \ell \frac{I^{xU}}{\;_x} \tag{4.6}$$

such, that $(\ell / \;_f; \ell / \;_x)$ define rational fractional families.

Under either (4.3 and 4) or (4.5 and 6), the *MKV* formulation takes the form

$$\;_{\ell}^{f} \mu_{f\ell} D_{a_\ell}(F_D) = \;_{\ell}^{x} \mu_{x\ell} D_{b_\ell}(x) \tag{4.7}$$

From the equation of motion, the damper force is:

$$F_D = S - M\ddot{x} - Kx \tag{4.8}$$

Given, that S is differintegrable, then most likely so too are $D_{a_\ell}(F_D)$ and $D_{b_\ell}(x)$. In this context (4.7 and 8) lead to the differintegrable equation

$$\begin{aligned} M \;_{\ell}^{f} \mu_{f\ell} D_{a_\ell}(\ddot{x}) + \;_{\ell}^{x} \mu_{x\ell} D_{b_\ell}(x) \\ + K \;_{\ell}^{f} \mu_{f\ell} D_{a_\ell}(x) = \;_{\ell}^{f} \mu_{f\ell} D_{a_\ell}(S) \end{aligned} \tag{4.9}$$

Since (4.7) yields a rational form in the Laplace and frequency (i.e, Fourier) domains, generally, to yield stable asymptotics, $I^{fU} > I^{xU}$ and $\;_f > \;_x$.

Earlier we saw that the composition rule for both arbitrary fractional and diophantine formulations has different long and small time characteristics. This can be seen from a rationing of (3.15 and 16). Specifically,

$$\frac{D_{2/q}(F)}{D_{1/q} \; D_{1/q}(F)} = 1 + \frac{1}{(\;)^2} \left\{ \frac{t^{-(1+1/q)}}{\Gamma(k/q+1)F} \right\} \tag{4.10}$$

As the system eigenvalue grows in size, the small time asymptotics diminish in importance. In this context, decomposition can be used in a formal sense to alter the fractional continuum formulation (4.9) into, what will be termed, the decomposed form.

For demonstration purposes, we consider the *KV* version, namely

$$M\ddot{x} + \;_{\ell}^{x} \mu_{x\ell} D_{b_\ell}(x) + Kx = S \tag{4.11}$$

where here for simplicity $_x$ and b_ℓ are chosen so, that

$$b_\ell = \frac{2\ell}{\;_x}; \quad \ell \in [1, \;_x] \tag{4.12}$$

Formally applying the composition rule, it follows that

$$D_{b_\ell}(x) = D_{\frac{2\ell}{x}}(x) = \prod_{k=1}^{\ell} D_{\frac{2}{x}}(\cdot)x \tag{4.13}$$

where

$$\prod_{k=1}^{\ell} D_{\frac{2}{x}}(\cdot) = D_{\frac{2}{x}} \; D_{\frac{2}{x}} \; \ldots D_{\frac{2}{x}}(\cdot) \ldots \tag{4.14}$$

Similarly,

$$\ddot{x} = \prod_{k=1}^{x} D_{\frac{2}{x}}(\cdot)x \tag{4.15}$$

Based on (4.13 and 15), (4.11) takes the following decomposed form

$$\left\{ M \prod_{k=1}^{x} D_{\frac{2}{x}}(\cdot) + \;_{\ell=1}^{x} \mu_{x\ell} \prod_{k=1}^{\ell} D_{\frac{2}{x}}(\cdot) + K \right\} x = S \tag{4.16}$$

Both (4.11) and (4.16) possess the same characteristic roots, i.e., eigenvalues. Employing the Laplace transform, it can be shown that the characteristic polynomial associated with these operators can be expressed as

$$P(\lambda) = M(\lambda)^2 + \;_{\ell=1}^{x} \mu_{x\ell}(\lambda)^{\frac{2\ell}{x}} + K \tag{4.17}$$

The roots then must satisfy

$$P(\lambda) = 0 \tag{4.18}$$

wherein $\lambda = \lambda_i; \; i \in [1, 2 \;_x]$. Letting

$$\lambda = \xi^{\;_x}, \tag{4.19}$$

the roots can be obtained from the integer expression

$$M(\xi)^{2 \;_x} + \;_{\ell=1}^{x} \mu_{x\ell}(\xi)^{2\ell} + K = 0 \tag{4.20}$$

In terms of the roots defined by Eq. (4.20), Eq. (4.16) can be recast in the following operator form, namely

$$\prod_{\ell=1}^{x} \; D_{\frac{2}{x}}(\cdot) - \lambda_\ell \;\; x = S \tag{4.21}$$

The solution to Eq. (4.21) involves homogeneous and particular parts, i.e.,

$$x = x_h + x_p \tag{4.22}$$

For the homogeneous case x_h must satisfy

$$\prod_{\ell=1}^{x} \; D_{\frac{2}{x}}(\cdot) - \lambda_\ell \;\; x_h = 0 \tag{4.23}$$

The solution to Eq. (4.23) can be obtained via successive substitutions. In particular, if we let

$$\begin{aligned} & D_{\frac{2}{x}}(\cdot) - \lambda_1 \;\; X_1 = 0 \\ & D_{\frac{2}{x}}(\cdot) - \lambda_2 \;\; X_2 = X_1 \\ & \quad \vdots \\ & D_{\frac{2}{x}}(\cdot) - \lambda_\ell \;\; X_\ell = X_{\ell-1} \\ & \quad \vdots \\ & D_{\frac{2}{x}}(\cdot) - \lambda_{\;_x} \;\; X_{\;_x} = X_{\;_x - 1} \end{aligned} \tag{4.24}$$

then $X_1, X_2, \ldots, X_{\nu_x}$ form the basis set for x_h. They can be established by successively applying the Laplace transform to Eq. (4.24). Given that they are nonhomogeneous, the solution would involve a series of convolution integrals yielding the expression

$$x_h = \sum_{k=1}^{\nu_x} C_k \prod_{\ell=k}^{\nu_x} \left(F\left(\lambda_\ell, \frac{\nu_x}{2}, t \right) * \right) \tag{4.25}$$

where C_k are constants, and

$$\prod_{\ell=k}^{\nu_x} \left(F\left(\lambda_\ell, \frac{\nu_x}{2}, t \right) * \right) = F(\lambda_\ell, \ldots) * F(\lambda_{\ell+1}, \ldots) * \cdots * F(\lambda_{\nu_x}, \ldots) \tag{4.26}$$

such, that "$*$" defines the convolution integral, namely

$$F\left(\lambda_\ell, \frac{\nu_x}{2}, t \right) * F\left(\lambda_{\ell+1}, \frac{\nu_x}{2}, t \right) = \int_0^t F\left(\lambda_\ell, \frac{\nu_x}{2}, t - \tau \right) F\left(\lambda_{\ell+1}, \frac{\nu_x}{2}, \tau \right) \mathrm{d}\tau \tag{4.27}$$

Given that the multiple convolutions appearing in (4.26) are somewhat awkward, an alternative form can be established due to functional format of $F\left(\lambda, \frac{\nu_x}{2}, t \right)$. In particular if $\lambda_\ell;\ \ell \in [1, \nu_x]$ are distinct, then the following identity can be used to substitute for operations such as (4.27), i.e.,

$$F\left(\lambda_\ell, \frac{\nu_x}{2}, t \right) * F\left(\lambda_{\ell+1}, \frac{\nu_x}{2}, t \right) = \frac{1}{\lambda_\ell - \lambda_{\ell+1}} \left(F\left(\lambda_\ell, \frac{\nu_x}{2}, t \right) - F\left(\lambda_{\ell+1}, \frac{\nu_x}{2}, t \right) \right) \tag{4.28}$$

Based on the foregoing, the particular solution can be written in terms of a convolution with $S(t)$. Such an operation can be formally expressed as

$$x_p(t) = S(t) * \prod_{\ell=1}^{\nu_x} \left(F\left(\lambda_\ell, \frac{\nu_x}{2}, t \right) * \right) \tag{4.29}$$

Again simplification can be achieved through the use of (4.28).

Recalling the work of Bagley and Calico (1991), it is instructive to seek a relationship with their stated solution involving Mittag-Leffler functions, Bagley and Calico (1991), Enelund and Josefson (1996), i.e.,

$$F_M(\lambda, q, t) = \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{\Gamma(1 + n/q)} \tag{4.30}$$

After a degree of reshuffling it follows that

$$F(\lambda, q, t) = (\lambda)^{q-1} F_M(\lambda, q, t) + (\lambda t)^{-1} \sum_{\ell=1}^{q-1} \frac{(\lambda t^{1/q})^\ell}{\Gamma(\ell/q)} \tag{4.31}$$

Asymptotically, i.e., for $t \rightarrow \textit{large}$, (4.31) reduces to the simpler form

$$F(\lambda, q, t) \sim (\lambda)^{q-1} F_M(\lambda, q, t) \tag{4.32}$$

In this context, the Bagley and Calico (1991) solution is a large time adaptation of the decompositional form.

5 Decompositional vector form

Because of the diophantine form of its powers, (4.9) can be converted to a more convenient form which will be called a decompositional vector form. After setting it up, we will seek its relationship with composed form, i.e., (4.9). To introduce the vector form, (4.16) is recursively transformed, that is

$$D_{\frac{2\nu_x}{\nu_x}}(X) = D_{\frac{2\nu_x - 1}{\nu_x}}\left(D_{\frac{1}{\nu_x}}(X) \right) = D_{\frac{2\nu_x - 1}{\nu_x}}(Y_1)$$

$$D_{\frac{2\nu_x - 1}{\nu_x}}(Y_1) = D_{\frac{2\nu_x - 2}{\nu_x}}\left(D_{\frac{1}{\nu_x}}(Y_1) \right) = D_{\frac{2\nu_x - 2}{\nu_x}}(Y_2)$$

$$\vdots$$

$$D_{\frac{2\nu_x - \ell}{\nu_x}}(Y_\ell) = D_{\frac{2\nu_x - \ell - 1}{\nu_x}}\left(D_{\frac{1}{\nu_x}}(Y_\ell) \right) = D_{\frac{2\nu_x - \ell - 1}{\nu_x}}(Y_{\ell+1}) \tag{5.1}$$

$$\vdots$$

$$D_{\frac{2\nu_x - 2\nu_x + 2}{\nu_x}}(Y_{2\nu_x - 2}) = D_{\frac{1}{\nu_x}}\left(D_{\frac{1}{\nu_x}}(Y_{2\nu_x - 2}) \right) = D_{\frac{1}{\nu_x}}(Y_{2\nu_x - 1})$$

Letting

$$X = Y_0, \tag{5.2}$$

use of (5.1) reduces (4.16) to the following vector form

$$D_{\frac{1}{\nu_x}}(\underline{Y}) = [\Theta]\underline{Y} + \underline{R} \tag{5.3}$$

where

$$\underline{Y}^T = (Y_0, Y_1, \ldots, Y_\ell, \ldots, Y_{2\nu_x - 1}) \tag{5.4}$$

$$\underline{R}^T = (0, 0, \ldots, 0, \ldots, S/M) \tag{5.5}$$

and the matrix coefficient $[\Theta]$ is given by the expression

$$[\Theta] = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & & 0 & & 0 & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & 0 & 0 & & 1 & & 0 & 0 \\ \vdots & & & & \vdots & & & \vdots \\ 0 & 0 & 0 & & 0 & & 0 & 1 \\ -\frac{K}{M} & -\frac{\mu_1}{M} & -\frac{\mu_2}{M} & \ldots & -\frac{\mu_\ell}{M} & \ldots & -\frac{\mu_{2\nu_x - 2}}{M} & -\frac{\mu_{2\nu_x - 1}}{M} \end{bmatrix} \tag{5.6}$$

A single element version of this equation was first generated by Bagley and Celico (1991).

Noting the form of (5.3–6), when ν_x is set to 1, the traditional state vector form of the dynamics equation can be extracted. Given that (4.16) is in matrix form, then each of the entries of (5.3–6) are either themselves subvectors or matrices. As can be seen, $2\nu_x$ defines the size of the fractional space. After more complex nomenclatural manipulations, a similar vector form can be derived for the full *MKV* version given by (4.9).

The solution of Eq. (5.3) can be established via the use of spatial transforms. Specifically, employing the eigenvalues of $[\Theta]$, (5.3) can be reduced to a Jordan canonical format. Given distinct eigenvalues it follows that

$$[\Theta] \equiv [T][\lambda][T]^{-1} \tag{5.7}$$

where

$$[\lambda] = \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_\ell & \\ & & & \ddots \end{bmatrix} \tag{5.8}$$

and $[T]$ is the basis formed by the eigenvectors associated with λ_i. Employing $[T]$, and letting

$$\underline{Y} = [T]\underline{Z} \ , \tag{5.9}$$

(5.3) can be reduced to the form

$$D_{\frac{1}{x}}(Z_i) = \lambda_i Z_i + ([T]\underline{R}) \ . \tag{5.10}$$

From the earlier sections, the solution to (5.10) is given by

$$Z_i = C_i F(\lambda_i, \quad _x, t) + F(\lambda_i, \quad _x, t) * ([T]\underline{R}) \tag{5.11}$$

The complete solution then follows from (5.9).

Equation (5.3) we recall is a decomposed version of the fractional continuum formulation. From an asymptotic point of view it is equivalent to the continuum version. Had integral operators been involved, as can be seen from the discussion of the decomposition property, equivalency would have held for all time. Note, a similar equivalency would hold true for homogeneous initial conditions. In such a situation (5.11) would reduce to

$$Z_i = F(\lambda_i, \quad _x, t) * ([T]\underline{R}) \tag{5.12}$$

From a numerical point of view (5.3) possess many advantages. For instance, the Grunwald formalism employed by Padovan (1987) could be used to directly integrate the vector form. For large time asymptotic situations, the Mittag-Leffler function, Enelund and Josefson (1996), could be employed in the convolution integral given in (5.12). Alternatively, it could be used to establish a workable algorithm to obtain a solution to (5.3), at least for large time problems.

6
Summary

This paper has introduced a generalized version of diophantine approximated fractional formulations of system damping. The various important properties of such a formulation have been developed. These include 1) the solution form, 2) a decomposed formulation, 3) a vector version of the decomposed formulation, 4) associated solutions, and 5) asymptotic properties and equivalencies. In its vector form the fractional representation appears to represent various algorithm advantages. In Part II of this series fitting schemes will be developed to handle such issues as frequency dependent damping, among several.

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