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Vertex coloring with forbidden subgraphs

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Vertex Coloring With Forbidden Subgraphs

By:

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Abstract

Given a set L of graphs, a graph G is L-free if G does not contain any graph in L as induced subgraph. A *hole* is an induced cycle of length at least 4. A *hole-twin* is a graph obtained by adding a vertex adjacent to three consecutive vertices in a *hole*. Hole-twins are closely related to the characterization of the line graphs in terms of forbidden subgraphs.

By using *clique-width* and *perfect graphs* theory, we show that (*claw*,$4K_1$,*hole-twin*)-free graphs and ($4K_1$,*hole-twin*,5-*wheel*)-free graphs are either perfect or have bounded clique-width. And thus the coloring of them can be done in polynomial time.

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Introduction

1.1 Background

In the 1700s, the idea of a graph was introduced during a Swiss mathematician Leonhard Euler's attempts to a famous problem: the Königsberg Bridge Problem. Königsberg is a city on the Pregolya river and it has four bodies of land connected by a total of seven bridges. The problem asks: Is there a way to walk through all lands by crossing each bridge exactly once?

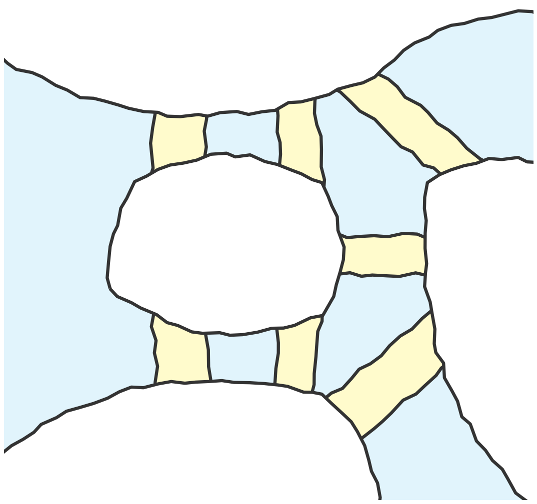
Definition 1.1.1 *A graph is an ordered pair $G = (V, E)$, where V is a set of vertices and E is a set of edges(a pair of vertices).*

Definition 1.1.2 *The number of edges incident to a vertex is called the degree of the vertex.*

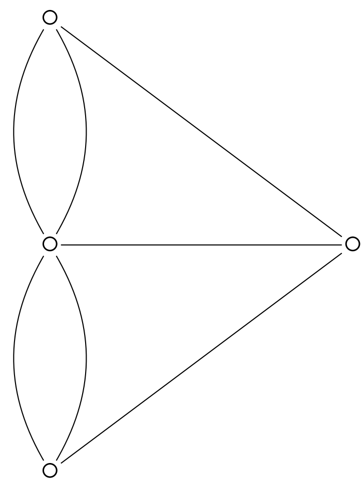
Definition 1.1.3 *A path is a sequence of distinct vertices v_i and edges of the form $v_0, \{v_0, v_1\}$, v_1, ... v_{k-1} ,$\{v_{k-1}, v_k\}$, v_k. And P_k represents a path of length k.*

Definition 1.1.4 *An Euler Path is a path that uses every edge of the graph exactly once.*

The four bodies of land can be seen as 4 vertices in the graph and there is an edge between two vertices if a bridge connects the two corresponding lands. Euler found that if G contains none or 2 odd degree vertices, then the graph has an Euler path. The discussion of the Königsberg problem in [7] is commonly considered to be the origin of graph theory.



(a) The Königsberg city



(b) A graph view of the Königsberg city

Figure 1.1: The landscape of Königsberg

One of the most important subfields of graph theory is the vertex coloring problem. It has been studied as an algorithmic problem and has attracted the interest of scientists since last century. A definition of vertex coloring is as follows:

Definition 1.1.5 *Vertex coloring is an assignment of colors to vertices of a graph G such that no two adjacent vertices are assigned with the same color.*

Definition 1.1.6 *A coloring using at most k colors is called a k-coloring and a graph that admits a k-coloring is called k-colorable.*

Definition 1.1.7 *The least number of colors needed to color a graph is called the chromatic number, denoted by $\chi(G)$.*

A famous example of graph coloring is the *Four Color Theorem*: Given any plane, partition the plane into contiguous regions and call it a map. Four colors are enough to color any map where no two regions sharing a common boundary are assigned with the same color. The *Four Color Theorem* was proved in 1976 with the use of computer by Kenneth Appel and Wolfgang Haken [1].

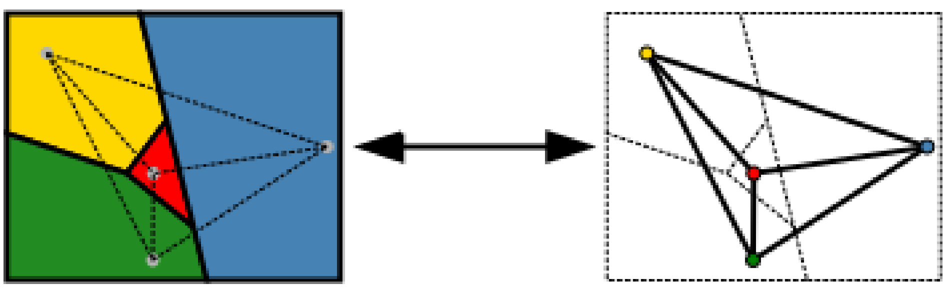


Figure 1.2: from the four coloring problem to graph coloring

Another application of graph coloring is scheduling. There is a set of tasks to be scheduled, and each task can be scheduled in any order, but tasks may share the same resources. Finding the best schedule to avoid conflicts can be seen as graph coloring problem, where each task represents a vertex in the graph and there is an edge between vertices if the tasks are in conflict.

Now we provide some useful definitions.

Definition 1.1.8 *The complement of a graph G is a graph that has the same set of* *vertices, and two vertices are adjacent if and only if they are not adjacent in* G*. It is denoted as* $\overline{G}$*.*

Definition 1.1.9 *A clique is a subset of vertices such that every two distinct vertices are adjacent.*

Definition 1.1.10 *A stable (or independent) set of a graph* G*, is a set of vertices of* G *where no two vertices are adjacent.*

The number of vertices in the largest stable set (or the stability number) in G is denoted as $\alpha(G)$.

Definition 1.1.11 *Given a graph* G *and partition* G *into cliques. The minimum number of cliques needed to cover all vertices of* G *is called the clique covering number of* G*, denoted as* $\theta(G)$*.*

The clique covering number of G is also equal to the chromatic number of its complement graph: $\theta(G) = \chi(\overline{G})$.

Definition 1.1.12 *The size of the largest clique in* G *is called the clique number of* G*, denoted as* $\omega(G)$*.*

Definition 1.1.13 *A hole is a cycle with length at least four. A* k*-hole is a hole with* k *vertices.*

Definition 1.1.14 *A decision problem is a problem with answer Yes or No.*

Definition 1.1.15 P *is a complexity class containing all decision problems that can be solved in polynomial time.*

Definition 1.1.16 *NP is a complexity class containing all decision problems that the answer of the problem can be verified in polynomial time.*

Definition 1.1.17 *NP-Complete is a complexity class containing all decision problems X in NP such that for any other NP problems Y, it is possible to reduce Y to X in polynomial time.*

It is well known that for any arbitrary graph, determining whether it is k-colorable or not is NP-Complete. Therefore, it is natural to study the complexity of vertex coloring when the graph is restricted with forbidden subgraphs.

Definition 1.1.18 *A subgraph of graph $G = (V, E)$ is another graph $G' = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. An induced subgraph of graph $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$ and E' is the set of all edges whose endpoints are both in V'.*

"H-Free" graphs refers to the class of all graphs that do not contain an induced subgraph H.

For a set L of graphs, a graph G is called L-free when G does not contain any graph of L as an induced subgraph.

In [31], Král, Kratochvíl, Tuza and Woeginger showed a complete characterization of H-free graphs for which the coloring can be done in polynomial time or NP-Complete:

Theorem 1.1.1 ([31]) *The coloring of H-free problem can be solved in polynomial time if H is an induced subgraph of P_4 or $P_3 \oplus K_1$(disjoint union of P_3 and K_1), and NP-Complete for any other H.*

The authors further studied the L-free graph coloring problem when L contains two sets of graphs.

The authors generalized L into fours types:

- **Type A:** Graphs containing a cycle
- **Type B:** Graphs containing a claw
- **Type C:** Graphs containing an induced subgraph of $2K_2$
- **Type D:** Graphs that are an induced subgraph of P_4 or $P_3 \oplus K_1$

Proposition 1.1.2 ([31]) *If at least one of graphs in L is of Type D, then L-free graphs coloring is polynomial time solvable. If all graphs of L are the same type A, B or C, then L-free graphs coloring is NP-Complete.*

Proposition 1.1.3 ([31]) *$(claw, C_k)$-free coloring is NP-Complete for any $k \geq 4$.*

1.2 Perfect Graph

In the 1950s, Claude Berge [4] introduced the concept of perfect graphs. He defined two kinds of perfectness:

- α-perfect: $\theta(G) = \alpha(G)$ (the smallest number of cliques that covers G equals the number of vertices in the largest stable set)
- χ-perfect: $\omega(G) = \chi(G)$ (the number of vertices in the largest clique equals the number of colors need to color G)

In 1963, Berge [5] was working on graphs that do not contain odd holes of length at least 5, or the complement of such a cycle. Such graphs are called Berge graphs.

Definition 1.2.1 *A Berge graph is a graph that contains neither odd holes nor odd antiholes(complement of holes) of length 5 or more.*

Then he published his two famous conjectures:

1. The *Weak Perfect Graph Conjecture*: α- and χ-perfect graphs are the same set of graphs.

2. The *Strong Perfect Graph Conjecture*: Berge Graphs are α-perfect.

The *Weak Perfect Graph conjecture* was proved in 1972 by Lovász [32], and because of the proof, there is no need to distinguish between $\alpha-$ and χ- perfect graphs. They have since been called perfect graphs by the graph theory community.

Theorem 1.2.1 *(Perfect Graph Theorem) If G is perfect, then $\overline{G}$ is perfect.*

Definition 1.2.2 *A perfect graph is a graph where the chromatic number of every induced subgraph is equal to the size of its largest clique.*

Chudnovsky, Robertson, Seymour and Thomas announced a proof of the *Strong Perfect Graph Conjecture* and the proof was published in 2006 [11]. Since then, the *Strong Perfect Graph Conjecture* was renamed the *Strong Perfect Graph Theorem.*

Theorem 1.2.2 *(Strong Perfect Graph Theorem) Perfect graphs are the graphs that contains neither odd holes nor odd anti-holes of length at least five.*

The Shannon capacity of a graph is defined by Shannon in [36]. The computational complexity of Shannon capacity is unknown. But the Lovász number, also known as the Lovász's theta function, can be computed in polynomial time and it is an upper bound of the Shannon capacity.

Theorem 1.2.3 *(Lovász's Sandwich Theorem) Lovász's theta function* $\theta(G)$ *satisfies* $\omega(G) \leq \theta(G) \leq \chi(G)$.

Perfect graphs satisfy $\omega(G) = \chi(G)$ for every induced subgraph, and thus $\omega(G) = \theta(G) = \chi(G)$. For any graph G, a polynomial time algorithm based on the ellipsoid method to find the Lovász number was given in [21]. Therefore, for all perfect graphs, the graph coloring problem can be solved in polynomial time.

1.3 Clique Width

The clique width of a graph G, denoted by $cwd(G)$, is a relatively new concept and it describes the structure of the graph. In 1990, Courcelle, Engelfriet and Rozenberg defined the construction sequences of clique width [15].

Consider the following operations to build a graph:

1. Create a vertex u labeled by integer ℓ.
2. Make the disjoint union of several graphs.
3. For some pair of distinct labels i and j, add all edges between vertices with label i and vertices with label j.
4. For some pair of distinct labels i and j, relabel all vertices of label i by label j.

The clique width of a graph G is the minimum number of labels used to generate G by the four operations. A k-expression for a graph G is the description of how G is recursively generated by repeatedly applying the four operations with k labels.

An example to build a path using 3 labels is presented in Figure 1.3.

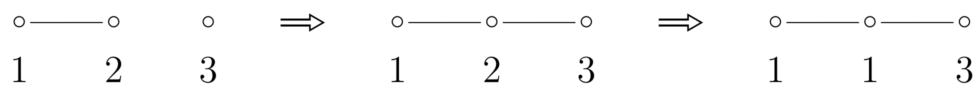
P1: P2:

Create a new vertex and label it 1.

Create a new vertex and label it 2.

Add all edges between vertices with label 1 and 2.

P3:

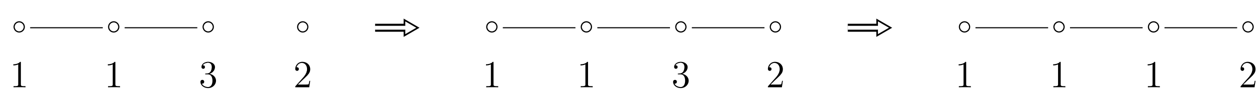


Create a new vertex and label it 3.

Add all edges between vertices with label 2 and 3.

Relabel all vertices with label 2 by label 1.

P4:



Create a new vertex and label it 2.

Add all edges between vertices with label 2 and 3.

Relabel all vertices with label 3 by label 1.

P5:

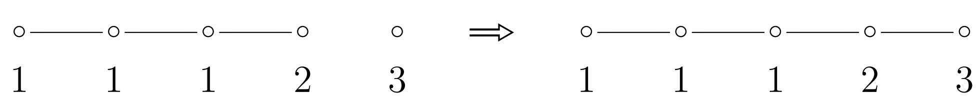


Figure 1.3: Create a path using 3 labels

Theorem 1.3.1 ([10]) *If G has a bounded clique width, then its complement $\overline{G}$ has a bounded clique width.*

Lemma 1.3.2 ([28]) *If G has n vertices, then $cwd(G) \leq n-k$ as long as $2^k+2k \leq n$.*

Thus, for any graph G with a finite number of vertices, G has bounded clique width.

Folklore 1.3.3 *Let G be a graph and let X be a finite set of vertices of G. Then G has bounded clique width if and only if $G-X$ has bounded clique width.*

Definition 1.3.1 *Given sets of vertices X, Y, the structure of X, Y is called co-join if there is no edges between any vertex in X and any vertex in Y, represented by $X \textcircled{0} Y$.*

Definition 1.3.2 *Given sets of vertices X, Y, the structure of X, Y is called join if there are all edges between vertices in X and Y, represented by $X \textcircled{1} Y$.*

Folklore 1.3.4 *Let G be a graph such that G is the join of two graphs G_1 and G_2. Then G has bounded clique width if and only if both G_i have bounded clique width, for $i = 1, 2$.*

Folklore 1.3.5 *Let G be a graph such that G is the co-join of two graphs G_1 and G_2. Then G has bounded clique width if and only if both G_i have bounded clique width, for $i = 1, 2$.*

Many graph problems that are NP-hard for arbitrary graphs can be solved in polynomial time on graphs with bounded clique width. In Rao [35], the following result is established.

Theorem 1.3.6 ([35]) *For any constant c,* VERTEX COLORING *is polynomial-time solvable in the class of graphs with clique-width at most c.*

1.3.1 Clique Width for 4-Vertex Forbidden Subgraph

A graph G is P_4-free if and only if $cwd(G) \leq 2$ and it is interesting to consider what other forbidden 4-vertex graphs will lead a graph G that avoid them to have a bounded clique width. In [9], the clique width of class (H, co-H)-free graphs for any 4-vertex graph has been proved to be bounded. In [10], the authors listed all the essential combinations of forbidden 4-vertex graphs and classified which make a graph have either a bounded or unbounded clique width. Figure 1.4 shows all 4-vertex graphs.

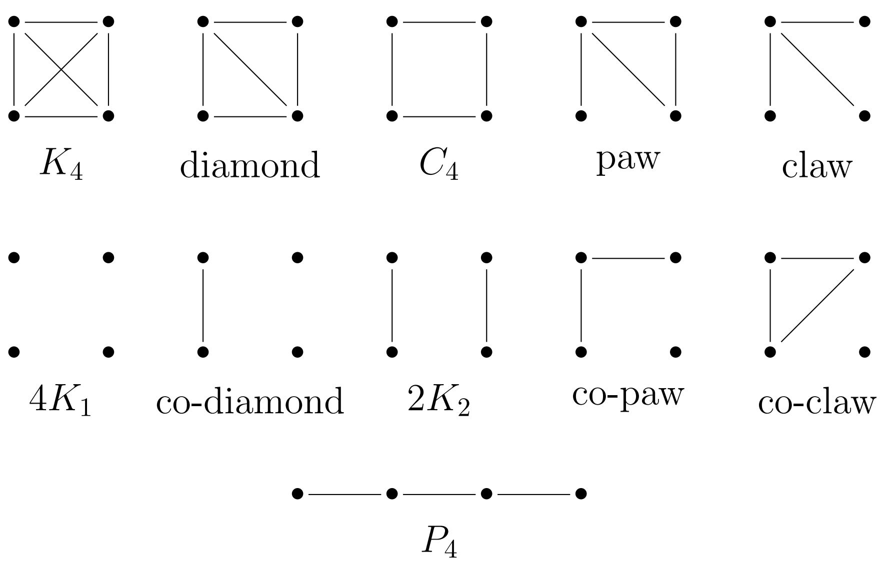


Figure 1.4: All 4-vertex graphs

Graphs defined by forbidding the following 4-vertex graphs have bounded clique width:

- K_4, co-paw
- K_4, co-diamond
- K_4, $4K_1$
- diamond, co-paw
- diamond, $2K_2$
- diamond, co-diamond

- C_4, co-paw
- paw, claw
- paw, co-paw
- claw, co-claw
- K_4, C_4, $2K_2$
- K_4, claw, $2K_2$
- C_4, claw, $2K_2$
- K_4, co-claw, $2K_2$

Graphs defined by forbidding the following 4-vertex graphs have a unbounded clique width:

- K_4, $2K_2$
- C_4, $2K_2$
- K_4, diamond, C_4, claw
- K_4, diamond, C_4, paw, co-claw

1.4 Line Graph

In graph theory, the line graph $L(G)$ represents the adjacencies between edges of the graph G. A formal definition of line graph is as follows:

Definition 1.4.1 ([24]) *The vertices of $L(G)$ are taken as the edges of G, and for each pair of vertices in $L(G)$, there is an edge if and only if the corresponding edges of G are adjacent.*

An example of building the line graph for C_4-twin is shown in Figure 1.5:

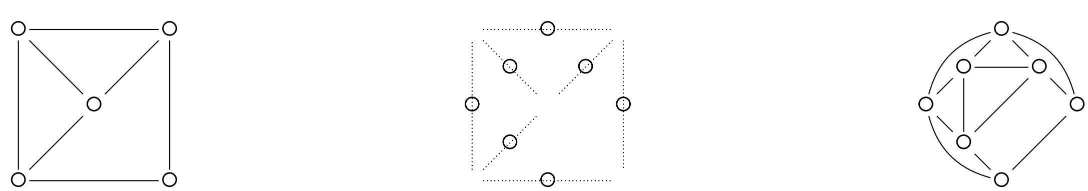


Figure 1.5: An example of a line graph

The recognition of line graphs and the reconstruction of their original graphs can be done in polynomial time. In [16], the authors described an efficient algorithm of recognizing line graphs using the *Whitney's Isomorphism Theorem*.

Theorem 1.4.1 *(Whitney's Isomorphism Theorem) If the line graphs of two connected graphs are isomorpic, then the underlying graphs are isomorpic, except for* K_3 *and claw.*

In 1970, Beineke [2] showed a characterization of line graphs in terms of nine forbidden subgraphs. In general, the coloring of line graphs is NP-Complete, but it is worth considering that whether the coloring can be done in polynomial time by forbidding some of the subgraphs in Figure: 1.6.

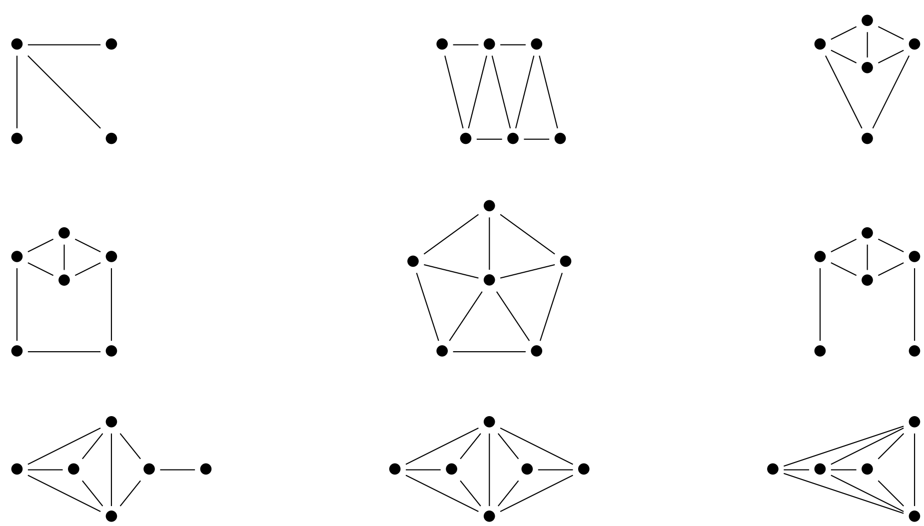


Figure 1.6: Nine forbidden subgraphs that characterize the line graph

Line graphs are *claw*-free, and *claw*-free graphs are considered to be an interesting generalization of line graphs.

Lemma 1.4.2 ([14]) *Let* G *be a connected claw-free graph with* $\alpha(G) \geq 3$. *If* G *contains an odd anti-hole then* G *contains a* C_5.

It is well known that vertex coloring is polynomial time solvable for graphs G with $\alpha(G) = 2$.

1.5 Motivation

It is interesting to think about the coloring on graphs whose forbidden list L contains graphs with 4 vertices. A recent paper by Lozin and Malyshev [33] discusses the computational complexity of vertex coloring on graphs defined by forbidden induced subgraphs with at most 4 vertices. For all classes except for three, they show the vertex coloring is either polynomial-time sovable or is NP-Complete. The three classes are: $(4K_1, C_4)$-free graphs, $(Claw, 4K_1)$-free graphs and $(Claw, 4K_1$,co$-$diamond)-free graphs. In the paper [18], the authors considered the vertex coloring problem of $(4K_1, C_4, C_5)$-free graphs, which is a slightly larger forbidden list of one of the three remaining classes: $(4K_1, C_4)$-free graphs. They showed that $(4K_1, C_4, C_5)$-free graphs have either bounded clique width or are perfect. Inspired by this approach, we consider what others graphs we can add into the forbidden list. In the paper [17], the authors proved the coloring of $(Claw, 4K_1, K_5 - e)$-free graphs can be done in polynomial time. The graph $K_5 - e$ is one of the nine forbidden subgraphs that characterize the line graph(see Figure: 1.6). We observed that three graphs in the nine forbidden subgraphs, namely, P_5-twin, C_5-twin and C_4-twin, share the same structure: all the graphs have a diamond attached to a path or a cycle. We enlarge the set to include hole-twins and consider whether forbidding hole-twin helps in the coloring of the remaining cases.

The tools we use in this thesis are perfect graph theory and clique width theory.

In Chapter 2, we show the problem of coloring (*claw*, $4K_1$, *hole-twin*)-free graphs can be solved in polynomial time. In Chapter 3, we design a polynomial time algorithm to color ($4K_1$, *hole-twin*, 5-*wheel*)-free graphs. In Chapter 4, we present our conclusions and open problems related to our work.

Chapter 2

(Claw, $4K_1$, *hole*-*twin*)-free graphs

In this section, we will prove that there exists a polynomial time algorithm to color (Claw, $4K_1$, *hole-twin*)-free.

We will assume that G is a connected (Claw, $4K_1$, *hole-twin*)-free graph. Since G is (C_4-twin)-free, G does not contain any odd antihole with length larger than 5. Thus we will focus on what happens when G contains an odd hole. We know G contains no hole C_k with $k \geq 9$ since G is $4K_1$-free. So we assume G contains a C_7 or a C_5, otherwise by the Strong Perfect Graph theorem, G is a perfect graph and the chromatic number and the optimal coloring can be found in polynomial time.

Definition 2.0.1 *A hole-twin is a graph obtained by adding a vertex adjacent to three consecutive vertices in a hole.*

Theorem 2.0.1 *There is a polynomial time algorithm to color (Claw, $4K_1$, hole-twin)-free graphs.*

Lemma 2.0.2 *Let G be a graph such that $V(G)$ can be covered by k (disjoint) cliques $X_1, \ldots, X_k$. For a vertex x, let X_{i_x} be the clique containing x, and let $N_F(x)$ be the* *set of neigbours y of x such that $y \in X_j$ for $j \neq i_x$. Suppose G satisfies the following conditions: (i) for every vertex x and any set X_j with $j \neq i_x$, x has at most one neighbor in X_j, and (ii) for any vertex x, $N_F(x)$ is a clique. Then G has clique width at most $2k$.*

Proof. By (i) and (ii), if some two vertices x, y are adjacent with $x \in X_i, y \in X_j, i \neq j$, then we have $N_F(x) - \{y\} = N_F(y) - \{x\}$; that is, x and y have the same neighbourhood in $V(G) - (X_i \cup X_j)$. It follows that we can partition the vertices of G into pairwise disjoint sets $Y_1, Y_2, \ldots, Y_t$, $Z = V(G) - (Y_1 \cup Y_2 \cup \ldots \cup Y_t)$, such that the following holds: (1) each Y_s is a clique with at least two vertices, (2) if two vertices x, y are adjacent with $x \in X_i, y \in X_j, i \neq j$, then x and y belong to some clique Y_s, and (3) every edge of G belongs to a clique X_i, or a clique Y_s.

The vertices of a set X_i will be associated with two labels $\ell_{i,new}, \ell_{i,old}$. We will label the vertices of G one by one. Suppose we are about to label a vertex x.

1. If there is a vertex with a label $\ell_{i,new}$, re-label it with label $\ell_{i,old}$ for all i.
2. Label x with label $\ell_{i_x,new}$ (X_{i_x} is the set containing x)
3. For each neighbour y of x in a set X_{i_y}, label y with label $\ell_{i_y,new}$
4. Add edges between vertices with *new* labels (building the clique Y_s)
5. Add edges between vertices of label $\ell_{i_x,new}$ and label $\ell_{i_x,old}$ (building the clique X_{i_x}).
6. Re-label all vertices of label $\ell_{i,new}$ with label $\ell_{i,old}$ for all i.

We repeat the above steps until all vertices are labeled. We will use $2k$ labels. This proves the lemma. □

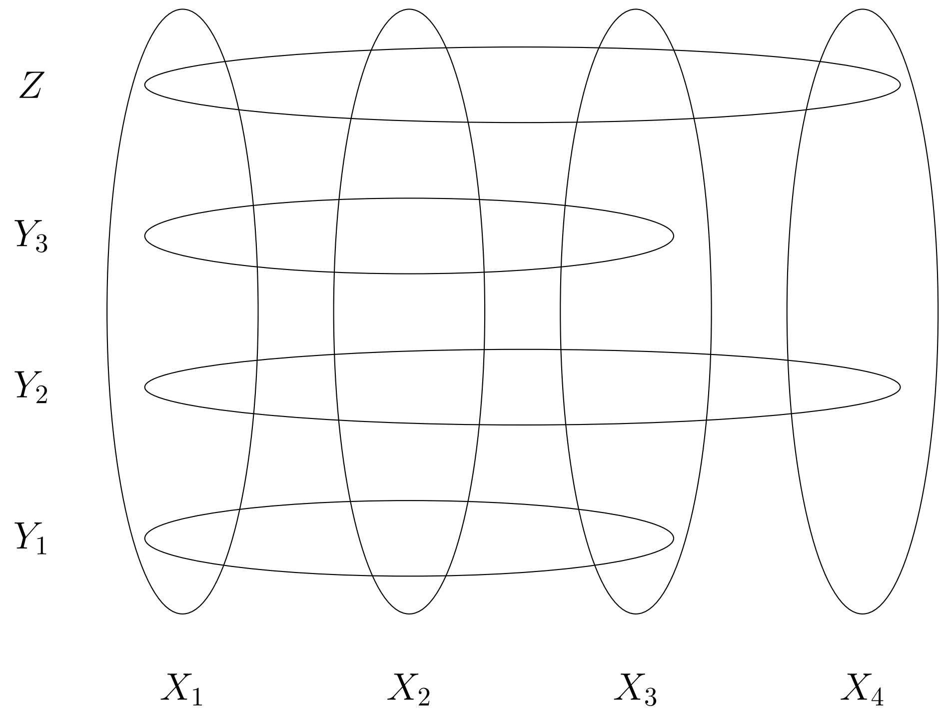


Figure 2.1: A visualization for Lemma 2.0.2

Next, we will establish a number of intermediate results before proving Theorem 2.0.1.

Lemma 2.0.3 *Let G be a connected (Claw, $4K_1$, hole-twin)-free graph. If G contains a C_7, then G has at most 21 vertices.*

Proof. Suppose that G contains a 7-hole H, with vertices $h_1, ..., h_7$ and edges h_ih_{i+1}, with the subscripts taken modulo 7. A vertex in $G - H$ is a k-vertex if it is adjacent to k vertices in H.

Let Y_i denote the set of 4-vertices adjacent to $h_i,\ h_{i+1},\ h_{i+2},\ h_{i+3}$. Let Z_i denote the sets of 4-vertices adjacent to $h_i,\ h_{i+1},\ h_{i+3},\ h_{i+4}$. It is easy to see that a 4-vertex must be of type Y_i, or Z_i.

Observation 2.0.4 *G has no k-vertex* $\forall\ k \in \{0, 1, 2, 3, 5, 6, 7\}$.

Proof. If G has a k-vertex, for $k \in \{0, 1, 2\}$, then G contains a $4K_1$. If G has a k-vertex, for $k \in \{5, 6, 7\}$, then G contains a claw. If there exists some 3-vertex, then G contains a C_7-twin, or a claw.

From the above observations, it follows that a vertex in $G - H$ must be of type Y_i, or Z_i.

Observation 2.0.5 *Y_i is a clique.*

Proof. If Y_i contains non-adjacent vertices u,v, then vertices $\{h_{i+3}, h_{i+4}, u, v\}$ induces a claw. □

Observation 2.0.6 *$|Y_i| \leq 1$ for any i.*

Proof. Suppose some Y_i has at least two vertices u, v. Then $\{h_{i+3}, h_{i+4}, h_{i+5}, h_{i+6}, h_i, u, v\}$ induces a C_6-twin. □

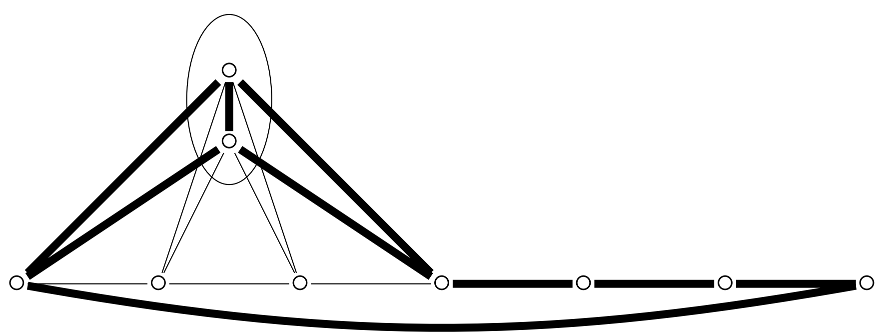


Figure 2.2: G contains a C_6-twin if some Y_i has at least two vertices

Observation 2.0.7 *Z_i is a clique.*

Proof. If Z_i contains non-adjacent vertices u,v, then $\{h_i, h_{i+6}, u, v\}$ induces a claw. □

Observation 2.0.8 *$|Z_i| \leq 1$ for any i.*

Proof. Suppose some Z_i has at least two vertices u, v. Then $\{h_{i+4}, h_{i+5}, h_{i+6}, h_i, u, v\}$ induces a C_5-twin. □

From the above observation, we have $|V(G)| = |V(H)| + \sum_{i=1}^{7} |Z_i| + \sum_{i=1}^{7} |Y_i| \leq 21$. We have established Lemma 2.0.3. □

Lemma 2.0.9 *Let G be a connected (Claw, $4K_1$, hole-twin)-free graph. If G contains a C_5, then either $\alpha(G) = 2$, or G has bounded clique width, or both.*

Proof. Suppose G contains a 5-hole H, with vertices $h_1, ..., h_5$, and edges $h_i h_{i+1}$ with the subscripts taken modulo 5. We define the following sets, for each $i \in \{1, ..., 5\}$.

- Let X_i be the set of 2-vertices adjacent to h_{i-2} and h_{i+2}.
- Let Y_i be the set of 4-vertices not adjacent to h_i.
- Let R be the set of 0-vertices.
- Let T be the set of 5-vertices.

We begin our proof of the Lemma with a number of observations. Let $Y = Y_1 \cup \ldots \cup Y_5$, and $X = X_1 \cup \ldots \cup X_5$.

Observation 2.0.10 *We have T ⓪ R.*

Proof. If there is an edge between a vertex $t \in T$ and a vertex $r \in R$, then G has a claw with t, r, and some two vertices in H. □

Observation 2.0.11 *We have T ⓪ X.*

Proof. If there is an edge between a vertex $t \in T$ and a vertex $x \in X$, then G has a claw with t, y, and some two vertices (that are non-neighbors of x) in H. □

Observation 2.0.12 *We have* $T \textcircled{1} Y$.

Proof. Suppose a vertex $t \in T$ is not adjacent to some vertex $y \in Y_i$ for some i. Then the set $\{y, h_{i-1}, t, h_{i+1}, h_i\}$ induces a C_4-twin. □

Observation 2.0.13 *We have* $R \textcircled{0} Y$.

Proof. If there is an edge between a vertex $r \in R$ and a vertex $y \in Y$, then G has a claw with r, y, and some two vertices in H. □

Observation 2.0.14 G *has no* k*-vertex* $\forall\, k \in \{1, 3\}$.

Proof. Suppose G has 1-vertex, then G contains a claw. If there exists some 3-vertex, then G contains a C_5-twin or a claw. □

Observation 2.0.15 X_i *is a clique.*

Proof. Let $u, v \in X_i$ and $uv \notin E$. Then $\{u, v, h_{i+1}, h_{i+2}\}$ induces a claw. □

Observation 2.0.16 Y_i *is a clique.*

Proof. Let $u, v \in Y_i$ and $uv \notin E$. Then $\{u, v, h_i, h_{i+1}\}$ induces a claw. □

Observation 2.0.17 $|Y_i| \leq 1$ *for* $i = 1, 2, \ldots, 5$.

Proof. Suppose some Y_i contains two vertices u, v. By Observation 2.0.16, uv is an edge of G. Now, $\{h_{i-1}, h_i, h_{i+1}, u, v\}$ induces a C_4-twin. □

Observation 2.0.18 R *is a clique.*

Proof. If R is not a clique, then some two non-adjacent vertices of R and some two non-adjacent vertices of H induce a $4K_1$. □

Observation 2.0.19 *A vertex u of X_i cannot be adjacent to two vertices in X_{i+1}, and by symmetry, u cannot be adjacent to two vertices of X_{i-1}.*

Proof. Let $u \in X_i$, $v, k \in X_{i+1}$ and $uv \in E$, $uk \in E$. Then $\{u, v, h_{i-1}, h_i, h_{i+1}, h_{i+2}, k\}$ induces a C_6-twin. □

Observation 2.0.20 *A vertex u of X_i cannot be adjacent to two vertices in X_{i+2}; and by symmetry, u cannot be adjacent to two vertices of X_{i-2}.*

Proof. Let $u \in X_i$, $v, k \in X_{i+2}$ and $uv \in E$, $uk \in E$. Then $\{u, v, h_i, h_{i+1}, h_{i+2}, k\}$ induces a C_5-twin. □

For a vertex $x \in X_i$ for some i, define $N_F(x)$ to be the set of vertices y such that xy is an edge, and $y \in X_j$ for some $j \neq i$. By Observations 2.0.19 and 2.0.20, for each $x \in X_i$, we have $|N_F(x)| \leq 4$.

Observation 2.0.21 *For any i and any vertex x in X_i, the set $N_F(x)$ is a clique.*

Proof. We prove by contradiction. Let x be a vertex in X_i for some i. Suppose $N_F(x)$ is not a clique, and so there are non-adjacent vertices $y, z \in N_F(x)$. First, let us suppose $y \in X_{i+1}$. If $z \in X_{i+2} \cup X_{i-2}$, then the set $\{x, y, z, h_{i+2}\}$ induces a claw. Thus, z belongs to X_{i-1}, but now $\{h_{i+1}, h_i, h_{i-1}, y, x, z, h_{i-2}\}$ induces a C_6-twin. So we know $\{y, z\} \cap (X_{i+1} \cup X_{i-1}) = \emptyset$. Thus, we may assume $y \in X_{i+2}$ and $z \in X_{i-2}$. Now, the set $\{x_i, y, z, h_{i+2}\}$ induces a claw. We have established the observation. □

We now continue the proof of Lemma 2.0.9. We know $\alpha(T) \leq 2$ for otherwise G has a claw with one vertex in H and some three vertices in T. Suppose T contains two non-adjacent vertices t_1, t_2. Then X has to be empty, for otherwise the set $\{h, x, t_1, t_2\}$ induces a claw, where x is a vertex in X, and h is a neighbour of x in H (by Observation 2.0.11, X has no neighbours in T). Now, R has to be empty, for otherwise there is an edge rz with $r \in R$ and $z \in Y \cup T$ (since G is connected); and thus G has a claw with r, z and some two vertices in H. Now, G is the join of T and $H \cup Y$ by Observation 2.0.12. The set $Y \cup H$ cannot contain a stable set S on three vertices, for otherwise S and a vertex in T induce a claw. It follows that $\alpha(G) = 2$, and we are done. So we may assume T is a clique. Note that cliques have clique width 2.

Let G_1 be the subgraph of G obtained by removing all vertices in $H \cup Y$. Since the set Y is finite (by Observations 2.0.17), by folklore 1.3.3, we only need to prove G_1 has bounded clique width. In G_1, there are no edges between T (if it is not empty) and $X \cup R$ by Observations 2.0.10 and 2.0.11. So, by folklore 1.3.5, we only need to prove the graph G_2 induced by $X \cup R$ has bounded clique width.

There is an edge between any vertex $r \in R$ and any vertex $x \in X$, for otherwise there is a $4K_1$ containing r, x, and some two vertices of H. So, G_2 is the join of R and X. By folklore 1.3.4, we only need to prove $G_3 = G_2 - R = X$ has bounded clique width. Recall Observation 2.0.21 that for each $x \in X_i$, $N_F(x)$ is a clique. Thus, G_3 satisfies the hypothesis of Lemma 2.0.2, and so it has bounded clique width. The proof of Lemma 2.0.9 is completed.

Now, we can prove the main theorem

Proof of Theorem 2.0.1. Let G be a (Claw, $4K_1$, *hole-twin*)-free graph. We may assume that G is connected and has $\alpha(G) \geq 3$. We may assume that G is not perfect, for otherwise we may use the algorithm of Hsu [27] to color a claw-free perfect graph in polynomial time. Thus G contains an odd hole or odd antihole. By Lemma 1.4.2, we know G must contain an odd hole H. Since $\alpha(G) < 4$, H is a 7-hole or a 5-hole. If H is a 7-hole, then by Lemma 2.0.3, G has a bounded number of vertices and we are done. So H is a 5-hole. By Lemma 2.0.9, G has bounded clique width and we are done. □

Chapter 3

($4K_1$,*hole-twin*,5-*wheel*)-free graphs

In the last section, we proved that the coloring (Claw, $4K_1$, *hole-twin*)-free graphs can be solved in polynomial time. A larger problem is the coloring of ($4K_1$, *hole-twin*)-free graphs. We list some known results that will be useful tools in this work.

Theorem 3.0.1 *[10] K_4-free co-chordal graphs have unbounded clique width.*

Lemma 3.0.2 *($4K_1$,hole-twin)-free graphs have unbounded clique width.*

Proof. By Theorem 3.0.1 and 1.3.1, $4K_1$-free chordal graphs have unbounded clique width. Since ($4K_1$, *hole-twin*)-free graphs is a super class of $4K_1$-free chordal graphs, ($4K_1$,*hole-twin*)-free graphs have unbounded clique width. □

In this section, we design a polynomial time algorithm to color ($4K_1$,*hole-twin*,5-*wheel*)-free graphs. We will prove that these graphs have bounded clique width, and so by Thereom 1.3.6, they can be colored in polynomial time.

First, we consider the case the graphs contain a C_7.

Lemma 3.0.3 *Let G be a ($4K_1$,hole-twin)-free graph. If G contains a C_7, then G has bounded clique width.*

We will prove a number of preliminary results before establishing Lemma 3.0.3. Consider a ($4K_1$, *hole-twin*, 5-*wheel*)-free graph G with a 7-hole H, with vertices $h_1, ..., h_7$, with the subscripts taken modulo 7. Let k-vertex denotes the set of vertices that are adjacent to k vertices in H. We will eventually show that the sets of k-vertices with $k < 7$ are finite, and the set of 7-vertices form a clique.

Observation 3.0.4 *G has no k-vertex $\forall\ k \in \{0, 1, 2\}$.*

Proof. Suppose G has k-vertex, x for $k \in \{0,\ 1,\ 2\}$. Then, G contains a $4K_1$. □

Now, we examine the sets of 3-vertices. We will show all 3-vertices belong to the set T_i defined below.

- Let T_i to denote the set of 3-vertices adjacent to $h_i,\ h_{i+1},\ h_{i+4}$.

Observation 3.0.5 *Any 3-vertex of G must belong to T_i.*

Proof. Suppose G has 3-vertex v. If v is adjacent to 3 consecutive vertices in H, then G contains a C_7-twin. If v is not adjacent to two consecutive vertices in H, then G contains a $4K_1$. So, we may assume v is adjacent to h_i, h_{i+1} and not to h_{i+2}, h_{i+6}. The edge vh_{i+4} must be present, for otherwise, $\{v, h_{i+2}, h_{i+4}, h_{i+6}\}$ induces a $4K_1$. Now, v clearly belongs to T_i. □

The next obvervations will show that the sets T are finite.

Observation 3.0.6 *$|T_i| \leq 1$ for any i.*

Proof. Suppose T_i contains at least two vertices u, v. If uv is not an edge, then G contains a $4K_1$. So we may assume uv is an edge. Now G contains a C_5-twin with vertices $\{h_{i+1}, h_{i+2}, h_{i+3}, h_{i+4}, u, v\}$. □

Now, we examine the sets of 4-vertices. We will show all 4-vertices belong to the sets Y defined below.

- Let Y_{1i} to denote the set of 4-vertices adjacent to $h_i,\ h_{i+1},\ h_{i+2},\ h_{i+3}$.
- Let Y_{2i} to denote the set of 4-vertices adjacent to $h_i,\ h_{i+1},\ h_{i+3},\ h_{i+4}$.
- Let Y_{3i} to denote the set of 4-vertices adjacent to $h_i,\ h_{i+1},\ h_{i+2},\ h_{i+4}$.

Observation 3.0.7 *Any 4-vertex of G must belong to $Y_{1i} \cup Y_{2i} \cup Y_{3i}$ for some i.*

Proof. Suppose G has 4-vertex v. If v is adjacent to 4 consecutive vertices in H, then v belongs to Y_{1i} for some i. Suppose v is adjacent to $h_i,\ h_{i+1},\ h_{i+2}$ and not to h_{i+3}, h_{i+6}. Now v is adjacent to h_{i+4}, or h_{i+5}, but not both. Then, v belongs to Y_{3i}. Finally, we may suppose v is adjacent to two, but not three, consecutive vertices of H. So, suppose v is adjacent to h_i, h_{i+1}, and not to h_{i+2}, h_{i+6}. Vertex v must be adjacent to h_{i+4}, for otherwise G has a $4K_1$ with vertices $v, h_{i+2}, h_{i+4}, h_{i+6}$. Now, v belongs to Y_{2i}. □

The next obvervations will show that the sets Y are finite.

Observation 3.0.8 *We have $|Y_{ji}| \leq 1$ for any j and any i ($j = 1, 2, 3$; $i = 1, 2, \ldots 7$)*

Proof. Consider the set Y_{1i}. If Y_{1i} contains non-adjacent vertices u,v, then $\{h_i, h_{i+1}, h_{i+3}u, v\}$ induces a C_4-twin. Now we assume Y_{1i} contains two adjacent vertices u,v, then $\{h_i, h_{i+3}, h_{i+4}, h_{i+5}, h_{i+6}, u, v\}$ induces a C_6-twin. So $|Y_{1i}| \leq 1$. □

Now, consider the set Y_{2i}. If Y_{2i} contains non-adjacent vertices u,v, then $\{h_i, h_{i+1}, h_{i+3}u, v\}$ induces a C_4-twin. Now we assume Y_{2i} contains two adjacent vertices u,v, then $\{h_i, h_{i+4}, h_{i+5}, h_{i+6}, u, v\}$ induces a C_5-twin. So, $|Y_{2i}| \leq 1$.

Finally, consider the set Y_{3i}. If Y_{3i} contains non-adjacent vertices u,v, then $\{h_i, h_{i+1}, h_{i+4}u, v\}$ induces a C_4-twin. Now we assume Y_{3i} contains at two adjacent vertices u,v, then $\{h_i, h_{i+4}, h_{i+5}, h_{i+6}, u, v\}$ induces a C_5-twin. So, $|Y_{3i}| \leq 1$. □

Now, we examine the set of 5-vertices.

- Let Z_{1i} to denote the set of 5-vertices adjacent to $h_i,\ h_{i+1},\ h_{i+2},\ h_{i+3},\ h_{i+4}$.
- Let Z_{2i} to denote the set of 5-vertices adjacent to $h_i,\ h_{i+1},\ h_{i+2},\ h_{i+3},\ h_{i+5}$.
- Let Z_{3i} to denote the set of 5-vertices adjacent to $h_i,\ h_{i+1},\ h_{i+2},\ h_{i+4},\ h_{i+5}$.

Observation 3.0.9 *Any 5-vertex of G must belong to $Z_{1i} \cup Z_{2i} \cup Z_{3i}$ for some i.*

Proof. Let v be a 5-vertex of H. Vertex v must be adjacent to at least three consecutive vertices of H. If v is adjacent to five consecutive vertices of H, then v belongs to some Z_{1i}. If v is adjacent to four consecutive vertices of H, but not to five consecutive vertices, the v belongs to Z_{2i}. Finally, if v is adjacent to three consecutive vertices of H, but not to four consecutive vertices, the v belongs to Z_{3i}. □

Observation 3.0.10 *We have $|Z_{ji}| \leq 1$ for $j = 1, 2, 3$.*

Proof. Consider Z_{1i}, the set of 5-vertex that are adjacent to $\{h_i, h_{i+1}, h_{i+2}, h_{i+3}, h_{i+4}\}$. Let u,v be two vertices in Z_i. If $uv \in E$, then $\{h_i, h_{i+6}, h_{i+5}, h_{i+4}, u, v\}$ induces a C_5-twin. Now if $uv \notin E$, then $\{h_i, h_{i+1}, h_{i+4}, u, v\}$ induces a C_4-twin. So we have $|Z_{1i}| \leq 1$.

Consider Z_{2i}, the set of 5-vertex that are adjacent to $\{h_i, h_{i+1}, h_{i+2}, h_{i+3}, h_{i+5}\}$. Let u,v be two vertices in Z_i. If $uv \in E$, then $\{h_i, h_{i+6}, h_{i+5}, u, v\}$ induces a C_4-twin. Now if $uv \notin E$, then $\{h_i, h_{i+1}, h_{i+5}, u, v\}$ induces a C_4-twin.

Finally, consider Z_{3i}, the set of 5-vertex that are adjacent to $\{h_i, h_{i+1}, h_{i+2}, h_{i+4}, h_{i+5}\}$. Let u,v be two vertices in Z_i.If $uv \in E$, then $\{h_i, h_{i+5}, h_{i+6}, u, v\}$ induces a C_4-twin. Now if $uv \notin E$, then $\{h_1, h_{i+1}, h_{i+5}, u, v\}$ induces a C_4-twin. □

Now, we examine the set of 6-vertex of G.

- Let M_i to denote the set of 6-vertices adjacent to $h_i,\ h_{i+1},\ h_{i+2},\ h_{i+3},\ h_{i+4},\ h_{i+5}$.

Clearly, every 6-vertex belongs to some M_i.

Observation 3.0.11 *We have $|M_i| \leq 1$ for all i.*

Proof. Consider M_i, the set of 6-vertex that are adjacent to $\{h_i, h_{i+1}, h_{i+2}, h_{i+3}, h_{i+4}, h_{i+5}\}$. Let u,v be two vertices in M_i. If $uv \in E$, then $\{h_i, h_{i+5}, h_{i+6}, u, v\}$ induces a C_4-twin. Now if $uv \notin E$, then $\{h_i, h_{i+1}, h_{i+5}, u, v\}$ induces a C_4-twin. □

Now, we examine the 7-vertices of G.

Observation 3.0.12 *The set of 7-vertices induces a clique.*

Proof. Let u,v be two 7-vertices. If $uv \notin E$, then $\{h_i, h_{i+1}, h_{i+3}, u, v\}$ induces a C_4-twin. □

Proof of Lemma 3.0.3. Let G be a ($4K_1$,*hole-twin*)-free graph G with a 7-hole H. By Observations 3.0.6,3.0.8, 3.0.10, 3.0.11, the sets of k-vertices with $k < 7$ are finite. So, we can remove them and the C_7 from consideration. That is, we only need to prove the set S of 7-vertices have bounded clique width. But S is a clique by Observation 3.0.12, and so it has clique width two. □

Now, we consider the case where the graphs contain a C_5 but not a C_7.

Lemma 3.0.13 *Let G be a ($4K_1$,hole-twin, 5-wheel)-free graph G that contains 5-hole H and does not contain a 7-hole. Then G has bounded clique width.*

We will need to establish a number of preliminary results before proving Lemma 3.0.13 We assume that G contains a 5-hole H, with vertices $h_1, ..., h_5$, with the subscripts taken modulo 5. Let k-vertex denotes the set of vertices that are adjacent to k vertices in H. We define the following sets, for each $i \in \{1, ..., 5\}$.

- Let O_i be the set of 1-vertices adjacent to h_i.

- Let X_i be the set of 2-vertices adjacent to h_i and h_{i+1}.
- Let Y_i be the set of 4-vertices adjacent to h_i,h_{i+1},h_{i+2} and h_{i+3}.
- Let R be the set of 0-vertices.

Observation 3.0.14 *There are at most ten* 2*-vertices that do not belong to some* X_i*.*

Proof. Suppose there are two 2-vertices x, y that have the same neighbours in the C_5, and x, y do not belong to some X_i. Without loss of generality, we may assume x, y are adjacent to h_1, h_3. It is easy to see that G contains a $4K_1$, or a C_4-twin. □

Observation 3.0.15 *Each set* Y_i *has at most one vertex.*

Proof. Let x, y be two vertices in Y_1. If xy is not an edge, the $\{h_1, x, y, h_4, h_3\}$ induces a C_4-twin. So, xy is an edge. Now $\{h_4, h_5, h_1, x, y\}$ induces a C_4-twin. So Y_i has at most one vertex for all i. □

Observation 3.0.16 G *has no* 3*-vertices and* 5*-vertices.*

Proof. Suppose G has 3-vertex, then G contains a C_5-Twin. If G has 5-vertex, then G contains a 5-*wheel*. □

Observation 3.0.17 R *is a clique.*

Proof. Let $u, v \in R$ and $uv \notin E$. Then $\{u, v, h_{i+1}, h_{i+3}\}$ induces a $4K_1$. □

Observation 3.0.18 *The entire set* $O_1 \cup O_2 \cup \ldots \cup O_5$ *of* 1*-vetices is a clique.*

Proof. If $O_1 \cup O_2 \cup \ldots \cup O_5$ is not a clique then G contains a $4K_1$. □

Observation 3.0.19 *X_i is a clique.*

Proof. Let $u, v \in X_i$ and $uv \notin E$. Then $\{u, v, h_{i+2}, h_{i+4}\}$ induces a $4K_1$. □

Observation 3.0.20 *Any vertex in X_i cannot be adjacent to 2 vertices in X_{i+1}.*

Proof. Let $u \in X_i$, $v, k \in X_{i+1}$ and $uv \in E$, $uk \in E$. Then $\{u, v, k, h_i, h_{i+2}, h_{i+3}, h_{i+4}\}$ induces a C_6-twin. □

Observation 3.0.21 *Any vertex in X_i cannot be adjacent to 2 vertices in X_{i+2}.*

Proof. Let $u \in X_i$, $v, k \in X_{i+2}$ and $uv \in E$, $uk \in E$. Then $\{u, v, k, h_i, h_{i+3}, h_{i+4}\}$ induces a C_5-twin. □

Observation 3.0.22 *If G contains X_i, X_{i+1}, X_{i+2}, let $u \in X_i$, $v \in X_{i+1}$, $k \in X_{i+2}$ and $uv \in E$, $vk \in E$, then $uk \in E$.*

Proof. Suppose $uk \notin E$, then $\{u,\ v,\ k,\ h_i,\ h_{i+1},\ h_{i+3}\ h_{i+4}\}$ induces a C_6-twin. □

Observation 3.0.23 *If G contains X_i, X_{i+1}, X_{i+2}, let $u \in X_i$, $v \in X_{i+1}$, $k \in X_{i+2}$ and $uk \in E$, $vk \in E$, then $uv \in E$.*

Proof. Suppose $uv \notin E$, then $\{u,\ v,\ k,\ h_{i+1},\ h_{i+2}\}$ induces a C_4-twin. □

Observation 3.0.24 *If G contains X_i, X_{i+1}, X_{i+3}, let $u \in X_i$, $v \in X_{i+1}$, $k \in X_{i+3}$ and $uv \in E$, $vk \in E$, then $uk \in E$.*

Proof. Suppose $uk \notin E$, then $\{u,\ v,\ k,\ h_i,\ h_{i+1},\ h_{i+4}\}$ induces a C_5-twin. □

Observation 3.0.25 *If G contains X_i, X_{i+1}, X_{i+3}, let $u \in X_i$, $v \in X_{i+1}$, $k \in X_{i+3}$ and $uv \in E$, $uk \in E$, then $vk \in E$.*

Proof. Suppose $vk \notin E$, then $\{u,\ v,\ k,\ h_{i+1},\ h_{i+2},\ h_{i+3}\}$ induces a C_5-twin. □

Observation 3.0.26 X_i Ⓘ O_i.

Proof. Let $u \in X_i$, $v \in O_i$ and $uv \notin E$, then $\{u, v, h_{i+2}, h_{i+4}\}$ induces a $4K_1$. □

Observation 3.0.27 *Any vertex in X_i cannot be adjacent to 2 vertices in O_{i+2} and vice versa.*

Proof. Let $u, v \in X_i$, $k, j \in O_{i+2}$. Suppose $uk \in E$ and $uj \in E$, then $\{u, k, j, h_{i+1}, h_{i+2}\}$ induces a C_4-twin. Now suppose $uk \in E$ and $vk \in E$, then $\{u, v, k, h_{i+1}, h_{i+2}\}$ induces a C_4-twin. □

Observation 3.0.28 X_i Ⓘ O_{i+3}.

Proof. Let $u \in X_i$, $v \in O_{i+3}$ and $uv \notin E$, then $\{u, v, h_{i+2}, h_{i+4}\}$ induces a $4K_1$. □

Observation 3.0.29 *Any vertex in X_i cannot be adjacent to 2 vertices in O_{i+3} and vice versa.*

Proof. Let $u, v \in X_i$, $k, j \in O_{i+3}$. Suppose $uk \in E$ and $uj \in E$, then $\{u, k, j, h_{i+1}, h_{i+2}, h_{i+3}\}$ induces a C_5-twin. Now suppose $uk \in E$ and $vk \in E$, then $\{u, v, k, h_{i+1}, h_{i+2}, h_{i+3}\}$ induces a C_5-twin. □

Observation 3.0.30 *If G contains X_i and O_{i+3}, then $|X_i| = |O_{i+3}| = 1$.*

Proof. The Observation follows from Observations 3.0.28 and 3.0.29. □

The following claim is easy to verify.

Claim 3.0.31 *Let G be a graph whose vertices can be partitioned into three cliques C_1, C_2, C_3 such that each vertex in C_j is adjacent to at most one vertex of C_k, $j \neq k$. Then G is diamond-free, in particular, G has bounded clique width.*

Lemma 3.0.32 *If G contains 1-vertices and 2-vertices only, then G has a bounded clique width.*

Proof. Let Z be the set of 2-vertices that do not belong to some X_i. Let G_1 be the graph obtained by removing from G the C_5, the set Z, and any sets O_i, X_i, Y_i, R that has less than 3 vertices. Note that all Y_i are removed by Observation 3.0.15. Since we remove a fixed number of vertices, by Theorem 1.3.3, G has bounded clique width if and only if G_1 does. □

Now, we may assume each of the sets O_i, X_i, R contains at least 3 vertices. Let $P = O_1 \cup \ldots O_5 \cup X_1 \cup \ldots X_5$. It is easy to see that for any $x \in R, y \in P$, xy is an edge, for otherwise G has a $4K_1$. So G_1 is the join of P and R. So G_1 has bounded clique width if and only if P has bounded clique width. The rest of the proof is devoted to proving that P has bounded clique width. We only need to consider the sets O_i and X_i.

Claim 3.0.33 *There does not exist three consecutive sets X_i, X_{i+1}, X_{i+2}.*

Proof. Suppose there are three consecutive sets X_i, X_{i+1}, X_{i+2}. By Observations 3.0.20 and 3.0.21, a vertex in X_j is adjacent to at most one vertex in X_k with $j \neq k, \{j, k\} \subset \{i, i+1, i+2\}$. So the set $X_i \cup X_{i+1} \cup X_{i+2}$ contains a stable set S on three vertices. Now S and v_{i+4} induces a $4K_1$, a contradiction. □

Claim 3.0.34 *If the sets O_i, X_i, X_{i+4} are all non-empty, then G contains a C_7-twin.*

Proof. Suppose that $X_1 \neq \emptyset$, $X_4 \neq \emptyset$ $X_5 \neq \emptyset$. Consider a vertex $u \in X_5$. Then by Observation 3.0.20, there are two vertices $x, y \in X_5$ that are not adjacent to u. Let v be a vertex in O_1. By Observations 3.0.26, v is adjacent to u, x, y. So $\{u, v, x, y, h_2, h_3, h_4, h_5\}$ induces a C_7-twin. The Claim is justified. □

Claim 3.0.35 *If the sets O_i, O_{i+1}, X_{i+1} are all non-empty, then G contains a C_7-twin.*

Proof. Suppose that $O_1 \neq \emptyset$, $O_2 \neq \emptyset$ $X_2 \neq \emptyset$. Consider a vertex $x \in O_1$. By Observation 3.0.27, there is a vertex $y \in X_2$ that is not adjacent to x. Let u, v be two vertices in O_2. Then $\{x, u, v, y, h_3, h_4, h_5, h_1\}$ induces a C_7-twin. □

Claim 3.0.36 *If the sets O_i, X_i, O_{i+2} are all non-empty, then G contains a C_5-twin.*

Proof. Suppose $O_1 \neq \emptyset, X_1 \neq \emptyset$ and $O_3 \neq \emptyset$. Consider a vertex $x \in X_1$. By Observation 3.0.27, there is a vertex $y \in O_3$ that is not adjacent to x. Consider a vertex $z \in O_1$. By Observations 3.0.18 and 3.0.26, vertex z is adjacent to x, y. Now $\{y, z, x, h_2, h_3, h_1\}$ induces a C_5-twin. □

To prove the lemma, we will distinguish among six cases.

Case 1: G contains no sets of 1-vertices.

By Claim 3.0.33, G contains at most 3 distinct sets X_i. And the three cliques X_i satisfy the hypothesis of Claim 3.0.31, and so G has bounded clique width.

Case 2: G contains one set of 1-vertices.

We may assume $O_1 \neq \emptyset$. By Observation 3.0.30, we have $X_3 = \emptyset$.

Suppose that $X_1 \neq \emptyset$ and $X_5 \neq \emptyset$. By Claim 3.0.34, G contains a C_7-twin, a contradiction.

Suppose that $X_1 \neq \emptyset$ and $X_5 = \emptyset$. The sets X_2 and X_4 could be non-empty. Let $X_1' = X_1 \cup O_1$. By Observation 3.0.26, X_1' is a clique. By Observations 3.0.27, 3.0.20

and 3.0.21, the three cliques X_1', X_2, X_4 satisfy the hypothesis of Claim 3.0.31 and so G has bounded clique width.

Now we may assume $X_1 = \emptyset$ and by symmetry, $X_5 = \emptyset$. The sets X_2 and X_4 could be non-empty. By Observations 3.0.27, 3.0.20 and 3.0.21, the three cliques O_1, X_2, X_4 satisfy the hypothesis of Claim 3.0.31 and so G has bounded clique width. Case 2 is settled.

Case 3: G contains two sets of 1-vertices.

Consider the case that $O_1 \neq \emptyset$ and $O_2 \neq \emptyset$. By Observations 3.0.30, we know $X_3 = \emptyset$ (because O_1 has at least three vertices). Similarly, by considering O_2, we know $X_4 = \emptyset$.

Suppose $X_2 \neq \emptyset$. Then by Claim 3.0.35, G contains a C_7-twin, a contradiction. So we may assume $X_2 = \emptyset$ and by symmetry, $X_5 = \emptyset$.

Suppose now that $X_1 \neq \emptyset$. By Observation 3.0.26, G is the join of X_1 and $O_1 \cup O_2$. By Observation 3.0.18, G is a clique and we are done. Now, we have $X_1 = \emptyset$. Since $G = O_1 \cup O_2$, it is a clique by Observation 3.0.18, and we are done.

To complete the analysis of this case, suppose $O_1 \neq \emptyset$ and $O_3 \neq \emptyset$. By Observation 3.0.30, we have $X_2 = X_5 = \emptyset$. Suppose $X_1 \neq \emptyset$. By Claim 3.0.36, G contains a C_5-twin, a contradiction. So we have $X_1 = \emptyset$, and by symmetry, $X_2 = \emptyset$. We must have $X_3 \neq \emptyset$, for otherwise G is a clique and we are done. Let $O = O_1 \cup O_3$. The two cliques O and X_4 satisfy the hypothesis of Claim 3.0.31 by Observation 3.0.27, and so G has bounded clique width.

Case 4: G contains at least 3 sets of 1-vertices.

Suppose G contains three consecutive sets O_1, O_2, O_3. By considering O_1, O_2 and Claim 3.0.35, we have $X_2 = \emptyset$, and by symmetry, $X_5 = \emptyset$. Similarly, by considering O_2, O_3 and Claim 3.0.35, we have $X_3 = \emptyset$, and by symmetry, $X_1 = \emptyset$. Finally, by considering O_2 and Observation 3.0.30, we have $X_4 = \emptyset$. Thus all the sets X_i are empty, and so G is a clique and we are done.

Now, we may assume $O_1 \neq \emptyset, O_2 \neq \emptyset, O_4 \neq \emptyset$. By considering O_1, O_2 and Claim 3.0.35, we have $X_2 = \emptyset$, and by symmetry, $X_5 = \emptyset$. Now, we will rely on Observation 3.0.30. By considering O_1 (respectively, O_2, O_4), we see that $X_3 = \emptyset$ (respectively, $X_4 = \emptyset$, $X_1 = \emptyset$). Thus all the sets X_i are empty, and so G is a clique and we are done. □

We are now ready to prove the following theorem.

Theorem 3.0.37 *Let G be a ($4K_1$, hole-twin, 5-wheel)-free graph. Then G has bounded clique width, or G is perfect.*

Proof. Let G be a ($4K_1$, *hole-twin*, 5-*wheel*)-free graph G. We may assume G contain an odd hole, or odd anti-hole, for otherwise G is perfect. Since G contains no C_4-twin, it contains no odd anti-hole on at least 7 vertices. So G contains a C_5 or C_7. Now by Lemma 3.0.3 and 3.0.13, G has bounded clique width. □

Theorem 3.0.37 implies the Corollary below.

Corollary 3.0.38 *There is a polynomial time algorithm to color a ($4K_1$, hole-twin, 5-wheel)-free graph.*

Chapter 4

Conclusion and Future Work

We showed that the problems of coloring (*claw*, $4K_1$, *hole-twin*)-free and ($4K_1$, *hole-twin*, 5-*wheel*)-free graphs can be solved in polynomial time by proving the graph G to be either perfect or have bounded clique width. Our two results are partial results to the two challenging problems of determining the complexity of coloring (*claw*, $4K_1$)-free graphs and ($4K_1$, C_4)-free graphs.

Besides what we have studied in the thesis, there are other works and open problems. We list some possible future research as follows.

- Can the coloring of (*claw*, $4K_1$, C_5)-free graphs be done in polynomial time?

  By using the same method we discussed in the thesis, we could assume there exists a C_7 in the (*claw*, $4K_1$, C_5)-free graph, otherwise G is perfect. 3-vertices and 4-vertices are the only vertices alone with the C_7 and they are cliques that adjacent to 3 and 4 consecutive vertices in the C_7. There can be at most 3 sets of different 4-vertices and up to 7 sets of 3-vertices. Instead of proving G has bounded clique width, we have tried to find the largest clique in G of size k with direct coloring of $k + c$ colors and prove it is the optimal coloring when G

has only 4-vertices. Since the number of cases is finite and we conjecture that the coloring of (*claw*, $4K_1$, C_5)-free graphs can be done in polynomial time.

- Determine the complexity of (*claw*,$4K_1$)-free graphs.

  Together with ($4K_1$,C_4)-free and (*claw*,$4K_1$,*co-diamond*)-free graphs, the complexity of (*claw*,$4K_1$)-free graphs is unknown. Some information related to this problem can be found in [33] and [17].

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