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Daniel J. Simon
Cleveland State University, d.j.simon@csuohio.edu

Hossny El-Sherief
TRW System Integration Group

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HYBRID KALMAN/MINIMAX FILTERING IN PHASE-LOCKED LOOPS

D. Simon* and H. El-Sherief**

**TRW Test Laboratory, 4051 N. Higley Road, Mesa, AZ 85215, USA (d.simon@ieee.org)*
***TRW Systems Integration Group, 1111 E. Mill Street, San Bernardino, CA 92408. USA*

Abstract. A method of combining Kalman filtering and minimax filtering is proposed and demonstrated in an application to phase-locked loop design. Kalman filtering suffers from a lack of robustness to departures from the assumed noise statistics. Minimax filtering, however, has the drawback of ignoring the engineer's (admittedly incomplete) knowledge of the noise statistics. It is shown in this paper that hybrid Kalman/minimax filtering can provide the "best of both worlds". Phase-locked loop filter design is used in this paper to demonstrate an application of hybrid estimation.

Key Words. H_∞ control; Kalman filters; Global positioning systems; Missiles; Aerospace trajectories

1. INTRODUCTION

H_2 filtering, also known as Kalman filtering, is a well-established technology which dates back to the 1960s and has its roots in the late 1700s (Sorenson, 1985). H_2 filtering is an estimation method which minimizes the variance of the estimation error, and assumes that the noisy inputs have known statistical properties.

Unfortunately, the assumption that the statistical properties of the noise are known limits the applicability of Kalman filters. This limitation has given rise to a recent interest in minimax estimation, also known as H_∞ filtering. The optimality measure which is used in H_∞ filtering is the magnitude of the maximum singular value of the transfer function from the noise to the estimation error. No knowledge of the noise statistics is assumed. H_∞ filtering appears to have first been introduced in 1987 (Grimble, 1987), with roots dating back to 1981 (Zames, 1981).

If the H_2 approach to filtering assumes too much, the H_∞ approach assumes too little. Generally, an engineer has less knowledge about the noise than an H_2 filter requires, but more knowledge than an H_∞ filter can use. This motivates an interest in designing an estimation filter which uses the best characteristics from each type of filter. This type of cross between H_2 and H_∞ filtering can be called a *hybrid* filter.

The motivation in this paper for using a hybrid filter is digital phase-locked loop (PLL) design. PLLs are used to track the phase and frequency of the carrier component of a sinusoidal signal (Lindsey, 1986; Simon, 1994). The development of digital PLLs began in the late 1960s, and reached a reasonable state of maturity by the early 1980s. Many different approaches have been taken in the past to PLL filter design. Perhaps the most successful approach for highly dynamic trajectories, based on comparisons in (Vilnrotter, 1988; p. 55), is the use of the Kalman filter.

PLLs are of particular interest to the Global Positioning System (GPS) community. GPS is a satellite-based navigation system developed and maintained by the United States Department of Defense. In its final configuration it will consist of 24 satellites in semi-geosynchronous orbit, which will provide position and velocity information to any user with a GPS receiver (Janiczek, 1980). The user position is obtained by tracking a known binary pseudo-random (PR) code transmitted by the GPS satellites, and the user velocity is obtained by tracking the sinusoidal carrier which modulates the PR code. Because of its continuous global coverage and the passive nature of the receiver, GPS is being used in a wide range of aerospace applications (Dougherty, 1993). It is clearly desirable to provide robust algorithms for the GPS receiver's PLL, or the user's velocity information may be lost.

This paper is organized as follows. Section 2 provides a brief overview of H_2, H_∞, and hybrid

H_2/H_∞ filtering. Section 3 discusses the details of the application of hybrid filtering to PLL design, and Section 4 provides simulation results. Section 5 provides concluding remarks.

2. OPTIMAL FILTERING FUNDAMENTALS

This section reviews some of the fundamental theory of optimal filtering. First the problem is defined, and then H_2 filtering and H_∞ filtering are discussed. Finally, a method of combining these two approaches is proposed.

2.1. *Problem Description*

Consider a linear, discrete, time-invariant system given by

$$\begin{aligned} x_{k+1} &= \phi x_k + v_k \\ y_k &= H x_k + n_k. \end{aligned} \tag{1}$$

$x_k \in R^n$ is the state vector, $y_k \in R^m$ is the measurement, and v_k and n_k are noise processes. An augmented noise vector is defined as

$$\omega_k = \begin{pmatrix} v_k \\ n_k \end{pmatrix}. \tag{2}$$

It is desired to find an estimate $\hat{x}_k$ for x_k based on measurements y_i, $i \leq k$. This is known as the *a posteriori* filtering problem. The estimator structure is assumed to be

$$\hat{x}_{k+1} = \phi \hat{x}_k + K_k(y_{k+1} - H\phi\hat{x}_k) \tag{3}$$

where K_k is a gain to be determined. Define the estimation error as

$$e_k = x_k - \hat{x}_k. \tag{4}$$

The transfer function matrix from the noise ω_k to the estimation error e_k is denoted $G_{e\omega}$. If $K_k = K$ is a constant, this transfer function is given by

$$\begin{aligned} G_{e\omega}(z) = {} & [zI - (I - KH)\phi]^{-1} \times \\ & \{(I - KH)[I_n \quad 0_{nm}] - K[0_{mn} \quad I_m]z\} \end{aligned} \tag{5}$$

where I_n is the $n \times n$ identity matrix and 0_{pq} is the $p \times q$ zero matrix.

2.2. *H_2 Filtering*

In H_2 filtering, also known as Kalman filtering, it is assumed that the noise processes v_k and n_k are zero-mean. The gain K_k is computed according to the formulas

$$\begin{aligned} K_k &= P_k(-)H^T(HP_k(-)H^T + R_k)^{-1} \\ P_k(+) &= (I - K_k H)P_k(-) \\ P_{k+1}(-) &= \lambda\phi P_k(+)\phi^T + Q_k \end{aligned} \tag{6}$$

where $R_k = E(n_k n_k^T)$, $Q_k = E(v_k v_k^T)$, and λ is a forgetting factor. If $\lambda = 1$, then the Kalman filter is the affine filter which minimizes $E(e_k S_k e_k^T)$ for any positive semidefinite weight matrix sequence $\{S_k\}$. This is commonly expressed by stating that the Kalman filter is the linear minimum variance estimator (Anderson, 1979; chap. 5). If w_k is white and wide-sense stationary with a power spectral density of $S_w(\omega) = S$, then the Kalman filter is also the affine filter which minimizes the S-weighted 2-norm of $G_{e\omega}$ (Khargonekar, 1992),(Kwakernaak, 1972; sec. 6.5),(Papaloulis, 1984; sec. 10.4). The S-weighted 2-norm of a discrete transfer function matrix is given by

$$\begin{aligned} \|G\|_S^2 &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \mathrm{tr}[G(e^{j\omega})SG^*(e^{j\omega})]\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \sum_{i=1}^{n} \lambda_i[G(e^{j\omega})SG^*(e^{j\omega})]\,d\omega \end{aligned} \tag{7}$$

where *tr* indicates *trace*, and G^* is the Hermitian transpose of G. If the system (1) is completely observable and completely controllable, the gain K_k will reach a unique steady state, denoted by $K^{(2)}$:

$$\lim_{k\to\infty} K_k = K^{(2)}. \tag{8}$$

In order to save computational expense, the steady-state gain (which can be computed off-line) is often used in real-time systems. The resulting filter is identical to the Wiener filter (Gelb, 1984; sec. 4.7). If $\lambda > 1$, then greater emphasis is given to more recent data (Anderson, 1979; sec. 6.2). This results in greater stability and improved performance in many practical cases.

2.3. *H_∞ Filtering*

The fact that the Kalman filter is the linear minimum variance estimator is a powerful and attractive result, but several facts may indicate against the use of a Kalman filter (Shaked, 1992):

1. The Kalman filter minimizes $E(e_k e_k^T)$, while the user may be more interested in minimizing the worst-case error.
2. The Kalman filter assumes that $E(n_k n_k^T)$ and $E(v_k v_k^T)$ are known.
3. The Kalman filter assumes that $E(n_k)$ and $E(v_k)$ are known.

These considerations have led to the statement of the H_∞ filtering problem. Several H_∞ problem formulations have been presented in the literature (Shaked, 1992). The problem which is considered in this paper can be posed as follows (Yaesh, 1991). Given the system in (1) and the estimator structure in (3), find a gain K_k such that $\|G_{e\omega}\|_\infty < \gamma$, where (as before) $G_{e\omega}$ is the transfer function matrix from the noise ω_k to the estimation error e_k, and $\|G\|_\infty$ is the magnitude of the largest singular value of G (over all frequencies).

$$\|G\|_\infty = \sup_{\omega\in[-\pi,\pi]} \lambda_{max}[G(e^{j\omega})G^*(e^{j\omega})]. \quad (9)$$

It can be shown that if this problem has a solution for a given γ, then it can be solved by a constant gain, denoted by $K^{(\infty)}$ (Zhou, 1988). Amazingly enough, as $\gamma \to \infty$, the solution of the H_∞ problem is identical to the steady-state Kalman filter when $R_k = Q_k = I$. The H_∞ filtering solution for a specified γ is given by

$$\begin{aligned} K^{(\infty)} &= (I + P/\gamma^2)^{-1}PH^T \\ P^{-1} &= M^{-1} - I/\gamma^2 + H^TH \\ M &= \phi P \phi^T + I. \end{aligned} \quad (10)$$

One method to solved these equations is given in (Yaesh, 1991):

1. Form the matrix

$$Z = \begin{bmatrix} \phi^{-T} & \phi^{-T}(H^TH - I/\gamma^2) \\ \phi^{-T} & \phi + \phi^{-T}(H^TH - I/\gamma^2) \end{bmatrix} \in R^{2n\times 2n}. \quad (11)$$

2. Find the eigenvectors of Z. Denote those eigenvectors corresponding to eigenvalues outside the unit circle as χ_i $(i = 1, \ldots, n)$.
3. Form the matrix

$$\begin{pmatrix} \chi_1 & \cdots & \chi_n \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \quad (12)$$

   where $X_1, X_2 \in R^{n\times n}$.
4. Compute $M = X_2X_1^{-1}$.

Alternatively, (10) can be solved iteratively. Note from the problem statement that as γ gets larger, the problem is "easier" to solve. If γ is too small, the problem will not have a solution, and X_1 will be singular.

2.4. *Hybrid H_2/H_∞ Filtering*

H_2 and H_∞ filters both have pros and cons. The H_2 filter assumes that the noise statistics are known. The H_∞ filter does not make this assumption, but further assumes that absolutely nothing is known about the noise characteristics.

Suppose that although the noise statistics are not perfectly known, the user does have a rough idea of these statistics. Also suppose that a user desires to minimize some combination of the H_2 and H_∞ objective functions. What could be done? Perhaps a hybrid H_2/H_∞ filter could be used.

Several approaches to hybrid filtering have been proposed in the literature. For instance, (Khargonekar, 1992) considers a system driven by two independent noise processes – one with known statistics and one with unknown statistics. Then a filter gain is found which simultaneously satisfies two objectives: (1) bounding the 2-norm of G_1, where G_1 is the transfer function from the noise with known statistics to the estimation error, and (2) bounding the ∞-norm of G_2, where G_2 is the transfer function from the noise with unknown statistics to the estimation error. Since this approach does not fit the problem considered in the following section, a more heuristic, hybrid filtering approach is proposed. Simply use a weighted combination of the steady-state H_2 and H_∞ gains in the estimator. That is,

$$K = d\,K^{(2)} + (1-d)K^{(\infty)} \quad (13)$$

where $[0, 1] \ni d =$ the relative weight given to H_2 performance. $K^{(2)}$ is given by the steady-state solution of (6) and $K^{(\infty)}$ is the solution of (10). The key design parameter in the hybrid filter is the weight d. This weight must be chosen so as to ensure stability. A convex combination of two stable estimators is not necessarily stable, as is shown in Section 4.1. So the first criterion for the choice of d is stability. The second criterion is the relative weight given by the user to H_2 performance versus H_∞ performance. This relative weight can be determined on the basis of the engineer's confidence in the *a priori* statistics.

3. APPLICATION TO PHASE-LOCKED LOOP DESIGN

Consider the problem of tracking a sinusoidal signal with an unknown, time-varying phase $\theta(t)$:

$$s(t) = A\cos\theta(t). \quad (14)$$

This signal is corrupted by noise. The device used to track such a signal is called a phase-locked loop. PLLs are of particular interest in Global Positioning System receivers. A GPS satellite transmits a sinusoidal signal modulated by a known pseudorandom binary code. After the PR code is removed from the signal, the receiver has access to the sinusoid. Since the sinusoid is transmitted at a known frequency, the frequency which the receiver tracks can be used to compute the doppler between the user and the satellite. The satellite orbit is known fairly accurately, so the doppler frequency can be used to obtain the user's velocity. A GPS receiver can therefore be used as a nav-

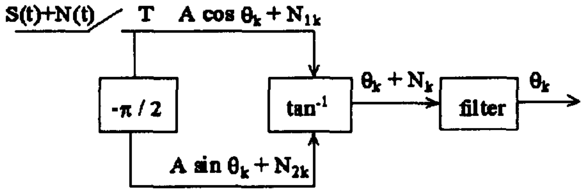

Fig. 1. Phase-locked loop architecture.

igational instrument in place of more expensive and complex inertial instruments. The receiver architecture considered in this paper is shown in Fig. 1.

Note from Fig. 1 that the output of the arctan phase discriminator is modulo 2π. That is, the phase discriminator does not know the difference between θ radians and $\theta+2\pi$ radians. If the phase estimation error suddenly goes from zero to some multiple of 2π, it is said that a *cycle slip* has occurred. So it is more important in a PLL to prevent cycle slips than it is to maintain a small phase error. If the PLL maintains lock on the phase, the PLL contribution to a GPS receiver's velocity error is small compared to other sources of velocity error (Simon, 1993a; Simon, 1993b). For instance, the velocity error due to a typical 4° RMS PLL tracking error may be of the order of 0.01 feet/second, but the velocity error due to all other sources may be of the order of 0.10 feet/second. If a cycle slip occurs, then the velocity error due to the PLL tracking error momentarily jumps to 0.90 feet/second. So undetected cycle slips can be catastrophic. In some cases, the noise is so high or the phase dynamics are so severe that the estimation error begins growing without bound. In this case it is said that *loss of lock* has occurred, and the user loses all velocity information from the GPS receiver. Therefore, for a GPS receiver, it is primarily loss of lock and secondarily cycle slips which are of greatest concern (rather than phase error).

Optimal filtering can be used in PLL design by a method similar to that used in (Vilnrotter, 1988). Create a state vector from successive derivatives of the incoming phase.

$$x_k = \begin{pmatrix} \theta_k & \omega_k & \alpha_k & \beta_k \end{pmatrix}^T \tag{15}$$

where

$$\begin{aligned} \theta_k &= \theta(t_k) \\ \omega_k &= \theta'(t_k) \\ \alpha_k &= \theta''(t_k) \\ \beta_k &= \theta'''(t_k). \end{aligned} \tag{16}$$

This leads to the state transition equations (Vilnrotter, 1988)

$$\beta_{k+1} = \beta_k + \int_{kT}^{(k+1)T} \theta^{(4)}(t)\,dt \tag{17}$$

$$\alpha_{k+1} \approx \alpha_k + T\beta_k + \int_{kT}^{(k+1)T} t\theta^{(4)}(t)\,dt \tag{18}$$

$$\omega_{k+1} \approx \omega_k + T\alpha_k + T^2\beta_k/2 + \int_{kT}^{(k+1)T} \frac{t^2}{2}\theta^{(4)}(t)\,dt \tag{19}$$

$$\theta_{k+1} \approx \theta_k + T\omega_k + T^2\alpha_k/2 + T^3\beta_k/6 + \int_{kT}^{(k+1)T} \frac{t^3}{6}\theta^{(4)}(t)\,dt \tag{20}$$

where the approximations are valid for a small sample period T. This gives rise to the system description

$$\begin{aligned} x_{k+1} &= \phi x_k + v_k \\ y_k &= Hx_k + n_k \end{aligned} \tag{21}$$

where the system matrices are given by

$$\phi = \begin{pmatrix} 1 & T & T^2/2 & T^3/6 \\ 0 & 1 & T & T^2/2 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{22}$$

$$H = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}. \tag{23}$$

As a further approximation, the fourth derivative of the phase is modelled as a white noise process with variance N.

$$E[\theta^{(4)}(t)\theta^{(4)}(\tau)] = N\delta(t-\tau). \tag{24}$$

Similarly, the continuous-time phase measurement noise is modelled as a white noise process with variance N_0, so the variance of the sampled version is N_0/T.

$$\begin{aligned} E[n(t)n(\tau)] &= N_0\delta(t-\tau) \\ E[n(t_k)n(t_j)] &= \frac{N_0}{T}\delta_{k-j}. \end{aligned} \tag{25}$$

These considerations lead to the assumed noise statistics

$$E(v_k v_j^T) = Q\delta_{k-j} = NT \times \begin{pmatrix} T^6/252 & T^5/72 & T^4/30 & T^3/24 \\ T^5/72 & T^4/20 & T^3/8 & T^2/6 \\ T^4/30 & T^3/8 & T^2/3 & T/2 \\ T^3/24 & T^2/6 & T/2 & 1 \end{pmatrix} \delta_{k-j} \tag{26}$$

$$E(n_k n_j) = R\delta_{k-j} = (N_0/T)\delta_{k-j}. \tag{27}$$

The filter structure used to obtain a state estimate is given by

$$\hat{x}_{k+1} = \phi\hat{x}_k + K(y_{k+1} - H\phi\hat{x}_k). \tag{28}$$

A constant gain K will be used due to real-time computational constraints. Note that the estimate $\hat{\theta}_k$ (the first component of $\hat{x}_k$) is actually an estimate of the phase modulo 2π. This estimate can be placed in the proper phase cycle by using the frequency estimate $\hat{\omega}_k$. Conceptually, this process can be viewed as follows:

$$i = \text{sign}(\hat{\theta}_{k-1} + T\hat{\omega}_{k-1} - \hat{\theta}_k)$$
$$\text{do while } \mid \hat{\theta}_{k-1} + T\hat{\omega}_{k-1} - \hat{\theta}_k \mid > \pi$$
$$\hat{\theta}_k = \hat{\theta}_k + 2\pi i.$$

If the noise processes n_k and v_k are zero-mean, Q_k and R_k are known, and the user wants to minimize the variance of the phase estimation error, then the steady-state Kalman filter gain can be used. If, on the other hand, the noise processes may or may not be zero-mean, Q_k and R_k are *not* known, and the user wants to minimize the worst-case effect of the noise on the phase estimation error, then the H_∞ filter gain can be used. If the user has some idea of the noise statistics (but does not know them exactly) and wants to minimize some combination of the H_2 and H_∞ objective functions, then the hybrid H_2/H_∞ filter discussed in Section 2.4 can be used.

4. SIMULATION RESULTS

The hybrid H_2/H_∞ filter discussed in this paper was simulated for a GPS receiver used for missile navigation. The simulated missile trajectory originated at Vandenberg Air Force Base in California, and ended in the South Pacific. The behavior of the H_2/H_∞ PLL was investigated by examining its ability to track the phase between the missile and one GPS satellite for the first 60 seconds of boost (i.e., during Stage I burn). The filter rate was fixed at 50 Hz. The satellite-to-missile range, range rate, and range acceleration are depicted in Figs. 2–4. The relationship between the phase θ and the range ρ is given by

$$\rho(t) = \frac{c\theta(t)}{2\pi f} \tag{29}$$

where c is the speed of light and f is the frequency of the sinusoid. This paper concentrates on tracking the GPS L1 carrier at a frequency of 1.575 GHz. It can be assumed without loss of generality (see Fig. 1) that the magnitude of the carrier is unity.

It was found, in agreement with (Vilnrotter, 1988), that $\lambda \approx 1.055$ resulted in the best performance for the Kalman filter. Typical carrier-to-noise ratios (CNRs) for GPS are around 30 to 40 dB-Hz (Hurd, 1987). But if atmospheric con-

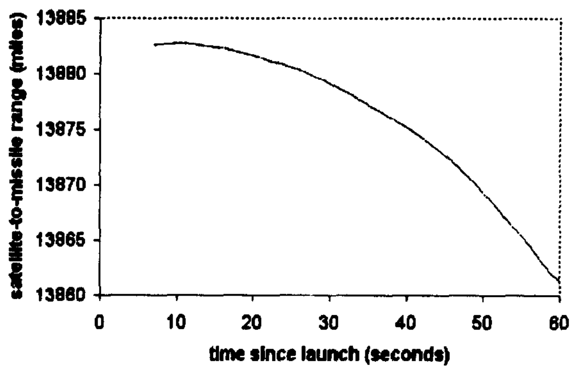

Fig. 2. Satellite-to-missile range.

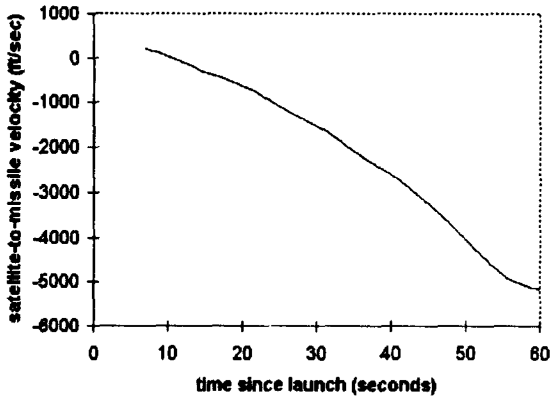

Fig. 3. Satellite-to-missile range rate.

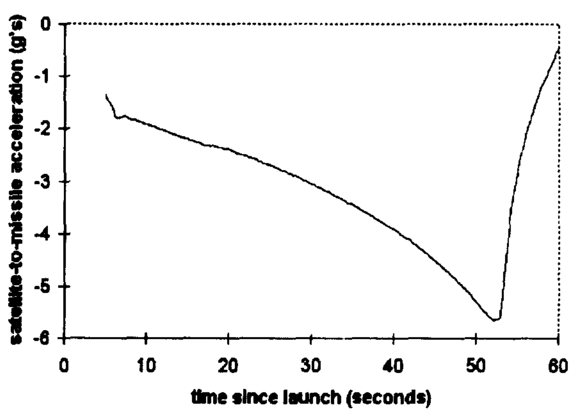

Fig. 4. Satellite-to-missile range acceleration.

ditions are severe or jamming is present, the CNR could drop into the 20's and biases could be introduced into the measurement noise. The CNR is related to the variance of the measurement noise (R) by

$$\mathrm{CNR} = \frac{1}{2TR} \tag{30}$$

where $T = 0.02$ seconds is the filter rate (Jeruchim, 1992; p. 282). The interest in this paper is tracking the GPS L1 carrier in spite of the fact that the CNR is significantly different than expected and the noise is not zero-mean. It is assumed in this section that there is a constant, unknown phase measurement bias of 1 radian. So the *true* measurement equation is

$$y_k = \theta_k + n_k + 1 \tag{31}$$

where n_k is zero-mean noise, but the filters are designed according to the incorrect measurement equation

$$y_k = \theta_k + n_k. \tag{32}$$

The simulated noise n_k was generated with a Laplacian (exponential) density, which has heavier "tails" than a Gaussian density.

4.1. *Assumed CNR = 30 dB-Hz*

The steady-state Kalman filter gain for CNR = 30 dB-Hz and $\lambda = 1.055$ was found to be

$$K^{(2)} = \begin{pmatrix} 0.5799 \\ 11.6510 \\ 132.2306 \\ 728.8681 \end{pmatrix}. \tag{33}$$

The H_∞ estimation problem was found numerically to be solvable for $\gamma > 1$, so $\gamma = 1.01$ was used in (10) to compute the H_∞ gain

$$K^{(\infty)} = \begin{pmatrix} 0.9127 \\ 1.8732 \\ 1.7398 \\ 0.6833 \end{pmatrix}. \tag{34}$$

As discussed in Section 2.4, the user needs to choose a value of the Kalman weight gain d such that the hybrid filter is stable. Figure 5 shows the magnitude of the largest eigenvalue of the hybrid estimator as a function of d. It is seen that the estimator is unstable for $0.01 < d < 0.31$. This shows that d must be chosen greater than 0.31 for satisfactory estimator performance. How much greater? For the pure H_∞ filter, $|\lambda_{max}| = 0.988$. Note from Fig. 5 that as d increases, $|\lambda_{max}|$ also increases at first, then begins decreasing. $|\lambda_{max}|$ drops back down to 0.988 at $d \approx 0.45$. So in this

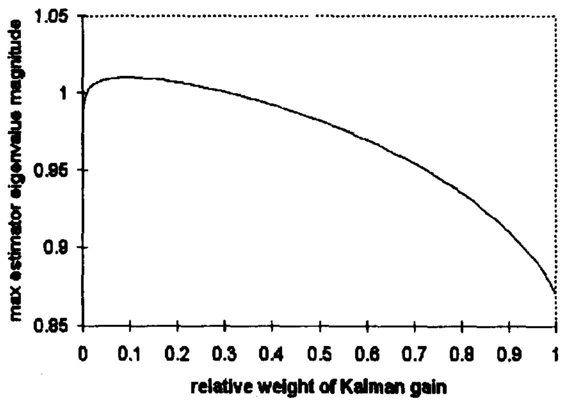

Fig. 5. The stability of the hybrid filter. The Kalman gain was designed for CNR = 30 db-Hz.

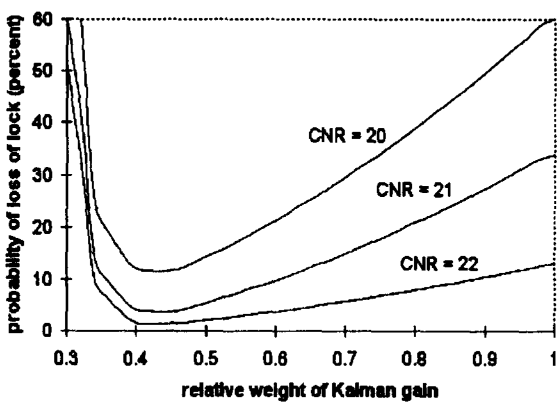

Fig. 6. Probability of loss of lock. The Kalman gain was designed for CNR = 30 db-Hz. Note that the filter is unstable for $d < 0.3$.

application, a rule of thumb for hybrid H_2/H_∞ filter design is to choose $d \geq 0.45$. This ensures that the hybrid filter is at least as stable as the pure H_∞ filter.

Recall that the primary goal is in maintaining lock during the mission. With this in mind, the probability of loss of lock was obtained experimentally for various values of the Kalman gain weight d in (13). Recall further that $d = 0$ corresponds to a pure H_∞ filter, while $d = 1$ corresponds to a pure H_2 filter. Probability of loss of lock was obtained by conducting 100 Monte Carlo samples for each data point. This probability is shown in Fig. 6 as a function of d for three values of CNR. It is seen that the use of hybrid H_2/H_∞ ($d < 1$) filtering results in a noticable improvement in phase lock over pure H_2 or pure H_∞ filtering. Furthermore, the advantage becomes more significant as the CNR decreases. For example, a pure Kalman filter with a CNR of 20 dB-Hz has a 55% chance of losing lock. But a hybrid H_2/H_∞ filter with a weight d around 0.4 or 0.5 only has a 10% chance of losing lock.

When the Kalman filter *does* maintain lock, it performs better than the hybrid filter. This is seen in

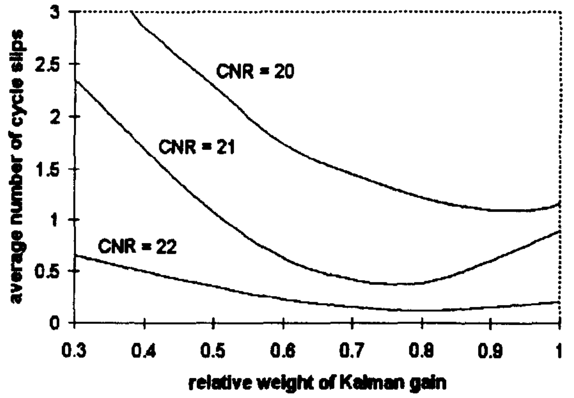

Fig. 7. Average number of cycle slips when the PLL maintains lock. The Kalman gain was designed for CNR = 30 db-Hz. Note that the filter is unstable for $d < 0.3$.

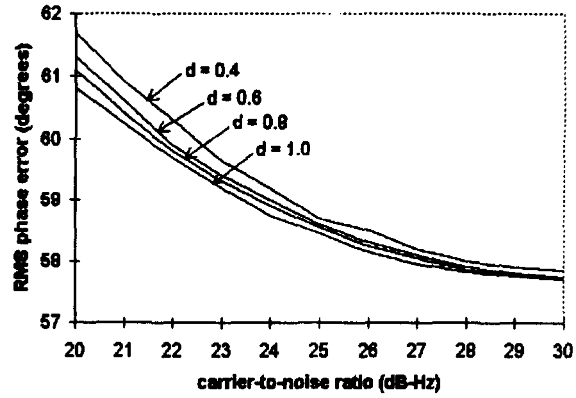

Fig. 9. RMS phase error when the PLL maintains lock. The Kalman gain was designed for CNR = 30 db-Hz. Note that the filter is unstable for $d < 0.3$.

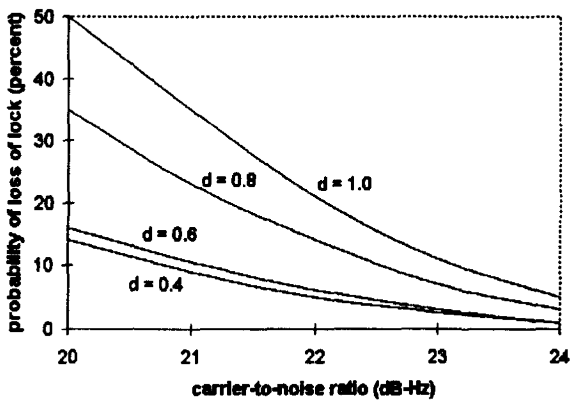

Fig. 8. Probability of loss of lock. The Kalman gain was designed for CNR = 30 db-Hz. Note that the filter is unstable for $d < 0.3$.

Fig. 7, which shows the average number of cycle slips as a function of d for various CNR. The numbers in Fig. 7 are derived only from those Monte Carlo samples which did not lose lock. So if the Kalman filter does not lose lock, the chances are it will have fewer cycle slips than the hybrid filter, but the advantage is not significant. For example, at a CNR of 20 dB-Hz, the Kalman filter (if it maintains lock) slips an average of one cycle, while a hybrid filter with d around 0.4 or 0.5 slips an average of two or three cycles.

Figure 8, which shows the probability of loss of lock as a function of CNR for various values of d, is another way of looking at the data. It is again seen that hybrid filtering is most advantageous when the CNR is low. For a CNR above 24 dB-Hz or so, none of the filters has a problem with loss of lock.

Figure 9 shows the RMS phase error as a function of CNR for various values of d. (Again, the numbers in Fig. 9 are derived only from those Monte Carlo samples which did not lose lock.) Although the Kalman filter has an advantage over the hybrid filter, this advantage is not pronounced. For example, at a CNR of 20 dB-Hz, the Kalman filter has an RMS phase error of 61.0°, while the hybrid ($d = 0.4$) filter has an RMS phase error of 61.8°. At the nominal CNR of 30 dB-Hz, the Kalman filter has even less of an advantage over the hybrid filter – 57.7° versus 57.8°.

4.2. *Assumed CNR = 20 dB-Hz*

The steady-state Kalman filter gain for CNR = 20 dB-Hz and $\lambda = 1.055$ was found to be

$$K^{(2)} = \begin{pmatrix} 0.4926 \\ 7.6403 \\ 65.9854 \\ 274.1215 \end{pmatrix}. \quad (35)$$

As before, an unknown bias of 1 radian corrupts the phase measurement. The PLL performance as a function of Kalman gain weight d and CNR is shown in Figs. 10 – 13. Results are essentially the same as seen in Figs. 6 – 9. Figure 10 shows that the use of a hybrid filter decreases the probability of loss of lock as compared to a pure Kalman filter, with the advantage being more pronounced for low CNRs. Figure 11 shows that when the Kalman filter maintains lock, it will slip fewer cycles on average than the hybrid filter. Figure 12 shows the decrease of the probability of loss of lock with decreasing d and increasing CNR. Finally, Fig. 13 shows that the Kalman filter has a slight advantage over the hybrid filter relative to RMS phase error (again, only when the Kalman filter maintains phase lock).

5. CONCLUSION

A hybrid H_2/H_∞ filtering approach has been proposed and applied to phase-locked loop design. This hybrid approach not only takes advantage of the noise statistics knowledge which is inherent

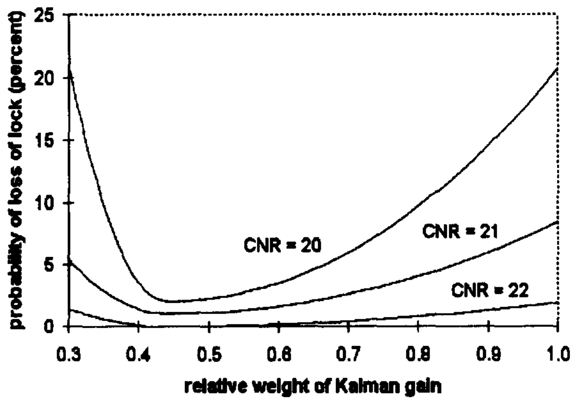

Fig. 10. Probability of loss of lock. The Kalman gain was designed for CNR = 20 db-Hz. Note that the filter is unstable for $d < 0.3$.

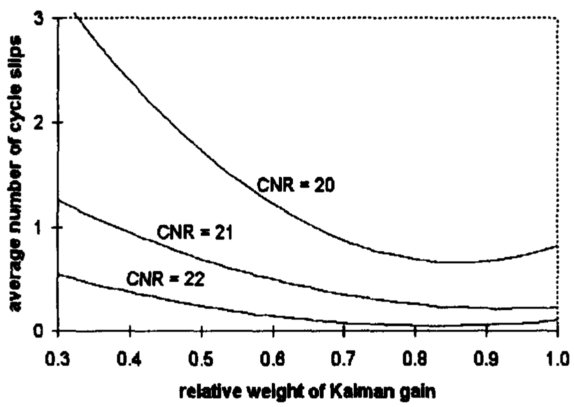

Fig. 11. Average number of cycle slips when the PLL maintains lock. The Kalman gain was designed for CNR = 20 db-Hz. Note that the filter is unstable for $d < 0.3$.

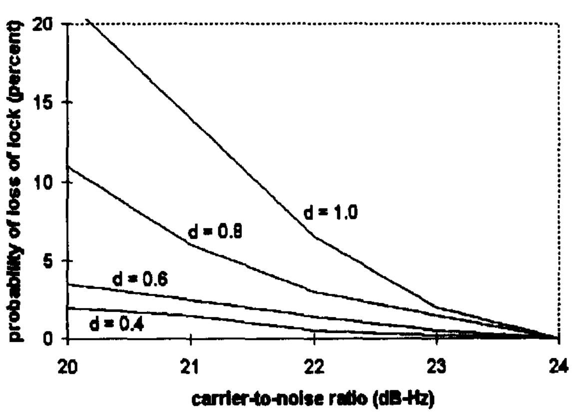

Fig. 12. Probability of loss of lock. The Kalman gain was designed for CNR = 20 db-Hz. Note that the filter is unstable for $d < 0.3$.

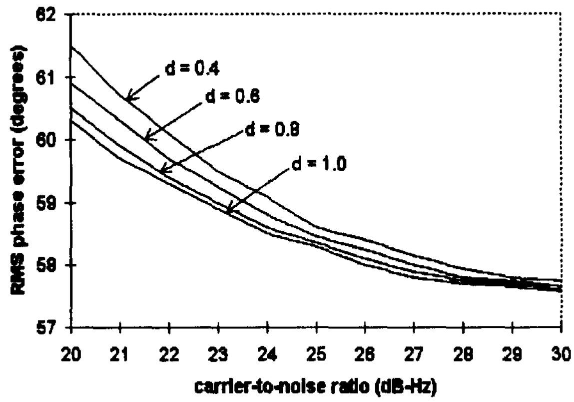

Fig. 13. RMS phase error when the PLL maintains lock. The Kalman gain was designed for CNR = 20 db-Hz. Note that the filter is unstable for $d < 0.3$.

in H_2 filter design, but also takes advantage of the robustness of H_∞ filtering. It is seen from the simulation data that, in general, the hybrid filter provides a large advantage over the pure Kalman filter and the pure H_∞ filter. This advantage is particularly noticable at low CNRs, even when the Kalman filter designer has perfect knowledge of the true CNR. The Kalman filter is not robust to departures from the assumed noise statistics, but the H_∞ filter does not take advantage of the designer's (albeit incomplete) knowledge of the noise properties. Hybrid filtering is an approach which combines the best of both worlds – or at least avoids the worst of both worlds. It is thus recommended that the hybrid H_2/H_∞ filter proposed in this paper be given serious consideration for PLL design in particular, and for state estimation in general.

6. ACKNOWLEDGEMENT

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