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Square-lattice Ising model in a weak uniform magnetic field: Renormalization-group analysis

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For the two-dimensional ferromagnetic Ising critical point, I show that the known values of the critical exponents imply the absence of logarithms of the reduced temperature in the leading contributions to any field derivative of the free energy at zero magnetic field. For the square-lattice Ising antiferromagnet in a weak magnetic field, I compute the critical line $T_c(H)=T_c^0(1-0.038\,023\,259H^2)$ and the leading contribution to the susceptibility $\chi=0.014\,718\,006\,6H^2\ln(1/|t|)$, where t is the reduced temperature.

I. INTRODUCTION

Historically, the square-lattice Ising model has played a central role in our understanding of phase transitions. Spins $s_i=\pm 1$, located at the vertices of the square lattice, interact through the Hamiltonian

$$-\mathcal{H}=\pm\sum_{\langle i,j\rangle} s_i s_j + H\sum_i s_i \ , \tag{1}$$

where the plus sign corresponds to the ferromagnetic problem and the minus sign to the antiferromagnetic problem. The free energy per spin is

$$f=\lim_{N\to\infty}\frac{1}{N}\ln Z \ , \tag{2}$$

where Z is the partition function

$$Z=\sum_{\text{conf.}}\exp\left[-\frac{\mathcal{H}}{kT}\right] .$$

In zero field, $H=0$, the free energy,[1] the spontaneous magnetization,[2] the susceptibility (ferromagnetic problem),[3] and various correlation functions[4] are known exactly.

I report here two results concerning the weak field critical behavior, results obtained by using the renormalization-group formalism. For the ferromagnetic problem, I prove a conjecture by Aharony and Fisher[5] that no logarithms of the reduced temperature appear in the leading contributions to any field derivative of the free energy at $H=0$. For the antiferromagnetic problem, I compute the critical curve and the amplitude of the leading contributions to the susceptibility and other thermodynamic quantities in a weak field $H\neq 0$. Essential to these calculations is the high precision (uncertainty $=10^{-10}$) computation of the amplitude of the zero-field antiferromagnetic susceptibility by Kong, Au-Yang, and Perk.[6]

II. FERROMAGNET IN A WEAK MAGNETIC FIELD

Recently Aharony and Fisher[5] stated the "minimal logarithms" conjecture concerning all field derivatives of the free energy f of the planar ferromagnetic Ising model close to the critical point. According to this conjecture, there are no logarithmic terms $\ln|t|$ multiplying the leading singular contribution to any field derivative of f, e.g., for the zero-field susceptibility $\chi\sim|t|^{-\gamma}$, rather than $\chi\sim|t|^{-\gamma}\ln|t|$, where t is the reduced temperature. By using the renormalization-group formalism, I show this conjecture follows from the critical-exponent values $\alpha=0$ and $\Delta=\frac{15}{8}$.

Logarithmic modifications of the power-law behavior are known[7,8] to occur when the critical exponents satisfy certain relationships for which the Taylor expansion of the regular part of the free energy breaks down. I consider the general case of a critical point with two scaling fields: a temperaturelike field τ and a magneticlike field h. Within a typical position-space renormalization-group scheme the following recursion equations hold close to the critical point:

$$\begin{aligned}\tau'&=b^{y_1}\tau \ ,\\ h'&=b^{y_2}h \ ,\\ f(\tau,h)&=g(\tau,h)+b^{-d}f(\tau',h') \ ,\end{aligned} \tag{3}$$

where f is the free energy per degree of freedom, b is the linear rescaling factor, d is the spatial dimension, and g is an analytic function of τ and h. The free energy f is the sum of a regular part f_r and a singular part f_s,

$$f_r=\sum_{m,n=0}^{\infty} f_{mn}\tau^m h^n \ , \tag{4}$$

and

$$f_s=|\tau|^{2-\alpha}A_{\pm}(h/|\tau|^{\Delta}) \ , \tag{5}$$

where $2-\alpha=d/y_1$ and $\Delta=y_2/y_1$.

The regular part f_r is a particular solution of Eq. (3) and the coefficients f_{mn} can be obtained from Eqs. (3), (4), and $g=\sum_{m,n=0}^{\infty} g_{mn}\tau^m h^n$ as

$$f_{mn}=g_{mn}/(1-b^{-d+my_1+ny_2}) \ . \tag{6}$$

If there exists a pair of integers m_0, n_0 satisfying

$$m_0 y_1+n_0 y_2=d \ ,$$

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or equivalently , (7)

$$m_0+n_0\Delta=2-\alpha \ ,$$

then the coefficient $f_{m_0n_0}$ is infinite, and Eqs. (4) and (5) are not valid. To discuss this special case we vary the exponents α and Δ by continuously changing an appropriate parameter such as the spatial dimension. In the limit $2-\alpha-m_0-n_0\Delta\rightarrow 0$, the singular contribution f_s is the sum of the usual $|\tau|^{2-\alpha}A_{\pm}(h/|\tau|^{\Delta})$ and $f_{m_0n_0}\tau^{m_0}h^{n_0}$ (formally part of the regular free energy)

$$f_s=\lim_{2-\alpha-m_0-n_0\Delta\rightarrow 0}[f_{m_0n_0}\tau^{m_0}h^{n_0}+|\tau|^{2-\alpha}A_{\pm}(h/|\tau|^{\Delta})] \ , \tag{8}$$

and the regular part is

$$f_r=\sum_{\substack{m,n=0\\ m\neq m_0\\ n\neq n_0}}^{\infty} f_{mn}\tau^m h^n \ . \tag{9}$$

It follows from Eq. (6) that for small $2-\alpha-m_0-n_0\Delta$, $f_{m_0n_0}$ behaves as

$$f_{m_0n_0}=\frac{a}{2-\alpha-m_0-n_0\Delta}+c+0(2-\alpha-m_0-n_0\Delta) \ . \tag{10}$$

This divergence has to be canceled by a similar term which is part of $A_{\pm}|\tau|^{2-\alpha}$ in Eq. (8). We then write the right-hand side of Eq. (8) as

$$f_{m_0n_0}\tau^{m_0}h^{n_0}+A_{\pm}|\tau|^{2-\alpha}=[f_{m_0n_0}+A_{n_0}^{(\pm)}(\text{sgn}\tau)^{m_0}|\tau|^{2-\alpha-m_0-n_0\Delta}]\tau^{m_0}h^{n_0}+[A_{\pm}-A_{n_0}^{(\pm)}(h/|\tau|^{\Delta})^{n_0}]|\tau|^{2-\alpha} \tag{11}$$

where $A_{n_0}^{(\pm)}$ is the coefficient of x^{n_0} in the power expansion of $A_{\pm}(x)$ for small x, i.e., $|h| \ll |\tau|^{\Delta}$. By using

$$|\tau|^{2-\alpha-m_0-n_0\Delta}=1+(2-\alpha-m_0-n_0\Delta)\ln|\tau|+0((2-\alpha-m_0-n_0\Delta)^2) \ , \tag{12}$$

we find

$$f_{m_0n_0}+A_{n_0}^{(\pm)}(\text{sgn}\tau)^{m_0}|\tau|^{2-\alpha-m_0-n_0\Delta}=[f_{m_0n_0}+A_{n_0}^{(\pm)}(\text{sgn}\tau)^{m_0}]+A_{n_0}^{(\pm)}(\text{sgn}\tau)^{m_0}(2-\alpha-m_0-n_0\Delta)\ln|\tau|+O(2-\alpha-m_0-n_0\Delta) \ . \tag{13}$$

In view of Eq. (10), to keep $f_{m_0n_0}+A_{n_0}^{(\pm)}(\text{sgn}\tau)^{m_0}$ finite when $2-\alpha-m_0-n_0\Delta\rightarrow 0$, the coefficient $A_{n_0}^{(\pm)}$ behaves as

$$A_{n_0}^{(\pm)}(\text{sgn}\tau)^{m_0}=-\frac{a}{2-\alpha-m_0-n_0\Delta}-c+c_{\pm}(\text{sgn}\tau)^{m_0}+O(2-\alpha-m_0-n_0\Delta) \ . \tag{14}$$

Then the coefficient of $\ln|\tau|$ in Eq. (13) behaves as

$$A_{n_0}^{(\pm)}(\text{sgn}\tau)^{m_0}(2-\alpha-m_0-n_0\Delta)=-a+O(2-\alpha-m_0-n_0\Delta) \ . \tag{15}$$

By substituting the results of Eqs. (10), (14), and (15) into Eq. (13), then this result into Eq. (11), and then finally taking the limit $2-\alpha-m_0-n_0\Delta\rightarrow 0$ [see Eq. (8)], we find the singular part of the the free energy

$$f_s=-a\tau^{m_0}h^{n_0}\ln|\tau|+|\tau|^{m_0+n_0\Delta}\hat{A}_{\pm}(h/|\tau|^{\Delta}) \ , \tag{16}$$

where $\hat{A}_{\pm}=A_{\pm}-(A_{n_0}^{(\pm)}-c_{\pm})(h/|\tau|^{\Delta})^{n_0}$. Since $f_{m_0n_0}$ is the same above and below the critical temperature, the constant a is also the same above and below the critical temperature, see Eq. (10). Note that Eq. (16) can be set in a form similar to Eq. (3.1) of Ref. 5, namely,

$$f_s=\tilde{A}_{\pm}(h/|\tau|^{\Delta})|\tau|^{2-\alpha}\ln(1/|\tau|)+\hat{A}_{\pm}(h/|\tau|^{\Delta})|\tau|^{2-\alpha} \ , \tag{17}$$

where $2-\alpha=m_0+n_0\Delta$, and

$$\tilde{A}_{\pm}=a(h/|\tau|^{\Delta})^{n_0}(\text{sgn}\tau)^{m_0} \ . \tag{18}$$

Equation (18), absent in Ref. (5), is the central result of this section. If more than one pair (m_0,n_0) satisfy Eq. (7), the right-hand side of Eq. (18) is a sum over all pairs.

In the case of the planar ferromagnetic Ising model, the critical exponents are $\alpha=0$ and $\Delta=\frac{15}{8}$. The only nonnegative integers m_0,n_0 satisfying Eq. (7) are $m_0=2$ and $n_0=0$. Hence the singular free energy as given in Eq. (16) is

$$f_s=a\tau^2\ln(1/|\tau|)+\tau^2\hat{A}_{\pm}(h/|\tau|^{15/8}) \ , \tag{19}$$

where the leading order[5] τ is the reduced temperature and h is the magnetic field. Equivalently, the function $\tilde{A}_{\pm}(h/|\tau|^{15/8})$ in Eq. (17) is equal to a constant. Therefore, the amplitude of the logarithmic contribution to f_s is independent of the field h. The leading contributions to the field derivatives of f_s at zero field come from the second term on the right-hand side of Eq. (19) and thus are logarithm free, in agreement with the conjecture of Aharony and Fisher.[5]

The consequences of Eq. (19) for the critical behavior on the critical isotherm were discussed in full detail by Aharony and Fisher.[5] In particular, since the scaling field τ is a function of the reduced temperature t and the magnetic field H, $\tau = t + uH^2 + \cdots$, the logarithmic term in Eq. (19) will give a contribution $H^4 \ln|H|$ to f_s at $t=0$. The leading singularity comes from the last term in Eq. (19), and it is $f_s \sim |H|^{16/15}$. The last result is the consequence of $h = H + \cdots$ and of the requirement that no singularity arises when $t \to 0$ at finite H, which is satisfied only if the scaling function behaves as $\hat{A}_\pm \sim |x|^{16/15}$ for $|x| \to \infty$.

III. ANTIFERROMAGNET IN A WEAK MAGNETIC FIELD

The square-lattice Ising antiferromagnet in zero field is equivalent to the zero-field ferromagnet. In the presence of a uniform field the two models differ substantially. While in the ferromagnetic problem only an isolated critical point occurs at zero field, in the antiferromagnetic case a line of critical points in the temperature, field plane separates the antiferromagnetic and paramagnetic phases. By using the renormalization-group formalism and a recent very accurate numerical result[6] on the zero-field susceptibility, I determine the critical curve and the leading singular contribution to the susceptibility in a weak uniform field.

The temperaturelike scaling field τ is relevant and the critical exponent is $\alpha=0$. The magnetic field is an irrelevant field $\Delta \le 0$. According to Eq. (17) the singular part of the free energy is

$$f_s = \tilde{A}_\pm \tau^2 \ln(1/|\tau|) + \hat{A}_\pm \tau^2 \ . \quad (20)$$

Besides the pair $m_0=2$ and $n_0=0$, there may be other pairs of non-negative integers satisfying the equation $2 = m_0 + n_0\Delta$, and this obviously depends on the actual value of Δ. In particular, if $\Delta=0$, i.e., the magnetic field is marginally irrelevant, then $m_0=2$ and n_0 equals any non-negative integer. In this case the amplitudes $\tilde{A}_\pm$ and $\hat{A}_\pm$ are functions of the field H. Since we consider here only the leading contributions in weak fields, it suffices to set $m_0=2$, $n_0=0$ in Eq. (18), and to ignore the second term in Eq. (20)

$$f_s = a\tau^2 \ln(1/|\tau|) \ . \quad (21)$$

The scaling field τ is a nonlinear function of the reduced temperature $t=(T-T_c)/T_c$ and of the field H. T_c is the critical temperature in the presence of the field H. Due to the up-down symmetry, no odd powers of H occur

$$\tau = t + O(tH^2, t^2, \cdots) = t_0 + uH^2 + O(t_0^2, t_0 H^2, H^4, \cdots) \ , \quad (22)$$

where t_0 is the zero-field reduced temperature. The leading contribution to susceptibility as obtained from Eqs. (21) and (22) is

$$\chi_s = (kT)^2 \frac{\partial^2 f_s}{\partial H^2} = D'H^2 \ln(1/|t|) + Dt \ln(1/|t|) \ , \quad (23)$$

where

$$D = 4au(kT_c^0)^2 \text{ and } D' = 8au^2(kT_c^0)^2 \ . \quad (24)$$

By using the value of the zero-field amplitude determined with high precision by Kong, Au-Yang, and Perk[6]: $D = 0.193\,595\,186\,3$ and the exactly known[1] $a = [\ln(1+\sqrt{2})]^2/\pi$ and $kT_c^0 = 2/\ln(1+\sqrt{2})$, we compute from Eqs. (24)

$$u = 0.038\,023\,259 \quad (25)$$

and

$$D' = 0.014\,718\,006\,6 \ . \quad (26)$$

A dependence as given in Eq. (23) has been predicted by Fisher[9] from the exact solution of the "superexchange" antiferromagnet. An immediate consequence of Eq. (23) is the following formula for the leading contribution to the zero-field fourth-order susceptibility,

$$\chi_s^{(4)} = (kT)^4 \frac{\partial^4 f_s}{\partial H^4}\bigg|_{H=0} = E \ln(1/|t_0|) \ , \quad (27)$$

where $E = 3D'(KT_c^0)^2 = 0.227\,357\,966\,6$. It would be interesting to verify this result with the high accuracy techniques[6] available for the zero-field problem.

The line of critical points starts at $T=0$ and $H=\pm 4$ and for small fields is given by $t = t_0 + uH^2 + \cdots = 0$, which implies

$$T_c = T_c^0[1 - uH^2 + O(H^4)] \ , \quad (28)$$

with $u = 0.038\,012\,325\,9$. This highly accurate, essentially exact result departs from the Müller-Hartmann and Zittartz[10] conjectured critical line, $u = 0.038\,95$, by 2.6%. It does confirm that the latter conjecture is not exact but is a good approximation. The critical temperature $T_c(H)$ from a recent position-space renormalization-group computation[11] is smaller than the conjectured[10] $T_c(H)$, which in turn is smaller than $T_c(H)$ from Eq. (28). Hence this renormalization-group scheme[11] is a worse approximation than the Müller-Hartmann and Zittartz[10] conjecture. I mention parenthetically that the exact[12] $T_c(H)$ on the diamond hierarchical lattice is also parabolic for small H as in Eq. (28). Finally, our u agrees (within uncertainties) with the estimate of Rapaport and Domb,[13] $u = 0.038\,0$. This last result was obtained[13] from high-temperature series for the staggered susceptibility analyzed by means of Griffiths's smoothness postulate.[14] Eqs. (21) and (22) are completely consistent with the smoothness postulate.

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