Spring 5-19-2018

Comparisons of Ad Valorem and Unit Tax in Two Different Industries

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COMPARISONS OF AD VALOREM AND UNIT TAX
IN TWO DIFFERENT INDUSTRIES

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COMPARISONS OF AD VALOREM AND UNIT TAX IN TWO DIFFERENT INDUSTRIES

A Dissertation Presented to the Graduate Faculty of the

Dedman College

Southern Methodist University

in

Partial Fulfillment of the Requirements

for the degree of

Doctor of Philosophy

with a

Major in Economics

by

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May 19, 2018

ACKNOWLEDGMENTS

I am grateful to all of those with whom I have had the pleasure to work during these five years. Each of the members of my Dissertation Committee has provided me extensive personal and professional guidance and taught me a great deal about both research and life in general. I would especially like to thank Dr. Bo Chen, my advisor. As my teacher and mentor, he has taught me more than I could ever give him credit for here.

Nobody has been more important to me in the pursuit of this degree than my parents. I would like to thank my parents, whose love and guidance are with me in whatever I pursue.

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Advisor: Dr. Bo Chen
Doctor of Philosophy degree conferred May 19, 2018
Dissertation completed April 19, 2018

This dissertation analyzes the comparisons between unit and ad valorem tax in two different industries.

Chapter 1 considers an industry with one upstream monopolist and a fixed number of asymmetric downstream firms. I compare market outcomes from unit and ad valorem tax under three pricing schemes that the upstream monopolist can use, uniform pricing, third-degree price discrimination, and two-part tariffs. The major finding is that ad valorem tax is always tax revenue preferred to unit tax when both are charged at the same layer as long as profit margin is positive. Downstream ad valorem tax improves total social welfare by relocating production from less efficient downstream firms to more efficient ones.

Chapter 2 investigates a similar tax comparison between ad valorem and unit tax in a two-market industry in which one market is generating negative aggregate externalities and being taxed. Firms monopolistically compete in each market by producing different varieties of the same product. Each consumer values both prices and varieties and chooses a single market to visit. I show that for a given tax revenue target, unit tax leads to less consumption of the good with negative externalities than ad valorem tax in both the short run and the long run. In the short run, the choice between these two taxes regarding total social welfare depends on the relative magnitudes of distortions from imperfect competition and externalities. In the long run, switching taxes does not affect the distortion from imperfect competition. Unit tax is preferred because it leads to less consumption of the good with negative externalities.

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This dissertation is dedicated to my parents and my grandmother.

Chapter 1

A COMPARISON OF AD VALOREM AND UNIT TAX IN A VERTICALLY STRUCTURED INDUSTRY

1.1. Introduction

Ad valorem tax and unit tax are commonly adopted in varies industries and countries. Ad valorem tax is a percentage tax that charged on the value of a transaction. A unit tax is a tax that is defined as a fixed amount of each unit of a good sold. Both are prevalently being used on, but not restricted to, vehicles, fuel, liquor and cigarette. In the United States, for example, both taxes are levied on the alcohol beverage industry. In 2012, alcohol beverages are taxed based on the alcohol content of these beverages. Beer's unit taxes range from $0.02 to $1.07/gallon, wine's unit taxes range from $0.11 to $2.50/gallon, and spirits' unit taxes range from $1.50 to $12.80/gallon. States also levy ad valorem tax on alcohol beverages. Beer ad valorem tax rates range from 1 to 17 percent. Wine and Distilled spirit rates range from 1.7 to 15 percent. Both taxes have important implications on the structure and the performance of the underlying market.

A large amount of studies have analyzed and compared these two taxes, starting from the 18th century. However, the existing literature has restricted attention almost entirely to downstream markets. Their results could be justified only if the input price was not affected by either of these taxes. However, this condition usually does not hold. To understand the full effects of ad valorem tax and unit tax, it is important to take into account of how these taxes affect intermediate product prices, which, in turn, can have further influence on downstream markets.

By introducing ad valorem tax and unit tax into a vertical structure, the following questions arise: Under the vertical structure, is ad valorem tax preferred over unit tax when both are charged upstream? What if both are charged downstream? Is there any difference between upstream and downstream taxation? If yes, is it more efficient to introduce a tax upstream or downstream? Are the answers to the previous questions different when the upstream manufacturer employs different pricing schemes? Is uniform pricing still preferred to third-degree price discrimination?

In this paper, I answer these questions. To do so, I set up a model with one upstream market and one downstream markets. In the upstream market, there is only one monopolist upstream manufacturer. In the downstream markets, a fixed number of asymmetric downstream firms compete in quantities. I assume that the fixed cost of entering either market is extremely high so that no entry will happen in both layers[1]. I consider four different taxes including upstream ad valorem tax, upstream unit tax, downstream ad valorem tax, and downstream unit tax, all of which can be charged simultaneously. Moreover, the upstream manufacturer can use three different pricing schemes, uniform pricing, third-degree price discrimination, and two-part tariffs.

I obtain the following results: (1) Switching from upstream unit tax t_U to upstream ad valorem tax τ_U while keeping total output level at the equilibrium unchanged, total social welfare remains unchanged for all three pricing schemes. Tax revenue is increasing under uniform pricing and third-degree price discrimination but keeps unchanged under two-part tariffs. (2) Switching from downstream unit tax t_D to downstream ad valorem tax τ_D while keeping total output level at the equilibrium unchanged, total social welfare and tax revenue are increasing for all three pricing schemes. (3) There is an ambiguous result for the comparison between downstream ad valorem tax and upstream ad valorem tax when the upstream manufacturer uses uniform pricing and third-degree price discrimination. In contrast, under a two-part tariff, downstream ad valorem tax is superior to the corresponding upstream ad valorem tax.

I also discuss three additional results. First, I show the reason why it is not possible to derive an unambiguous result of tax revenue comparison under a general demand function

[1]I only focus on the short run where no free entry and exit occurs in either market.

by keeping total social welfare level unchanged. Second, I consider a special case when the upstream manufacturer is restricted to uniform pricing. In this case, the upstream manufacturer might have incentive to stop selling to several inefficient downstream firms so as to increase input price and profit. This is the corner solution to the general model. The result implies that forcing the upstream manufacturer to use uniform pricing may not work as expected to improve total social welfare. In the end, I show that under two-part tariffs, the upstream manufacturer has incentive to only serve the most efficient downstream firm and becomes the monopolist in the downstream market.

This paper has several contributions. First, this paper considers the case of asymmetric downstream markets, which is commonly seen in reality. This fills the gap in the existing literature that compare indirect taxes in vertical structures. And I show that asymmetry makes downstream ad valorem tax generate higher total social welfare, even if output level remain unchanged. Second, comparing the results of three different pricing schemes in one model can help us better understand the difference between them. Finally and most importantly, this paper has policy implications. The results suggest that a government concerning tax revenue should always use ad valorem tax instead of unit tax in a vertical structure. This result does not depend on which pricing scheme the upstream manufacturer uses and which layer the tax is being charged. In addition, when the upstream manufacturer is using two-part tariffs, the optimal tax for government to use is downstream ad valorem tax.

1.1.1. Literature Review

The existing literature comparing the effects of indirect taxation on market structure has mainly focused on downstream markets. The equivalence of the two taxes regimes under perfect competition has long been recognized. Under conditions of monopoly, these two taxes require separate consideration. The first to recognize that the two kinds of tax had potentially different effects was Cournot (1960), writing in the 1830s. Wicksell (1959) developed Cournot's work by comparing these two taxes in monopoly market. Suits and

Musgrave (1955) go one step further by comparing tax revenue collected by these two taxes. Suits and Musgrave (1955) find that given an output level, unit tax always generates less tax revenue compared to ad valorem tax. In contrast, given a tax revenue target, the final good price is higher under unit tax, whereas the total output level is lower.

Delipalla and Keen (1992) is the first paper to compare effects of ad valorem and unit taxes outside the polar cases of monopoly and perfect competition. They extend the comparison between two taxes to intermediate market forms, with and without free entry. They find that ad valorem taxation dominates unit taxation by implying a relatively low consumer price, high tax revenue, and high welfare.

The comparison of unit and ad valorem taxes has also been extended to different pricing schemes. Yang (1993) shows that Suits and Musgrave's result in the traditional single market monopoly model still holds in the same model with third-degree price discrimination. At a given output level, an ad valorem tax generates more tax revenue than does its corresponding unit tax. On the other hand, the profit level under the unit tax exceeds that under ad valorem tax by exactly the same amount. Thus, the welfare measure is identical under both tax systems for a given output level. Jensen and Schjelderup (2009) compare ad valorem and specific taxation under heterogeneous demand when a monopolist offers a menu of two-part tariffs. They show that an increase in either tax rate leads to a higher unit price for all consumers, whereas the fixed fee under reasonable assumptions will fall. If the government changes the mix of taxes in such a way that the firm's behavior is unchanged, a system of only ad valorem taxation generates higher tax revenue than does a system of only specific taxes. Tax reform designed to leave tax revenue constant leads to a lower per unit fee and a higher fixed fee for all consumers. It also increases market coverage, profits, tax revenue, and the consumer surplus.

The literature mentioned above has only compared ad valorem tax and unit tax in downstream markets while assuming upstream markets are perfectly competitive. To the best of my knowledge, the only paper that compares ad valorem tax and unit tax in a vertical market structure is Peitz and Reisinger (2014). They analyze the effect of unit and ad valorem taxation in an industry with downstream and upstream oligopolies and endogenous entry in both layers. They find that it is typically more efficient to levy an ad valorem tax downstream than upstream, while it does not matter which layer a specific tax is levied. They also show that tax revenue should be raised only through ad valorem taxes. My model is different than Peitz and Reisinger (2014) in three ways. First, I consider the case of the short run with no entry in either layer and I only assume one upstream monopolist in the upper layer. Thus, firms at both layers have positive profits. Second, the downstream firms in my model are asymmetric. This setting facilitates us to better understand how the downstream firms react differently to different taxes. Finally, I consider three different pricing schemes the upstream manufacturer can use, including uniform pricing, third-degree price discrimination and two-part tariffs. By analyzing these schemes in one model, I can directly compare my results with results from previous literature and provide policy implications.

The rest of this chapter is organized as follows: Section 1.2 provides a simple numerical example to show some basic results of this model. Section 1.3 sets up the model. Section 1.4 characterizes the equilibrium for each pricing regime. Section 1.5 determines whether ad valorem tax is superior to unit tax, for each pricing scheme, if both are charged on the same layer. Section 1.6 compares ad valorem tax charged on the upstream manufacturer with that charged on the downstream firms. Section 1.7 provides some extensions. Section 1.8 concludes and provides several policy implications. All proofs are relegated to an Appendix.

1.2. An Example

The purpose of this section is to introduce a simple numerical example that explores the comparison between unit and ad valorem tax. Consider a two-layer industry with an upstream manufacturer and downstream firm 1 and 2. The upstream manufacturer produces a homogeneous intermediate product at constant a marginal cost $c = 0.05$ and it can meet any demand volume from the downstream firms. In the downstream market, two downstream firms with different marginal costs compete in the same market. The downstream firms transform the intermediate product into final product at constant marginal costs $\beta_1 = 0.1$

and $\beta_2 = 0.2$ respectively. Without loss of generality, the downstream firms need one unit of input for one unit of output. The inverse demand function for the final product is denoted as $P(Q) = 1 - Q$, with $Q = q_1 + q_2$, where $Q \in [0, 1]$.

There are two different types of taxes, unit tax and ad valorem tax. Both taxes can be imposed on both layers. I denote the downstream unit tax by $t_D \geq 0$ and the upstream unit tax by $t_U \geq 0$. The unit tax is a tax that needs to be paid per unit sold. I denote the downstream ad valorem tax $\tau_D \geq 0$ and the upstream ad valorem tax $\tau_U \geq 0$.

I consider uniform pricing, third-degree price discrimination and two-part tariffs. Under uniform pricing, the upstream manufacturer is charging the sams intermediate good's price $w_i = w$ for all downstream firms. Under third-degree price discrimination, the upstream manufacturer is charging different intermediate good's price $w_i = w$ for downstream firm i. Under two-part tariffs, the upstream manufacturer charges a fixed fee F_i in addition to a unit price w_i for downstream firm i. Thus, the upstream manufacturer has three pricing schemes.

1.2.1. Example of Uniform Pricing

First considering the case that the upstream manufacturer is charging the same price for each unit of intermediate goods sold to the downstream firms. There is no price discrimination. Solving the game by backward induction, the profit function of downstream firm i can be written as

$$\pi_i (q_i, Q_{-i}) = q_i \left[(1 - \tau_D)(1 - Q) - t_D - w - \beta_i \right]$$

where $Q_{-i} = Q - q_i$. The resulting first-order condition of profit maximization is

$$\frac{\partial \pi_i (q_i, Q_{-i})}{\partial q_i} = (1 - \tau_D)(1 - Q - q_i) - t_D - w - \beta_i = 0$$

Combining two first-order conditions, the intermediate good's price can be represented as

$$w = (1 - \tau_D)\left(1 - \frac{3Q}{2}\right) - t_D - 0.15$$

The output level for each downstream firm can be rewritten as

$$
\begin{aligned}
q_1 &= \frac{0.05}{(1-\tau_D)} + \frac{Q}{2} \\
q_2 &= -\frac{0.05}{(1-\tau_D)} + \frac{Q}{2}
\end{aligned}
$$

According to this equation, the more efficient a downstream firm is, the more this downstream firm is producing. In addition, downstream ad valorem tax can amplify this relocation effect.

The profit function of the upstream manufacturer can be written as

$$
\begin{aligned}
\Pi(w) &= \left[(1-\tau_U)\,w - c - t_U\right] Q \\
&= \left[\begin{array}{c} (1-\tau_U)(1-\tau_D)\left(1-\frac{3Q}{2}\right) \\ -(1-\tau_U)\,t_D - 0.15\,(1-\tau_U) - 0.05 - t_U \end{array} \right] Q
\end{aligned}
$$

This yields the first-order condition

$$
1 - 3Q = \frac{t_D + 0.15 + \frac{0.05+t_U}{1-\tau_U}}{1-\tau_D}
$$

Tax revenue can be written as

$$
\begin{aligned}
T &= \tau_U \left((1-\tau_D)\left(1 - \frac{3Q}{2}\right) - t_D - 0.15 \right) Q \\
&\quad + \tau_D (1-Q)\, Q + t_U Q + t_D Q
\end{aligned}
$$

Total social welfare includes consumer surplus, producer surplus, and tax revenue. It can be written as

$$W = \frac{4}{5}Q - \frac{1}{2}Q^2 + \frac{1}{200\,(1-\tau_D)}$$

Consider the simplest case where only one tax is being charged each time. The comparison of these four taxes can be summarized as the table below. Row 2 shows the results with no

τ_U	t_U	τ_D	t_D	q_1	q_2	Q	T	W
–	–	–	–	0.18335	0.08335	0.2667	0	0.1828
0.3077	–	–	–	0.1611	0.0611	0.2222	**0.0353**	0.1581
–	0.0222	–	–	0.1611	0.0611	0.2222	0.0049	0.1581
–	–	0.1	–	0.1667	0.0555	0.2222	**0.0173**	**0.1586**
–	–	–	0.0222	0.1611	0.0611	0.2222	0.0049	0.1581

Table 1.1. Uniform Pricing

tax being charged. Row 3 shows the results when only upstream ad valorem tax is being charged. Row 4 shows the results when only upstream unit tax is being charged. Row 5 shows the results when only downstream ad valorem tax is being charged. Row 6 shows the results when only downstream unit tax is being charged.

Results show that keeping total output level at the equilibrium level unchanged, ad valorem tax always leads to higher tax revenue than unit tax. This is because that unit tax is equivalent to corresponding ad valorem tax charged on marginal cost. In contrast, ad valorem tax is charged on price, which is strictly higher than marginal cost. Downstream ad valorem tax is total social welfare preferred to other taxes because it amplifies the relocation effect by letting efficient downstream firm produce even more and inefficient downstream firm produce even less. This can be seen from the value of q_1 and q_2.

1.2.2. Example of Third-Degree Price Discrimination

Next considering the case that the upstream manufacturer can perfectly identify each downstream firm and price discriminates among them. Similarly, solving the game by backward induction, the comparison of these four taxes can be summarized as the table below. Results are very similar to the uniform pricing case except two things. First, total social

τ_U	t_U	τ_D	t_D	q_1	q_2	Q	T	W
–	–	–	–	0.15835	0.10835	0.2667	0	0.1803
0.3077	–	–	–	0.1361	0.0861	0.2222	**0.0357**	0.1556
–	0.0222	–	–	0.1361	0.0861	0.2222	0.0049	0.1556
–	–	0.1	–	0.1389	0.0833	0.2222	**0.0173**	**0.1559**
–	–	–	0.0222	0.1361	0.0861	0.2222	0.0049	0.1556

Table 1.2. Third-Degree Price Discrimination

welfare levels under price discrimination are lower than those under uniform pricing. This is because that upstream monopolist favors inefficient downstream firms by charging lower intermediate good price. As a result, efficiency is lower. Second, when ad valorem tax is charging on upstream layer, tax revenue under price discrimination is higher than that under uniform pricing. Even though the intermediate good price for the inefficient downstream firm is reducing and the output level for the efficient downstream firm is shrinking, these changes are outweigh by the increase in intermediate good price for the efficient downstream firm and the increase in output level of the inefficient downstream firm. Thus, tax revenue is increasing.

1.2.3. Example of Two-Part Tariffs

Lastly, I assume that the upstream manufacturer uses two-part tariffs. The upstream manufacturer optimally determining a fixed fee and a unit price pair (F_i, w_i) for each different downstream firm i. Again, solving the game by backward induction, the comparison of these four taxes can be summarized as the table below. Again, for downstream taxation, results

τ_U	t_U	τ_D	t_D	q_1	q_2	Q	T	W
–	–	–	–	0.31665	0.21665	0.5333	0	0.2894
0.3077	–	–	–	0.30925	0.20925	0.5185	0.0115	0.2854
–	0.0222	–	–	0.30925	0.20925	0.5185	0.0115	0.2854
–	–	0.1	–	0.3148	0.2037	0.5185	**0.0250**	**0.2859**
–	–	–	0.0222	0.30925	0.20925	0.5185	0.0115	0.2854

Table 1.3. Two-Part Tariffs

are very similar to those from uniform pricing case and third-degree price discrimination case. However, different from previous cases, upstream ad valorem tax generates the same amount of tax revenue compared to upstream unit tax. This is because that, under two-part tariffs, intermediate good's price is at the marginal cost level. Thus, upstream ad valorem tax and upstream unit tax are equivalent.

In next section, I consider a more general model and show that all the results derived here can be extended to the general case.

1.3. Model

I consider a two-layer industry with an upstream manufacturer and $n \in \mathbb{Z}^+$ independent downstream firms. The upstream manufacturer produces a homogeneous intermediate product at constant marginal cost $c > 0$ with a sufficiently large production capacity to. In other words, there is no production limitation. In the downstream market, n asymmetric downstream firms compete in quantities. The downstream firms transform the intermediate product into final product at constant marginal costs $\beta_i > 0 \; \forall i = 1, \dots, n$. Without loss of generality, the downstream firms need one unit of input for one unit of output. The inverse demand function for the final product is denoted by $P(Q)$, with $P'(Q) < 0$ and $Q = \sum_{i=1}^{n} q_i$, where $Q \in \left[0, \overline{Q}\right)$ with $\overline{Q} \in \mathbb{R}^+$ defined by $P(Q) > 0$ fo all $Q \in \left[0, \overline{Q}\right)$ and $\lim_{Q \uparrow \overline{Q}} P(Q) = 0$. For simplicity, I assume that $P(Q)$ is thrice continuously differentiable.

There are two different forms of taxes, unit tax and ad valorem tax. Both taxes can be imposed on both layers, implying that there can be four taxes overall. I denote the downstream unit tax by $t_D \geq 0$ and the upstream unit tax by $t_U \geq 0$. The unit tax is a tax that needs to be paid per unit sold. I denote the downstream ad valorem tax $\tau_D \geq 0$ and the upstream ad valorem tax $\tau_U \geq 0$.

I consider uniform pricing, third-degree price discrimination and two-part tariffs. Under uniform pricing, the upstream manufacturer is charging the sams intermediate good's price $w_i = w$ for all downstream firms. Under third-degree price discrimination, the upstream manufacturer is charging different intermediate good's price $w_i = w$ for downstream firm i. Under two-part tariffs, the upstream manufacturer charges a fixed fee F_i in addition to a unit price w_i for downstream firm i. Thus, the upstream manufacturer has three pricing schemes.

1.4. Equilibrium Characterization

1.4.1. Uniform Pricing

First considering the case that the upstream manufacturer is charging the same price for each unit of intermediate goods sold to the downstream firms. There is no price discrimination. Solving the game by backward induction for a given number of firms at the two layers, the profit function of downstream firm i can be written as

$$\pi_i \left(q_i, Q_{-i}\right) = q_i \left[\left(1 - \tau_D\right) P\left(Q\right) - t_D - w - \beta_i\right]$$

where $Q_{-i} = Q - q_i$. The resulting first-order condition of profit maximization is

$$\frac{\partial \pi_i \left(q_i, Q_{-i}\right)}{\partial q_i} = \left(1 - \tau_D\right)\left(P\left(Q\right) + q_i P'\left(Q\right)\right) - t_D - w - \beta_i = 0 \tag{1.1}$$

Combining all first-order conditions, the intermediate good's price can be represented as

$$w = (1-\tau_D)\left(P(Q) + \frac{Q}{n}P'(Q)\right) - t_D - \frac{1}{n}\sum_{j=1}^{n}\beta_j$$

Taking this condition back into equation (1.1), the output level for downstream firm i can be denoted as

$$q_i = \frac{\beta_i - \frac{1}{n}\sum_{j=1}^{n}\beta_j}{(1-\tau_D)P'(Q)} + \frac{Q}{n}$$

According to this equation, more efficient a downstream firm is, higher output this downstream firm is producing. In addition, downstream ad valorem tax can amplify this relocation effect.

To ensure that each downstream firm's problem is concave and that there exists a unique solution, I need two conditions: $2P'(Q) + q_i P''(Q) < 0$ and $\lim_{Q\downarrow 0}(P(Q) + q_i P'(Q)) > \frac{w+\beta_i+t_D}{1-\tau_D}$ for $\forall i = 1, \ldots, n$. Replacing the equation of w in, I can rewrite it as Assumption 1.1

Assumption 1.1. *For any downstream firm i, $0 > 2P'(Q)+q_i P''(Q)$ and $\lim_{q_i\downarrow 0}\left(\left(q_i - \frac{Q}{n}\right)P'(Q)\right) > \frac{\beta_i - \frac{1}{n}\sum_{j=1}^{n}\beta_j}{1-\tau_D}$.*

The first part of Assumption 1.1 is a key condition for existence and uniqueness of Cournot equilibrium: it guarantees that the solution to equation (1.1) is unique and, thus, that the inverse demand function faced by the upstream manufacturer is unique as well. The second part ensures that it is indeed optimal for the downstream firms to produce a positive quantity.

The profit function of the upstream manufacturer can be written as

$$\begin{aligned}\Pi(w) &= [(1-\tau_U)w - c - t_U]\,Q \\ &= \begin{bmatrix} (1-\tau_U)(1-\tau_D)\left(P(Q) + \frac{Q}{n}P'(Q)\right) \\ -(1-\tau_U)t_D - (1-\tau_U)\frac{1}{n}\sum_{j=1}^{n}\beta_j - c - t_U \end{bmatrix} Q\end{aligned}$$

This yields the first-order condition

$$\begin{aligned} & P(Q) + \frac{(n+2)}{n} Q P'(Q) + \frac{1}{n} Q^2 P''(Q) \\ = \quad & \frac{t_D + \frac{1}{n}\sum_{i=1}^{n}\beta_i + \frac{c+t_U}{1-\tau_U}}{1-\tau_D} \end{aligned} \tag{1.2}$$

To ensure that the upstream manufacturer's problem is concave and that there exists a unique solution I make the following assumption:

Assumption 1.2. *For the upstream manufacturer,* $0 > \frac{2n+2}{n} P'(Q) + \frac{n+4}{n} Q P''(Q) + \frac{1}{n} Q^2 P'''(Q)$ *and* $\lim_{Q \downarrow 0} \left[P(Q) + \frac{(n+2)Q}{n} P'(Q) + \frac{Q^2}{n} P''(Q) \right] > \frac{t_D + \frac{1}{n}\sum_{j=1}^{n}\beta_j + \frac{c+t_U}{1-\tau_U}}{1-\tau_D}$.

Assumption 1.2 is the counterpart to Assumption 1.1 for the upstream manufacturer. The first part guarantees concavity and uniqueness while the second part ensures that the equilibrium quantity is positive.

Total social welfare includes four parts: consumer surplus, the upstream manufacturer's producer surplus, the downstream firms' producer surplus and tax revenue from all four taxes. Consumer surplus can be written as

$$CS = \int_0^{Q^*} \left[P(Q) - P^* \right] dQ$$

The upstream manufacturer's producer surplus is the same as the upstream manufacturer's profit

$$PS^U = \sum_{i=1}^{n} \left[(1-\tau_U) w_i - c - t_U \right] q_i$$

The downstream firms' producer surplus is the same as the downstream firms' profits

$$PS^D = \sum_{i=1}^{n} q_i \left[(1-\tau_D) P(Q) - t_D - w_i - \beta_i \right]$$

Tax revenue can be written as

$$T = \tau_U w^* Q^* + \tau_D P(Q^*) Q^* + t_U Q^* + t_D Q^*$$

Hence, by summing up all these four parts, total social welfare can be written as

$$\begin{aligned} W &= CS + PS^U + PS^D + T \\ &= \int_0^{Q^*} P(Q)\, dQ - cQ^* - \sum_{i=1}^{n} \beta_i q_i \end{aligned}$$

1.4.2. Third-Degree Price Discrimination

Considering the case that the upstream manufacturer can perfectly identify each downstream firm and price discriminates among them. Solving the game by backward induction for a given number of firms at two layers, I can write the profit function of downstream firm i as

$$\pi_i(q_i, Q_{-i}) = q_i [(1 - \tau_D) P(Q) - t_D - w_i - \beta_i]$$

where $Q_{-i} = Q - q_i$. The resulting first-order condition of profit maximization is

$$\frac{\partial \pi_i(q_i, Q_{-i})}{\partial q_i} = (1 - \tau_D)(P(Q) + q_i P'(Q)) - t_D - w_i - \beta_i = 0 \tag{1.3}$$

To ensure that each downstream firm's problem is concave and that there exists a unique solution, I need two conditions: $2P'(Q) + q_i P''(Q) < 0$ and $\lim_{Q \downarrow 0}(P(Q) + q_i P'(Q)) > \frac{w_i + \beta_i + t_D}{1 - \tau_D}$ for $\forall i = 1, \ldots, n$.

The profit function of the upstream manufacturer can be written as

$$\begin{aligned}
\Pi\left(w_1,\ldots,w_n\right) &= \sum_{i=1}^{n}\left[\left(1-\tau_U\right)w_i - c - t_U\right]q_i \\
&= \left(1-\tau_U\right)\left(1-\tau_D\right)\left(P\left(Q\right)Q + P'\left(Q\right)\sum_{i=1}^{n}q_i^2\right) \\
&\quad -\left(1-\tau_U\right)t_D Q - \left(1-\tau_U\right)\sum_{i=1}^{n}\beta_i q_i - cQ - t_U Q
\end{aligned}$$

This yields the first-order condition for q_i

$$\begin{aligned}
&\left(1-\tau_U\right)\left(1-\tau_D\right)\left(P\left(Q\right) + P'\left(Q\right)Q + P''\left(Q\right)\sum_{j=1}^{n}q_j^2 + P'\left(Q\right)2q_i\right) \\
&-\left(1-\tau_U\right)\left(t_D+\beta_i\right) - \left(c+t_U\right) \\
= \;& 0
\end{aligned}$$

Combining all first-order conditions:

$$\sum_{j=1}^{n}q_j^2 = \frac{\frac{nt_D+\sum_{i=1}^{n}\beta_i}{1-\tau_D} + \frac{n(c+t_U)}{(1-\tau_U)(1-\tau_D)} - \left(n+2\right)P'\left(Q\right)Q - nP\left(Q\right)}{nP''\left(Q\right)}$$

Taking the condition above back into the first-order condition, it can be rewritten as

$$\begin{aligned}
q_i &= \frac{\left(1-\tau_U\right)\left(t_D+\beta_i\right) + \left(c+t_U\right)}{2P'\left(Q\right)\left(1-\tau_U\right)\left(1-\tau_D\right)} - \left[\frac{Q}{2} + \frac{P\left(Q\right)}{2P'\left(Q\right)} + \frac{P''\left(Q\right)}{2P'\left(Q\right)}\sum_{j=1}^{n}q_j^2\right] \\
&= \frac{\beta_i - \frac{1}{n}\sum_{j=1}^{n}\beta_j}{2P'\left(Q\right)\left(1-\tau_D\right)} + \frac{Q}{n}
\end{aligned}$$

According to this equation, more efficient a downstream firm is, higher output this downstream firm is producing. In addition, downstream ad valorem tax can amplify this relocation effect.

By taking q_i into the equation of the combination of all first-order conditions, I can derive an equation for Q:

$$\begin{aligned} & P(Q) + \frac{(n+2)}{n} Q P'(Q) + \frac{1}{n} Q^2 P''(Q) \\ = & \frac{t_D + \frac{1}{n}\sum_{i=1}^{n}\beta_i + \frac{c+t_U}{1-\tau_U}}{1-\tau_D} - \frac{\sum_{i=1}^{n}\beta_i^2 - \frac{1}{n}\left(\sum_{j=1}^{n}\beta_j\right)^2}{4\left[P'(Q)\right]^2 (1-\tau_D)^2} P''(Q) \end{aligned} \tag{1.4}$$

Taking q_i into the equation of w_i

$$\begin{aligned} w_i &= (1-\tau_D)\left(P(Q) + \frac{Q}{n}P'(Q) + \frac{\beta_i - \frac{1}{n}\sum_{j=1}^{n}\beta_j}{2(1-\tau_D)}\right) - t_D - \beta_i \\ &= (1-\tau_D)\left(P(Q) + \frac{Q}{n}P'(Q)\right) - t_D - \frac{\beta_i + \frac{1}{n}\sum_{j=1}^{n}\beta_j}{2} \end{aligned}$$

One thing worth to mention is that, under third-degree price discrimination, the upstream manufacturer favors inefficient downstream firms by charging lower intermediate good's price and hurts efficient downstream firms by charging higher intermediate good's price.

Taking w_i back into the assumption in this section, I can rewrite it as Assumption 1.3

Assumption 1.3. *For any downstream firm i, $0 > 2P'(Q) + q_i P''(Q)$ and $\lim_{q_i \downarrow 0}\left(\left(q_i - \frac{Q}{n}\right)P'(Q)\right) > \frac{\beta_i - \frac{1}{n}\sum_{j=1}^{n}\beta_j}{2(1-\tau_D)}$ for $\forall i = 1, \ldots, n$.*

This assumption is similar to Assumption 1.1. The first part guarantees concavity and uniqueness while the second part ensures that the equilibrium quantity is positive.

To ensure that the upstream manufacturer's problem is concave and that there exists a unique solution I make the following assumption:

Assumption 1.4. *For the upstream manufacturer,* $0 > 4P'(Q) + (4q_i + Q)P''(Q) + \left(\sum_{j=1}^{n} q_j^2\right) P'''(Q)$ *and* $\lim_{q_i \downarrow 0}\left(P(Q) + P'(Q)Q + P''(Q)\sum_{j=1}^{n} q_j^2 + P'(Q)2q_i\right) > \frac{t_D + \beta_i + \frac{c+t_U}{1-\tau_U}}{1-\tau_D}$ *for* $\forall i = 1, \ldots, n$.

Assumption 1.4 is the counterpart to Assumption 1.3 for the upstream manufacturer. The first part guarantees concavity and uniqueness while the second part ensures that the equilibrium quantity is positive.

Total social welfare includes four parts: Consumer surplus, the upstream manufacturer's producer surplus, the downstream firms' producer surplus and tax revenue from all four taxes. Consumer surplus can be written as

$$CS = \int_0^{Q^*} \left[P(Q) - P^*\right] dQ$$

The upstream manufacturer's producer surplus is the same as the upstream manufacturer's profit

$$PS^U = \sum_{i=1}^{n} \left[(1-\tau_U) w_i - c - t_U\right] q_i$$

The downstream firms' producer surplus is the same as the downstream firms' profits

$$PS^D = \sum_{i=1}^{n} q_i \left[(1-\tau_D) P(Q) - t_D - w_i - \beta_i\right]$$

Tax revenue can be written as

$$T = \tau_U \sum_{i=1}^{n} w_i q_i + \tau_D P(Q^*) Q^* + t_U Q^* + t_D Q^*$$

Hence, summing up all four parts, total social welfare can be written as

$$\begin{aligned} W &= CS + PS^U + PS^D + T \\ &= \int_0^{Q^*} P(Q)\, dQ - cQ^* - \sum_{i=1}^{n} \beta_i q_i^* \end{aligned}$$

1.4.3. Two-Part Tariffs

Now I assume that the upstream manufacturer uses two-part tariffs. The upstream manufacturer optimally determining a fixed fee and a unit price pair (F_i, w_i) for each different downstream firm i. Solving the game by backward induction for a given number of firms at the two layers, I can write the profit function of downstream firm i as

$$\pi_i(q_i, Q_{-i}) = q_i\left[(1-\tau_D)P(Q) - t_D - w_i - \beta_i\right] - F_i$$

where $Q_{-i} = Q - q_i$. The resulting first-order condition of profit maximization is

$$\frac{\partial \pi_i(q_i, Q_{-i})}{\partial q_i} = (1-\tau_D)\left(P(Q) + q_i P'(Q)\right) - t_D - w_i - \beta_i = 0 \tag{1.5}$$

To ensure that each downstream firm's problem is concave and that there exists a unique solution, I need two conditions: $2P'(Q) + q_i P''(Q) < 0$ and $\lim_{Q\downarrow 0}\left(P(Q) + q_i P'(Q)\right) > \frac{w_i+\beta_i+t_D}{1-\tau_D}$ for $\forall i = 1, \ldots, n$.

To maximize profit, the upstream manufacturer has to extract as more downstream firms' producer surplus as possible as fixed fee F_i. Hence, F_i need to be equal to π_i. Correspondingly, w_i is the upstream firm's cost of producing a unit of intermediate goods. These two conditions can be summarized as

$$\begin{aligned} F_i &= \pi_i \\ (1-\tau_U)\,w_i &= c + t_U \end{aligned}$$

Taking the equation of w_i back to (5)

$$q_i = \frac{t_D + \frac{c+t_U}{1-\tau_U} + \beta_i}{(1-\tau_D)P'(Q)} - \frac{P(Q)}{P'(Q)}$$

According to this equation, more efficient a downstream firm is, higher output this downstream firm is producing. In addition, downstream ad valorem tax can amplify this relocation effect.

Combining all first-order conditions, I have the following condition

$$P(Q) + \frac{Q}{n} P'(Q) = \frac{t_D + \frac{c+t_U}{1-\tau_U} + \frac{1}{n}\sum_{i=1}^{n} \beta_i}{1-\tau_D} \tag{1.6}$$

Taking w_i back into the assumption in this section, I can rewrite it as Assumption 1.5

Assumption 1.5. *For any downstream firm* i, $0 > 2P'(Q) + q_i P''(Q)$ *and* $\lim_{q_i \downarrow 0} \left(P(Q) + q_i P'(Q)\right) > \frac{c+t_U+(\beta_i+t_D)(1-\tau_U)}{(1-\tau_U)(1-\tau_D)}$ *for* $\forall i = 1, \ldots, n$.

This assumption is again similar to Assumption 1.1. The first part guarantees concavity and uniqueness while the second part ensures that the equilibrium quantity is positive.

The upstream manufacturer's profit function can be written as

$$\begin{aligned} \Pi &= \sum_{i=1}^{n} \left[\left[(1-\tau_U) w_i - t_U - c\right] q_i + (1-\tau_U) F_i\right] \\ &= \left[(1-\tau_U)(1-\tau_D) P(Q) - (1-\tau_U) t_D - t_U - c\right] Q - (1-\tau_U) \sum_{i=1}^{n} q_i \beta_i \end{aligned}$$

Total social welfare includes four parts: consumer surplus, the upstream manufacturer's producer surplus, the downstream firms' producer surplus and tax revenue from all four taxes. Consumer surplus can be written as

$$CS = \int_0^{Q^*} \left[P(Q) - P^*\right] dQ$$

The upstream manufacturer's producer Surplus is the same as the upstream manufacturer's profit

$$PS^U = \left[(1-\tau_U)(1-\tau_D)P^* - (1-\tau_U)t_D - t_U - c\right]Q^* - (1-\tau_U)\sum_{i=1}^{n} q_i^*\beta_i$$

The downstream firms' producer surplus is the same as the downstream firms' profits

$$PS^D = 0$$

Tax revenue can be written as

$$T = \tau_U w^* Q^* + \tau_D P(Q^*)Q^* + t_U Q^* + t_D Q^*$$

Hence, summing up all four parts, total social welfare can be written as

$$\begin{aligned} W &= CS + PS^U + PS^D + T \\ &= \int_0^{Q^*} P(Q)\,dQ - cQ^* - \sum_{i=1}^{n} \beta_i q_i^* \end{aligned}$$

1.5. Compare Different Tax Regimes

For all three pricing schemes, I first compare ad valorem tax and unit tax in each layer. To facilitate the comparison of the two taxes in the same layer, I start by following the approach of Suits and Musgrave (1959) in which two tax revenues are compared at a given output level. At a given output level and when both ad valorem tax and unit tax are charged on the upstream manufacturer, I can derive the following proposition for three different pricing schemes.

Proposition 1.1. *For a given output level, switching from upstream unit tax t_U to upstream ad valorem tax τ_U,*

(1) under Uniform Pricing and Third-Degree Price Discrimination, total social welfare keeps unchanged and tax revenue increases;

(2) under Two-Part Tariffs, total social welfare keeps unchanged and tax revenue also keeps unchanged.

When comparing upstream ad valorem tax and upstream unit tax, total social welfare only depends on total output level. Total social welfare is unchanged as long as the total output level is unchanged. On the other hand, tax revenue is changing under uniform pricing and third-degree price discrimination but keeps unchanged under two-part tariffs. The reason is that upstream unit tax is equivalent to upstream ad valorem tax charged on actual the upstream manufacturer's marginal cost $\frac{c+t_U}{1-\tau_U}$ when total output level at the equilibrium is fixed. However, under uniform pricing and third-degree price discrimination, upstream ad valorem tax is charged on intermediate good's price w, which is higher than c. Hence, switching to upstream ad valorem tax leads to higher tax revenue under uniform pricing and third-degree price discrimination. On the contrary, under two-part tariffs, tax revenue keeps unchanged.

Next, I compare ad valorem tax and unit tax charged on the downstream firms.

Proposition 1.2. *For a given output level, switching from downstream unit tax t_D to downstream ad valorem tax τ_D, under Uniform Pricing, Thrid-Degree Price Discrimination and Two-Part Tariffs, tax revenue and total social welfare increase.*

Under all three cases, the more efficient downstream firms are producing more output than the less efficient downstream firms. Downstream ad valorem tax amplify this relocation effect compared to downstream unit tax. As a result, total social welfare increases when switching to downstream ad valorem tax, even though total output level remain unchanged. For tax revenue, downstream unit tax t_D is equivalent to downstream ad valorem tax τ_D charged on actual total marginal cost when total output level at the equilibrium is fixed.

Downstream ad valorem tax, in contrast, is charged on final good's equilibrium price. Under all three cases, final good's equilibrium price P is always greater than actual average total marginal cost $\frac{\frac{c+t_U}{1-\tau_U}+\frac{1}{n}\sum_{i=1}^{n}\beta_i+t_D}{1-\tau_D}$ and hence downstream ad valorem tax τ_D leads to higher tax revenue than downstream unit tax.

These two propositions have important policy implications. They suggest that government should switch from unit tax to ad valorem tax.

1.6. Upstream Ad Valorem Tax or Downstream Ad Valorem Tax

Based on those results derived from previous section, I know that, for all three pricing schemes, ad valorem tax is at least as good as unit tax at both layers. These results imply that government should always use ad valorem tax rather than unit tax. However, there is one more question remain unclear. Which one of these two, downstream ad valorem tax or upstream ad valorem tax, is better than the other from the social planner's point of view? In other words, should government levy ad valorem tax on the upstream layer or the downstream layer?

I have already shown that downstream ad valorem tax can improve efficiency by relocating intermediate goods to the more efficient downstream firms so that the more efficient firms can produce more under downstream ad valorem tax than under downstream unit tax. However, this result cannot guarantee that downstream ad valorem tax is superior to upstream ad valorem tax. To compare these two ad valorem taxes, I use Suit and Musgrave's method again by keeping total output level at the equilibrium unchanged.

Proposition 1.3. *For a given output level, switching from upstream ad valorem tax* τ_U *to downstream ad valorem tax* τ_D*,*

(1) under uniform pricing and third-degree price discrimination, total social welfare increases but the effect on tax revenue is ambiguous;

(2) under two-part tariffs, total social welfare and tax revenue increase.

Downstream ad valorem tax welfare dominates upstream ad valorem tax because it amplifies the relocation effect in which the more efficient downstream firms tend to produce more than the less efficient downstream firms. This result holds for all three pricing schemes. Under uniform pricing and third-degree price discrimination, markup on both layers are positive. The deterministic result cannot be derived unless a specific demand function is provided. On the other hand, under two-part tariffs, markup only exist at the downstream layer because the equilibrium price of the intermediate good w is set to be equal to the upstream manufacturer's marginal cost. Hence, tax revenue is increasing in a switch from upstream ad valorem tax to downstream ad valorem tax.

These results have important policy implications in the sense that government should always levy ad valorem tax on the downstream firms rather than the upstream manufacturer whenever the upstream manufacturer is using two-part tariffs. In contrast, I cannot make any suggestion between upstream ad valorem tax and downstream ad valorem tax when the upstream manufacturer uses linear pricing unless a specific demand function is provided.

1.7. Discussions

In this section, I discuss some additional results from this model. In the first part, I show that why a deterministic result of tax revenue comparison cannot be derived when total social welfare is fixed unchanged. Next, I show the possibility that upstream monopolist has incentive to drop some inefficient downstream firms under uniform pricing. Finally, I consider a two-part tariff case with only one downstream firm so that the upstream monopolist becomes the monopolist in the downstream market.

1.7.1. Equalizing Welfare

In this part, I show that, different from previous researches, it is not possible to derive an deterministic result of tax revenue comparison by fixing total social welfare level unless a specific demand function is provided.

To analyze the comparison between unit and ad valorem tax in terms of tax revenue and welfare, this paper applies the same method used by a lot of existing literatures by fixing the total output level unchanged. The advantage of this method is that the effect from the shape of the demand function can be ignored and therefore the results can be derived with a general demand function. In those existing literatures, total social welfare levels are equal as long as total output levels are equal in different tax scenarios. Thus, keeping total output level unchanged is the same as keeping total social welfare unchanged. Thus, they can derive an unambiguous result of tax revenue comparison for the same level of total social welfare. However, different from these literatures, I consider an unsymmetric case where each downstream firm has a unique marginal cost. Keeping total output level unchanged does not necessarily fix total social welfare level. This can be shown by the following equations. Under uniform pricing, the change in total social welfare can be represented as

$$
\begin{aligned}
dW &= \left[P\left(Q^{*}\right) - c - \frac{1}{n}\sum_{i=1}^{n}\beta_i\right] dQ \\
&\quad + \frac{\sum_{j=1}^{n}\beta_j^2 - \frac{1}{n}\left(\sum_{j=1}^{n}\beta_j\right)^2}{\left(1-\tau_D\right)} \frac{P''\left(Q^{*}\right)}{\left[P'\left(Q^{*}\right)\right]^2} dQ \\
&\quad - \frac{\sum_{j=1}^{n}\beta_j^2 - \frac{1}{n}\left(\sum_{j=1}^{n}\beta_j\right)^2}{\left(1-\tau_D\right)^2 P'\left(Q^{*}\right)} d\tau_D
\end{aligned}
$$

Under third-degree price discrimination, the change in total social welfare can be represented as

$$
\begin{aligned}
dW &= \left[P\left(Q^{*}\right) - c - \frac{1}{n}\sum_{i=1}^{n}\beta_i\right] dQ \\
&\quad + \frac{\sum_{j=1}^{n}\beta_j^2 - \frac{1}{n}\left(\sum_{j=1}^{n}\beta_j\right)^2}{2\left(1-\tau_D\right)} \frac{P''\left(Q^{*}\right)}{\left[P'\left(Q^{*}\right)\right]^2} dQ \\
&\quad - \frac{\sum_{j=1}^{n}\beta_j^2 - \frac{1}{n}\left(\sum_{j=1}^{n}\beta_j\right)^2}{2\left(1-\tau_D\right)^2 P'\left(Q^{*}\right)} d\tau_D
\end{aligned}
$$

Under two-part tariffs, the change in total social welfare can be represented as

$$
\begin{aligned}
dW &= \left[P\left(Q^{*}\right) - c - \frac{1}{n}\sum_{i=1}^{n}\beta_i\right] dQ \\
&\quad + \frac{\sum_{j=1}^{n}\beta_j^2 - \frac{1}{n}\left(\sum_{j=1}^{n}\beta_j\right)^2}{\left(1-\tau_D\right)} \frac{P''\left(Q^{*}\right)}{\left[P'\left(Q^{*}\right)\right]^2} dQ \\
&\quad - \frac{\sum_{j=1}^{n}\beta_j^2 - \frac{1}{n}\left(\sum_{j=1}^{n}\beta_j\right)^2}{\left(1-\tau_D\right)^2 P'\left(Q^{*}\right)} d\tau_D
\end{aligned}
$$

For all three cases, the change in total social welfare depends on the change in total output level and the change in downstream ad valorem tax. The first terms in each of these three equations represent how the change in total output level affect total social welfare when all the downstream firms are symmetric. This is the part that previous literatures covered. Since the term in the bracket of the first term is always positive, total social welfare moves the same direction as total output level.

The second terms in each of these three equations represent the relocation effect. If the demand function is neither strictly concave nor strictly convex with $P''(Q) = 0$, the second term is zero. In this case, market share of each firm is not changing with a change in total output level. If the demand function is strictly convex with $P''(Q) > 0$, an increase in total output level increases (decreases) the market share of firms with higher (lower) than average efficiency and hence average marginal cost decreases. It benefits total social welfare and the second term is positive. On the other hand, if the demand function is strictly concave with $P''(Q) < 0$, an increase in total output level decreases (increases) the market share of firms with higher (lower) than average efficiency and hence average marginal cost increases. As a result, it hurts total social welfare and the second term is negative. In some extreme cases, the summation of first and second term can be negative such that an increase in total output level actually hurts total social welfare. I provide a simple numerical example under uniform pricing. Consider the demand function to be $P(Q) = 1 - Q^{\alpha}$ with $c = 0.1$, $n = 2$, $\beta_1 = 0.1$,

$\beta_2 = 0.5$, $t_D = \tau_D = t_U = \tau_U = 0$. To simplify my analysis, I assume the third term to be zero by imposing $d\tau_D = 0$. The equilibrium is defined by equation (2.2)

$$\frac{1.2}{[\alpha^2 + 3\alpha + 2]} = Q^\alpha$$

Taking this equilibrium back into the change in total social welfare,

$$\frac{dW}{dQ} = \frac{(\alpha+1)(\alpha+2) - 2}{\frac{5}{3}(\alpha+1)(\alpha+2)} - \frac{(\alpha-1)(\alpha+1)(\alpha+2)}{15 * \alpha}$$

Suppose the demand function is strictly concave by assuming $\alpha = 10$

$$\left.\frac{dW}{dQ}\right|_{\alpha=10} = 0.590909 - 7.92 = -7.32909$$

The second term is negative. The summation of the first and second terms is negative and hence total output level and total social welfare moves to the opposite direction. In contrast, suppose the demand function is strictly convex by assuming $\alpha = \frac{1}{10}$

$$\left.\frac{dW}{dQ}\right|_{\alpha=\frac{1}{10}} = 0.08052 + 1.386 = 1.46652$$

The second term is positive. The summation of the first and second term is positive and hence total output level and total social welfare moves to the same direction. Finally, suppose the demand function is neither strictly concave nor strictly convex by assuming $\alpha = 1$

$$\left.\frac{dW}{dQ}\right|_{\alpha=1} = 0.4 - 0 = 0.4$$

The second term is zero. Total output level and total social welfare moves to the same direction.

The third terms in each of these three equations represent how the change in downstream ad valorem tax rate affect total social welfare. As shown in previous sections, downstream ad valorem tax can amplify the relocation effect where the more efficient firms tend to produce more outputs. Hence, an increase in downstream ad valorem tax rate benefits total social welfare by reducing the downstream firms' average marginal cost.

By imposing no change in total social welfare, $dQ \neq 0$ as long as $d\tau_D \neq 0$. If $dQ \neq 0$, the result of tax revenue comparison depends on the shape of the demand function. Thus, a deterministic result cannot be derived with general demand function $P(Q)$.

1.7.2. Corner Solution of Uniform Pricing

From the upstream manufacturer's point of view, it can always obtain higher profit from third-degree price discrimination. If government does not allow the upstream manufacturer to price discriminate, and if the upstream manufacturer has enough market power to choose how many downstream firms and which specific ones to serve with, the upstream manufacturer may not want to cover the entire downstream market. In other words, the upstream manufacturer might only serve some downstream firms but drop off others in order to increase its profit.

It is quite complicated to show this part with our general model and it is obviously out of the range of this paper. But we can provide a simple example to illustrate this case. To simplify our original model, we assume that there are only 2 downstream firms and the demand function is $P(Q) = 1 - Q$.

First considering the case that the upstream manufacturer is charging the same price for each unit of intermediate goods sold to both downstream firms. We have already known the results for the general model. Hence, we can simply take $n = 2$ and $P(Q) = 1 - Q$ into the results. The upstream manufacturer's profit for serving both downstream firms under uniform pricing is

$$\Pi_2 = \frac{(1-\tau_U)(1-\tau_D)}{6}\left[1 - \frac{t_D + \frac{\beta_1+\beta_2}{2}}{(1-\tau_D)} - \frac{c + t_U}{(1-\tau_U)(1-\tau_D)}\right]^2$$

Considering the case that the upstream manufacturer only serve the more efficient downstream firm. We can take $n = 1$ and $P(Q) = 1 - Q$ into the result. The upstream manufacturer's profit then becomes

$$\Pi_1 = \frac{(1-\tau_U)(1-\tau_D)}{8}\left[1 - \frac{t_D+\beta_1}{(1-\tau_D)} - \frac{c+t_U}{(1-\tau_U)(1-\tau_D)}\right]^2$$

The profit from serving only one downstream firm is greater than that from serving both downstream firms if and only if the inequality condition

$$\Pi_1 > \Pi_2$$

is satisfied. The solution of this inequality

$$\begin{cases} 0 < \frac{t_D+\beta_1}{1-\tau_D} < 1 \\ \left(2-\sqrt{3}\right) + \left(\sqrt{3}-1\right)\frac{t_D+\beta_1}{1-\tau_D} < \frac{t_D+\beta_2}{1-\tau_D} < 1 \quad \text{or} \\ 0 < \frac{c+t_U}{(1-\tau_U)(1-\tau_D)} < 1 - \frac{t_D+\beta_2}{1-\tau_D} \end{cases}$$

$$\begin{cases} 0 < \frac{t_D+\beta_1}{1-\tau_D} < 1 \\ \frac{t_D+\beta_1}{1-\tau_D} < \frac{t_D+\beta_2}{1-\tau_D} \leq \left(2-\sqrt{3}\right) + \left(\sqrt{3}-1\right)\frac{t_D+\beta_1}{1-\tau_D} \\ \left(\sqrt{3}+1\right)\frac{t_D+\beta_1}{1-\tau_D} - \left(2+\sqrt{3}\right)\frac{t_D+\beta_2}{1-\tau_D} + 1 < \frac{c+t_U}{(1-\tau_U)(1-\tau_D)} < 1 - \frac{t_D+\beta_2}{1-\tau_D} \end{cases}$$

Hence, if β_1, β_2, and c satisfy either of these two conditions above, then the upstream manufacturer is better off by dropping off the less efficient downstream firm. Otherwise, the upstream manufacturer does not have an incentive to drop the less efficient downstream firm off. However, if parameters and demand function are such that the upstream manufacturer is profitable to drop the less downstream firm, then total social welfare might be even lower than third-degree price discrimination with both downstream firms being served.

1.7.3. Two-Part Tariffs with Only One Downstream Firm

A similar argument also holds in the case of two-part tariffs. There is no double marginalization problem under two-part tariffs because the upstream firm can effectively extract the entire downstream firms' profits as long as it can perfectly identify each downstream firm. Thus, the optimal strategy under two-part tariffs is to serve only one downstream firm and extracts all downstream monopolist's producer surplus. In addition, the upstream manufacturer should only serve the most efficient downstream firm in order to maximize monopoly profit. Therefore, if the upstream manufacturer has enough market power, it will drop off all the downstream firms except the most efficient one.

To consider this case, I suppose $n = 1$ and β_i ($\beta_i < \beta_j \ \forall j$). We take these two conditions back into the original model under two-part tariffs. All the results from the original model still hold. But now, the upstream manufacturer gets strictly higher profit compared to the case that it serves all n firms. This can be shown as the following

$$
\begin{aligned}
& \Pi_1 - \Pi_n \\
= \; & \left[(1-\tau_U)(1-\tau_D)P(Q_1) - (1-\tau_U)t_D - t_U - c\right]Q_1 - (1-\tau_U)Q_1\beta_1 \\
& - \left[\begin{array}{c} \left[(1-\tau_U)(1-\tau_D)P(Q_n) - (1-\tau_U)t_D - t_U - c\right]Q_n \\ -(1-\tau_U)\sum_{i=1}^{n} q_i\beta_i \end{array}\right] \\
> \; & \left[(1-\tau_U)(1-\tau_D)P(Q_1) - (1-\tau_U)t_D - t_U - c - (1-\tau_U)\beta_1\right]Q_1 \\
& - \left[(1-\tau_U)(1-\tau_D)P(Q_n) - (1-\tau_U)t_D - t_U - c - (1-\tau_U)\beta_1\right]Q_n \\
> \; & \left[(1-\tau_U)(1-\tau_D)P(Q_1) - (1-\tau_U)t_D - t_U - c - (1-\tau_U)\beta_1\right]Q_1 \\
& - \left[(1-\tau_U)(1-\tau_D)P(Q_n) - (1-\tau_U)t_D - t_U - c - (1-\tau_U)\beta_1\right]Q_1 \\
> \; & (1-\tau_U)(1-\tau_D)Q_1\left[P(Q_1) - P(Q_n)\right] > 0
\end{aligned}
$$

Similarly, we can show that if the upstream manufacturer can choose the firm it wants to serve, it will always choose the most efficient one because it can lower the marginal cost. Therefore, under two-part tariffs, the upstream firm always prefers to serve only one downstream firm and that downstream firm is always the most efficient one. Obviously, total social welfare under this case is less than when the upstream manufacturer has to cover the entire downstream market. This is because that when the upstream manufacturer only serves one downstream firm, the upstream manufacturer becomes the monopolist in the downstream market.

1.8. Conclusion

I consider a vertical structure model with one upstream manufacturer and n asymmetric downstream firms under three different pricing schemes: uniform pricing, third-degree price discrimination and two-part tariffs. Four taxes are compared including upstream ad valorem tax, upstream unit tax, downstream ad valorem tax and downstream unit tax. I show that keeping total output level at the equilibrium unchanged, downstream ad valorem tax welfare dominates all other taxes under all three cases because it improves efficiency by relocating production toward the more efficient firms. Downstream ad valorem tax always leads to higher total tax revenue than downstream unit tax. Switching from upstream unit tax to upstream ad valorem tax, tax revenue is increasing under uniform pricing and third-degree price discrimination and not changing under two-part tariffs. In short, ad valorem tax is at least as good as unit tax. It is unclear whether downstream ad valorem tax is tax revenue superior to upstream ad valorem tax or the opposite under uniform pricing and third-degree price discrimination. Under two-part tariffs, downstream ad valorem tax always leads to higher tax revenue than upstream ad valorem tax. This paper tries to compare ad valorem and unit taxes in a two-layer industry so as to provide a more complete analysis when the intermediate goods' prices are not pre-determined. It shows that ad valorem tax is preferred in both layers, which is consistent with existing literatures. Hence, this result can be extended to a two-layer industry. This paper has important policy implications. It suggests that policy makers should always use ad valorem tax in industries with similar structure.

Chapter 2

A COMPARISON OF AD VALOREM AND UNIT TAX WITH NEGATIVE EXTERNALITIES IN A TWO-MARKET CASE

2.1. Introduction

In 2011, Denmark is the first country to charge 'fat tax'. Meat, dairy products and cooking oil with more than 2.3% saturated fat are charged \$2.56/kg unit 'fat tax'. Similarly, Kerala, a state of India, has started charging a 14.5% ad valorem 'fat tax' only on junk food since June 2016, while keeping other food 'fat tax' exempt. Recently, two US cities, Berkeley and Philadelphia, passed a 'sugar tax' on sugary drinks with unit tax rates of 1 cent/oz and 1.5 cent/oz respectively. Taxes on non-sugar drink remain unchanged. In these examples, the levying of taxes not only aims to raise revenue for governments, but also to correct inefficiency.

In the present paper, I compare two indirect taxes, ad valorem and unit tax, to answer the following research questions: For a given tax revenue target, which tax, ad valorem or unit tax, is more effective in reducing consumption of harmful products? Which tax leads to higher total social welfare? Which tax leads to lower equilibrium prices? What is the optimal subsidy strategy if tax revenue is used to subsidize the healthier market? Are the results from short-run case different from the long-run case in which free entry and exit is allowed?

Existing literature compared the effect of ad valorem and unit tax in correcting externalities. Based on Delipalla and Keen (1992), Pirttila (2002) compares unit and ad valorem taxes under imperfect competition and when the production creates negative externalities. Social welfare favors ad valorem (unit) taxation if the distortion from negative externalities is smaller (larger) than that from imperfect competition. Instead of using an equal tax yield criterion, Droge and Schroder (2009) compare the unit and ad valorem taxation with respect to equal corrective-effect criterion in a Dixit-Stiglitz setting with consumption negative externalities. They find that unit taxes lead to more firms in the industry, less output per firm, less tax revenue, but higher welfare compared to ad valorem taxes.

These existing papers limit their analysis in one market in which consumers stop buying if the price is high enough. However, this is generally not the case if there exists an alternative choice. This paper is trying to capture a more general scenario where consumers are also optimizing in choosing substitute goods when their original choices are taxed and become more expensive. The purpose of this paper is to study how taxes affect consumption of the negative-externalities-generating goods when the alternative non-externalities-generating goods available.

To be specific, I assume that there are two distinctive markets, located at the two opposite ends of a unit interval. Consumers are uniformly distributed along the line and have to incur increasing transportation costs over distance to visit either market. Hence, the location of a consumer represents her preference toward those two markets. In each market, several firms are monopolistically competing by producing slightly different varieties of the same product. Specifically, I use the Dixit-Stiglitz monopolistic competition model in each market so that consumers value both product variety and quantity consumed. Without loss of generality, I assume that products from one market generate negative externalities, which can be either consumption externalities or production externalities. Ad valorem tax and unit tax can only be levied on products from the externalities-generating market. I consider the short-run case, in which the number of firms in each market is fixed and the firms are earning positive profits, and the long-run case, in which free entry and exit is allowed and the firms are earning zero profits.

I show that, starting from zero tax rates, for an equal tax yield criterion, unit tax leads to less consumption of the negative-externalities-generating goods than ad valorem tax in both the short run and the long run. Ad valorem tax always results in a lower equilibrium price than unit tax. In addition, in the short run, ad valorem tax can lower distortion from imperfect competition but intensify externalities by increasing production. Unit tax has exact the opposite effects. Hence, the choice between unit and ad valorem taxation depends on the relative magnitudes of those two distortions. In the long run, the firms are earning zero profits and switching taxes does not affect the distortion from imperfect competition. Unit tax is preferred because it leads to less total consumption in the negative-externalities-generating market and hence less negative externalities. In addition, if tax revenue collected from the externalities-generating-market is used to subsidize the consumption (or production) in the substitute market, unit subsidy leads to lower total consumption in the externalities-generating market and higher total social welfare than ad valorem subsidy in the short run. In the long run, both subsidies have the same effects. A combination of unit and ad valorem tax cannot be optimal unless the equilibrium is such that a switch between taxes does not affect total distortions.

I also consider a more general setting where positive tax rates already exist in the industry. The results show that, depending on the elasticity of demand function, either tax can be effective in reducing total consumption in negative-externalities-generating market. Since the industry is no longer symmetric with existing positive tax rates, there are three types of distortions: imperfect competition, negative externalities and transportation costs. For a given tax revenue target, the optimal tax is the one that leads to less total distortions.

2.1.1. Literature Review

The comparison between ad valorem tax and unit tax has been talked for a long time. It is well known that ad valorem tax and unit tax are equivalent in a perfectly competitive market. Suits and Musgrave (1955) show that in the monopoly case, ad valorem tax dominates unit tax. Delipalla and Keen (1992) filled the gap between perfectly competitive market and monopoly market by showing that ad valorem tax dominates unit tax in the n identical firms' oligopoly case with homogeneous product.

The extension from a homogeneous good market to a market with product differentiation started from Anderson. Anderson et al (2001a) and Anderson et al (2001b) compared these two taxes in an imperfectly competitive market with product differentiation and show that ad valorem tax is associated with lower consumer prices, lower firms' profits and higher total social welfare. Schroder has a couple of papers discussing this comparison under monopolistic competition using Dixit and Stiglitz type setting with an explicit utility function. Schroder (2004) shows that, while unit tax leads to higher profits and more entry which increases consumers' utility by adding varieties, ad valorem tax reduces the distortion caused by mark-up pricing and displays the lower tax over-shift. The latter effect dominates and therefore the general result that ad valorem tax is welfare superior can extend to the monopolistic competition case.

Both ad valorem and unit tax can be used as Pigouvian tax to correct inefficient market outcomes, which include imperfect competition and negative externalities. Myles (1996) explores the optimal combination of ad valorem and unit taxation in the presence of imperfect competition. His results show how combining ad valorem and unit tax can eliminate welfare loss due to imperfect competition by inducing profit-maximizing firms to charge the appropriate Ramsey price. Dillen (1995) examined the possibility of using tax-subsidy schemes to correct an imperfectly competitive economy in which subsidies are financed by taxes. It also showed that a competitive equilibrium can be achieved by a combination of ad valorem taxes and unit output subsidies. On the other hand, Droge and Schroder (2009) focus on correcting negative externalities. Their paper compares two tax instruments with respect to equal corrective effect in a Dixit-Stiglitz setting with love of variety, entry, exit, and redistribution of tax revenues. Their results show that unit taxes lead to more firms in the industry, less output per firm, less tax revenue, but higher welfare compared to ad valorem taxes. Pirttila (2002) considers both imperfect competition and externalities in one model. Specifically, he examines the choice between unit and ad valorem taxes when the production of a homogeneous good, produced under imperfect competition, creates harmful externalities. It turns out that the optimal tax choice depends on the magnitude of those two effects. If the valuation of distortion arising from the harmful externalities is smaller than that from imperfect competition, ad valorem tax dominates unit tax. Otherwise, unit tax is superior.

2.2. An Example

The purpose of this section is to introduce a simple numerical example that explores the comparison between unit and ad valorem tax. Consider an industry comprising 2 firms, firm X and firm Y. These two firms are located at opposite extremes on a line of unit length, and hence produce differentiated goods. Consumers are uniformly distributed along the line and incur increasing transportation costs over distance to visit firms. Each consumer only visits one firm, either firm X or firm Y. Without loss of generality, I assume that firm X is generating negative externalities when producing goods. All goods produced by firm X is subject to taxation, and taxes are levied through either unit tax t or ad valorem tax τ.

I describe a consumer's utility by $u(z) = z - \frac{1}{2}z^2$, where $z \in \{x, y\}$ is the level of consumption from the visited firm. Inverse demand for a consumer buying from either firm can be represented as

$$p(z) = u'(z) = 1 - z$$

Let $s > 0$ denote consumer transportation cost per unit distance. I assume that consumer's transportation cost is low enough so that the entire line is fully covered. A consumer at a distance of $\delta \in (0, 1)$ from firm X could achieve surplus of $x - \frac{1}{2}x^2 - \delta s$ by purchasing from that firm. The surplus available by purchasing from the rival firm is $y - \frac{1}{2}y^2 - (1 - \delta) s$. Let δ denote the location of the consumer who is indifferent between these two markets. Then δ^* solves $x - \frac{1}{2}x^2 - \delta s = y - \frac{1}{2}y^2 - (1 - \delta) s$, or

$$\delta^* = \frac{1}{2} + \frac{1}{2s}\left[\left(x - \frac{1}{2}x^2\right) - \left(y - \frac{1}{2}y^2\right)\right]$$

In this example, I assume $s = \frac{1}{2}$ so that this condition can be rewrite as

$$\delta^* = \frac{1}{2} + \left(x - \frac{1}{2}x^2\right) - \left(y - \frac{1}{2}y^2\right)$$

All consumers located at a distance of $\delta \leq \delta^*$ prefer to shop with firm X and more distant consumers prefer to shop with firm Y. Thus, aggregate demand facing firm X is

$$q_X = x\delta^* = x\left[\frac{1}{2} + \left(x - \frac{1}{2}x^2\right) - \left(y - \frac{1}{2}y^2\right)\right]$$

Now, consider the firms' problems. Each firm has a constant marginal cost of $c = 0.1$ to produce each unit of product. All products produced by firm X is taxed, and taxes are levied through some combination of ad valorem tax rates τ and unit tax rates t. Firm X and Y are symmetric except that no products in firm Y is taxed. Profit for firm X is

$$\pi_X = [(1-\tau)(1-x) - c - t]\, x\left[\frac{1}{2} + \left(x - \frac{1}{2}x^2\right) - \left(y - \frac{1}{2}y^2\right)\right]$$

The first-order necessary condition for firm X's profit maximization is

$$\begin{aligned} 0 \;=\; & \left[(1-2x) - \frac{c+t}{1-\tau}\right]\left[\frac{1}{2} + \left(x - \frac{1}{2}x^2\right) - \left(y - \frac{1}{2}y^2\right)\right] \\ & + \left[(1-x) - \frac{c+t}{1-\tau}\right] x\,(1-x) \end{aligned}$$

Similarly, Profit for firm Y is

$$\pi_Y = [(1-y) - c]\, y\left[\frac{1}{2} + \left(y - \frac{1}{2}y^2\right) - \left(x - \frac{1}{2}x^2\right)\right]$$

The first-order necessary condition for firm Y's profit maximization is

$$\begin{aligned} 0 \;=\; & [(1-2y) - c]\left[\frac{1}{2} + \left(y - \frac{1}{2}y^2\right) - \left(x - \frac{1}{2}x^2\right)\right] \\ & + [(1-y) - c]\, y\,(1-y) \end{aligned}$$

The equilibrium is determined by both first-order necessary conditions.

Consumption (or production) in market X generates negative externalities $E = -\xi x \delta^*$, where ξ is a multiplier. Total social welfare includes consumer surplus, producer surplus, tax revenue, transportation cost, and externalities. It can be rewritten as

$$\begin{aligned} W = & \left[\left(x - \frac{1}{2}x^2\right) + \left[(1-x) - 0.1\right]x\right]\left[\frac{1}{2} + \left(x - \frac{1}{2}x^2\right) - \left(y - \frac{1}{2}y^2\right)\right] \\ & + \left[\left(y - \frac{1}{2}y^2\right) + \left[(1-y) - 0.1\right]y\right]\left[\frac{1}{2} + \left(y - \frac{1}{2}y^2\right) - \left(x - \frac{1}{2}x^2\right)\right] \\ & -0.5 * \left[\left[\frac{1}{2} + \left(x - \frac{1}{2}x^2\right) - \left(y - \frac{1}{2}y^2\right)\right]^2 + \left(y - \frac{1}{2}y^2\right) - \left(x - \frac{1}{2}x^2\right)\right] \\ & -\xi x\left[\frac{1}{2} + \left(x - \frac{1}{2}x^2\right) - \left(y - \frac{1}{2}y^2\right)\right] \end{aligned}$$

Tax revenue can be written as

$$T = \left[\tau\left(1-x\right) + t\right]x\left[\frac{1}{2} + \left(x - \frac{1}{2}x^2\right) - \left(y - \frac{1}{2}y^2\right)\right]$$

Consider the case that government has a tax revenue target to be reached, I compare unit and ad valorem tax in two scenarios. First, keeping tax revenue target to be reached, which tax is more efficiently in reducing consumption from the negative-externalities-generating firm. Second, keeping tax revenue target to be reached, which tax is total social welfare superior. Assuming the tax revenue target is 0.025, the outcomes are reported in Table 1 below.

The second column reports the results when tax revenue target is reached solely by unit tax. The third column reports the results when tax revenue target is reached solely by ad valorem tax.

Row 2 reports the tax revenue target level, which is 0.025. Row 3 and 4 reports the tax rates that are required to reach that tax revenue target level. Results in row 5 show that unit tax is more effectively in reducing consumption from negative-externalities-generating

	Unit	Ad Valorem
T	0.025	0.025
τ	0	0.203
t	0.111	0
x	0.47709	0.525413
y	0.535666	0.538686
δ	0.471086	0.493789
$CS + PS + T$	0.576848	0.586238
ΔDWL from tax	-0.011626	-0.002236
$Trans$	-0.125418	-0.125019
E	-0.224750ξ	-0.259443ξ
$W\,(\xi = 0.1)$	0.428955	**0.4352747**
$W\,(\xi = 0.5)$	**0.339055**	0.3314975

Table 2.1. Unit vs. Ad Valorem

firm. The reason firm X tend to produce more under ad valorem than unit tax is that firm X has incentive to charge a lower price under ad valorem tax in order to pay less total tax to the government. It has no incentive to do so under unit tax since unit tax affect firm X's maximization problem the same way as marginal cost. Depending on the value of externalities parameter ξ, either tax can be welfare preferred. There are three types of distortions: Imperfect competition, transportation costs and negative externalities. Comparing to unit tax, ad valorem tax is more preferred to reduce the first two distortions. As what I have shown, ad valorem tax has less impact on quantity consumed from firm X, which leads to relatively higher consumer surplus and producer surplus than unit tax. These are summarized in row 8. Row 9 shows the additional deadweight loss resulted from tax. Unit tax leads to more deadweight loss than ad valorem tax since the output level under unit tax is lower. In addition, as shown in row 7, consumers are less likely to switch from firm X to firm Y because firm X's equilibrium price is not increasing too much under ad valorem tax, compared to the case under unit tax. Hence, total transportation cost is lower under ad valorem tax as shown in row 10. However, unit tax is superior to ad valorem tax in terms of reducing total negative externalities because it leads to less consumption from the negative-externalities-generating firm. This result is shown in row 11. Therefore, the optimal choice of taxes depends on the magnitude of these three distortions. In row 12, I assume that the negative externalities parameter $\xi = 0.1$ such that the main distortions of the economy come from imperfect competition and transportation costs. Thus, ad valorem tax should be used. In row 13, I consider the case that negative externalities is the main distortion by assuming $\xi = 0.5$. Then, unit tax is welfare superior. In fact, whichever tax that leads to lower total distortions is the one that maximizes total social welfare.

In next section, I consider a more general model and show that all the results derived here can be extended to the general case.

2.3. Model

Consider an industry comprising 2 markets, market X and market Y. These two markets are located at opposite extremes on a line of unit length, and hence produce differentiated goods. These two markets are differentiated in terms of their spatial proximity to consumers. Consumers are uniformly distributed along the line and incur increasing transportation costs over distance to visit markets. This setup is based on Hotelling's (1929) linear city model.

In each market, there are a number of competing firms, each of which produces a different variety of the same good. I consider the class of preferences analyzed by Dixit and Stiglitz (1977) and subsequently pursued by Hamilton (2008). Specifically, I describe preferences by a utility function $U(z_v) = G(z_v)$, where z is a composite commodity and $v \in \{x, y\}$. $G(z_v)$ is an increasing function with constant elasticity $(1 - \varepsilon) \in (0, 1)$, and the consumption level of the composite commodity z_v is determined by the subutility function $z_v = \int_{i=0}^{\infty} f(v_i)\, di$, where v_i is the amount consumed of variety i and $f(v)$ is a smooth, increasing, and strictly concave function for all $v > 0$.

Inverse demand for variety i for a consumer is

$$p(z_v, v_i) = G'(z_v) f'(v_i).$$

Aggregate demand facing firm i depends on the decision made by consumers at each point on the line regarding where to shop. Let $s > 0$ denote consumer transportation cost per unit distance. I assume that consumer's transportation cost s is low enough so that the entire line is fully covered[1]. A consumer at a distance of $\delta \in (0,1)$ from market X could achieve surplus of $U(z_x) - \delta s$ by purchasing from that market. The surplus available by purchasing from the rival market is $U(z_y) - (1-\delta)s$. Let δ denote the location of the consumer who is indifferent between these two markets. Then δ^* solves $U(z_x) - \delta s = U(z_y) - (1-\delta)s$, or

$$\delta^* = \frac{1}{2} + \frac{1}{2s}\left[U(z_x) - U(z_y)\right]$$

All consumers located at a distance of $\delta \leq \delta^*$ prefer to shop with market X and more distant consumers prefer to shop with market Y. Thus, aggregate demand facing firm i is

$$q_i = x_i\delta^* = x_i\left[\frac{1}{2} + \frac{1}{2s}\left[U(z_x) - U(z_y)\right]\right]$$

Now, consider firm i's problem in market X. Each firm in market X pays a fixed set-up cost, F, and a constant unit cost of c to produce an individual product. All products in market X is subject to taxation, and taxes are levied through some combination of ad valorem tax rates τ and unit tax rates t. Profit for firm i in market X is

$$\pi_i(x_i) = \left[(1-\tau) G'(z_x) f'(x_i) - c - t\right] x_i\delta^* - F$$

Here, I make an assumption that the number of firms in market X, denoted by m, and the number of firms in market Y, denoted by n, are sufficiently large such that no firm can

[1] To make sure the entire line is fully covered, $s < U(z_x^*) + U(z_y^*)$, where z_x^* and z_y^* are the equilibrium values of the composite commodity, has be to satisfied.

affect the market price individually. The first-order necessary condition for firm i's profit maximization is

$$G'(z_x)\left[f''(x_i)x_i + f'(x_i)\right] = \frac{c+t}{1-\tau}$$

For symmetric allocation, the equilibrium price in market X can be represented as

$$p(m,x) = G'(mf(x))\,f'(x)$$

and the first-order necessary condition can be rewritten as

$$G'(mf(x))\left[f''(x)x + f'(x)\right] = \frac{c+t}{1-\tau} \tag{2.1}$$

Similarly, consider firm j's problem in market Y. I assume market X and Y are symmetric except that no products in market Y is taxed. The first-order necessary condition for profit maximization problem in market Y can be written as

$$G'(nf(y))\left[f''(y)y + f'(y)\right] = c \tag{2.2}$$

The equilibrium output of each firm x_i and y_i in market X and Y in the short run are determined by simultaneous equation (2.1) and (2.2). The long run equilibrium is determined by these two equations and two zero profit conditions, which implies equation (2.3) and (2.4)

$$\left[(1-\tau)G'(mf(x))f'(x) - c - t\right]x\left[\frac{1}{2} + \frac{1}{2s}\left[G(mf(x)) - G(nf(y))\right]\right] = F \tag{2.3}$$

$$\left[G'(nf(y))f'(y) - c\right]y\left[\frac{1}{2} + \frac{1}{2s}\left[G(nf(y)) - G(mf(x))\right]\right] = F \tag{2.4}$$

It is well known that the effects of taxes on equilibrium depend on the shape of the demand function, and therefore it is helpful to make some assumptions about function $f(\cdot)$:

Assumption 2.1. $-\frac{zf''(z)}{f'(z)} < 1$

Assumption 2.2. $-\frac{zf'''(z)}{f''(z)} < 2$

Assumption 2.1 makes sure the first-order necessary condition for profit maximization can be satisfied. Assumption 2.2 rules out cases in which second order derivative of firm i's profit function is greater or equal to zero, and hence it ensures profit function to be strictly concave. These two assumptions together guarantee the existence of an interior equilibrium.

Similar to Pirttila (2002) and Droge and Schroder (2009), I assume that the size of externalities is proportional to total output level of a market. Specifically, consumption (or production) in market X generates negative externalities $E = -\xi Q_X$, where Q_X is the total consumption in market X and ξ is a multiplier.

Hamilton (2008) also compares ad valorem and unit tax in a model with a combination of horizontal differentiation and Dixit-Stiglitz's monopolistic competition. There are two main differences between his model and mine. His paper focuses on multi-product transactions. The firms are equally spaced out on Salop (1979)'s circle model and each firm is selling fixed number of multiple products. Entry and exit affect the distance between each firm on the circle. I focus on single product transaction. Both markets are fixed at two extreme points of the linear city. The firms in each market are located at the same extreme point on the hotelling's linear city and each firm is producing a single variety of the same product. Entry and exit do not affect the distance between these two markets but instead change the number of varieties in each market. Second, in his model, all firms are taxed the same way such that consumers do not switch from one firm to another. On the contrary, my model is not necessarily always symmetric because I only consider taxing the externalities-generating market and leave the other market untaxed.

2.4. Main Results

2.4.1. Changes in Tax Rates and Tax Pass-Through

The first result that can be derived from the model is how taxes affect equilibrium value of each variable. I consider an arbitrary tax reform between ad valorem and unit tax instruments that preserves the existing tax yield to see how a switching from unit to ad valorem tax affect the equilibrium level of consumption, price, the number of varieties and the number of consumers in market X.

For any combination of excise tax rates, the total tax yield is $T = (\tau p_X + t)\, m x \delta^*$, where p_X, m, x, and δ^* are at the equilibrium. Differentiating these expressions and evaluating terms at an initial zero tax position[2], a revenue-neutral shift from unit taxes to ad valorem taxes satisfies

$$\left.\frac{dt}{d\tau}\right|_{\tau=t=0} = -G'\left(mf\left(x\right)\right) f'\left(x\right) = -p_X$$

In addition, at an initial zero tax position, the equilibrium at market X and market Y are exactly the same and the industry is symmetric such that $\delta^* = \frac{1}{2}$.

In the short run, the equilibrium in market X is determined by equation (2.1). Using implicit function, chain rule and making difference, the effect of a switch from unit to ad valorem tax on per variety consumption for each consumer can be represented as

$$\frac{dx}{p_X d\tau} - \frac{dx}{dt} = -\frac{\frac{1}{1-\tau}\frac{G'(mf(x))f'(x)-\frac{c+t}{1-\tau}}{[G'(mf(x))f'(x)]^2}x}{-\varepsilon\frac{xf'(x)}{f(x)}\left[\frac{f''(x)x}{f'(x)}+1\right]+\frac{xf''(x)}{f'(x)}\left[\frac{f'''(x)x}{f''(x)}+2\right]} > 0$$

Similarly, the effect of a switch from unit to ad valorem tax on the equilibrium price in market X can be represented as

$$\frac{dp_X}{p_X d\tau_x} - \frac{dp_X}{dt_x} = \frac{\frac{1}{1-\tau}\frac{f''(x)x}{f'(x)}\left[-\varepsilon\frac{xf'(x)}{f(x)}+\frac{xf''(x)}{f'(x)}\right]}{-\varepsilon\frac{xf'(x)}{f(x)}\left[\frac{f''(x)x}{f'(x)}+1\right]+\frac{xf''(x)}{f'(x)}\left[\frac{f'''(x)x}{f''(x)}+2\right]} < 0$$

and the effect of a switch on the number of consumers consuming in market X is

$$\frac{d\delta^*}{p_X d\tau} - \frac{d\delta^*}{dt} = -\frac{\frac{1}{1-\tau}\frac{1}{2s}mx\frac{G'(mf(x))f'(x)-\frac{c+t}{1-\tau}}{G'(mf(x))f'(x)}}{-\varepsilon\frac{xf'(x)}{f(x)}\left[\frac{f''(x)x}{f'(x)}+1\right]+\frac{xf''(x)}{f'(x)}\left[\frac{f'''(x)x}{f''(x)}+2\right]} > 0$$

[2]Evaluating equal-yield tax switch at the initial zero tax position is consistent with the "P-Shift" approach used by Delipalla and Keen (1992) and is also used by Hamilton (2009). It ignores second-order effects on tax revenue that arise from equilibrium responses to marginal changes in taxes. In the discussion section, I provide some basic results about equal-yield tax reform at positive tax revenue levels.

In summary, a switch from unit to ad valorem tax in the short run leads to higher per variety consumption for each consumer, lower equilibrium prices for each variety and more consumers in market X.

In the long run, the number of firms in each market is endogenous determined because of free entry and exit. The equilibrium is determined by equation (2.1), (2.2), (2.3) and (2.4) simultaneously. Combining all four equations and using implicit function and chain rule, the effect of a switch from unit tax to ad valorem tax on per variety consumption for each consumer can be represented as

$$\frac{dx}{p_X d\tau} - \frac{dx}{dt} = \frac{x}{G'\left(mf\left(x\right)\right) f'\left(x\right)\left[2 + \frac{xf'''(x)}{f''(x)}\right]} > 0$$

Similarly, the effect of a switch on the number of firms in market X is denoted as

$$\frac{dm}{p_X d\tau} - \frac{dm}{dt} = -\frac{m\frac{xf'(x)}{f(x)}}{G'\left(mf\left(x\right)\right) f'\left(x\right)\left[2 + \frac{xf'''(x)}{f''(x)}\right]} < 0$$

and the effect of a switch on the equilibrium price in market X is

$$\frac{dp_X}{p_X d\tau_x} - \frac{dp_X}{dt_x} = \frac{\frac{xf''(x)}{f'(x)}}{\left[2 + \frac{xf'''(x)}{f''(x)}\right]} < 0$$

and the effect of a switch on the number of consumers consuming in market X is

$$\frac{d\delta^*}{p_X d\tau} - \frac{d\delta^*}{dt} = 0$$

In summary, a switch from unit to ad valorem tax in the long run leads to higher per variety consumption for each consumer, fewer varieties, and lower equilibrium prices for each variety in market X. The number of consumers consuming in market X keeps unchanged. All these results can be summarized by the following proposition.

Proposition 2.1. *For a given tax revenue target:*

(i) in the short run, it leads to higher per variety consumption for each consumer, lower equilibrium prices for each variety and more consumers in market X;

(ii) in the long run, it leads to higher per variety consumption for each consumer, lower equilibrium prices for each variety, fewer varieties, and no change in the number of consumers in market X.

Proposition 2.1 shows that tax pass-through is always lower under ad valorem tax, which is consistent with previous literature. With unit tax being charged, it works as an additional marginal cost. On the contrary, under ad valorem tax, the amount of money the firms need to pay as tax depends on price and quantity. The firms have incentive to lower prices in order to pay less tax. This difference drives the equilibrium price low under ad valorem tax. In addition, since the equilibrium price is lower under ad valorem tax, consumers tend to consume more quantities for each variety compared to that under unit tax.

In the short run, the number of varieties in both markets are fixed and price is the only factor that affects consumers' market choice decisions. Since the equilibrium price is lower under ad valorem tax, more consumers are willing to purchase from market X. Different from the short run, in the long run, the firms enter and exit both markets freely. Since the equilibrium price is higher under unit tax, each firm's profit margin is higher in market X. A larger profit margin under unit tax than under ad valorem tax attracting more firms to enter market X, which results in more varieties in the equilibrium. Even though the equilibrium price is higher under unit tax, which discourage consumers from purchasing in market X, more varieties encourage consumers to switch from market Y to market X. The number of consumers in market X is determined by these two effects, which have opposite directions. Under the condition that tax revenue is fixed and starting from 0 tax position, these two effects canceled out exactly. A tax switch does not affect consumers' utilities. In turn, consumers' market choice decisions remain unchanged. As a result, switching tax does not change the number of consumers in each market.

2.4.2. Welfare Analysis

Welfare is one of the most important issues in the study of the comparison of ad valorem and unit tax. In homogeneous product oligopoly markets, it is well known that ad valorem taxes welfare dominate unit taxes. The reason is that ad valorem taxes make after-tax residual demand function more elastic, and this narrows equilibrium price-cost margins in the market. In differentiated product environments, Anderson et al (2001a) show that ad valorem taxes have adverse effects on product variety relative to unit taxes when the firms operate under a fixed aggregate output constraint. In my model, total social welfare includes consumer surplus, producer surplus, tax revenue, transportation costs and externalities.

Another important issue this paper tries to understand is which tax is more efficient in reducing total quantity consumed in market X. This is because that in some cases social planner's goal is not to maximize total social welfare but to reduce total consumption of the negative-externalities-generating goods without sacrificing tax revenue. Therefore, in this section, I provide results to answer these two issues.

Total social welfare includes consumer surplus, producer surplus, tax revenue, transportation costs and externalities. In the short run, it can be simplified as

$$\begin{aligned} W &= G\left(mf\left(x\right)\right)\delta^{*} + \left[G'\left(mf\left(x\right)\right)f'\left(x\right) - c\right]mx\delta^{*} - \xi mx\delta^{*} \\ &\quad + G\left(nf\left(y\right)\right)\left(1-\delta^{*}\right) + \left[G'\left(nf\left(y\right)\right)f'\left(y\right) - c\right]ny\left(1-\delta^{*}\right) \\ &\quad - s\int_{0}^{\delta^{*}}\delta d\delta - s\int_{\delta^{*}}^{1}\left(1-\delta\right)d\delta - \left(m+n\right)F \end{aligned}$$

Total quantity consumed in market X in the short run can be written as

$$Q_X = mx\delta^{*}$$

Total social welfare in the long run can be simplified as

$$W = G\left(mf\left(x\right)\right)\delta^{*} - \xi mx\delta^{*} + G\left(nf\left(y\right)\right)\left(1-\delta^{*}\right) + \left(\tau G'\left(mf\left(x\right)\right)f'\left(x\right) + t\right)mx\delta^{*} - s\int_{0}^{\delta^{*}}\delta d\delta - s\int_{\delta^{*}}^{1}\left(1-\delta\right)d\delta$$

Total quantity consumed in market X in the long run can be written as

$$Q_X = mx\delta^{*}$$

Again, I consider an arbitrary tax reform between ad valorem and unit tax instruments that preserves existing tax yield such that the following condition has to hold

$$\left.\frac{dt}{d\tau}\right|_{\tau=t=0} = -G'\left(mf\left(x\right)\right)f'\left(x\right) = -p_X$$

In the short run, the change in total consumption in market X can be decomposed into the change in per variety consumption for each consumer and the change in the number of consumers in market X

$$\frac{dQ_X}{p_X d\tau} - \frac{dQ_X}{dt} = m\delta^{*}\left(\frac{dx}{p_X d\tau} - \frac{dx}{dt}\right) + mx\left(\frac{d\delta^{*}}{p_X d\tau} - \frac{d\delta^{*}}{dt}\right)$$

Taking the equal tax yield condition and the equilibrium values' changes under this condition into this equation, it can be simplified as

$$\frac{dQ_X}{p_X d\tau} - \frac{dQ_X}{dt} = -\frac{mx\frac{G'(mf(x))f'(x)-c}{G'(mf(x))f'(x)}\left[\frac{1}{2}\frac{1}{G'(mf(x))f'(x)} + \frac{1}{2s}mx\right]}{-\varepsilon\frac{xf'(x)}{f(x)}\left[\frac{f''(x)x}{f'(x)} + 1\right] + \frac{xf''(x)}{f'(x)}\left[\frac{f'''(x)x}{f''(x)} + 2\right]} > 0$$

Thus, switching from unit to ad valorem tax leads to higher total consumption in market X since it leads to both higher per variety consumption for each consumer and more number of consumers in market X.

Total social welfare is the summation of consumer surplus, producer surplus, tax revenue, total transportation costs and externalities. With a switch from unit to ad valorem tax while keeping tax yield unchanged, the change in total social welfare can be decomposed as several parts. Consumer surplus in market X is increasing because of higher per variety consumption for each consumer and more consumers in market X. Consumer surplus in market Y is decreasing because of fewer consumers. Producer surplus in market Y is decreasing since the equilibrium price is unchanged while total quantity consumed shrinks. Total transportation costs are supposed to increase but the absolute value stays the same because of symmetry at the initial 0 tax rates position. Total externalities in absolute value is increasing because total consumption in market X is increasing. The only unclear part is producer surplus in market X, which depends on both the equilibrium price, which is decreasing, and total output, which is increasing. These decomposed effects can be summarized as below

$$\begin{aligned}
\frac{dW}{p_X d\tau} - \frac{dW}{dt} \quad = \quad & \underset{(>0)}{\left(\frac{dCS_x}{p_X d\tau} - \frac{dCS_x}{dt}\right)} + \underset{(<0)}{\left(\frac{dCS_y}{p_X d\tau} - \frac{dCS_y}{dt}\right)} \\
& + \underset{(?)}{\left(\frac{dPS_x}{p_X d\tau} - \frac{dPS_x}{dt}\right)} + \underset{(<0)}{\left(\frac{dPS_y}{p_X d\tau} - \frac{dPS_y}{dt}\right)} \\
& + \underset{(=0)}{\left(\frac{dT}{p_X d\tau} - \frac{dT}{dt}\right)} + \underset{(=0)}{\left(\frac{dTRANS}{p_X d\tau} - \frac{dTRANS}{dt}\right)} \\
& - \underset{(>0)}{\left(\frac{dE}{p_X d\tau} - \frac{dE}{dt}\right)}
\end{aligned}$$

Taking all calculations from previous section in, this equation can be rewrite as

$$\frac{dW}{p_X d\tau} - \frac{dW}{dt} = \left\{ \left[1 - \varepsilon \frac{f'(x)\, x}{f(x)} \right] - \xi_x \left[\frac{1}{G'(mf(x))\, f'(x)} + \frac{1}{s} mx \right] \right\}$$
$$\cdot \frac{\frac{1}{2} mx \frac{G'(mf(x))f'(x) - c}{G'(mf(x))f'(x)}}{\varepsilon \frac{xf'(x)}{f(x)} \left[\frac{f''(x)x}{f'(x)} + 1 \right] - \frac{xf''(x)}{f'(x)} \left[\frac{f'''(x)x}{f''(x)} + 2 \right]}$$

Since $\varepsilon \in (0,1)$ and $\frac{f'(x)x}{f(x)} \in (0,1)$ by assumptions, there exist a range of ξ_x such that ad valorem tax leads to higher total social welfare than unit tax when

$$0 \leq \xi_x < M$$

where

$$M = \frac{1 - \varepsilon \frac{xf'(x)}{f(x)}}{\frac{1}{G'(mf(x))f'(x)} + \frac{1}{s} mx}$$

Otherwise, unit tax is welfare superior.

Proposition 2.2. *For a given tax revenue target,*

(i) Unit tax t is more efficient in reducing total consumption of good X than ad valorem tax τ;

(ii) Unit tax t leads to higher total social welfare than ad valorem tax τ when $\xi \in (M, \infty)$;

(iii) Ad valorem tax τ leads to higher total social welfare than unit tax t when $\xi \in [0, M)$;

(iv) These two taxes are indifferent in increasing total social welfare when $\xi = M$.

Total quantity consumed in market X is a product of the number of varieties, per variety consumption for each consumer, and the number of consumers.The number of varieties is fixed in the short run. It has been shown that ad valorem tax leads to more per variety consumption for each consumer because the equilibrium price is lower under ad valorem tax. Lower equilibrium price also attracts more consumers. Therefore, ad valorem tax leads to higher total quantity consumed than unit tax. Total social welfare is affected by two types of distortions here. The first distortion comes from imperfect competition. Comparing to unit tax, ad valorem tax leads to lower equilibrium price and hence less deadweight loss. Thus, ad valorem tax leads to less distortion from imperfect competition than unit tax. The second distortion comes from negative externalities. By assumption, negative externalities are proportional to total quantity consumed in market X. Since unit tax is more effective in reducing total consumption in market X, it leads to less negative externalities compared to ad valorem tax. A tax switch leads to both change in imperfect competition and change in negative externalities. Depending on the magnitude of the changes of these two distortions, either tax can be superior to the other. If externalities in market X is not too large, ad valorem tax leads to higher total social welfare because distortion from imperfect competition is lower under ad valorem tax. However, if externalities in market X is large enough such that it generates more distortions than imperfect competition, unit tax is preferred.

In the long run, the change in total consumption in market X can be decomposed as three parts including the change in per variety consumption for each consumer, the change in the number of consumers in market X and the change in the number of active firms in market X

$$\frac{dQ_X}{p_X d\tau} - \frac{dQ_X}{dt} = m\delta^* \left(\frac{dx}{p_X d\tau} - \frac{dx}{dt} \right) + x\delta^* \left(\frac{dm}{p_X d\tau} - \frac{dm}{dt} \right) + mx \left(\frac{d\delta^*}{p_X d\tau} - \frac{d\delta^*}{dt} \right)$$

I have already shown in previous section that switching from unit to ad valorem tax increases per variety consumption for each consumer, reduces varieties and does not affect the number of consumers in market X. The change in total quantity consumed in market X depends on the combination of these two changes. Taking these conditions into the equation above, I have

$$\frac{dQ_X}{p_X d\tau} - \frac{dQ_X}{dt} = \frac{\frac{1}{2} mx \left[1 - \frac{xf'(x)}{f(x)} \right]}{G'(mf(x)) f'(x) \left[2 + \frac{xf'''(x)}{f''(x)} \right]} > 0$$

Since $f(x)$ is assumed to be strictly concave, $\frac{xf'(x)}{f(x)}$ is always less than 1 for any value of $x > 0$. Thus, the increase in per variety consumption for each consumer outweigh the decrease in varieties. Ad valorem tax is less effective in reducing consumption compared to unit tax.

Total social welfare in the long run is the summation of consumer surplus, tax revenue, total transportation costs and externalities. With a switch from unit to ad valorem tax while keeping tax yield unchanged, the change in total social welfare can be decomposed as several parts. Total transportation cost is not changing since the model is still symmetric as long as the absolute values of taxes are zero. Consumer surplus in market X stays unchanged because the increase in per variety consumption for each consumer is canceled out by the decrease in varieties. Consumer surplus in market Y is not changing either. Total externalities in absolute value is increasing because total consumption in market X is increasing. These analysis can be summarized as below

$$\begin{aligned}\frac{dW}{p_X d\tau} - \frac{dW}{dt} &= \underset{(=0)}{\left(\frac{dCS_x}{p_X d\tau} - \frac{dCS_x}{dt}\right)} + \underset{(=0)}{\left(\frac{dCS_y}{p_X d\tau} - \frac{dCS_y}{dt}\right)} \\ &+ \underset{(=0)}{\left(\frac{dT}{p_X d\tau} - \frac{dT}{dt}\right)} + \underset{(=0)}{\left(\frac{dTRANS}{p_X d\tau} - \frac{dTRANS}{dt}\right)} - \underset{(>0)}{\left(\frac{dE}{p_X d\tau} - \frac{dE}{dt}\right)}\end{aligned}$$

Taking all the calculations from previous section in, this equation can be rewritten as

$$\frac{dW}{p_X d\tau} - \frac{dW}{dt} = -\xi_x \frac{\frac{1}{2}mx\left[1 - \frac{xf'(x)}{f(x)}\right]}{G'\left(mf\left(x\right)\right)f'\left(x\right)\left[2 + \frac{xf'''(x)}{f''(x)}\right]} < 0$$

Result show that switching from unit to ad valorem tax hurts total social welfare by increasing total quantity consumed in market X.

Proposition 2.3. *For a given tax revenue target,*

(i) Unit tax t is more efficient in reducing total consumption of good X than ad valorem tax τ;

(ii) Unit tax t leads to higher total social welfare than ad valorem tax τ.

Total quantity consumed in market X is a product of the number of varieties, per variety consumption for each consumer, and the number of consumers. Switching from unit to ad valorem tax leads to lower equilibrium price, which has two effects. First, it leads to more per variety consumption for each consumer in market X. Second, it attracts less firms activating in the market because of lower profit margin. Hence, the number of varieties is lower under ad valorem tax. It has been shown that these two effects canceled out exactly in consumers' utilities and hence consumers' market choice decisions remain unchanged. However, these two effects cannot be canceled out in terms of total quantity consumed in market X. Since $f(x)$ is assumed to be strictly concave, the marginal utility a consumer can get from consuming the same good is decreasing as she consuming more of that good. The change in x has to be greater than the change in m such that consumer's utility $G(mf(x))$ is unchanged. Thus, even though switching to ad valorem tax leads to fewer varieties, the increase in per variety consumption for each consumer outweigh the decrease in varieties. Hence, Ad valorem tax leads to more total quantity consumed in market X than unit tax. Total social welfare is affected by both imperfect competition and negative externalities. Different from the short run, switching tax does not change distortion from imperfect competition. Producer surplus is not changing because the firms are earning zero profit in the long run. Consumers surplus is not changing either since a switch from unit to ad valorem tax leads to higher per variety consumption and fewer varieties, both of which affect consumers' utility the opposite way and cancel out exactly. Hence, switching from unit to ad valorem tax does not affect distortion from imperfect competition but only reduce distortion from negative externalities. Thus, unit tax leads to higher total social welfare than ad valorem tax in the long run because it leads to less total distortions.

2.4.3. Optimal Tax

Another interesting question need to answer is what the optimal taxation is in this industry. Is it possible that a combination of unit and ad valorem tax maximizes total social welfare? To find the optimal taxation, I use Lagrangian method.

In the short run, government maximizes total social welfare subject to the tax revenue budget constraint $T \geq \overline{R}$ and the non-negativity constraints on the tax rates $t \geq 0$ and

$\tau \geq 0$. The Lagrangian is as follows:

$$
\begin{aligned}
L &= \left[G\left(mf\left(x\right)\right) + \left[G'\left(mf\left(x\right)\right)f'\left(x\right) - c - \xi\right]mx\right]\delta^* \\
&\quad + \left[G\left(nf\left(y\right)\right) + \left[G'\left(nf\left(y\right)\right)f'\left(y\right) - c\right]ny\right]\left(1-\delta^*\right) \\
&\quad - s\int_0^{\delta^*}\delta d\delta - s\int_{\delta^*}^{1}\left(1-\delta\right)d\delta + \mu\left[T - \overline{R}\right] + \zeta_t t + \zeta_\tau \tau
\end{aligned}
$$

where $\mu, \zeta_t, \zeta_\tau \geq 0$ are the Lagrangian multipliers. The first-order conditions with respect to unit and ad valorem tax rates are, respectively:

$$
\begin{aligned}
0 &= \left[G'\left(mf\left(x\right)\right)f'\left(x\right) + G''\left(mf\left(x\right)\right)mf'\left(x\right)f'\left(x\right)x - \xi\right]m\delta^*\frac{dx}{dt} \\
&\quad + \left[\begin{array}{c} G\left(mf\left(x\right)\right) - G\left(nf\left(y\right)\right) + \left[G'\left(mf\left(x\right)\right)f'\left(x\right) - c - \xi\right]mx \\ - \left[G'\left(nf\left(y\right)\right)f'\left(y\right) - c\right]ny \end{array}\right]\frac{d\delta^*}{dt} \\
&\quad - s\left(2\delta^* - 1\right)\frac{d\delta^*}{dt} + \mu\frac{dT}{dt} + \zeta_t
\end{aligned} \tag{2.5}
$$

$$
\begin{aligned}
0 &= \left[G'\left(mf\left(x\right)\right)f'\left(x\right) + G''\left(mf\left(x\right)\right)mf'\left(x\right)f'\left(x\right)x - \xi\right]m\delta^*\frac{dx}{d\tau} \\
&\quad + \left[\begin{array}{c} G\left(mf\left(x\right)\right) - G\left(nf\left(y\right)\right) + \left[G'\left(mf\left(x\right)\right)f'\left(x\right) - c - \xi\right]mx \\ - \left[G'\left(nf\left(y\right)\right)f'\left(y\right) - c\right]ny \end{array}\right]\frac{d\delta^*}{d\tau} \\
&\quad - s\left(2\delta^* - 1\right)\frac{d\delta^*}{d\tau} + \mu\frac{dT}{d\tau} + \zeta_\tau
\end{aligned} \tag{2.6}
$$

First, I show that tax revenue budget constraint has to be binding. I impose that $\phi\frac{dx}{dt} = \frac{dx}{d\tau}$ where $\phi = \frac{c+t}{1-\tau}$. After substituting derivatives and combining equation (2.5) and (2.6), I have the following condition

$$\left[G'\left(mf\left(x\right)\right)f'\left(x\right)-\frac{c+t}{1-\tau}\right]\mu mx\delta^{*}=\frac{c+t}{1-\tau}\zeta_{t}-\zeta_{\tau}$$

To show that the budget constraint has to be binding, I need to show that $\mu > 0$.

Proof: Suppose the opposite that $\mu = 0$ is true, then the left hand side of the equation above is 0, which in turn means the right hand side should also be 0. There are two possibilities:

(1) $\frac{c+t}{1-\tau}\zeta_t = \zeta_\tau$ and $\zeta_t, \zeta_\tau > 0$: If that's the case, then $t = \tau = 0$, which cannot satisfies the tax revenue condition that $T > \overline{R}$. Contradiction.

(2) $\zeta_t = \zeta_\tau = 0$: If that's the case, then we have $t, \tau > 0$. However, if we take $\mu = \zeta_t = \zeta_\tau = 0$ into those two first-order conditions, we have

$$0=\left\{\begin{array}{c}\left[G'\left(mf\left(x\right)\right)f'\left(x\right)+G''\left(mf\left(x\right)\right)mf'\left(x\right)f'\left(x\right)x-\xi\right]m\delta^{*}\\ +\left[\begin{array}{c}G\left(mf\left(x\right)\right)+\left[G'\left(mf\left(x\right)\right)f'\left(x\right)-c-\xi\right]mx\\ -G\left(nf\left(y\right)\right)-\left[G'\left(nf\left(y\right)\right)f'\left(y\right)-c\right]ny\end{array}\right]\frac{1}{2s}G'\left(mf\left(x\right)\right)mf'\left(x\right)\\ -s\left(2\delta^{*}-1\right)\frac{1}{2s}G'\left(mf\left(x\right)\right)mf'\left(x\right)\end{array}\right\}\frac{dx}{dt}$$

$$0=\left\{\begin{array}{c}\left[G'\left(mf\left(x\right)\right)f'\left(x\right)+G''\left(mf\left(x\right)\right)mf'\left(x\right)f'\left(x\right)x-\xi\right]m\delta^{*}\\ +\left[\begin{array}{c}G\left(mf\left(x\right)\right)+\left[G'\left(mf\left(x\right)\right)f'\left(x\right)-c-\xi\right]mx\\ -G\left(nf\left(y\right)\right)-\left[G'\left(nf\left(y\right)\right)f'\left(y\right)-c\right]ny\end{array}\right]\frac{1}{2s}G'\left(mf\left(x\right)\right)mf'\left(x\right)\\ -s\left(2\delta^{*}-1\right)\frac{1}{2s}G'\left(mf\left(x\right)\right)mf'\left(x\right)\end{array}\right\}\frac{dx}{d\tau}$$

To satisfy both equations, we have to have the condition that $\frac{dx}{dt} = \frac{dx}{d\tau} = 0$, which is a contradiction. Therefore, μ has to be greater than 0, which means the tax revenue condition has to be binding. □

Next, I show the optimal tax choice and the optimal tax rate when starting from positive tax rates. Combining equation (2.5) and (2.6), and imposing tax revenue unchanged, the following condition can be derived

$$\zeta_\tau \frac{dT}{dt} - \zeta_t \frac{dT}{d\tau} = \left\{ \begin{array}{c} \left[1 - \varepsilon \frac{f'(x)x}{f(x)}\right] \delta^* - \xi \left[\frac{\delta^*}{G'(mf(x))f'(x)} + mx\frac{1}{2s}\right] \\ + \frac{1}{2s}\left[G\left(mf\left(x\right)\right) + \left[G'\left(mf\left(x\right)\right)f'\left(x\right) - c\right]mx\right] \\ - \frac{1}{2s}\left[G\left(nf\left(y\right)\right) + \left[G'\left(nf\left(y\right)\right)f'\left(y\right) - c\right]ny\right] \\ + \left(\frac{1}{2} - \delta^*\right) \end{array} \right\} \cdot \frac{mx\left[G'\left(mf\left(x\right)\right)f'\left(x\right) - \frac{c+t}{1-\tau}\right]mx\delta^*}{(1-\tau)\left\{-\varepsilon\frac{xf'(x)}{f(x)}\left[\frac{f''(x)x}{f'(x)} + 1\right] + \frac{xf''(x)}{f'(x)}\left[\frac{f'''(x)x}{f''(x)} + 2\right]\right\}}$$

Assuming tax rates are zero and industry is symmetric initially, this condition can be simplified as

$$\begin{aligned} & \left[-\varepsilon\frac{xf'\left(x\right)}{f\left(x\right)}\left(\frac{f''\left(x\right)x}{f'\left(x\right)} + 1\right) + \frac{xf''\left(x\right)}{f'\left(x\right)}\left(\frac{f'''\left(x\right)x}{f''\left(x\right)} + 2\right)\right]\delta^*\left[\zeta_\tau - G'\left(mf\left(x\right)\right)f'\left(x\right)\zeta_t\right] \\ = & \left\{\left[1 - \varepsilon\frac{f'\left(x\right)x}{f\left(x\right)}\right]\delta^* - \xi\left[\frac{\delta^*}{G'\left(mf\left(x\right)\right)f'\left(x\right)} + \frac{1}{2s}mx\right]\right\}\left[G'\left(mf\left(x\right)\right)f'\left(x\right) - \frac{c+t}{1-\tau}\right]mx\delta^* \end{aligned}$$

When $\xi < \frac{\left[1-\varepsilon\frac{f'(x)x}{f(x)}\right]\delta^*}{\left[\frac{\delta^*}{G'(mf(x))f'(x)} + \frac{1}{2s}mx\right]}$, $\zeta_t > 0$ and $t = 0$. Ad valorem tax is the optimal tax type to maximize total social welfare. On the other hand, when $\xi > \frac{\left[1-\varepsilon\frac{f'(x)x}{f(x)}\right]\delta^*}{\left[\frac{\delta^*}{G'(mf(x))f'(x)} + \frac{1}{2s}mx\right]}$, $\zeta_\tau > 0$ and $\tau = 0$. Unit tax is the optimal tax type to maximize total social welfare. These results are consistent with that of the original model. Assuming tax rates are not necessarily zero, then again the results depend on the value of ξ. To simplify the notations, I denote $\overline{\xi}$, at the equilibrium as the followings

$$\bar{\xi} = \frac{\left\{ \begin{array}{c} \left[1 - \varepsilon \frac{f'(x)x}{f(x)}\right] \delta^* + \left(\frac{1}{2} - \delta^*\right) \\ + \frac{1}{2s} \left[G\left(mf\left(x\right)\right) + \left[G'\left(mf\left(x\right)\right) f'\left(x\right) - c\right] mx\right] \\ - \frac{1}{2s} \left[G\left(nf\left(y\right)\right) + \left[G'\left(nf\left(y\right)\right) f'\left(y\right) - c\right] ny\right] \end{array} \right\}}{\left[\frac{\delta^*}{G'(mf(x))f'(x)} + \frac{1}{2s} mx\right]}$$

Proposition 2.4. *The optimal tax rates that maximize total social welfare are*

(i) $t^* = 0$ *and* τ^* *solves* $\tau^* G'\left(mf\left(x\right)\right) f'\left(x\right) mx \left[\frac{1}{2} + \frac{1}{2s}\left[G\left(mf\left(x\right)\right) - G\left(nf\left(y\right)\right)\right]\right] = \overline{R}$ *when the equilibrium satisfies either of these two conditions:* $\xi < \bar{\xi}$ *and* $\frac{dT}{d\tau} > 0$ *or* $\xi > \bar{\xi}$ *and* $\frac{dT}{d\tau} < 0$*;*

(ii) $\tau^* = 0$ *and* t^* *solves* $t^* mx \left[\frac{1}{2} + \frac{1}{2s}\left[G\left(mf\left(x\right)\right) - G\left(nf\left(y\right)\right)\right]\right] = \overline{R}$ *when the equilibrium satisfies either of these two conditions:* $\xi < \bar{\xi}$ *and* $\frac{dT}{dt} < 0$ *or* $\xi > \bar{\xi}$ *and* $\frac{dT}{dt} > 0$*;*

(iii) (τ^*, t^*) *solves* $\left(\tau^* G'\left(mf\left(x\right)\right) f'\left(x\right) + t^*\right) mx \left[\frac{1}{2} + \frac{1}{2s}\left[G\left(mf\left(x\right)\right) - G\left(nf\left(y\right)\right)\right]\right] = \overline{R}$ *only at the knife-edge case where the equilibrium satisfies* $\xi = \bar{\xi}$*.*

Similarly, the Lagrangian problem that minimizing total output level in market X subject to the tax revenue budget constraint $T \geq \overline{R}$ and the non-negativity constraints on tax rates $t \geq 0$ and $\tau \geq 0$ can be represented as

$$L = -mx\delta^* + \mu\left[T - \overline{R}\right] + \zeta_t t + \zeta_\tau \tau$$

The first-order conditions with respect to unit and ad valorem tax rates are, respectively:

$$0 = -m\delta^* \frac{dx}{dt} - mx\frac{d\delta^*}{dt} + \mu\frac{dT}{dt} + \zeta_t \tag{2.7}$$

$$0 = -m\delta^* \frac{dx}{d\tau} - mx\frac{d\delta^*}{d\tau} + \mu\frac{dT}{d\tau} + \zeta_\tau \tag{2.8}$$

Again, tax revenue budget constraint has to be binding. Next, I show the optimal tax choice and the optimal tax rate when starting from positive tax rates. Combining equation (2.7) and (2.8), and imposing tax revenue unchanged, the following condition can be derived

$$\frac{dT}{dt}\zeta_\tau - \frac{dT}{d\tau}\zeta_t = -\frac{\left[\frac{\delta^*}{G'(mf(x))f'(x)} + \frac{1}{2s}mx\right]mx\left[G'\left(mf\left(x\right)\right)f'\left(x\right) - \frac{c+t}{1-\tau}\right]mx\delta^*}{\left(1-\tau\right)\left\{-\varepsilon\frac{xf'(x)}{f(x)}\left[\frac{f''(x)x}{f'(x)}+1\right] + \frac{xf''(x)}{f'(x)}\left[\frac{f'''(x)x}{f''(x)}+2\right]\right\}}$$

The right hand side is always positive.

Proposition 2.5. *The optimal tax rates that minimize total quantity consumed in market X*

(i) are $t^* = 0$ and τ^* solves $\tau^* G'\left(mf\left(x\right)\right)f'\left(x\right)mx\left[\frac{1}{2} + \frac{1}{2s}\left[G\left(mf\left(x\right)\right) - G\left(nf\left(y\right)\right)\right]\right] = \overline{R}$ when the equilibrium satisfies $\frac{dT}{d\tau} < 0$;

(ii) are $\tau^* = 0$ and t^* solves $t^* mx\left[\frac{1}{2} + \frac{1}{2s}\left[G\left(mf\left(x\right)\right) - G\left(nf\left(y\right)\right)\right]\right] = \overline{R}$ when the equilibrium satisfies $\frac{dT}{dt} > 0$;

(iii) cannot be a combination of unit and ad valorem tax.

In the long run, the government maximizes total social welfare subject to the tax revenue budget constraint $T \geq \overline{R}$ and the non-negativity constraints on the tax rates $t \geq 0$ and $\tau \geq 0$. The Lagrangian is as follows:

$$\begin{aligned} L &= \left[G\left(mf\left(x\right)\right) - \xi mx\right]\delta^* + G\left(nf\left(y\right)\right)\left(1-\delta^*\right) \\ &\quad - s\int_0^{\delta^*}\delta d\delta - s\int_{\delta^*}^1\left(1-\delta\right)d\delta + \mu\left[T - \overline{R}\right] + \zeta_t t + \zeta_\tau \tau \end{aligned}$$

where $\mu, \zeta_t, \zeta_\tau \geq 0$ are Lagrangian multipliers. The first-order conditions with respect to unit and ad valorem tax rates are, respectively:

$$\begin{aligned}
0 &= \left[G'\left(mf\left(x\right)\right)\left(f\left(x\right)\frac{dm}{dt}+mf'\left(x\right)\frac{dx}{dt}\right)-\xi\left(\frac{dm}{dt}x+m\frac{dx}{dt}\right)\right]\delta^{*} \\
&+\left[G'\left(nf\left(y\right)\right)\left(f\left(y\right)\frac{dn}{dt}+nf'\left(y\right)\frac{dy}{dt}\right)\right]\left(1-\delta^{*}\right) \\
&+\left[G\left(mf\left(x\right)\right)-G\left(nf\left(y\right)\right)-\xi mx-s\left(2\delta^{*}-1\right)\right]\frac{d\delta^{*}}{dt} \\
&+\mu\frac{dT}{dt}+\zeta_{t} \qquad (2.9)
\end{aligned}$$

$$\begin{aligned}
0 &= \left[G'\left(mf\left(x\right)\right)\left(f\left(x\right)\frac{dm}{d\tau}+mf'\left(x\right)\frac{dx}{d\tau}\right)-\xi\left(\frac{dm}{d\tau}x+m\frac{dx}{d\tau}\right)\right]\delta^{*} \\
&+\left[G'\left(nf\left(y\right)\right)\left(f\left(y\right)\frac{dn}{d\tau}+nf'\left(y\right)\frac{dy}{d\tau}\right)\right]\left(1-\delta^{*}\right) \\
&+\left[G\left(mf\left(x\right)\right)-G\left(nf\left(y\right)\right)-\xi mx-s\left(2\delta^{*}-1\right)\right]\frac{d\delta^{*}}{d\tau} \\
&+\mu\frac{dT}{d\tau}+\zeta_{\tau} \qquad (2.10)
\end{aligned}$$

First, I show that tax revenue budget constraint has to be binding. I impose that $\phi\frac{dx}{dt}=\frac{dx}{d\tau}$ where $\phi=\frac{c+t}{1-\tau}$. After substituting derivatives and combining equation (2.9), (2.10), I have the following condition

$$\left[G'\left(mf\left(x\right)\right)f'\left(x\right)-\frac{c+t}{1-\tau}\right]\mu mx\delta^{*}=\frac{c+t}{1-\tau}\zeta_{t}-\zeta_{\tau}$$

To show that the budget constraint has to be binding, I need to show that $\mu>0$.

Proof: Suppose the opposite that $\mu=0$ is true, then the left hand side of the equation above is 0, which in turn means the right hand side should also be 0. There are two possibilities:

(1) $\frac{c+t}{1-\tau}\zeta_t=\zeta_\tau$ and $\zeta_t,\zeta_\tau>0$: If that's the case, then $t=\tau=0$, which cannot satisfies the tax revenue condition that $T>\overline{R}$. Contradiction.

(2) $\zeta_t = \zeta_\tau = 0$: If that's the case, then we have $t, \tau > 0$. However, if we take $\mu = \zeta_t = \zeta_\tau = 0$ into those two first-order conditions, we have

$$\begin{aligned} 0 &= \left[G'\left(mf\left(x\right)\right)\left(f\left(x\right)\frac{dm}{dt} + mf'\left(x\right)\frac{dx}{dt}\right) - \xi\left(\frac{dm}{dt}x + m\frac{dx}{dt}\right)\right]\delta^* \\ &+ \left[G'\left(nf\left(y\right)\right)\left(f\left(y\right)\frac{dn}{dt} + nf'\left(y\right)\frac{dy}{dt}\right)\right]\left(1-\delta^*\right) \\ &+ \left[G\left(mf\left(x\right)\right) - G\left(nf\left(y\right)\right) - \xi mx\right]\frac{d\delta^*}{dt} \end{aligned}$$

$$\begin{aligned} 0 &= \left[G'\left(mf\left(x\right)\right)\left(f\left(x\right)\frac{dm}{d\tau} + mf'\left(x\right)\frac{dx}{d\tau}\right) - \xi\left(\frac{dm}{d\tau}x + m\frac{dx}{d\tau}\right)\right]\delta^* \\ &+ \left[G'\left(nf\left(y\right)\right)\left(f\left(y\right)\frac{dn}{d\tau} + nf'\left(y\right)\frac{dy}{d\tau}\right)\right]\left(1-\delta^*\right) \\ &+ \left[G\left(mf\left(x\right)\right) - G\left(nf\left(y\right)\right) - \xi mx\right]\frac{d\delta^*}{d\tau} \end{aligned}$$

Since the negative externalities parameter ξ can be any positive value, these two equations do not hold in general. Therefore, μ has to be greater than 0, which means the tax revenue condition has to be binding. □

Next, I show the optimal tax choice and the optimal tax rate when starting from positive tax rates. Combining equation (2.9) and (2.10), and imposing tax revenue unchanged, the following condition can be derived

$$
\begin{aligned}
&\zeta_\tau \frac{dT}{dt} - \zeta_t \frac{dT}{d\tau} \\
= \quad & \frac{\left[\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{1}{2s}\frac{[G'(nf(y))]^2}{G''(nf(y))}\frac{\frac{f'(x)}{xf''(x)}\delta^*}{\frac{f'(y)}{yf''(y)}(1-\delta^*)} - \frac{f'(x)}{xf''(x)}\frac{[G'(mf(x))]^2}{G''(mf(x))}(\delta^*)^2\right]}{(1-\tau)\left[\frac{1}{2s}\frac{[G'(mf(x))]^2}{G''(mf(x))} - \frac{f'(x)}{xf''(x)}\delta^* + \frac{1}{2s}\frac{[G'(nf(y))]^2}{G''(nf(y))}\frac{\frac{f'(x)}{xf''(x)}\delta^*}{\frac{f'(y)}{yf''(y)}(1-\delta^*)}\right]}\left[\frac{dT}{G'(mf(x))f'(x)d\tau} - \frac{dT}{dt}\right] \\
& -\xi\delta^*\left\{\begin{array}{c}\dfrac{\left[\frac{\frac{1}{2s}\frac{[G'(nf(y))]^2}{G''(nf(y))}}{\frac{f'(y)}{yf''(y)}(1-\delta^*)}-1\right]\frac{\frac{f'(x)}{xf''(x)}\frac{xf'(x)}{f(x)}\delta^*}{G'(mf(x))f'(x)}+mx\frac{1}{2s}\frac{f'(x)}{xf''(x)}\left[\frac{\left(1-\frac{xf'(x)}{f(x)}\right)\left(1+\frac{xf''(x)}{f'(x)}\right)}{\left(2+\frac{xf'''(x)}{f''(x)}\right)}-1\right]}{\frac{G''(mf(x))}{[G'(mf(x))]^2}(1-\tau)\left[\frac{1}{2s}\frac{[G'(mf(x))]^2}{G''(mf(x))}-\frac{f'(x)}{xf''(x)}\delta^*+\frac{1}{2s}\frac{[G'(nf(y))]^2}{G''(nf(y))}\frac{\frac{f'(x)}{xf''(x)}\delta^*}{\frac{f'(y)}{yf''(y)}(1-\delta^*)}\right]}\left[\frac{dT}{G'(mf(x))f'(x)d\tau}-\frac{dT}{dt}\right] \\ -\dfrac{mx\left(1-\frac{xf'(x)}{f(x)}\right)}{(1-\tau)\left(2+\frac{xf'''(x)}{f''(x)}\right)}\dfrac{dT}{G'(mf(x))f'(x)d\tau}\end{array}\right\}
\end{aligned}
$$

Assuming tax rates are zero and industry is symmetric initially, this condition can be simplified as

$$\zeta_\tau - G'(mf(x))f'(x)\zeta_t = \xi\frac{\left(1-\frac{xf'(x)}{f(x)}\right)mx\delta^*}{(1-\tau)\left[2+\frac{xf'''(x)}{f''(x)}\right]}$$

The right hand side is positive and therefore $\zeta_\tau > 0$ and $\tau = 0$. Unit tax is the optimal tax type to maximize total social welfare. These results are consistent with that of the original model. Assuming tax rates are not necessarily zero, then again the results depend on the value of ξ. To simplify, I denote the right hand side of the equation as RHS.

Proposition 2.6. *The optimal tax rates that maximize total social welfare are*

(i) $t^* = 0$ *and* τ^* *solves* $\tau^* G'(mf(x))f'(x)mx\left[\frac{1}{2}+\frac{1}{2s}[G(mf(x))-G(nf(y))]\right] = \overline{R}$ *when the equilibrium satisfies either of these two conditions:* $RHS > 0$ *and* $\frac{dT}{d\tau} < 0$ *or* $RHS < 0$ *and* $0 < \frac{dT}{d\tau}$*;*

(ii) $\tau^* = 0$ *and* t^* *solves* $t^* mx\left[\frac{1}{2}+\frac{1}{2s}[G(mf(x))-G(nf(y))]\right] = \overline{R}$ *when the equilibrium satisfies either of these two conditions:* $RHS > 0$ *and* $\frac{dT}{dt} > 0$ *or* $RHS < 0$ *and* $\frac{dT}{dt} < 0$*;*

(iii) (τ^*, t^*) *solves* $(\tau^* G'(mf(x))f'(x) + t^*)mx\left[\frac{1}{2}+\frac{1}{2s}[G(mf(x))-G(nf(y))]\right] = \overline{R}$

only at the knife-edge case where the equilibrium satisfies $RHS = 0$.

Similarly, the Lagrangian problem that minimizing total output level in market X subject to the tax revenue budget constraint $T \geq \overline{R}$ and the non-negativity constraints on tax rates $t \geq 0$ and $\tau \geq 0$ can be represented as

$$L = -mx\delta^* + \mu \left[T - \overline{R}\right] + \zeta_t t + \zeta_\tau \tau$$

The first-order conditions with respect to unit and ad valorem tax rates are, respectively:

$$0 = -m\delta^* \frac{dx}{dt} - x\delta^* \frac{dm}{dt} - mx\frac{d\delta^*}{dt} + \mu \frac{dT}{dt} + \zeta_t \tag{2.11}$$

$$0 = -m\delta^* \frac{dx}{d\tau} - x\delta^* \frac{dm}{d\tau} - mx\frac{d\delta^*}{d\tau} + \mu \frac{dT}{d\tau} + \zeta_\tau \tag{2.12}$$

Again, tax revenue budget constraint has to be binding. Next, I show the optimal tax choice and the optimal tax rate when starting from positive tax rates. Combining equation (2.11) and (2.12), and imposing tax revenue unchanged, the following condition can be derived

$$\begin{aligned} &\zeta_\tau \frac{dT}{dt} - \zeta_t \frac{dT}{d\tau} \\ = \quad &-\delta^* \left\{ \begin{array}{c} \dfrac{\left[\dfrac{\frac{1}{2s}\frac{\left[G'(nf(y))\right]^2}{G''(nf(y))}}{\frac{f'(y)}{yf''(y)}(1-\delta^*)} - 1\right] \dfrac{\frac{f'(x)}{xf''(x)}\frac{xf'(x)}{f(x)}\delta^*}{G'(mf(x))f'(x)} + mx\frac{1}{2s}\frac{f'(x)}{xf''(x)}\left[\dfrac{\left(1-\frac{xf'(x)}{f(x)}\right)\left(1+\frac{xf''(x)}{f'(x)}\right)}{\left(2+\frac{xf'''(x)}{f''(x)}\right)} - 1\right]}{\frac{G''(mf(x))}{\left[G'(mf(x))\right]^2}(1-\tau)\left[\frac{1}{2s}\frac{\left[G'(mf(x))\right]^2}{G''(mf(x))} - \frac{f'(x)}{xf''(x)}\delta^* + \frac{1}{2s}\frac{\left[G'(nf(y))\right]^2}{G''(nf(y))}\dfrac{\frac{f'(x)}{xf''(x)}\delta^*}{\frac{f'(y)}{yf''(y)}(1-\delta^*)}\right]} \left[\dfrac{dT}{G'(mf(x))f'(x)d\tau} - \dfrac{dT}{dt}\right] \\ - \dfrac{mx\left(1-\frac{xf'(x)}{f(x)}\right)}{(1-\tau)\left(2+\frac{xf'''(x)}{f''(x)}\right)}\dfrac{dT}{G'(mf(x))f'(x)d\tau} \end{array} \right\} \end{aligned}$$

Assuming tax rates are zero and industry is symmetric initially, this condition can be simplified as

$$\zeta_\tau - G'\left(mf\left(x\right)\right)f'\left(x\right)\zeta_t = \frac{\left(1 - \frac{xf'(x)}{f(x)}\right)mx\delta^*}{\left[2 + \frac{xf'''(x)}{f''(x)}\right]}$$

Since the right hand side of the equation is positive, $\zeta_\tau > 0$ and $\tau = 0$. To minimize total output level in market X, the optimal tax is unit tax t and the optimal tax rate t^* satisfies $t^* mx\left[\frac{1}{2} + \frac{1}{2s}\left[G\left(mf\left(x\right)\right) - G\left(nf\left(y\right)\right)\right]\right] = \overline{R}$. This result is consistent with that of the original model. Assuming tax rates are not necessarily zero, I denote the right hand side of this equation as RHS'. Then again the results depends on the sign of $\frac{dT}{dt}$, $\frac{dT}{d\tau}$ and RHS'.

Proposition 2.7. *The optimal tax rates that minimize total quantity consumed in market X are*

(i) $t^* = 0$ and τ^* solves $\tau^* G'\left(mf\left(x\right)\right) f'\left(x\right) mx\left[\frac{1}{2} + \frac{1}{2s}\left[G\left(mf\left(x\right)\right) - G\left(nf\left(y\right)\right)\right]\right] = \overline{R}$ when the equilibrium satisfies either of these two conditions: $RHS' > 0$ and $\frac{dT}{d\tau} < 0$ or $RHS' < 0$ and $0 < \frac{dT}{d\tau}$;

(ii) $\tau^* = 0$ and t^* solves $t^* mx\left[\frac{1}{2} + \frac{1}{2s}\left[G\left(mf\left(x\right)\right) - G\left(nf\left(y\right)\right)\right]\right] = \overline{R}$ when the equilibrium satisfies either of these two conditions: $RHS' > 0$ and $\frac{dT}{dt} > 0$ or $RHS' < 0$ and $\frac{dT}{dt} < 0$;

(iii) (τ^*, t^*) solves $\left(\tau^* G'\left(mf\left(x\right)\right) f'\left(x\right) + t^*\right) mx\left[\frac{1}{2} + \frac{1}{2s}\left[G\left(mf\left(x\right)\right) - G\left(nf\left(y\right)\right)\right]\right] = \overline{R}$ only at the knife-edge case where the equilibrium satisfies $RHS' = 0$.

In this section, I show that, when the industry is not symmetric, there are three types of distortions: imperfect competition, externalities and transportation costs. The tax type that leads to less total distortions is the one maximize total social welfare. No combination of both unit and ad valorem tax maximizes total social welfare unless the equilibrium is such that a switch between taxes does not affect total distortions. In other words, changes in distortions from imperfect competition, externalities and transportation costs are canceled out exactly.

2.5. Discussion

In this section, some additional results are provided. First, I provide an example with an explicit utility function to verify all the results I derived from the general model. Then, I compare the effects of taxes when tax revenue is used to subsidize production in the normal market. Finally, I compare unit and ad valorem tax in a more general scenario in which positive tax rates are being charged and the industry is not symmetric initially.

2.5.1. Example: CES Preference

I consider here the case in which preferences for the composite commodity are characterized by constant elasticity of substitution (CES), $z = \int_{i=0}^{\infty} v_i^{\alpha} di$ with $0 < \alpha < 1$ given. I examine equal-yield tax positions for a subutility function $G(z) = \left(\int_{i=0}^{\infty} v_i^{\alpha} di\right)^{1-\varepsilon}$ with $0 < (1-\varepsilon) < 1$. In the short run, switching from unit to ad valorem tax leads to higher total quantity consumed in market X

$$\frac{dQ_X}{p_X d\tau} - \frac{dQ_X}{dt} = \frac{(1-\alpha)\, m^2 x^2 \left[\frac{1}{2} + \frac{1}{2s}\alpha(1-\varepsilon)(mx^{\alpha})^{1-\varepsilon}\right]}{\alpha^2 (1-\varepsilon)\left[1-\alpha(1-\varepsilon)\right](mx^{\alpha})^{(1-\varepsilon)}} > 0$$

and has ambiguous effect on total social welfare

$$\frac{dW}{p_X d\tau} - \frac{dW}{dt} = \frac{\frac{1}{2}\alpha(1-\varepsilon)(1-\alpha\varepsilon)(mx^{\alpha})^{-\varepsilon} x^{\alpha-1} - \xi\left[\frac{1}{2} + \frac{1}{2s}\alpha(1-\varepsilon)(mx^{\alpha})^{1-\varepsilon}\right]}{\frac{\alpha^2}{(1-\alpha)}(1-\varepsilon)\left[1-\alpha(1-\varepsilon)\right](mx^{\alpha})^{-\varepsilon-1} x^{2\alpha-2}}$$

If ξ is between $\left[0, \frac{\frac{1}{2}\alpha(1-\varepsilon)(1-\alpha\varepsilon)(mx^{\alpha})^{-\varepsilon}x^{\alpha-1}}{\frac{1}{2}+\frac{1}{2s}\alpha(1-\varepsilon)(mx^{\alpha})^{1-\varepsilon}}\right)$, then ad valorem tax leads to higher total social welfare than unit tax. Otherwise, unit tax leads to higher total social welfare.

Switching from unit to ad valorem subsidy leads to more total consumption in market X

$$\frac{dQ_X}{p_y d\tau_y} - \frac{dQ_X}{dt_y} = \frac{\frac{1}{2s} m^2 x^2 (1-\alpha)}{\left[1-\alpha(1-\varepsilon)\right]\alpha} > 0$$

and lower total social welfare

$$\frac{dW}{p_y d\tau_y} - \frac{dW}{dt_y} = -\frac{\left[\frac{1}{2}(1-\varepsilon\alpha) + \frac{1}{2s}\xi m x\right](1-\alpha)\, m x}{\left[1-\alpha(1-\varepsilon)\right]\alpha} < 0$$

for any value of ξ.

In the long run, switching from unit tax to ad valorem tax leads to higher total quantity consumed in market X

$$\frac{dQ_X}{p_X d\tau} - \frac{dQ_X}{dt} = \frac{m(1-\alpha)}{2\alpha^2(1-\varepsilon)(mx^\alpha)^{-\varepsilon}x^{\alpha-2}} > 0$$

and leads to lower total social welfare

$$\frac{dW}{p_X d\tau} - \frac{dW}{dt} = -\xi\frac{m(1-\alpha)}{2\alpha^2(1-\varepsilon)(mx^\alpha)^{-\varepsilon}x^{\alpha-2}} < 0$$

In addition, both subsidies have the same effect on total quantity consumed in market X and total social welfare.

For both the short run and the long run, all the results derived from the original general model can be verified by this specific CES example.

2.5.2. Subsidy

In previous sections, I assumed that tax revenue collected from market X is not returned to this industry. However, in some cases, tax revenue is used to subsidize production (or consumption) in market Y where no negative externalities are generating. In this section, I consider two ways of subsidy: subsidy strategies and direct transfer.I assume that all tax revenue collected from market X is used to subsidize the firms' production in market Y.

2.5.2.1. Subsidy Strategies

I denote unit tax and ad valorem tax in market X as t_x and τ_x and unit subsidy and ad valorem subsidy in market Y as t_y and τ_y. The first-order necessary condition for profit maximization problem in market X and Y can be written as

$$G'(mf(x))[f''(x)x + f'(x)] = \frac{c+t_x}{1-\tau_x} \tag{2.13}$$

$$G'(nf(y))[f''(y)y + f'(y)] = \frac{c-t_y}{1+\tau_y} \tag{2.14}$$

The equilibrium output of each firm x_i and y_i in market X and Y in the short-run allocation are determined simultaneously by equation (2.13) and (2.14). The long run equilibrium is determined by these two equations and the free entry and exit conditions, which requires the firms are earning zero profits. In symmetric case, this implies

$$[(1-\tau_x) G'(mf(x)) f'(x) - c - t_x] x \left[\frac{1}{2} + \frac{1}{2s}[G(mf(x)) - G(nf(y))]\right] = F \qquad (2.15)$$

$$[(1+\tau_y) G'(nf(y)) f'(y) - c + t_y] y \left[\frac{1}{2} + \frac{1}{2s}[G(nf(y)) - G(mf(x))]\right] = F \qquad (2.16)$$

I have already found the better tax strategies in both short-run and long-run cases. The question here is that which subsidy strategy is preferred in both cases if all tax revenues collected from market X are used to subsidize the production process in market Y. Specifically, in the short run, the change in total consumption in market X can be represented as

$$\frac{dQ_X}{p_y d\tau_y} - \frac{dQ_X}{dt_y} = \frac{\frac{1}{2s} m^2 x^2 \frac{[G'(mf(x))f'(x)-c]}{G'(mf(x))f'(x)}}{\varepsilon \frac{xf'(x)}{f(x)}\left[\frac{f''(x)x}{f'(x)} + 1\right] - \frac{xf''(x)}{f'(x)}\left[2 + \frac{f'''(x)x}{f''(x)}\right]} > 0$$

Ad valorem subsidy leads to higher level of total consumption in market X by pushing consumers from market Y to market X.

The best subsidy strategy to improve total social welfare can be found by the equation below

$$\frac{dW}{p_y d\tau_y} - \frac{dW}{dt_y} = -\frac{\left[\frac{1}{2}\left(1 - \varepsilon \frac{xf'(x)}{f(x)}\right) + \frac{1}{2s}\xi m x\right] m x \frac{G'(mf(x))f'(x)-c}{G'(mf(x))f'(x)}}{\varepsilon \frac{xf'(x)}{f(x)}\left[\frac{f''(x)x}{f'(x)} + 1\right] - \frac{xf''(x)}{f'(x)}\left[2 + \frac{f'''(x)x}{f''(x)}\right]} < 0$$

Since $1 - \varepsilon \frac{xf'(x)}{f(x)} > 0$, for any value of externalities parameter ξ, unit subsidy is always welfare superior to ad valorem subsidy. These results can be summarized into corollary 2.1.

Corollary 2.1. *When tax revenue is used to subsidize the production in market Y in the short run,*

(i) unit subsidy is more efficient in reducing total consumption in market X;

(ii) unit subsidy is always welfare superior to ad valorem subsidy.

In contrast to tax comparison, the firms in market Y tend to charge higher price under ad valorem subsidy than under unit subsidy in order to obtain a larger amount of total subsidy. Higher equilibrium price under ad valorem subsidy discourage consumers from consuming in market Y. Thus, more consumers tend to switch to market X, which results in higher total quantity consumed in market X. Therefore, unit subsidy is relatively more effective in reducing total consumption in market X compared to ad valorem subsidy. Total social welfare is affected by two types of distortions here. The first distortion comes from imperfect competition. Comparing to unit subsidy, ad valorem subsidy leads to higher equilibrium price in market Y and hence more deadweight loss. Thus, ad valorem subsidy leads to more distortion from imperfect competition than unit subsidy. The second distortion comes from negative externalities. By assumption, negative externalities are proportional to total quantity consumed in market X. Since ad valorem subsidy is less effective in reducing total consumption in market X, it leads to more negative externalities compared to unit subsidy. These two effects move the same direction. A switch from unit to ad valorem subsidy leads to increase in both imperfect competition and negative externalities. Hence, unit subsidy is welfare superior to ad valorem subsidy.

In the long run, the change in total consumption in market X when switching from unit subsidy to ad valorem subsidy in market Y is

$$\frac{dQ_X}{p_y d\tau_y} - \frac{dQ_X}{dt_y} = 0$$

Both subsidy strategies are equivalently effective in reducing consumption in market X. The change in total social welfare can be represented as

$$\frac{dW}{p_y d\tau_y} - \frac{dW}{dt_y} = 0$$

Both subsidy strategies lead to the same total social welfare level. These results can be summarized into the following corollary.

Corollary 2.2. *When tax revenue is used to subsidize the production in market Y in the long run,*

(i) Both types of subsidies have the same effect on total consumption in market X;

(ii) Both types of subsidies have the same effect on total social welfare.

Total quantity consumed in market X is a product of the number of varieties, per variety consumption for each consumer, and the number of consumers. Subsidy strategies in market Y can only affect total consumptions in market X through changing the number of consumers in market X. Switching from unit to ad valorem subsidy leads to higher equilibrium price in market Y, which has two effects. First it leads to less per variety consumption for each consumer in market Y. Second, it attracts more firms activating in market Y because of higher profit margin. This increases the number of varieties. These two effects canceled out exactly in consumers' utilities and hence consumers' market choice decisions remain unchanged. Since a subsidy switch does not affect any one of those three terms, these two subsidy strategies are indifferent in reducing total quantity consumed in market X.

Total social welfare is affected by both imperfect competition and negative externalities. Different from the short run, switching subsidy does not change distortion from imperfect competition. Producer surplus is not changing because the firms are earning zero profit in the long run. Consumers surplus is not changing either since a switch from unit to ad valorem leads to lower per variety consumption and more varieties, both of which affect consumers' utility the opposite way and cancel out exactly. Hence, switching from unit to ad valorem subsidy does not affect the distortion from imperfect competition. Since I have already shown that switching subsidy does not affect total quantity consumed in market X either, these two subsidies are indifferent in increasing total social welfare.

2.5.2.2. Direct Transfer

Here, I assume that all tax revenue collected from market X is directly transferred to the firms in market Y. Each firm in market Y obtains an equal share of such transfer. Unit and ad valorem taxes are compared under this scenario.

Since each firm in market Y obtains an equal share of a direct transfer that is equal to T, each firm's profit in equilibrium has an additional $\frac{T}{n}$. Equation (2.4) can be rewritten as

$$\left[G'\left(nf\left(y\right)\right)f'\left(y\right)-c\right]y\left[\frac{1}{2}+\frac{1}{2s}\left[G\left(nf\left(y\right)\right)-G\left(mf\left(x\right)\right)\right]\right]+\frac{T}{n}=F$$

In addition, total social welfare in the long run changes to

$$\begin{aligned} W &= G\left(mf\left(x\right)\right)\delta^{*}-\xi mx\delta^{*}+G\left(nf\left(y\right)\right)\left(1-\delta^{*}\right) \\ &\quad -s\int_{0}^{\delta^{*}}\delta d\delta-s\int_{\delta^{*}}^{1}\left(1-\delta\right)d\delta \end{aligned}$$

This is because that tax revenue returns back to the industry and the firms are still earning zero profits.

Again, I consider an equal tax yield switch from unit to ad valorem tax. I show that Proposition 2.2 and Proposition 2.3 still hold.

Corollary 2.3. *When tax revenue are directly transferred to the firms in market Y, Proposition 2.2 and Proposition 2.3 still hold.*

Starting from 0 tax position and symmetric industry, such a direct transfer does not change the equilibrium. An equal tax yield switch does not change the value of tax revenue. Hence, all the results derived previously still hold.

2.5.3. Starting from Positive Tax Rate(s)

Throughout this paper, I assume that initial tax rates are zero and the industry is starting from symmetric so as to simplify the analysis. In this section, I provide some basic results when positive tax rates are being charged and the equilibrium is not symmetric. Again, tax revenue can be represented as

$$T=\left(\tau p_{X}+t\right)Q_{X}$$

where $p_X = G'(mf(x)) f'(x)$ and $Q_X = mx\delta^*$. To keep tax revenue unchanged by assuming $dT = 0$ (and not assuming $\tau = t = 0$)

$$(p_X d\tau + dt) Q_X + \tau Q_X dp_X + (\tau p_X + t) dQ_X = 0$$

This condition can be rewritten as

$$p_X d\tau + dt + \frac{\left[\tau p_X \frac{1}{E_x} + (\tau p_X + t)\right]}{Q_X} dQ_X = 0$$

where $E_x = \frac{dQ_X}{dp_X}\frac{p_X}{Q_X}$ is the elasticity of demand in market X and it is negative. Taking this condition into the change in total quantity consumed in market X

$$\frac{dQ_X}{dt}dt + \frac{dQ_X}{d\tau}d\tau = -2\frac{dQ_X}{dt}p_X \left[1 + \frac{\left[\tau p_X \frac{1}{E_x} + (\tau p_X + t)\right]}{p_X Q_X}\frac{dQ_X}{d\tau}\right] d\tau$$

In the short run, both $\frac{dQ_X}{d\tau}$ and $\frac{dQ_X}{dt}$ are negative since an increase in tax leads to less per variety consumption for each consumer and fewer consumers in market X. As E_x goes to infinity, the sign of $\left[1 + \frac{\left[\tau p_X \frac{1}{E_x} + (\tau p_X + t)\right]}{Q_X}\frac{dQ_X}{p_X d\tau}\right]$ is more likely to be positive and thus unit tax is more likely to be the efficient one to reduce total consumption in market X. This result is consistent with when initial tax rates are assumed to be zero. However, as E_x goes close to 0, the sign of $\left[1 + \frac{\left[\tau p_X \frac{1}{E_x} + (\tau p_X + t)\right]}{Q_X}\frac{dQ_X}{p_X d\tau}\right]$ is more likely to be negative and thus ad valorem tax is more likely to be the efficient one to reduce total consumption in market X. That is, if the absolute value of elasticity of demand is large enough, consumers will purchase no matter how much the firms charge. Hence, even though the firms have incentive to reduce price under ad valorem tax than under unit tax, price will not drop significantly. In addition, since the change in quantity is small, to keep tax revenue unchanged, the firms have to increase prices. Because of these two reasons, prices under ad valorem tax might exceed those under unit tax. Thus, total quantity consumed would be lower under ad valorem tax. This result is different than what I derived when assuming tax rates are zero initially.

In the long run, the sign of $\frac{dQ_X}{dt}$ and $\frac{dQ_X}{d\tau}$ are indeterministic. Different to the short run, a change in tax affects product varieties in addition to per variety consumption under the long run. As a tax increases while keeping everything else unchanged, it has two effects. First, it leads to higher equilibrium price, which in turn leads to less per variety consumption for each consumer. Second, it pushes down the profit margin and drives the firms to exit the market. Consumers tend to substitute variety for quantity consumed and hence per variety consumption for each consumer is increasing. Hence, effects of taxes on per variety consumption for each consumer is ambiguous. In addition, since the number of varieties depends on both tax and per variety consumption, it is unclear whether the number of variety is increasing or decreasing. It turns out that the sign of $\frac{dQ_X}{dt}$ and $\frac{dQ_X}{d\tau}$ are indeterministic.

For total social welfare in the short run, the change can be represented as

$$\begin{aligned}
&\frac{dW}{d\tau}d\tau + \frac{dW}{dt}dt \\
= &\left\{\begin{array}{c}
\delta^* \left[1 - \varepsilon \frac{f'(x)x}{f(x)}\right] - \xi \left[\frac{\delta^*}{G'(mf(x))f'(x)} + \frac{1}{2s}mx\right] \\
+\frac{1}{2s}\left[G\left(mf\left(x\right)\right) + \left[G'\left(mf\left(x\right)\right)f'\left(x\right) - c\right]mx\right] \\
-\frac{1}{2s}\left[G\left(nf\left(y\right)\right) + \left[G'\left(nf\left(y\right)\right)f'\left(y\right) - c\right]ny\right] \\
+\left(\frac{1}{2} - \delta^*\right)
\end{array}\right\} \\
&\cdot \frac{G'\left(mf\left(x\right)\right)f'\left(x\right)\left(\frac{dQ_X}{d\tau}d\tau + \frac{dQ_X}{dt}dt\right)}{\left[\delta^* + \frac{1}{2s}G'\left(mf\left(x\right)\right)mxf'\left(x\right)\right]}
\end{aligned}$$

The first thing to notice is that when both taxes are zero such that market X and market Y are symmetric, the result is exactly the same as that from previous sections. When either tax is not zero, as long as ξ is small enough such that the sign of the bracket $\{\}$ is positive, the tax that leads to higher total quantity consumed in market X is the one which has welfare superiority. Otherwise, if ξ is large enough, the tax that leads to lower total quantity consumed in market X is the one which has welfare superiority.

For total social welfare in the long run, the change can be represented as

$$
\begin{aligned}
& \frac{dW}{d\tau}d\tau + \frac{dW}{dt}dt \\
= \quad & -\xi \frac{\delta^* \frac{1}{1-\tau} xm \left[\frac{xf'(x)}{f(x)} - 1 \right]}{G'\left(mf\left(x\right)\right) f'\left(x\right) \left[2 + \frac{xf'''(x)}{f''(x)} \right]} dt \\
& + \frac{\frac{1}{1-\tau} \left\{ \begin{array}{c} \xi \left[\begin{array}{c} \frac{f'(x)}{xf''(x)} \left(\frac{\delta^*}{G''(mf(x))f(x)} + \frac{1}{2s} m \right) x\delta^* \\ -x \frac{1}{2s} \frac{[G'(nf(y))]^2}{G''(nf(y))} \frac{\frac{f'(x)}{xf''(x)}\delta^*}{\frac{f'(y)}{yf''(y)}(1-\delta^*)} \left[\frac{\delta^*}{G''(mf(x))f(x)} + \frac{1}{2s} m \left(1 - \frac{G'(mf(x))}{G''(mf(x))} \right) \right] \\ + \frac{1}{2s} \frac{G'(mf(x))}{G''(mf(x))} m\delta^* \left(\frac{xf'(x)}{f(x)} - 1 \right) \frac{[f''(x)x + f'(x)]}{[2f''(x) + xf'''(x)]} \end{array} \right] \\ - \left[\frac{f'(x)}{xf''(x)} \left(\delta^*\right)^2 \frac{G'(mf(x))}{G''(mf(x))} - \frac{1}{2s} \frac{[G'(nf(y))]^2}{G''(nf(y))} \frac{\frac{f'(x)}{xf''(x)}\delta^*}{\frac{f'(y)}{yf''(y)}(1-\delta^*)} \frac{G'(mf(x))}{G''(mf(x))} \right] \end{array} \right\}}{\frac{Q_X f'(x)}{\tau Q_X \frac{dp_X}{dt} + (\tau p_X + t) \frac{dQ_X}{dt}} \left[\frac{f'(x)}{xf''(x)} \delta^* - \frac{1}{2s} \frac{[G'(nf(y))]^2}{G''(nf(y))} \frac{\frac{f'(x)}{xf''(x)}\delta^*}{\frac{f'(y)}{yf''(y)}(1-\delta^*)} - \frac{1}{2s} \frac{[G'(mf(x))]^2}{G''(mf(x))} \right]} dt
\end{aligned}
$$

Again, when both taxes are zero and hence market X and market Y are symmetric, the result is exactly the same as that from main results section. However, when either tax is not zero, the effect of switching taxes is indeterministic unless an explicit utility function is provided. Total social welfare is increasing with a tax switch that leads to less total distortions.

2.5.4. Tax Revenue Comparison

Throughout this paper, I assume that social planner has a tax revenue target need to be reached. The comparison between unit and ad valorem tax is based on the condition that fixing tax revenue unchanged. However, in some cases, it is interesting to see which one of these two taxes is generating higher tax revenue when total social welfare is fixed. In this part, I provide some results regarding this question for both the short run and the long run. Again, to simplify the analysis, I assume tax rates are zero initially such that the industry is symmetric.

In the short run, the change in total social welfare can be represented as the following

$$dW = \frac{1}{2} G'\left(mf\left(x\right)\right) f'\left(x\right) m \begin{bmatrix} \left(1 - \varepsilon \frac{f'(x)x}{f(x)}\right) + \left(\frac{f''(x)x}{f'(x)} + 1\right) \\ -\frac{c}{G'(mf(x))f'(x)} - \xi \left(\frac{1}{G'(mf(x))f'(x)} + \frac{mx}{s}\right) \end{bmatrix} dx$$

The change in total social welfare dW only depends on the change in per variety consumption for each consumer dx. To keep total social welfare unchanged, dx has to be equal to zero. Hence, the following condition that keeps $dx = 0$ can be derived

$$dt = -\frac{c+t}{1-\tau} d\tau$$

Taking this condition into the change in tax revenue

$$\frac{dT}{d\tau} = \left[G'\left(mf\left(x\right)\right) f'\left(x\right) - c\right] mx\delta > 0$$

the result shows that ad valorem tax is tax revenue superior to unit tax while keeping total social welfare fixed. This result does not depend on the value of externalities parameter ξ. The intuition here is that, as shown by previous literature, unit tax is equivalent to ad valorem tax times marginal cost under perfect competition. Keeping everything else the same, tax revenue collected from unit tax is a product of ad valorem tax rate, quantity and marginal cost. In contrast, tax revenue collected from ad valorem tax is a product of ad valorem tax rate, quantity and equilibrium price. As long as there exist imperfect competition, the equilibrium price is higher than the marginal cost, and therefore ad valorem tax generates more tax revenue than unit tax.

In the long run, the change in total social welfare can be represented as the follwoing

$$
\begin{aligned}
dW &= \frac{1}{2}G'\left(mf\left(x\right)\right)\left[f\left(x\right)\left(dm+dn\right)+mf'\left(x\right)\left(dx+dy\right)\right] \\
&\quad +\frac{1}{2}\left(\tau G'\left(mf\left(x\right)\right)f'\left(x\right)+t-\xi\right)\left[\begin{array}{c} xdm+mdx \\ +\frac{mxG'(mf(x))}{s}\left[\begin{array}{c} f\left(x\right)\left(dm+dn\right) \\ +mf'\left(x\right)\left(dx+dy\right)\end{array}\right]\end{array}\right] \\
&\quad +\frac{1}{2}\left(\begin{array}{c}\tau G''\left(mf\left(x\right)\right)\left[f\left(x\right)dm+mf'\left(x\right)dx\right]f'\left(x\right) \\ +\tau G'\left(mf\left(x\right)\right)f''\left(x\right)dx+G'\left(mf\left(x\right)\right)f'\left(x\right)d\tau+dt\end{array}\right)mx
\end{aligned}
$$

The change in total social welfare dW depends on the change in per variety consumption for each consumer in both markets dx, dy, the change in the number of varieties in both markets dm, dn and the change in tax rates $dt, d\tau$. To keep total social welfare unchanged, the following condition between unit and ad valorem taxes can be derived

$$
\begin{aligned}
&\left\{\begin{array}{c}
\frac{1}{2}\left[\frac{G'(mf(x))}{G''(mf(x))f'(x)}+mx\right]+\xi\frac{mx\frac{1}{2s}\frac{\left[G'(mf(x))\right]^2}{G''(mf(x))}\frac{1}{G'(mf(x))f'(x)}\delta^*}{\left[\frac{1}{s}\frac{\left[G'(mf(x))\right]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right]} \\
-\xi\frac{\left[m\left(1-\frac{xf'(x)}{f(x)}\right)-\frac{xG'(mf(x))\left[f'''(x)x+2f''(x)\right]}{G''(mf(x))\left[f''(x)x+f'(x)\right]f(x)}\right]\left[\frac{1}{2}-\frac{1}{2s}\frac{\left[G'(mf(x))\right]^2}{G''(mf(x))}\left[\frac{xf''(x)}{f'(x)}-1\right]\right]}{G'(mf(x))\frac{f'(x)}{xf''(x)}\left[\frac{1}{s}\frac{\left[G'(mf(x))\right]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right]\left[2f''(x)+xf'''(x)\right]} \\
-\xi\frac{x}{G''(mf(x))\left[f''(x)x+f'(x)\right]f(x)}
\end{array}\right\}dt \\
=&\ -\left\{\begin{array}{c}
\frac{1}{2}\left[\frac{G'(mf(x))}{G''(mf(x))f'(x)}+mx\right]+\xi\frac{mx\frac{1}{2s}\frac{\left[G'(mf(x))\right]^2}{G''(mf(x))}\frac{1}{G'(mf(x))f'(x)}\delta^*}{\left[\frac{1}{s}\frac{\left[G'(mf(x))\right]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right]} \\
-\xi\frac{\left[m\left(1-\frac{xf'(x)}{f(x)}\right)-\frac{xG'(mf(x))\left[f'''(x)x+2f''(x)\right]}{G''(mf(x))\left[f''(x)x+f'(x)\right]f(x)}\right]\frac{1}{2s}\frac{\left[G'(mf(x))\right]^2}{G''(mf(x))}\left[1+\frac{xf''(x)}{f'(x)}\right]}{G'(mf(x))\frac{f'(x)}{xf''(x)}\left[\frac{1}{s}\frac{\left[G'(mf(x))\right]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right]\left[2f''(x)+xf'''(x)\right]} \\
-\xi\frac{x\left(1+\frac{xf''(x)}{f'(x)}\right)}{G''(mf(x))\left[f''(x)x+f'(x)\right]f(x)}
\end{array}\right\}G'\left(mf\left(x\right)\right)f'\left(x\right)d\tau
\end{aligned}
$$

Taking this condition into the change in tax revenue

$$\frac{dT}{d\tau} = \frac{\xi \frac{mx\left[1-\frac{xf'(x)}{f(x)}\right]}{\left[2+\frac{xf'''(x)}{f''(x)}\right]}}{\left\{\begin{array}{c} \frac{1}{2}\left[\frac{G'(mf(x))}{G''(mf(x))f'(x)} + mx\right] + \xi \frac{mx\frac{1}{2s}\frac{\left[G'(mf(x))\right]^2}{G''(mf(x))}\frac{1}{G'(mf(x))f'(x)}\delta^*}{\left[\frac{1}{s}\frac{\left[G'(mf(x))\right]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right]} \\ -\xi \frac{\left[m\left(1-\frac{xf'(x)}{f(x)}\right)-\frac{xG'(mf(x))\left[f'''(x)x+2f''(x)\right]}{G''(mf(x))\left[f''(x)x+f'(x)\right]f(x)}\right]\left[\frac{1}{2}-\frac{1}{2s}\frac{\left[G'(mf(x))\right]^2}{G''(mf(x))}\left[\frac{xf''(x)}{f'(x)}-1\right]\right]}{G'(mf(x))\frac{f'(x)}{xf''(x)}\left[\frac{1}{s}\frac{\left[G'(mf(x))\right]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right]\left[2f''(x)+xf'''(x)\right]} \\ -\xi \frac{x}{G''(mf(x))\left[f''(x)x+f'(x)\right]f(x)} \end{array}\right\}}$$

the sign is ambiguous. Different from the short run, fixing total social welfare does not fix the equilibrium value of x, y, m and n. The changes in these equilibrium values with a tax switch depends on the shape of the utility function. Hence, the intuition from the short run does not apply here. A deterministic result can be derived only if a specific utility function and the value of externalities parameter ξ are provided.

2.6. Conclusion

In this paper I compare ad valorem and unit tax in a model with two monopolistic competitive markets, one of which is generating negative externalities and being taxed. Under both short-run and long-run cases, ad valorem tax leads to lower equilibrium price than unit tax. Keeping tax revenue unchanged, unit tax is more efficient in reducing total consumption in the externalities-generating market. In the short run, unit tax is welfare superior only when externalities parameter is large enough such that the distortion from externalities is larger than that from imperfect competition. These results are consistent with previous literature. However, in the long run, unit tax always leads to relatively higher total social welfare than ad valorem tax. The major driving force behind this finding is that switching taxes does not change the distortion from imperfect competition when the firms are earning zero profits in the long run. The tax that is more efficient in reducing distortion from externalities is the one leading to welfare superior. When tax revenue is used to subsidize the production in the alternative market, switching from unit to ad valorem subsidy results in higher total quantity consumed in the externalities-generating market and lower total social welfare in the short run and has no effect in the long run. A combination of unit and ad valorem tax cannot be optimal unless a tax switch does not affect total distortions. My findings provide some important policy implications in areas related to health. If social planners only care about distortion from negative externalities and their main purpose is to discourage people from consuming negative-externalities-generating goods like tobacco, alcohol drinks, sugary drinks, and junk food, then unit tax should be used. If they would like to maximize total social welfare, ad valorem tax might be optimal.

Appendix A

Omitted Proofs in Chapter 1

Proof: [Proof of Proposition 1.1] **(Part I: Uniform Pricing)** Suppose that total output level Q^* unchanged by imposing $dQ = 0$, the following condition can be derived

$$dt_U = -\frac{c + t_U}{1 - \tau_U} d\tau_U$$

Taking this condition into the change in total social welfare

$$\begin{aligned} dW_U^{UP} &= -d\left(\sum_{i=1}^{n} \beta_i q_i\right) \\ &= -\sum_{i=1}^{n} \beta_i dq_i \\ &= -\sum_{i=1}^{n} \beta_i d\left(\frac{\beta_i - \frac{1}{n}\sum_{j=1}^{n} \beta_j}{(1-\tau_D) P'(Q)} + \frac{Q}{n}\right) \\ &= 0 \end{aligned}$$

and the change in total tax revenue

$$\begin{aligned} dT_U^{UP} &= w^* Q^* d\tau_U + \tau_U Q^* dw^* + Q^* dt_U \\ &= w^* Q^* d\tau_U + \tau_U Q^* d\left((1-\tau_D)\left(P(Q) + \frac{Q}{n} P'(Q)\right) - t_D - \frac{1}{n}\sum_{j=1}^{n} \beta_j\right) + Q^* dt_U \\ &= w^* Q^* d\tau_U + Q^* dt_U \\ &= \left[w^* - \frac{c + t_U}{1 - \tau_U}\right] Q^* d\tau_U > 0 \end{aligned}$$

(Part II: third degree price discrimination) Suppose that total output level Q^* unchanged by imposing $dQ = 0$, the following condition can be derived

$$dt_U = -\frac{c + t_U}{1 - \tau_U} d\tau_U$$

Taking this condition into the change in total social welfare

$$\begin{aligned} dW_U^{PD} &= -\sum_{i=1}^{n} \beta_i dq_i^* \\ &= -\sum_{i=1}^{n} \beta_i d\left(\frac{\beta_i - \frac{1}{n}\sum_{j=1}^{n}\beta_j}{2P'(Q)(1-\tau_D)} + \frac{Q}{n}\right) \\ &= 0 \end{aligned}$$

and the change in total tax revenue

$$\begin{aligned} dT_U^{PD} &= \sum w_i^* q_i^* d\tau_U + \tau_U d\left(\sum w_i^* q_i^*\right) + Q^* dt_U + t_U dQ^* \\ &= \sum w_i^* q_i^* d\tau_U + Q^* dt_U \\ &= \sum w_i^* \left(\frac{\beta_i - \frac{1}{n}\sum_{j=1}^{n}\beta_j}{2P'(Q)(1-\tau_D)} + \frac{Q}{n}\right) d\tau_U + \sum \left(\frac{\beta_i - \frac{1}{n}\sum_{j=1}^{n}\beta_j}{2P'(Q)(1-\tau_D)} + \frac{Q}{n}\right) dt_U \\ &= \sum w_i^* \left(\frac{\beta_i - \frac{1}{n}\sum_{j=1}^{n}\beta_j}{2P'(Q)(1-\tau_D)} + \frac{Q}{n}\right) d\tau_U - \sum \left(\frac{\beta_i - \frac{1}{n}\sum_{j=1}^{n}\beta_j}{2P'(Q)(1-\tau_D)} + \frac{Q}{n}\right) \frac{(c+t_U)}{(1-\tau_U)} d\tau_U \\ &= \sum \left[\left(w_i^* - \frac{(c+t_U)}{(1-\tau_U)}\right) q_i\right] d\tau_U > 0 \end{aligned}$$

(Part III: A Two-Part Tariff) Suppose that total output level Q^* unchanged by imposing $dQ = 0$, the following condition can be derived

$$dt_U = -\frac{c + t_U}{1 - \tau_U} d\tau_U$$

Taking this condition into the change in total social welfare

$$\begin{aligned} dW_U^{TT} &= -\sum_{i=1}^{n}\beta_i dq_i^* \\ &= -\sum_{i=1}^{n}\beta_i d\left[\frac{t_D+\beta_i+\frac{c+t_U}{1-\tau_U}}{(1-\tau_D)\,P'(Q)}-\frac{P(Q)}{P'(Q)}\right] \\ &= -\frac{\sum_{i=1}^{n}\beta_i}{(1-\tau_D)\,P'(Q)}\left[\frac{dt_U}{1-\tau_U}+\frac{c+t_U}{1-\tau_U}\frac{d\tau_U}{1-\tau_U}\right] \\ &= 0 \end{aligned}$$

and the change in total tax revenue

$$\begin{aligned} dT_U^{TT} &= \sum_{i=1}^{n}(q_iw_i)\,d\tau_U+\tau_U\sum_{i=1}^{n}[d\,(q_iw_i)]+Qdt_U \\ &= \sum_{i=1}^{n}(q_iw_i)\,d\tau_U+\tau_U\sum_{i=1}^{n}[d\,(q_iw_i)]-\frac{c+t_U}{1-\tau_U}\sum_{i=1}^{n}q_id\tau_U \\ &= \sum_{i=1}^{n}(q_iw_i)\,d\tau_U-\sum_{i=1}^{n}\left(q_i\frac{c+t_U}{1-\tau_U}\right)d\tau_U+\tau_U\sum_{i=1}^{n}[d\,(q_iw_i)] \\ &= \sum_{i=1}^{n}\left[q_i\left(w_i-\frac{c+t_U}{1-\tau_U}\right)\right]d\tau_U+\tau_U\sum_{i=1}^{n}[d\,(q_iw_i+F_i)] \\ &= \sum_{i=1}^{n}\left[q_i\left(\frac{c+t_U}{1-\tau_U}-\frac{c+t_U}{1-\tau_U}\right)\right]d\tau_U=0 \end{aligned}$$

□

Proof: [Proof of Proposition 1.2] **(Part I: Uniform Pricing)** Suppose that total output level Q^* unchanged by imposing $dQ=0$, the following condition can be derived

$$dt_D=-\frac{t_D+\frac{1}{n}\sum_{j=1}^{n}\beta_j+\frac{c+t_U}{1-\tau_U}}{(1-\tau_D)}d\tau_D$$

Taking this condition into the change in total social welfare

$$
\begin{aligned}
dW_D^{UP} &= -d\left(\sum_{i=1}^{n}\beta_i q_i\right) \\
&= -\sum_{i=1}^{n}\beta_i dq_i \\
&= -\sum_{i=1}^{n}\beta_i d\left(\frac{\beta_i - \frac{1}{n}\sum_{j=1}^{n}\beta_j}{(1-\tau_D)\,P'(Q)} + \frac{Q}{n}\right) \\
&= -\frac{\sum_{i=1}^{n}\beta_i^2 - \frac{1}{n}\left(\sum_{j=1}^{n}\beta_j\right)^2}{(1-\tau_D)^2\,P'(Q)}d\tau_D > 0
\end{aligned}
$$

and the change in total tax revenue

$$
\begin{aligned}
dT_D^{UP} &= P^*Q^*d\tau_D + Q^*dt_D \\
&= \left[P^* - \frac{(1-\tau_U)\,t_D + (1-\tau_U)\,\frac{1}{n}\sum_{j=1}^{n}\beta_j + c + t_U}{(1-\tau_U)\,(1-\tau_D)}\right]Qd\tau_D > 0
\end{aligned}
$$

(Part II: third degree price discrimination) Suppose that total output level Q^* unchanged by imposing $dQ = 0$, the following condition can be derived

$$
dt_D = -\left[\frac{t_D + \frac{1}{n}\sum_{i=1}^{n}\beta_i + \frac{c+t_U}{1-\tau_U}}{(1-\tau_D)} - \frac{\sum_{i=1}^{n}\beta_i^2 - \frac{1}{n}\left(\sum_{j=1}^{n}\beta_j\right)^2}{2\left[P'(Q)\right]^2(1-\tau_D)^2}P''(Q)\right]d\tau_D
$$

Taking this condition into the change in total social welfare

$$
\begin{aligned}
dW_D^{PD} &= -\sum_{i=1}^{n} \beta_i dq_i^* \\
&= -\sum_{i=1}^{n} \beta_i d\left(\frac{\beta_i - \frac{1}{n}\sum_{j=1}^{n}\beta_j}{2P'(Q)(1-\tau_D)} + \frac{Q}{n}\right) \\
&= -\sum_{i=1}^{n} \beta_i d\left(\frac{\beta_i - \frac{1}{n}\sum_{j=1}^{n}\beta_j}{2P'(Q)(1-\tau_D)}\right) \\
&= -\sum_{i=1}^{n} \beta_i \left(\frac{\beta_i - \frac{1}{n}\sum_{j=1}^{n}\beta_j}{2P'(Q)(1-\tau_D)^2}\right) d\tau_D \\
&= -\frac{\sum_{i=1}^{n}\beta_i^2 - \frac{1}{n}\left(\sum_{j=1}^{n}\beta_j\right)^2}{2P'(Q)(1-\tau_D)^2} d\tau_D > 0
\end{aligned}
$$

and the change in total tax revenue

$$
\begin{aligned}
dT_D^{PD} &= P^* Q^* d\tau_D + Q^* dt_D \\
&= \left[P^* d\tau_D + dt_D\right] Q^* \\
&= \left[P^* - \left(\frac{t_D + \frac{1}{n}\sum_{i=1}^{n}\beta_i + \frac{c+t_U}{1-\tau_U}}{(1-\tau_D)} - \frac{\sum_{i=1}^{n}\beta_i^2 - \frac{1}{n}\left(\sum_{j=1}^{n}\beta_j\right)^2}{2\left[P'(Q)\right]^2(1-\tau_D)^2} P''(Q)\right)\right] Q^* d\tau_D
\end{aligned}
$$

The sign of the bracket [] is positive. The reason is that the term in the bracket () is the average marginal cost. Since I assume that the downstream market is fully covered, final good price has to be greater than the average marginal cost. In the extreme case where the market share of the most inefficient downstream firm goes to 1, the average marginal cost goes to the marginal cost of the most inefficient firm. Even in that case, the sign is still positive because the final good price is still greater than the average marginal cost.

(Part III: A Two-Part Tariff) Suppose that total output level Q^* unchanged by imposing $dQ = 0$, the following condition can be derived

$$dt_D = -\frac{t_D + \frac{1}{n}\sum_{i=1}^{n}\beta_i + \frac{c+t_U}{1-\tau_U}}{(1-\tau_D)}d\tau_D$$

Taking this condition into the change in total social welfare

$$\begin{aligned}
dW_D^{TT} &= -\sum_{i=1}^{n}\beta_i dq_i^* \\
&= -\sum_{i=1}^{n}\beta_i d\left[\frac{t_D + \beta_i + \frac{c+t_U}{1-\tau_U}}{(1-\tau_D)P'(Q)} - \frac{P(Q)}{P'(Q)}\right] \\
&= -\frac{1}{P'(Q)}\sum_{i=1}^{n}\beta_i\left[d\left[\frac{t_D + \beta_i + \frac{c+t_U}{1-\tau_U}}{(1-\tau_D)}\right]\right] \\
&= -\frac{1}{(1-\tau_D)P'(Q)}\sum_{i=1}^{n}\beta_i\left[dt_D + \frac{t_D + \beta_i + \frac{c+t_U}{1-\tau_U}}{(1-\tau_D)}d\tau_D\right] \\
&= -\frac{1}{(1-\tau_D)P'(Q)}\sum_{i=1}^{n}\beta_i\left[dt_D + \frac{t_D + \beta_i + \frac{c+t_U}{1-\tau_U}}{(1-\tau_D)}d\tau_D\right] \\
&= -\frac{1}{(1-\tau_D)^2 P'(Q)}\left[\sum_{i=1}^{n}\beta_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n}\beta_i\right)^2\right]d\tau_D > 0
\end{aligned}$$

and the change in total tax revenue

$$\begin{aligned}
dT_D^{TT} &= P^*Q^*d\tau_D + Q^*dt_D \\
&= \left[P - \frac{\left[\frac{c+t_U}{1-\tau_U} + \frac{1}{n}\sum_{i=1}^{n}\beta_i + t_D\right]}{1-\tau_D}\right]Qd\tau_D > 0
\end{aligned}$$

□

Proof: [Proof of Proposition 1.3] **(Part I: Uniform Pricing)** Suppose that total output level Q^* unchanged by imposing $dQ = 0$, the following condition can be derived

$$d\tau_U = -\frac{(1-\tau_U)}{\frac{c+t_U}{(1-\tau_U)}} \frac{t_D + \frac{1}{n}\sum_{j=1}^{n}\beta_j + \frac{c+t_U}{1-\tau_U}}{(1-\tau_D)} d\tau_D$$

Taking this condition into the change in total social welfare

$$\begin{aligned}
dW_\tau^{UP} &= -d\left(\sum_{i=1}^{n}\beta_i q_i\right) \\
&= -\sum_{i=1}^{n}\beta_i d\left(\frac{\beta_i - \frac{1}{n}\sum_{j=1}^{n}\beta_j}{(1-\tau_D)P'(Q)} + \frac{Q}{n}\right) \\
&= -\frac{\sum_{i=1}^{n}\beta_i^2 - \frac{1}{n}\left(\sum_{j=1}^{n}\beta_j\right)^2}{(1-\tau_D)^2 P'(Q)} d\tau_D > 0
\end{aligned}$$

and the change in total tax revenue

$$\begin{aligned}
dT_\tau^{UP} &= PQd\tau_D + wQd\tau_U - \tau_U\left(P(Q) + \frac{Q}{n}P'(Q)\right)Qd\tau_D \\
&= \left\{P - w\frac{(1-\tau_U)t_D + (1-\tau_U)\frac{1}{n}\sum_{j=1}^{n}\beta_j + c + t_U}{\left[\frac{c+t_U}{1-\tau_U}\right](1-\tau_D)} - \tau_U\left(\frac{w + t_D + \frac{1}{n}\sum_{j=1}^{n}\beta_j}{(1-\tau_D)}\right)\right\}Qd\tau_D \\
&= \left\{P - w\frac{(1-\tau_U)\left[t_D + \frac{1}{n}\sum_{j=1}^{n}\beta_j\right]}{(1-\tau_D)\left[\frac{c+t_U}{1-\tau_U}\right]} - w\frac{1}{(1-\tau_D)} - \tau_U\frac{t_D + \frac{1}{n}\sum_{j=1}^{n}\beta_j}{(1-\tau_D)}\right\}Qd\tau_D \\
&= \frac{1}{(1-\tau_D)}\left\{\left[(1-\tau_D)P - w - t_D - \frac{1}{n}\sum_{j=1}^{n}\beta_j\right] - (1-\tau_U)\left[w - \frac{c+t_U}{1-\tau_U}\right]\frac{t_D + \frac{1}{n}\sum_{j=1}^{n}\beta_j}{\frac{c+t_U}{1-\tau_U}}\right\}Qd\tau_D
\end{aligned}$$

(Part II: third degree price discrimination) Suppose that total output level Q^* unchanged by imposing $dQ = 0$, the following condition can be derived

$$d\tau_U = \frac{(1-\tau_U)^2}{c+t_U}\left[\frac{\sum_{i=1}^n \beta_i^2 - \frac{1}{n}\left(\sum_{j=1}^n \beta_j\right)^2}{2\left[P'(Q)\right]^2(1-\tau_D)^2}P''(Q) - \frac{t_D + \frac{1}{n}\sum_{i=1}^n \beta_i + \frac{c+t_U}{1-\tau_U}}{(1-\tau_D)}\right]d\tau_D$$

Taking this condition into the change in total social welfare

$$\begin{aligned}
dW_\tau^{PD} &= -\sum_{i=1}^n \beta_i dq_i^* \\
&= -\sum_{i=1}^n \beta_i d\left(\frac{\beta_i - \frac{1}{n}\sum_{j=1}^n \beta_j}{2P'(Q)(1-\tau_D)} + \frac{Q}{n}\right) \\
&= -\sum_{i=1}^n \beta_i d\left(\frac{\beta_i - \frac{1}{n}\sum_{j=1}^n \beta_j}{2P'(Q)(1-\tau_D)}\right) \\
&= -\sum_{i=1}^n \beta_i \left(\frac{\beta_i - \frac{1}{n}\sum_{j=1}^n \beta_j}{2P'(Q)(1-\tau_D)^2}\right)d\tau_D \\
&= -\frac{\sum_{i=1}^n \beta_i^2 - \frac{1}{n}\left(\sum_{j=1}^n \beta_j\right)^2}{2P'(Q)(1-\tau_D)^2}d\tau_D > 0
\end{aligned}$$

and the change in total tax revenue

$$
\begin{aligned}
dT_{\tau}^{PD} &= d\left[\sum_{i=1}^{n}\left(\tau_D P+\tau_U w_i\right)q_i\right] \\
&= \sum_{i=1}^{n}\left[q_i\left[Pd\tau_D+w_i d\tau_U+\tau_U dw_i\right]+\left(\tau_D P+\tau_U w_i\right)dq_i\right] \\
&= \sum_{i=1}^{n}\left[q_i\left[\begin{array}{c}\left(P-\frac{t_D+\frac{1}{n}\sum w_i+\frac{1}{n}\sum\beta_i}{1-\tau_D}\right) \\ +\frac{(1-\tau_U)}{(1-\tau_D)}\left[-\left(\frac{w_i-\frac{c+t_U}{1-\tau_U}}{\frac{c+t_U}{1-\tau_U}}\right)\left(t_D+\frac{1}{n}\sum_{i=1}^{n}\beta_i\right)+\frac{1}{n}\sum_{i=1}^{n}w_i-w_i\right] \\ +\frac{(1-\tau_U)}{(1-\tau_D)}\frac{w_i}{\frac{c+t_U}{1-\tau_U}}\left[\frac{\sum_{i=1}^{n}\beta_i^2-\frac{1}{n}\left(\sum_{j=1}^{n}\beta_j\right)^2}{2[P'(Q)]^2(1-\tau_D)}P''(Q)\right]\end{array}\right]\right]d\tau_D \\
&\quad +\sum_{i=1}^{n}\left[\tau_U w_i\left(\frac{\beta_i-\frac{1}{n}\sum_{j=1}^{n}\beta_j}{2P'(Q)\left(1-\tau_D\right)^2}\right)\right]d\tau_D
\end{aligned}
$$

(Part III: A Two-Part Tariff) Suppose that total output level Q^* unchanged by imposing $dQ=0$, the following condition can be derived

$$
d\tau_U=-\frac{\left(1-\tau_U\right)^2}{c+t_U}\frac{\frac{c+t_U}{1-\tau_U}+\frac{1}{n}\sum_{i=1}^{n}\beta_i+t_D}{\left(1-\tau_D\right)}d\tau_D
$$

Taking this condition into the change in total social welfare

$$
\begin{aligned}
dW_{\tau}^{TT} &= -\sum_{i=1}^{n}\beta_i dq_i^* \\
&= -\sum_{i=1}^{n}\beta_i d\left[\frac{t_D+\beta_i+\frac{c+t_U}{1-\tau_U}}{\left(1-\tau_D\right)P'(Q)}-\frac{P(Q)}{P'(Q)}\right] \\
&= -\frac{1}{P'(Q)}\sum_{i=1}^{n}\beta_i\left[d\left[\frac{t_D+\beta_i+\frac{c+t_U}{1-\tau_U}}{\left(1-\tau_D\right)}\right]\right] \\
&= -\frac{1}{P'(Q)\left(1-\tau_D\right)}\sum_{i=1}^{n}\beta_i\left[\frac{t_D+\beta_i+\frac{c+t_U}{1-\tau_U}}{\left(1-\tau_D\right)}d\tau_D+\frac{c+t_U}{\left(1-\tau_U\right)^2}d\tau_U\right] \\
&= -\frac{1}{P'(Q)\left(1-\tau_D\right)^2}\left[\sum_{i=1}^{n}\beta_i^2-\frac{1}{n}\left(\sum_{i=1}^{n}\beta_i\right)^2\right]d\tau_D>0
\end{aligned}
$$

and the change in total tax revenue

$$
\begin{aligned}
dT_{\tau}^{TT} &= PQd\tau_D + \frac{c+t_U}{1-\tau_U}Qd\tau_U + \tau_U\frac{c+t_U}{(1-\tau_U)^2}Qd\tau_U \\
&= PQd\tau_D + \frac{1}{(1-\tau_U)}\frac{c+t_U}{1-\tau_U}Qd\tau_U \\
&= \left[P - \frac{\frac{c+t_U}{1-\tau_U} + \frac{1}{n}\sum_{i=1}^{n}\beta_i + t_D}{(1-\tau_D)}\right]Qd\tau_D > 0
\end{aligned}
$$

□

Appendix B

Omitted Proofs in Chapter 2

Proof: [Proof of Proposition 2.1] In the short run case, the effect of tax rates on the change of equilibrium quantity for each variety can be derived straightly from the implicit function of (2.1).

$$\begin{aligned}\frac{dx}{dt} &= \frac{\frac{1}{1-\tau}}{G''\left(mf\left(x\right)\right)m\left[f'\left(x\right)\right]^{2}\left[\frac{f''(x)x}{f'(x)}+1\right]+G'\left(mf\left(x\right)\right)f''\left(x\right)\left[\frac{f'''(x)x}{f''(x)}+2\right]}<0\\ \frac{dx}{d\tau} &= \frac{\frac{c+t}{1-\tau}\frac{1}{1-\tau}}{G''\left(mf\left(x\right)\right)m\left[f'\left(x\right)\right]^{2}\left[\frac{f''(x)x}{f'(x)}+1\right]+G'\left(mf\left(x\right)\right)f''\left(x\right)\left[\frac{f'''(x)x}{f''(x)}+2\right]}<0\end{aligned}$$

With equal tax yield assumption, the change in consumption per variety for each consumer can be represented as

$$\frac{dx}{p_X d\tau}-\frac{dx}{dt}=-\frac{\frac{G'(mf(x))f'(x)-\frac{c+t}{1-\tau}}{(1-\tau)G'(mf(x))f'(x)}}{G''\left(mf\left(x\right)\right)m\left[f'\left(x\right)\right]^{2}\left[\frac{f''(x)x}{f'(x)}+1\right]+G'\left(mf\left(x\right)\right)f''\left(x\right)\left[\frac{f'''(x)x}{f''(x)}+2\right]}>0$$

Differentiate equation $p\left(m,x\right)=G'\left(mf\left(x\right)\right)f'\left(x\right)$ with respect to x give us

$$\frac{dp_X}{dx}=\left[G''\left(mf\left(x\right)\right)m\left[f'\left(x\right)\right]^{2}+G'\left(mf\left(x\right)\right)f''\left(x\right)\right]$$

By chain rule, the effect of a change of tax rates on the change of equilibrium price of good X can be represented as

$$\frac{dp_X}{dt} = \frac{\frac{1}{1-\tau}\left[G''(mf(x))\,m\,[f'(x)]^2 + G'(mf(x))\,f''(x)\right]}{G''(mf(x))\,m\,[f'(x)]^2\left[\frac{f''(x)x}{f'(x)}+1\right] + G'(mf(x))\,f''(x)\left[\frac{f'''(x)x}{f''(x)}+2\right]} > 0$$

$$\frac{dp_X}{d\tau} = \frac{\frac{c+t}{1-\tau}\frac{1}{1-\tau}\left[G''(mf(x))\,m\,[f'(x)]^2 + G'(mf(x))\,f''(x)\right]}{G''(mf(x))\,m\,[f'(x)]^2\left[\frac{f''(x)x}{f'(x)}+1\right] + G'(mf(x))\,f''(x)\left[\frac{f'''(x)x}{f''(x)}+2\right]} > 0$$

With equal tax yield assumption,

$$\frac{dp_X}{p_X d\tau_x} - \frac{dp_X}{dt_x} = \frac{\frac{f''(x)x}{f'(x)}\left[f'(x)\,G''(mf(x))\,mf'(x) + G'(mf(x))\,f''(x)\right]\frac{1}{1-\tau}}{G''(mf(x))\,m\,[f'(x)]^2\left[\frac{f''(x)x}{f'(x)}+1\right] + G'(mf(x))\,f''(x)\left[\frac{f'''(x)x}{f''(x)}+2\right]} < 0$$

In addition, the effect of a change of tax rates on the change of the total number of consumers visiting market X can be represented as

$$\frac{d\delta^*}{dt} = \frac{\frac{1}{1-\tau}\frac{1}{2s}G'(mf(x))\,mf'(x)}{G''(mf(x))\,m\,[f'(x)]^2\left[\frac{f''(x)x}{f'(x)}+1\right] + G'(mf(x))\,f''(x)\left[\frac{f'''(x)x}{f''(x)}+2\right]} < 0$$

$$\frac{d\delta^*}{d\tau} = \frac{\frac{c+t}{1-\tau}\frac{1}{1-\tau}\frac{1}{2s}G'(mf(x))\,mf'(x)}{G''(mf(x))\,m\,[f'(x)]^2\left[\frac{f''(x)x}{f'(x)}+1\right] + G'(mf(x))\,f''(x)\left[\frac{f'''(x)x}{f''(x)}+2\right]} < 0$$

With equal tax yield assumption

$$\frac{d\delta^*}{p_X d\tau} - \frac{d\delta^*}{dt} = -\frac{\frac{1}{1-\tau}\frac{1}{2s}m\left[G'(mf(x))\,f'(x) - \frac{c+t}{1-\tau}\right]}{G''(mf(x))\,m\,[f'(x)]^2\left[\frac{f''(x)x}{f'(x)}+1\right] + G'(mf(x))\,f''(x)\left[\frac{f'''(x)x}{f''(x)}+2\right]} > 0$$

In the long run case, since both equilibrium quantity of each variety and the number of variety existing in each market is changing with a change in tax rates, the first step is to combine equation (2.1), (2.2), (2.3) and (2.4) to eliminate m and n:

$$(c+t)\frac{-x^2 f''(x)}{[f'(x)+xf''(x)]}\left\{\frac{1}{2}+\frac{1}{2s}\left[\begin{array}{c} G\left([G']^{-1}\left(\frac{c+t}{1-\tau}\frac{1}{[f'(x)+xf''(x)]}\right)\right) \\ -G\left([G']^{-1}\left(c\frac{1}{[f'(y)+yf''(y)]}\right)\right)\end{array}\right]\right\}=F$$

$$c\frac{-y^2 f''(y)}{[f'(y)+yf''(y)]}\left\{\frac{1}{2}+\frac{1}{2s}\left[\begin{array}{c} G\left([G']^{-1}\left(c\frac{1}{[f'(y)+yf''(y)]}\right)\right) \\ -G\left([G']^{-1}\left(\frac{c+t}{1-\tau}\frac{1}{[f'(x)+xf''(x)]}\right)\right)\end{array}\right]\right\}=F$$

Using implicit function on these two equations above, we can derive the effect of taxes on equilibrium consumption per variety for a representative consumer at the initial zero tax rate

$$\begin{cases} \frac{dx}{dt}=\frac{\frac{xf''(x)}{G'(mf(x))f'(x)}\left[\frac{1}{2}-\frac{1}{2s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\left[\frac{xf''(x)}{f'(x)}-1\right]\right]}{\left[\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right][2f''(x)+xf'''(x)]} \\ \frac{dx}{d\tau}=\frac{\frac{xf''(x)}{f'(x)}\frac{1}{2s}\frac{[G'(mf(x))]^2}{G''(mf(x))}[f'(x)+xf''(x)]}{\left[\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right][2f''(x)+xf'''(x)]}\end{cases}$$

With equal tax yield assumption,

$$\frac{dx}{p_X d\tau}-\frac{dx}{dt}=\frac{xf''(x)}{G'(mf(x))f'(x)[2f''(x)+xf'''(x)]}>0$$

Using implicit function to differentiate equation (2.1), the change of equilibrium number of firms in market X with a change of tax rates are

$$\frac{dx}{dt} = \frac{\frac{xf''(x)}{G'(mf(x))f'(x)}\left[\frac{1}{2}-\frac{1}{2s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\left[\frac{xf''(x)}{f'(x)}-1\right]\right]}{\left[\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right][2f''(x)+xf'''(x)]}$$

$$\frac{dx}{d\tau} = \frac{\frac{xf''(x)}{f'(x)}\frac{1}{2s}\frac{[G'(mf(x))]^2}{G''(mf(x))}[f'(x)+xf''(x)]}{\left[\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right][2f''(x)+xf'''(x)]}$$

With equal tax yield assumption,

$$\frac{dm}{p_X d\tau} - \frac{dm}{dt} = -\frac{mf'(x)}{f(x)} \frac{xf''(x)}{G'(mf(x)) f'(x) [2f''(x) + xf'''(x)]} < 0$$

Differentiating equation $p(m, x) = G'(mf(x)) f'(x)$ in the long run and using chain rule

$$\begin{aligned}
\frac{dp_X}{dt} &= G''(mf(x)) f'(x) f(x) \frac{dm}{dt} + [G''(mf(x)) f'(x) mf'(x) + G'(mf(x)) f''(x)] \frac{dx}{dt} \\
\frac{dp_X}{d\tau} &= G''(mf(x)) f'(x) f(x) \frac{dm}{d\tau} + [G''(mf(x)) f'(x) mf'(x) + G'(mf(x)) f''(x)] \frac{dx}{d\tau}
\end{aligned}$$

With equal tax yield assumption,

$$\frac{dp_X}{p_X d\tau_x} - \frac{dp_X}{dt_x} = \frac{f''(x)}{[2f''(x) + xf'''(x)]} \frac{xf''(x)}{f'(x)} < 0$$

Finally, to see the effect of switching taxes on the number of consumers in market X, I have to show first the effect of switching taxes on the consumption per variety for each consumer in market Y and the effect of switching taxes on the number of firms in market Y below. For the consumption per variety for each consumer in market Y

$$\begin{aligned}
\frac{dy}{dt} &= -\frac{1}{G'(mf(x)) f'(x)} \frac{xf''(x)}{f'(x) [2f''(x) + xf'''(x)]} \frac{\frac{1}{2s} \frac{[G'(mf(x))]^2}{G''(mf(x))} [f'(x) + xf''(x)]}{\left[\frac{1}{s} \frac{[G'(mf(x))]^2}{G''(mf(x))} \frac{xf''(x)}{f'(x)} - \frac{1}{2}\right]} \\
\frac{dy}{d\tau} &= -\frac{xf''(x)}{f'(x) [2f''(x) + xf'''(x)]} \frac{\frac{1}{2s} \frac{[G'(mf(x))]^2}{G''(mf(x))} [f'(x) + xf''(x)]}{\left[\frac{1}{s} \frac{[G'(mf(x))]^2}{G''(mf(x))} \frac{xf''(x)}{f'(x)} - \frac{1}{2}\right]}
\end{aligned}$$

With equal tax yield assumption

$$\frac{dy}{p_X d\tau} - \frac{dy}{dt} = 0$$

For the number of firms in market Y

$$\frac{dn}{dt} = \left[mf'(x) \frac{[f'(x) + xf''(x)]}{[2f''(x) + xf'''(x)]} + \frac{G'(mf(x))}{G''(mf(y))} \right] \frac{\frac{1}{2s} \frac{[G'(mf(x))]^2}{G''(mf(x))} \frac{xf''(x)}{f'(x)}}{f(x) \left[\frac{1}{s} \frac{[G'(mf(x))]^2}{G''(mf(x))} \frac{xf''(x)}{f'(x)} - \frac{1}{2} \right]} \frac{1}{G'(mf(x)) f'(x)}$$

$$\frac{dn}{d\tau} = \left[mf'(x) \frac{[f'(x) + xf''(x)]}{[2f''(x) + xf'''(x)]} + \frac{G'(mf(x))}{G''(mf(y))} \right] \frac{\frac{1}{2s} \frac{[G'(mf(x))]^2}{G''(mf(x))} \frac{xf''(x)}{f'(x)}}{f(x) \left[\frac{1}{s} \frac{[G'(mf(x))]^2}{G''(mf(x))} \frac{xf''(x)}{f'(x)} - \frac{1}{2} \right]}$$

With equal tax yield assumption

$$\frac{dn}{p_X d\tau} - \frac{dn}{dt} = 0$$

For the number of consumers in market X

$$\frac{d\delta^*}{dt} = \frac{1}{2s} G'(mf(x)) \left[f(x) \frac{dm}{dt} + mf'(x) \frac{dx}{dt} \right] - \frac{1}{2s} G'(nf(y)) \left[f(y) \frac{dn}{dt} + nf'(y) \frac{dy}{dt} \right]$$

$$\frac{d\delta^*}{d\tau} = \frac{1}{2s} G'(mf(x)) \left[f(x) \frac{dm}{d\tau} + mf'(x) \frac{dx}{d\tau} \right] - \frac{1}{2s} G'(nf(y)) \left[f(y) \frac{dn}{d\tau} + nf'(y) \frac{dy}{d\tau} \right]$$

taking all the conditions I derived above,

$$\frac{d\delta^*}{p_X d\tau} - \frac{d\delta^*}{dt} = 0$$

That is, the number of consumers is not changing. □

Proof: [Proof of Proposition 2.2] Suppose tax revenue is unchanged when switching taxes, total differentiate tax revenue equation

$$\begin{aligned}
dT &= \left[G'(mf(x))\, f'(x)\, d\tau + \tau G''(mf(x))\, f'(x)\, mf'(x)\, dx + \tau G'(mf(x))\, f''(x)\, dx + dt\right] mx\delta^* \\
&\quad + \left[\tau G'(mf(x))\, f'(x) + t\right] m\delta^* dx \\
&\quad + \left[\tau G'(mf(x))\, f'(x) + t\right] mx\frac{1}{2s}\left[G'(mf(x))\, mf'(x)\, dx - G'(nf(y))\, nf'(y)\, dy\right] \\
&= 0
\end{aligned}$$

Imposing the assumption that $t = \tau = 0$, the tax revenue unchanged condition becomes

$$G'(mf(x))\, f'(x)\, d\tau + dt = 0$$

which can be rewritten as

$$\left.\frac{dt}{d\tau}\right|_{\tau=t=0} = -G'(mf(x))\, f'(x)$$

Taking the condition above to the change in total quantity consumed in market X

$$\frac{dQ_X}{p_X d\tau} - \frac{dQ_X}{dt} = -\frac{m\left[G'(mf(x))\, f'(x) - \frac{c+t}{1-\tau}\right]\left[\frac{1}{2}\frac{1}{G'(mf(x))f'(x)} + \frac{1}{2s}mx\right]}{G''(mf(x))\, m\left[f'(x)\right]^2\left[\frac{f''(x)x}{f'(x)} + 1\right] + G'(mf(x))\, f''(x)\left[\frac{f'''(x)x}{f''(x)} + 2\right]} > 0$$

and total social welfare

$$\begin{aligned}
\frac{dW}{p_X d\tau} - \frac{dW}{dt} &= \left\{\begin{array}{c}\frac{1}{2}\left[2G'(mf(x))\, f'(x) + \left[G''(mf(x))\, mf'(x)\, f'(x) + G'(mf(x))\, f''(x)\right]x - c\right] \\ -\frac{1}{2}\xi - \frac{1}{2s}\xi mxG'(mf(x))\, f'(x)\end{array}\right\} \\
&\quad \cdot \frac{\frac{1}{G'(mf(x))f'(x)} m\left[G'(mf(x))\, f'(x) - \frac{c+t}{1-\tau}\right]}{-G''(mf(x))\, m\left[f'(x)\right]^2\left[\frac{f''(x)x}{f'(x)} + 1\right] - G'(mf(x))\, f''(x)\left[\frac{f'''(x)x}{f''(x)} + 2\right]}
\end{aligned}$$

The results are shown. □

Proof: [Proof of Proposition 2.3] Suppose tax revenue is unchanged when switching taxes, total differentiate tax revenue equation $T = [\tau G'(mf(x)) f'(x) + t]\, mx\delta^*$, the following condition can be derived

$$\begin{aligned} dT &= \left[\begin{array}{c} G'(mf(x)) f'(x)\, d\tau + \tau G''(mf(x)) [mf'(x)\, dx + f(x)\, dm] f'(x) \\ +\tau G'(mf(x)) f''(x)\, dx + dt \end{array} \right] mx\delta^* \\ &\quad + [\tau G'(mf(x)) f'(x) + t]\, x\delta^* dm + [\tau G'(mf(x)) f'(x) + t]\, m\delta^* dx \\ &\quad + [\tau G'(mf(x)) f'(x) + t]\, mx d\delta^* \\ &= 0 \end{aligned}$$

Imposing the assumption that $t = \tau = 0$, the tax revenue unchanged condition becomes

$$G'(mf(x)) f'(x)\, d\tau + dt = 0$$

which can be rewritten as

$$\left.\frac{dt}{d\tau}\right|_{\tau=t=0} = -G'(mf(x)) f'(x)$$

The change in total consumption in market X can be represented as

$$\frac{dQ_X}{p_X d\tau} - \frac{dQ_X}{dt} = m\left[1 - \frac{xf'(x)}{f(x)}\right] \frac{1}{2} \frac{xf''(x)}{G'(mf(x)) f'(x) [2f''(x) + xf'''(x)]} > 0$$

For the total social welfare

$$\frac{dW}{p_X d\tau} - \frac{dW}{dt} = -\xi \frac{1}{2} m \left[1 - \frac{xf'(x)}{f(x)}\right] \frac{xf''(x)}{G'(mf(x)) f'(x) [2f''(x) + xf'''(x)]} < 0$$

The results are shown. □

Proof: [Proof of Corollary 2.1] In the short run case, the equilibrium is determined by 2 equations below

$$
\begin{aligned}
G'(mf(x))\left[f'(x)+xf''(x)\right] &= \frac{c+t_x}{1-\tau_x}\\
G'(nf(y))\left[f'(y)+yf''(y)\right] &= \frac{c-t_y}{1+\tau_y}
\end{aligned}
$$

Total differentiate these two equations above and the tax revenue condition

$$
\begin{aligned}
dx &= \frac{\frac{1}{1-\tau_x}dt_x+G'(mf(x))\left[f'(x)+xf''(x)\right]\frac{1}{1-\tau_x}d\tau_x}{\left\{G''(mf(x))\,mf'(x)\left[f'(x)+xf''(x)\right]+G'(mf(x))\left[2f''(x)+xf'''(x)\right]\right\}}\\
dy &= -\frac{\frac{1}{1+\tau_y}dt_y+G'(nf(y))\left[f'(y)+yf''(y)\right]\frac{1}{1+\tau_y}d\tau_y}{\left\{G''(nf(y))\,nf'(y)\left[f'(y)+yf''(y)\right]+G'(nf(y))\left[2f''(y)+yf'''(y)\right]\right\}}\\
0 &= \left[G'(mf(x))\,f'(x)\,d\tau_x+dt_x\right]-\left[G'(nf(y))\,f'(y)\,d\tau_y+dt_y\right]
\end{aligned}
$$

Combining all three differentiated equations and taking these conditions into the change in total quantity consumed in market X

$$
\frac{dQ_X}{p_y d\tau_y}-\frac{dQ_X}{dt_y}=\frac{\frac{1}{2s}m^2x\left[G'(mf(x))\,f'(x)-c\right]}{-G''(mf(x))\,mf'(x)\left[f''(x)\,x+f'(x)\right]-G'(mf(x))\left[2f''(x)+f'''(x)\,x\right]}>0
$$

and taking these conditions into the change in total social welfare

$$
\frac{dW}{p_y d\tau_y}-\frac{dW}{dt_y}=-\frac{\left[\frac{1}{2}\left(1-\varepsilon\frac{xf'(x)}{f(x)}\right)+\frac{1}{2s}\xi mx\right]m\left[G'(mf(x))\,f'(x)-c\right]}{-G''(mf(x))\,mf'(x)\left[f''(x)\,x+f'(x)\right]-G'(mf(x))\left[2f''(x)+f'''(x)\,x\right]}<0
$$

□

Proof: [Proof of Corollary 2.2] In the long run case, the equilibrium is determined by 4 equations below

$$
\begin{aligned}
G'(mf(x))\left[f'(x)+xf''(x)\right] &= \frac{c+t_x}{1-\tau_x} \\
\left[(1-\tau_x)G'(mf(x))f'(x)-c-t_x\right]x\left[\frac{1}{2}+\frac{1}{2s}\left[G(mf(x))-G(nf(y))\right]\right] &= F \\
G'(nf(y))\left[f'(y)+yf''(y)\right] &= \frac{c-t_y}{1+\tau_y} \\
\left[(1+\tau_y)G'(nf(y))f'(y)-c+t_y\right]y\left[\frac{1}{2}+\frac{1}{2s}\left[G(nf(y))-G(mf(x))\right]\right] &= F
\end{aligned}
$$

Combining four equations to eliminate m and n

$$
(c+t_x)\frac{-x^2f''(x)}{\left[f'(x)+xf''(x)\right]}\left[\frac{1}{2}+\frac{1}{2s}\left[\begin{array}{c} G\left([G']^{-1}\left(\frac{c+t_x}{1-\tau_x}\frac{1}{[f'(x)+xf''(x)]}\right)\right) \\ -G\left([G']^{-1}\left(\frac{c-t_y}{1+\tau_y}\frac{1}{[f'(y)+yf''(y)]}\right)\right)\end{array}\right]\right]=F
$$

$$
(c-t_y)\frac{-y^2f''(y)}{\left[f'(y)+yf''(y)\right]}\left[\frac{1}{2}+\frac{1}{2s}\left[\begin{array}{c} G\left([G']^{-1}\left(\frac{c-t_y}{1+\tau_y}\frac{1}{[f'(y)+yf''(y)]}\right)\right) \\ -G\left([G']^{-1}\left(\frac{c+t_x}{1-\tau_x}\frac{1}{[f'(x)+xf''(x)]}\right)\right)\end{array}\right]\right]=F
$$

Total differentiate these two equations above and the tax revenue equilibrium equation

$$
\begin{aligned}
0 \quad = \quad & \frac{1}{(c+t_x)}\left[\frac{1}{2}+\frac{1}{2s}\left[G(mf(x))-G(nf(y))\right]\right]dt_x \\
& +\frac{\left[2f''(x)+xf'''(x)\right]}{\left[f'(x)+xf''(x)\right]}\frac{f'(x)}{xf''(x)}\left[\frac{1}{2}+\frac{1}{2s}\left[G(mf(x))-G(nf(y))\right]\right]dx \\
& +\frac{1}{2s}\left[\begin{array}{c} \frac{G'(mf(x))}{G''(mf(x))}\frac{1}{[f'(x)+xf''(x)]}\left(\frac{dt_x}{1-\tau_x}+\frac{c+t_x}{1-\tau_x}\frac{d\tau_x}{1-\tau_x}-\frac{c+t_x}{1-\tau_x}\frac{[2f''(x)+xf'''(x)]}{[f'(x)+xf''(x)]}dx\right) \\ +\frac{G'(nf(y))}{G''(nf(y))}\frac{1}{[f'(y)+yf''(y)]}\left(\frac{dt_y}{1+\tau_y}+\frac{c-t_y}{1+\tau_y}\frac{d\tau_y}{1+\tau_y}+\frac{c-t_y}{1+\tau_y}\frac{[2f''(y)+yf'''(y)]}{[f'(y)+yf''(y)]}dy\right)\end{array}\right]
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\;& \frac{1}{(c-t_y)}\left[\frac{1}{2}+\frac{1}{2s}\left[G\left(nf\left(y\right)\right)-G\left(mf\left(x\right)\right)\right]\right]dt_y \\
&-\frac{\left[2f''\left(y\right)+yf'''\left(y\right)\right]}{\left[f'\left(y\right)+yf''\left(y\right)\right]}\frac{f'\left(y\right)}{yf''\left(y\right)}\left[\frac{1}{2}+\frac{1}{2s}\left[G\left(nf\left(y\right)\right)-G\left(mf\left(x\right)\right)\right]\right]dy \\
&+\frac{1}{2s}\left[\begin{array}{c} \frac{G'(nf(y))}{G''(nf(y))}\frac{1}{[f'(y)+yf''(y)]}\left(\frac{dt_y}{1+\tau_y}+\frac{c-t_y}{1+\tau_y}\frac{d\tau_y}{1+\tau_y}+\frac{c-t_y}{1+\tau_y}\frac{[2f''(y)+yf'''(y)]}{[f'(y)+yf''(y)]}dy\right) \\ +\frac{G'(mf(x))}{G''(mf(x))}\frac{1}{[f'(x)+xf''(x)]}\left(\frac{dt_x}{1-\tau_x}+\frac{c+t_x}{1-\tau_x}\frac{d\tau_x}{1-\tau_x}-\frac{c+t_x}{1-\tau_x}\frac{[2f''(x)+xf'''(x)]}{[f'(x)+xf''(x)]}dx\right) \end{array}\right]
\end{aligned}
$$

In addition, total tax revenue is equal to total subsidy

$$
\left[\tau_x G'\left(mf\left(x\right)\right)f'\left(x\right)+t_x\right]mx\delta^* = \left[\tau_y G'\left(nf\left(y\right)\right)f'\left(y\right)+t_y\right]ny\left(1-\delta^*\right)
$$

Total differentiate and assuming that starting from 0 tax position by assuming $\tau_x = \tau_y = t_x = t_y = 0$, then market X and market Y are symmetric with $m = n$, $x = y$, and $\delta^* = \frac{1}{2}$. Thus I have

$$
G'\left(mf\left(x\right)\right)f'\left(x\right)d\tau_x + dt_x = G'\left(nf\left(y\right)\right)f'\left(y\right)d\tau_y + dt_y
$$

Combining all three differentiated equations,

$$
\begin{aligned}
dx \;=\;& \frac{\frac{1}{G'(mf(x))}\left[\frac{1}{2}+\frac{1}{2s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\left(1-\frac{xf''(x)}{f'(x)}\right)\right]}{\left[2f''\left(x\right)+xf'''\left(x\right)\right]\left\{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right\}\frac{f'(x)}{xf''(x)}}dt_x \\
&+\frac{\frac{1}{2s}\frac{G'(mf(x))}{G''(mf(x))}\left(1+\frac{xf''(x)}{f'(x)}\right)}{\left[2f''\left(x\right)+xf'''\left(x\right)\right]\left\{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right\}\frac{f'(x)}{xf''(x)}}dt_y \\
&+\frac{\frac{1}{2s}\frac{G'(mf(x))}{G''(mf(x))}G'\left(mf\left(x\right)\right)\left[f'\left(x\right)+xf''\left(x\right)\right]}{\left[2f''\left(x\right)+xf'''\left(x\right)\right]\left\{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right\}\frac{f'(x)}{xf''(x)}}\left(d\tau_x+d\tau_y\right)
\end{aligned}
$$

$$
\begin{aligned}
dy \;=\; & -\frac{\frac{1}{2s}\frac{[G'(mf(x))]}{G''(mf(x))}\left(1+\frac{xf''(x)}{f'(x)}\right)}{\left[2f''(x)+xf'''(x)\right]\left\{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right\}\frac{f'(x)}{xf''(x)}}dt_x \\
& -\frac{\frac{1}{G'(mf(x))}\left[\frac{1}{2}+\frac{1}{2s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\left(1-\frac{xf''(x)}{f'(x)}\right)\right]}{\left[2f''(x)+xf'''(x)\right]\left\{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right\}\frac{f'(x)}{xf''(x)}}dt_y \\
& -\frac{\frac{1}{2s}\frac{G'(mf(x))}{G''(mf(x))}G'\left(mf\left(x\right)\right)\left[f'\left(x\right)+xf''\left(x\right)\right]}{\left[2f''(x)+xf'''(x)\right]\left\{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right\}\frac{f'(x)}{xf''(x)}}\left(d\tau_x+d\tau_y\right)
\end{aligned}
$$

Taking these two conditions back, the effects of taxes and subsidies on equilibrium values can be represented. Specifically, the effects of changes of taxes/subsidies on changes of per variety consumption for each consumer in market X can be denoted as

$$
\begin{aligned}
\frac{dx}{dt_x} &= \frac{\frac{1}{G'(mf(x))}\left[\frac{1}{2}+\frac{1}{2s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\left(1-\frac{xf''(x)}{f'(x)}\right)\right]}{\left[2f''(x)+xf'''(x)\right]\left\{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right\}\frac{f'(x)}{xf''(x)}} \\
\frac{dx}{d\tau_x} &= \frac{\frac{1}{2s}\frac{G'(mf(x))}{G''(mf(x))}G'\left(mf\left(x\right)\right)\left[f'\left(x\right)+xf''\left(x\right)\right]}{\left[2f''(x)+xf'''(x)\right]\left\{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right\}\frac{f'(x)}{xf''(x)}} \\
\frac{dx}{dt_y} &= \frac{\frac{1}{2s}\frac{G'(mf(x))}{G''(mf(x))}\left(1+\frac{xf''(x)}{f'(x)}\right)}{\left[2f''(x)+xf'''(x)\right]\left\{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right\}\frac{f'(x)}{xf''(x)}} \\
\frac{dx}{d\tau_y} &= \frac{\frac{1}{2s}\frac{G'(mf(x))}{G''(mf(x))}G'\left(mf\left(x\right)\right)\left[f'\left(x\right)+xf''\left(x\right)\right]}{\left[2f''(x)+xf'''(x)\right]\left\{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)}-\frac{1}{2}\right\}\frac{f'(x)}{xf''(x)}}
\end{aligned}
$$

The effects of changes of taxes/subsidies on changes of number of varieties in market X can be denoted as

$$
\begin{aligned}
\frac{dm}{dt_x} &= \frac{\frac{1}{[f'(x)+xf''(x)]}}{(1-\tau_x)\,G''(mf(x))\,f(x)} - \left\{\frac{mf'(x)}{f(x)} + \frac{G'(mf(x))\,[2f''(x)+xf'''(x)]}{G''(mf(x))\,[f'(x)+xf''(x)]\,f(x)}\right\}\frac{dx}{dt_x} \\
\frac{dm}{d\tau_x} &= \frac{G'(mf(x))}{(1-\tau_x)\,G''(mf(x))\,f(x)} - \left\{\frac{mf'(x)}{f(x)} + \frac{G'(mf(x))\,[2f''(x)+xf'''(x)]}{G''(mf(x))\,[f'(x)+xf''(x)]\,f(x)}\right\}\frac{dx}{d\tau_x} \\
\frac{dm}{dt_y} &= \frac{dm}{dx}\frac{dx}{dt_y} = -\frac{\{G''(mf(x))\,[f'(x)+xf''(x)]\,mf'(x) + G'(mf(x))\,[2f''(x)+xf'''(x)]\}}{G''(mf(x))\,[f'(x)+xf''(x)]\,f(x)}\frac{dx}{dt_y} \\
\frac{dm}{d\tau_y} &= \frac{dm}{dx}\frac{dx}{d\tau_y} = -\frac{\{G''(mf(x))\,[f'(x)+xf''(x)]\,mf'(x) + G'(mf(x))\,[2f''(x)+xf'''(x)]\}}{G''(mf(x))\,[f'(x)+xf''(x)]\,f(x)}\frac{dx}{d\tau_y}
\end{aligned}
$$

The effects of changes of taxes/subsidies on changes of per variety consumption for each consumer in market Y can be denoted as

$$
\begin{aligned}
\frac{dy}{dt_x} &= -\frac{\frac{1}{2s}\frac{[G'(mf(x))]}{G''(mf(x))}\left(1+\frac{xf''(x)}{f'(x)}\right)}{[2f''(x)+xf'''(x)]\left\{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)} - \frac{1}{2}\right\}\frac{f'(x)}{xf''(x)}} \\
\frac{dy}{d\tau_x} &= -\frac{\frac{1}{2s}\frac{G'(mf(x))}{G''(mf(x))}G'(mf(x))\,[f'(x)+xf''(x)]}{[2f''(x)+xf'''(x)]\left\{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)} - \frac{1}{2}\right\}\frac{f'(x)}{xf''(x)}} \\
\frac{dy}{dt_y} &= -\frac{\frac{1}{G'(mf(x))}\left[\frac{1}{2}+\frac{1}{2s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\left(1-\frac{xf''(x)}{f'(x)}\right)\right]}{[2f''(x)+xf'''(x)]\left\{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)} - \frac{1}{2}\right\}\frac{f'(x)}{xf''(x)}} \\
\frac{dy}{d\tau_y} &= -\frac{\frac{1}{2s}\frac{G'(mf(x))}{G''(mf(x))}G'(mf(x))\,[f'(x)+xf''(x)]}{[2f''(x)+xf'''(x)]\left\{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)} - \frac{1}{2}\right\}\frac{f'(x)}{xf''(x)}}
\end{aligned}
$$

The effects of changes of taxes/subsidies on changes of number of varieties in market Y can be denoted as

$$
\begin{aligned}
\frac{dn}{dt_x} &= \frac{dn}{dy}\frac{dy}{dt_x} = -\frac{\{G''(nf(y))[f'(y)+yf''(y)]nf'(y)+G'(nf(y))[2f''(y)+yf'''(y)]\}}{G''(nf(y))[f'(y)+yf''(y)]f(y)}\frac{dy}{dt_x} \\
\frac{dn}{d\tau_x} &= \frac{dn}{dy}\frac{dy}{d\tau_x} = -\frac{\{G''(nf(y))[f'(y)+yf''(y)]nf'(y)+G'(nf(y))[2f''(y)+yf'''(y)]\}}{G''(nf(y))[f'(y)+yf''(y)]f(y)}\frac{dy}{d\tau_x} \\
\frac{dn}{dt_y} &= -\frac{\frac{1}{[f'(y)+yf''(y)]}}{(1+\tau_y)G''(nf(y))f(y)} - \left\{\frac{nf'(y)}{f(y)} + \frac{G'(nf(y))[2f''(y)+yf'''(y)]}{G''(nf(y))[f'(y)+yf''(y)]f(y)}\right\}\frac{dy}{dt_y} \\
\frac{dn}{d\tau_y} &= -\frac{G'(nf(y))}{(1+\tau_y)G''(nf(y))f(y)} - \left\{\frac{nf'(y)}{f(y)} + \frac{G'(nf(y))[2f''(y)+yf'''(y)]}{G''(nf(y))[f'(y)+yf''(y)]f(y)}\right\}\frac{dy}{d\tau_y}
\end{aligned}
$$

The effects of changes of taxes/subsidies on changes of number of consumers in market X can be denoted as

$$
\begin{aligned}
\frac{d\delta^*}{dt_x} &= \frac{1}{2s}\left[G'(mf(x))\left[f(x)\frac{dm}{dt_x}+mf'(x)\frac{dx}{dt_x}\right] - G'(nf(y))\left[f(y)\frac{dn}{dt_x}+nf'(y)\frac{dy}{dt_x}\right]\right] \\
\frac{d\delta^*}{d\tau_x} &= \frac{1}{2s}\left[G'(mf(x))\left[f(x)\frac{dm}{d\tau_x}+mf'(x)\frac{dx}{d\tau_x}\right] - G'(nf(y))\left[f(y)\frac{dn}{d\tau_x}+nf'(y)\frac{dy}{d\tau_x}\right]\right] \\
\frac{d\delta^*}{dt_y} &= \frac{1}{2s}\left[G'(mf(x))\left[f(x)\frac{dm}{dt_y}+mf'(x)\frac{dx}{dt_y}\right] - G'(nf(y))\left[f(y)\frac{dn}{dt_y}+nf'(y)\frac{dy}{dt_y}\right]\right] \\
\frac{d\delta^*}{d\tau_y} &= \frac{1}{2s}\left[G'(mf(x))\left[f(x)\frac{dm}{d\tau_y}+mf'(x)\frac{dx}{d\tau_y}\right] - G'(nf(y))\left[f(y)\frac{dn}{d\tau_y}+nf'(y)\frac{dy}{d\tau_y}\right]\right]
\end{aligned}
$$

Taking the condition that tax revenue is fixed back, a subsidy switch has no effect on the number of consumers in market X as shown below

$$\frac{d\delta^*}{p_y d\tau_y} - \frac{d\delta^*}{dt_y} = \frac{1}{2s}\left[\begin{array}{c} G'(mf(x))\left[f(x)\left(\frac{dm}{p_y d\tau_y} - \frac{dm}{dt_y}\right) + mf'(x)\left(\frac{dx}{p_y d\tau_y} - \frac{dx}{dt_y}\right)\right] \\ -G'(nf(y))\left[f(y)\left(\frac{dn}{p_y d\tau_y} - \frac{dn}{dt_y}\right) + nf'(y)\left(\frac{dy}{p_y d\tau_y} - \frac{dy}{dt_y}\right)\right] \end{array}\right]$$

$$= \frac{1}{2s}\left[\begin{array}{c} -G'(mf(x))\frac{G'(mf(x))[2f''(x)+xf'''(x)]}{G''(mf(x))[f'(x)+xf''(x)]}\left(\frac{dx}{p_y d\tau_y} - \frac{dx}{dt_y}\right) \\ -G'(nf(y))\left[\begin{array}{c} \frac{\frac{1}{[f'(y)+yf''(y)]}}{(1+\tau_y)G''(nf(y))} - \frac{\frac{1}{f'(y)}}{(1+\tau_y)G''(nf(y))} \\ +\frac{G'(nf(y))[2f''(y)+yf'''(y)]}{G''(nf(y))[f'(y)+yf''(y)]}\left(\frac{dy}{dt_y} - \frac{dy}{p_y d\tau_y}\right) \end{array}\right] \end{array}\right]$$

$$= -\frac{1}{2s}\frac{G'(nf(y))}{G''(nf(y))}\left[\frac{-xf''(x)}{[f'(x)+xf''(x)]f'(x)} + \frac{xf''(x)}{[f'(x)+xf''(x)]f'(x)}\right] = 0$$

Also, a subsidy switch has no effect on per variety consumption for each consumer in market X either.

$$\frac{dx}{p_y d\tau_y} - \frac{dx}{dt_y} = \frac{\frac{1}{2s}\frac{G'(mf(x))}{G''(mf(x))}\left(1 + \frac{xf''(x)}{f'(x)}\right)}{[2f''(x)+xf'''(x)]\left\{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)} - \frac{1}{2}\right\}\frac{f'(x)}{xf''(x)}}$$

$$-\frac{\frac{1}{2s}\frac{G'(mf(x))}{G''(mf(x))}\left(1 + \frac{xf''(x)}{f'(x)}\right)}{[2f''(x)+xf'''(x)]\left\{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\frac{xf''(x)}{f'(x)} - \frac{1}{2}\right\}\frac{f'(x)}{xf''(x)}}$$

$$= 0$$

In addition, a subsidy switch has no effect on the number of varieties in market X.

$$\frac{dm}{p_y d\tau_y} - \frac{dm}{dt_y} = -\left[\frac{mf'(x)}{f(x)} + \frac{G'(mf(x))[2f''(x)+xf'''(x)]}{G''(mf(x))[f'(x)+xf''(x)]f(x)}\right]\left(\frac{dx}{p_y d\tau_y} - \frac{dx}{dt_y}\right)$$

$$= 0$$

Thus, a subsidy switch has no effect on total quantity consumed in market X.

$$
\begin{aligned}
\frac{dQ_X}{p_y d\tau_y} - \frac{dQ_X}{dt_y} &= x\delta^* \left(\frac{dm}{p_y d\tau_y} - \frac{dm}{dt_y} \right) + m\delta^* \left(\frac{dx}{p_y d\tau_y} - \frac{dx}{dt_y} \right) + mx \left(\frac{d\delta^*}{p_y d\tau_y} - \frac{d\delta^*}{dt_y} \right) \\
&= 0
\end{aligned}
$$

For total social welfare $W = G(mf(x))\delta^* - \xi m x \delta^* + G(nf(y))(1-\delta^*) - s\int_0^{\delta^*} \delta d\delta - s\int_{\delta^*}^1 (1-\delta)\, d\delta$, a subsidy switch keeping tax revenue unchanged can be denoted as

$$
\begin{aligned}
\frac{dW}{p_y d\tau_y} - \frac{dW}{dt_y} &= G'(mf(x)) \left[f(x) \left(\frac{dm}{p_y d\tau_y} - \frac{dm}{dt_y} \right) + mf'(x) \left(\frac{dx}{p_y d\tau_y} - \frac{dx}{dt_y} \right) \right] \delta^* \\
&\quad + G'(nf(y)) \left[f(y) \left(\frac{dn}{p_y d\tau_y} - \frac{dn}{dt_y} \right) + nf'(y) \left(\frac{dy}{p_y d\tau_y} - \frac{dy}{dt_y} \right) \right] (1-\delta^*) \\
&= G'(nf(y)) \left[\begin{array}{c} \frac{\frac{1}{[f'(y)+yf''(y)]}}{G''(nf(y))} - \frac{\frac{1}{f'(y)}}{G''(nf(y))} \\ + \frac{G'(nf(y))[2f''(y)+yf'''(y)]}{G''(nf(y))[f'(y)+yf''(y)]} \left(\frac{dy}{dt_y} - \frac{1}{G'(nf(y))f'(y)} \frac{dy}{d\tau_y} \right) \end{array} \right] (1-\delta^*) \\
&= 0
\end{aligned}
$$

Therefore, a subsidy switch has no effect on total social welfare level. □

Proof: [Proof of Corollary 2.3] In the long run case, the equilibrium is determined by these four equations below

$$
\begin{aligned}
G'(mf(x))\left[f'(x) + xf''(x)\right] &= \frac{c+t}{1-\tau} \\
G'(nf(y))\left[f'(y) + yf''(y)\right] &= c \\
\left[(1-\tau)G'(mf(x))f'(x) - c - t\right] x \left[\frac{1}{2} + \frac{1}{2s}\left[G(mf(x)) - G(nf(y))\right] \right] &= F \\
\left[G'(nf(y))f'(y) - c\right] y \left[\frac{1}{2} + \frac{1}{2s}\left[G(nf(y)) - G(mf(x))\right] \right] + \frac{T}{n} &= F
\end{aligned}
$$

Eliminating m and n by combining these four equations, the equilibrium is determined by two equations below

$$-\left(c+t\right)\frac{x^{2}f''\left(x\right)}{\left[f'\left(x\right)+xf''\left(x\right)\right]}\left[\frac{1}{2}+\frac{1}{2s}\left[\begin{array}{c} G\left(\left[G'\right]^{-1}\left(\frac{c+t}{1-\tau}\frac{1}{\left[f'(x)+xf''(x)\right]}\right)\right)\\ -G\left(\left[G'\right]^{-1}\left(c\frac{1}{\left[f'(y)+yf''(y)\right]}\right)\right)\end{array}\right]\right]=F$$

$$\begin{aligned}&-c\frac{y^{2}f''\left(y\right)}{\left[f'\left(y\right)+yf''\left(y\right)\right]}\left[\frac{1}{2}+\frac{1}{2s}\left[\begin{array}{c} G\left(\left[G'\right]^{-1}\left(c\frac{1}{\left[f'(y)+yf''(y)\right]}\right)\right)\\ -G\left(\left[G'\right]^{-1}\left(\frac{c+t}{1-\tau}\frac{1}{\left[f'(x)+xf''(x)\right]}\right)\right)\end{array}\right]\right]\\ &+\frac{1}{n}\left[\tau\frac{c+t}{1-\tau}\frac{f'\left(x\right)}{\left[f'\left(x\right)+xf''\left(x\right)\right]}+t\right]mx\left[\frac{1}{2}+\frac{1}{2s}\left[\begin{array}{c} G\left(\left[G'\right]^{-1}\left(\frac{c+t}{1-\tau}\frac{1}{\left[f'(x)+xf''(x)\right]}\right)\right)\\ -G\left(\left[G'\right]^{-1}\left(c\frac{1}{\left[f'(y)+yf''(y)\right]}\right)\right)\end{array}\right]\right]\\ =\;&F\end{aligned}$$

Total differentiate these two equations

$$\begin{aligned}0\;=\;&\frac{1}{\left(c+t\right)}\left\{\frac{1}{2}+\frac{1}{2s}\left[G\left(mf\left(x\right)\right)-G\left(nf\left(y\right)\right)\right]\right\}dt\\ &+\frac{\left[2f''\left(x\right)+xf'''\left(x\right)\right]}{\left[f'\left(x\right)+xf''\left(x\right)\right]}\frac{f'\left(x\right)}{xf''\left(x\right)}\left\{\frac{1}{2}+\frac{1}{2s}\left[G\left(mf\left(x\right)\right)-G\left(nf\left(y\right)\right)\right]\right\}dx\\ &+\frac{1}{2s}\left[\begin{array}{c}\frac{G'(mf(x))}{G''(mf(x))}\frac{1}{\left[f'(x)+xf''(x)\right]}\left[\frac{dt}{1-\tau}+\frac{c+t}{1-\tau}\frac{d\tau}{1-\tau}-\frac{c+t}{1-\tau}\frac{\left[2f''(x)+xf'''(x)\right]}{\left[f'(x)+xf''(x)\right]}dx\right]\\ +\frac{G'(nf(y))}{G''(nf(y))}c\frac{\left[2f''(y)+yf'''(y)\right]}{\left[f'(y)+yf''(y)\right]^{2}}dy\end{array}\right]\end{aligned}$$

$$
\begin{aligned}
0 \;=\; & -c\frac{yf'(y)}{[f'(y)+yf''(y)]}\frac{[2f''(y)+yf'''(y)]}{[f'(y)+yf''(y)]}\left[\frac{1}{2}+\frac{1}{2s}\left[G(nf(y))-G(mf(x))\right]\right]dy \\
& +c\frac{y^2 f''(y)}{[f'(y)+yf''(y)]}\frac{1}{2s}\left[\begin{array}{c} \frac{G'(nf(y))}{G''(nf(y))}c\frac{[2f''(y)+yf'''(y)]}{[f'(y)+yf''(y)]^2}dy \\ +\frac{G'(mf(x))}{G''(mf(x))}\frac{1}{[f'(x)+xf''(x)]}\left(\frac{dt}{1-\tau}+\frac{c+t}{1-\tau}\frac{d\tau}{1-\tau}-\frac{c+t}{1-\tau}\frac{[2f''(x)+xf'''(x)]}{[f'(x)+xf''(x)]}dx\right) \end{array}\right] \\
& +\left[\begin{array}{c} \frac{c+t}{1-\tau}\frac{f'(x)}{[f'(x)+xf''(x)]}d\tau+\tau\frac{dt}{1-\tau}\frac{f'(x)}{[f'(x)+xf''(x)]} \\ +\tau\frac{c+t}{1-\tau}\frac{d\tau}{1-\tau}\frac{f'(x)}{[f'(x)+xf''(x)]} \\ +\tau\frac{c+t}{1-\tau}\frac{f''(x)[f'(x)+xf''(x)]-f'(x)[2f''(x)+xf'''(x)]}{[f'(x)+xf''(x)]^2}dx+dt \end{array}\right]\frac{1}{n}mx\left[\frac{1}{2}+\frac{1}{2s}\left[G(mf(x))-G(nf(y))\right]\right] \\
& +\left[\tau\frac{c+t}{1-\tau}\frac{f'(x)}{[f'(x)+xf''(x)]}+t\right]\frac{f'(y)}{nf(y)}mx\left[\frac{1}{2}+\frac{1}{2s}\left[G(mf(x))-G(nf(y))\right]\right]dy \\
& +\left[\tau\frac{c+t}{1-\tau}\frac{f'(x)}{[f'(x)+xf''(x)]}+t\right]f(y)\frac{c\frac{[2f''(y)+yf'''(y)]}{[f'(y)+yf''(y)]^2}mx}{G''(nf(y))[nf(y)]^2}\left[\frac{1}{2}+\frac{1}{2s}\left[G(mf(x))-G(nf(y))\right]\right]dy \\
& +\left[\tau\frac{c+t}{1-\tau}\frac{f'(x)}{[f'(x)+xf''(x)]}+t\right]\frac{1}{n}\frac{-xf'(x)}{f(x)}m\left[\frac{1}{2}+\frac{1}{2s}\left[G(mf(x))-G(nf(y))\right]\right]dx \\
& +\left[\tau\frac{c+t}{1-\tau}\frac{f'(x)}{[f'(x)+xf''(x)]}+t\right]\frac{1}{n}\frac{\frac{x\left[\frac{1}{2}+\frac{1}{2s}[G(mf(x))-G(nf(y))]\right]}{f(x)G''(mf(x))}}{[f'(x)+xf''(x)]}\left(\begin{array}{c} \frac{dt}{1-\tau}+\frac{c+t}{1-\tau}\frac{d\tau}{1-\tau} \\ -\frac{c+t}{1-\tau}\frac{[2f''(x)+xf'''(x)]}{[f'(x)+xf''(x)]}dx \end{array}\right) \\
& +\left[\tau\frac{c+t}{1-\tau}\frac{f'(x)}{[f'(x)+xf''(x)]}+t\right]\frac{1}{n}m\left[\frac{1}{2}+\frac{1}{2s}\left[G(mf(x))-G(nf(y))\right]\right]dx \\
& +\left[\tau\frac{c+t}{1-\tau}\frac{f'(x)}{[f'(x)+xf''(x)]}+t\right]\frac{1}{n}mx\frac{1}{2s}\left[\begin{array}{c} \frac{G'(mf(x))}{G''(mf(x))}\left(\begin{array}{c} \frac{dt}{1-\tau}\frac{1}{[f'(x)+xf''(x)]} \\ +\frac{c+t}{1-\tau}\frac{d\tau}{1-\tau}\frac{1}{[f'(x)+xf''(x)]} \\ -\frac{c+t}{1-\tau}\frac{[2f''(x)+xf'''(x)]}{[f'(x)+xf''(x)]^2}dx \end{array}\right) \\ +\frac{G'(nf(y))}{G''(nf(y))}c\frac{[2f''(y)+yf'''(y)]}{[f'(y)+yf''(y)]^2}dy \end{array}\right]
\end{aligned}
$$

Again, to simplify the analysis, I impose the condition that tax rates t and τ are zero initially, these two equations can be simplified as

$$0 = \frac{1}{G'(mf(x))[f'(x)+xf''(x)]}\frac{1}{2}dt + \frac{[2f''(x)+xf'''(x)]}{[f'(x)+xf''(x)]}\frac{f'(x)}{xf''(x)}\frac{1}{2}dx$$
$$+\frac{1}{2s}\frac{G'(mf(x))}{G''(mf(x))}\frac{1}{[f'(x)+xf''(x)]}\left[dt + cd\tau - c\frac{[2f''(x)+xf'''(x)]}{[f'(x)+xf''(x)]}dx + c\frac{[2f''(x)+xf'''(x)]}{[f'(x)+xf''(x)]}dy\right]$$

$$0 = -\frac{f'(x)}{xf''(x)}\frac{[2f''(x)+xf'''(x)]}{[f'(x)+xf''(x)]}\frac{1}{2}dy$$
$$+\frac{1}{2s}\frac{G'(mf(x))}{G''(mf(x))}\frac{1}{[f'(x)+xf''(x)]}\left[dt + cd\tau - c\frac{[2f''(x)+xf'''(x)]}{[f'(x)+xf''(x)]}dx + c\frac{[2f''(x)+xf'''(x)]}{[f'(x)+xf''(x)]}dy\right]$$
$$+\frac{1}{G'(mf(x))x^2f''(x)}[G'(mf(x))f'(x)d\tau + dt]\frac{1}{n}mx\frac{1}{2}$$

Combining these two simplified equations, dy can be represented by $d\tau$, dt and dx.

$$dy = -dx - \frac{xf''(x)}{[2f''(x)+xf'''(x)]G'(mf(x))f'(x)}dt$$
$$+\frac{[f'(x)+xf''(x)]}{[2f''(x)+xf'''(x)]G'(mf(x))f'(x)}[G'(mf(x))f'(x)d\tau + dt]$$

Taking this condition back into either of those two equations, the relationship between dx, dt, and $d\tau$ can be summarized by

$$dx = \frac{\left[\frac{1}{2}+\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\right]\frac{1}{G'(mf(x))[f'(x)+xf''(x)]}}{\frac{[2f''(x)+xf'''(x)]}{[f'(x)+xf''(x)]}\left[\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}-\frac{1}{2}\frac{f'(x)}{xf''(x)}\right]}dt + \frac{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}}{\frac{[2f''(x)+xf'''(x)]}{[f'(x)+xf''(x)]}\left[\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}-\frac{1}{2}\frac{f'(x)}{xf''(x)}\right]}d\tau$$

Hence, keeping everything else unchanged, the effects of taxes on equilibrium values can be summarized as following. Specifically, the effect of a change in taxes on the equilibrium values of per variety consumption for each consumer in market X and Y can be represented as

$$\frac{dx}{dt} = \frac{\left[\frac{1}{2}+\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}\right]\frac{1}{G'(mf(x))[f'(x)+xf''(x)]}}{\frac{[2f''(x)+xf'''(x)]}{[f'(x)+xf''(x)]}\left[\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}-\frac{1}{2}\frac{f'(x)}{xf''(x)}\right]}$$

$$\frac{dx}{d\tau} = \frac{\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}}{\frac{[2f''(x)+xf'''(x)]}{[f'(x)+xf''(x)]}\left[\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}-\frac{1}{2}\frac{f'(x)}{xf''(x)}\right]}$$

$$\frac{dy}{dt} = -\frac{\frac{1}{2}\frac{f'(x)}{xf''(x)}\frac{[f'(x)+xf''(x)]}{[2f''(x)+xf'''(x)]}\frac{1}{G'(mf(x))f'(x)}}{\left[\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}-\frac{1}{2}\frac{f'(x)}{xf''(x)}\right]}$$

$$\frac{dy}{d\tau} = -\frac{\frac{1}{2}\frac{f'(x)}{xf''(x)}\frac{[f'(x)+xf''(x)]}{[2f''(x)+xf'''(x)]}}{\left[\frac{1}{s}\frac{[G'(mf(x))]^2}{G''(mf(x))}-\frac{1}{2}\frac{f'(x)}{xf''(x)}\right]}$$

The effect of a change in taxes on the equilibrium values of the number of varieties in market X and Y can be represented as

$$\frac{dm}{dt} = \frac{1}{G''(mf(x))[f'(x)+xf''(x)]f(x)} - \left[\frac{mf'(x)}{f(x)}+\frac{G'(mf(x))[2f''(x)+xf'''(x)]}{G''(mf(x))[f'(x)+xf''(x)]f(x)}\right]\frac{dx}{dt}$$

$$\frac{dm}{d\tau} = \frac{\frac{c+t}{1-\tau}}{G''(mf(x))[f'(x)+xf''(x)]f(x)} - \left[\frac{mf'(x)}{f(x)}+\frac{G'(mf(x))[2f''(x)+xf'''(x)]}{G''(mf(x))[f'(x)+xf''(x)]f(x)}\right]\frac{dx}{d\tau}$$

$$\frac{dn}{dt} = -\left[\frac{nf'(y)}{f(y)}+\frac{G'(nf(y))[2f''(y)+yf'''(y)]}{G''(nf(y))[f'(y)+yf''(y)]f(y)}\right]\frac{dy}{dt}$$

$$\frac{dn}{d\tau} = -\left[\frac{nf'(y)}{f(y)}+\frac{G'(nf(y))[2f''(y)+yf'''(y)]}{G''(nf(y))[f'(y)+yf''(y)]f(y)}\right]\frac{dy}{d\tau}$$

which further depends on the effect of a change in taxes on the equilibrium values of per variety consumption for each consumer in market X and Y.

Considering the case that tax revenue remains unchanged in equilibrium, a switch from unit to ad valorem tax increases per variety consumption for each consumer in market X

$$\frac{dx}{p_X d\tau} - \frac{dx}{dt} = \frac{\frac{xf''(x)}{f'(x)}}{G'(mf(x))\left[2f''(x) + xf'''(x)\right]}$$

decreases the number of varieties in market X

$$\frac{dm}{p_X d\tau} - \frac{dm}{dt} = -\frac{mf'(x)}{f(x)} \frac{\frac{xf''(x)}{f'(x)}}{G'(mf(x))\left[2f''(x) + xf'''(x)\right]}$$

has no effects on per variety consumption for each consumer in market Y

$$\frac{dy}{p_X d\tau} - \frac{dy}{dt} = 0$$

has no effects on the number of varieties in market Y

$$\frac{dn}{p_X d\tau} - \frac{dn}{dt} = 0$$

and has no effects on the number of conusmers in each market

$$\begin{aligned} \frac{d\delta^*}{p_X d\tau} - \frac{d\delta^*}{dt} &= \frac{1}{2s}\left[\begin{array}{c} G'(mf(x))\left(f(x)\left(\frac{dm}{p_X d\tau} - \frac{dm}{dt}\right) + mf'(x)\left(\frac{dx}{p_X d\tau} - \frac{dx}{dt}\right)\right) \\ -G'(nf(y))\left(f(y)\left(\frac{dn}{p_X d\tau} - \frac{dn}{dt}\right) + nf'(y)\left(\frac{dy}{p_X d\tau} - \frac{dy}{dt}\right)\right) \end{array}\right] \\ &= 0 \end{aligned}$$

Taking all these conditions into the change in total quantity consumed in market X

$$\frac{dQ_X}{p_X d\tau} - \frac{dQ_X}{dt} = m\delta^* \left(\frac{dx}{p_X d\tau} - \frac{dx}{dt} \right) + x\delta^* \left(\frac{dm}{p_X d\tau} - \frac{dm}{dt} \right) + mx \left(\frac{d\delta^*}{p_X d\tau} - \frac{d\delta^*}{dt} \right)$$

$$= \frac{\frac{1}{2} mx \left[1 - \frac{xf'(x)}{f(x)} \right]}{G' \left(mf \left(x \right) \right) f' \left(x \right) \left[2 + \frac{xf'''(x)}{f''(x)} \right]} > 0$$

total quantity consumed in market X is increasing in a switch from unit to ad valorem tax. Taking all these conditions into the change in total social welfare

$$\frac{dW}{p_X d\tau} - \frac{dW}{dt} = -\xi \left(\frac{dQ_X}{p_X d\tau} - \frac{dQ_X}{dt} \right)$$

$$= -\xi \frac{\frac{1}{2} mx \left[1 - \frac{xf'(x)}{f(x)} \right]}{G' \left(mf \left(x \right) \right) f' \left(x \right) \left[2 + \frac{xf'''(x)}{f''(x)} \right]} < 0$$

total social welfare is decreasing in a switch from unit to ad valorem tax. These results are consistent with those from proposition 2.3. □

BIBLIOGRAPHY

[1] Anderson, S. P., De Palma, A., and Kreider, B. The efficiency of indirect taxes under imperfect competition. *Journal of Public Economics 81*, 2 (2001), 231–251.

[2] Anderson, S. P., De Palma, A., and Kreider, B. Tax incidence in differentiated product oligopoly. *Journal of Public Economics 81*, 2 (2001), 173–192.

[3] Delipalla, S., and Keen, M. The comparison between ad valorem and specific taxation under imperfect competition. *Journal of Public Economics 49*, 3 (1992), 351–367.

[4] Diamond, P. A. Consumption externalities and imperfect corrective pricing. *The Bell Journal of Economics and Management Science* (1973), 526–538.

[5] Dierickx, I., Matutes, C., and Neven, D. Indirect taxation and cournot equilibrium. *International Journal of Industrial Organization 6*, 3 (1988), 385–399.

[6] Dixit, A. K., and Stiglitz, J. E. Monopolistic competition and optimum product diversity. *The American Economic Review 67*, 3 (1977), 297–308.

[7] Dröge, S., and Schröder, P. J. The welfare comparison of corrective ad valorem and unit taxes under monopolistic competition. *International Tax and Public Finance 16*, 2 (2009), 164–175.

[8] Hamilton, S. F. Excise taxes with multiproduct transactions. *American Economic Review 99*, 1 (2009), 458–71.

[9] Hotelling, H. Stability in competition. In *The Collected Economics Articles of Harold Hotelling*. Springer, 1990, pp. 50–63.

[10] Inderst, R., and Shaffer, G. Market power, price discrimination, and allocative efficiency in intermediate-goods markets. *The RAND Journal of Economics 40*, 4 (2009), 658–672.

[11] Keen, M. The balance between specific and ad valorem taxation. *Fiscal studies 19*, 1 (1998), 1–37.

[12] Myles, G. D. Imperfect competition and the optimal combination of ad valorem and specific taxation. *International Tax and Public Finance 3*, 1 (1996), 29–44.

[13] Peitz, M., and Reisinger, M. Indirect taxation in vertical oligopoly. *The Journal of Industrial Economics 62*, 4 (2014), 709–755.

[14] Pirttilä, J. Specific versus ad valorem taxation and externalities. *Journal of Economics 76*, 2 (2002), 177–187.

[15] Schröder, P. J. The comparison between ad valorem and unit taxes under monopolistic competition. *Journal of Economics 83*, 3 (2004), 281–292.

[16] Skeath, S. E., and Trandel, G. A. A pareto comparison of ad valorem and unit taxes in noncompetitive environments. *Journal of Public Economics 53*, 1 (1994), 53–71.

[17] Suits, D. B., and Musgrave, R. A. Ad valorem and unit taxes compared. *The Quarterly Journal of Economics 67*, 4 (1953), 598–604.

[18] Yang, C. W. Taxation and the output and welfare of third-degree price discrimination. *Southern Economic Journal* (1993), 371–384.

[19] Yoshida, Y. Third-degree price discrimination in input markets: output and welfare. *American Economic Review 90*, 1 (2000), 240–246.