DOE/ER/53223--180

DE92 019960

Printed in U.S.A.

Available from

National Technical Information Service
U.S. Department of Commerce
5285 Port Royal Road
Springfield, VA 22161

UC 20

UNCLASSIFIED

New York University
Courant Institute of Mathematical Sciences
Magneto-Fluid Dynamics Division

MF-124 DOE/ER/53223-180

TOKAMAK EQUILIBRIA AND TRANSPORT BASED ON GRAD'S THIRTEEN MOMENT DESCRIPTION

Michael K. Tippett

U.S. Department of Energy
Grant No. DE-FG02-86ER53223

June 1992

UNCLASSIFIED

Tokamak Equilibria and Transport Based on Grad's Thirteen Moment Description

Michael Kelley Tippett

October 1992

Tokamak Equilibria and Transport

Based on Grad's Thirteen Moment Description

Author: Michael Kelley Tippett

Advisor: Harold Weitzner

Abstract

In this thesis, I study collisional transport of a hot magnetically confined plasma in a tokamak. The weakly collisional plasma is modeled by Grad's two-fluid thirteen moment equations. This model provides a better treatment of the stresses and the heat fluxes than do collisional fluid models such as Braginski's. Using physical parameters for a typical tokamak, I estimate the orders of magnitude of various effects. I obtain a reduced system by neglecting small terms in the two-fluid thirteen moment equations. This reduced model includes small particle flows, pressure anisotropy and temperature variation within flux surfaces. The reduced model is compared with standard fluid models. To understand better the behavior of solutions of this system, I expand the solution in a formal series in powers of the small parameter $(m_e/m_i)^{1/4}$. Flux coordinates are used to solve the equations in a general axisymmetric geometry. In lowest order, the equilibrium solution consists of a number of arbitrary flux functions together with a Grad-Shafranov equation relating the poloidal flux and the toroidal current. The energy dynamics of the system is complicated and requires determining the solution to high order. As corrections to the lowest order solution are calculated, the equilibrium is extended to successively longer time scales until on the time scale $\tau_e m_i/m_e$, time independent solutions are in general not possible. I calculate the time evolution of the lowest order solution on the time scale $\tau_e m_i/m_e$, a time scale consistent with experiment.

CONTENTS

List of Figures

INTRODUCTION

In this thesis, I study the transport of particles and energy of a hot magnetically confined plasma in a tokamak. Sufficiently hot and well confined plasmas are required to achieve controlled nuclear fusion. The tokamak is a particular toroidally symmetric magnetic confinement device. Magnetic confinement systems use the property that charged particles tend to move along magnetic field lines rather than across them. The magnetic field lines in a tokamak spiral around a torus and are confined in a finite volume. In an ideal plasma, particles follow the field lines and are perfectly confined. Transport across field lines is caused by non-ideal effects such as collisions, waves, instabilities, trapped particles and turbulence. This work considers transport due to collisions in a plasma represented by Grad's two-fluid thirteen moment equations. The work in this thesis consists of two parts. First, a suitable mathematical model is found. I begin with Grad's two-fluid thirteen moment equations for a plasma extract from this equation set a reduced model containing the essential physics. In the second part of my thesis, I analyze the solutions of the reduced equations. I find and describe solutions that vary slowly in time. The solutions evolve on a time scale comparable to that seen in experiment.

The Thirteen Moment Model

A first step in studying a physical system is to choose a mathematical description appropriate to the problem. Here, I will use Grad's thirteen moment description to approximate a kinetic model of a plasma [1]. In a kinetic description, each species of charged particles in the plasma is represented by a distribution function, $f(\mathbf{x}, \boldsymbol{\xi}, t)$. The distribution function gives the number probability density of particles at time t, at the position $\mathbf{x}$, with velocity $\boldsymbol{\xi}$. The evolution of the distribution functions is given by a Fokker-Planck equation [2]. The Fokker-Planck equation includes the effects of the electromagnetic fields and particle collisions on the particle distribution. The Fokker-Planck equation itself is an approximation but it is reasonable to assume that it gives a sufficiently accurate representation of a hot plasma. The electromagnetic fields are described by Maxwell's equations with source terms from the plasma.

The kinetic model described above provides detailed information about the plasma. The state space of the system $(\mathbf{x}, \boldsymbol{\xi})$ is six dimensional. Mathematical or numerical analysis of the model is difficult in general because the distribution function may have a complicated structure in velocity space. Experimental measurements give information only about a few low order velocity moments of the distribution function, such as the particle number density, the average particle velocity, and the temperature. Since experimental data are limited to a finite number of moments, it is reasonable to attempt to extract from the kinetic model a set of equations that describe the evolution of a finite number of moments. One can calculate equations for the evolution of the moments of the distribution function by taking moments of the Fokker-Planck equation. However, this procedure does not yield a closed set of equations; the equation for the moment of order N contains moments of order $(N+1)$.

The same difficulty of going from a kinetic description to a moment description is encountered in gas dynamics. There, the gas is described by the kinetic Boltzmann equation. There are a variety of methods of closing the system of moment equations; the best known are the collisional closures of Hilbert and Enskogg (for a discussion of these closure methods see [3]). These closure methods use formal calculations based on the assumption that the effect of collisions on the system is large. A measure of the collisionality of a gas is the ratio of the system dimension to the mean distance a particle travels between collisions. In a strongly collisional gas, a particle has many collisions while traveling a distance the order of the system length. A collisional system is easier to describe because collisions force the distribution function to be close to a local Maxwellian distribution, f_M. That is, the distribution function is approximately

$$f_M = n(\frac{m}{2\pi T})^{3/2} exp(-m(\boldsymbol{\xi} - \mathbf{u})^2/2T)$$

where the number density n, the temperature T and the fluid velocity $\mathbf{u}$ depend on $\mathbf{x}$ and t; the particle mass is m. In the limit of large collisionality the deviation of the distribution function from a Maxwellian can be formally found in terms of n, T, u and their gradients. In the limit of large collisionality the velocity space structure of the distribution function is greatly simplified. However, a hot plasma is weakly collisional; for the plasma I consider here, a particle travels on the order of 1000 system lengths between collisions. Hence, using a collisional closure like the Hilbert or Enskogg closures is questionable in a model representing a fusion plasma.

A method of closing the system of moment equations when the system is weakly collisional was introduced by Grad to describe a rarifed gas. In this method the distribution function is expanded about a local Maxwellian in an infinite series of Hermite polynomials in velocity. From the kinetic equation one can calculate equations for the evolution of the coefficients of the Hermite series. With some reasonable assumptions on the distribution function, the infinite set of Hermite coefficients and equations is equivalent to the original system. A level of approximation to the kinetic model is introduced by truncating the series with a finite number of terms. A closed system for the coefficients is found using the orthogonality of the expansion. By using enough terms in the series one hopes to approximate solutions to the kinetic equation. In the thirteen moment approximation all the Hermite coefficients through second order and part of the third order coefficient are used. The Hermite coefficients can be expressed as linear combinations of the velocity moments of the distribution. Since the velocity moments of the distribution function have standard physical interpretations, they are used as the unknowns rather than the Hermite coefficients. This representation leads to equations for thirteen (scalar) velocity moments: density (1), fluid velocity (3), pressure tensor (6), heat flow vector (3). The thirteen moment model contains the minimum complexity needed to include anisotropy and skewness in the velocity space structure of the distribution function. In this work I use a two-fluid thirteen moment approximation of the Fokker-Planck kinetic theory to model a tokamak plasma. Similar models were presented in [4,5].

The Transport Problem

I now discuss the application of the thirteen moment model to the problem of tokamak transport. I am considering the behavior of a hot plasma in a tokamak. A tokamak is a toroidally symmetric device as shown in Figure 1. The generated magnetic field has toroidal and poloidal components. The magnetic field lines spiral around the the tokamak, sweeping out surfaces called flux surfaces. The innermost degenerate flux surface is called the magnetic axis. One would like to confine a hot plasma to the vicinity of the magnetic axis. Since particles tend to move along field lines rather than across then, particles will tend to move in flux surfaces rather than across them. I use the thirteen moment model to examine the transport of particles and energy across flux surfaces.

The earliest calculations of transport in plasmas considered the effect of collisions on an equilibrium plasma in a uniform magnetic field. The effects of the electric field, the spatial variation of the magnetic field, and the pressure anisotropy were neglected. The results of these calculations, usually referred to as "classical" transport theory, give estimates for the rate at which energy and particles diffuse across flux surfaces. A calculation of this type is included in Section (1.2). The actual transport of particles and energy seen in experiment exceeds that predicted by classical transport theory by several orders of magnitude. In addition, classical transport predicts that the electron energy is much better confined than that of the ions, which is not seen in experiment.

Later work known as neoclassical transport theory included the effects of the electric field, the pressure anisotropy and the spatial variation of the magnetic field. The nonuniformity of the magnetic field was found to be important. These descriptions included the effects of trapped particles. Initially these calculations were done in the framework of a kinetic theory (see for example [6]); later moment methods were used (see for example [7]). Neoclassical transport theory predicts transport that is larger than in the classical theory but still smaller than what is measured.

A key difference between the analysis here and usual neoclassical calculations is the scaling of the distribution function. In neoclassical calculations the distribution function is taken to be a local Maxwellian plus an extremely small non-Maxwellian part. For many systems this may be appropriate. However, for a tokamak plasma there are reasons suggesting that other scalings should be investigated. One reason is the low collisionality of the system. The order of magnitude of the deviation of the distribution function from a local Maxwellian can be estimated to be the product of the Mach number and the mean free path. The Mach number of the flows in the system is quite small but the mean free path is very long. Another reason for the system to not be so close to a local Maxwellian is that there are sources driving the system. These sources include the magnetic field, and various forms of plasma heating. The size of these source terms are key to determining the character of the system. In this work the non-Maxwellian part of the distribution is taken to be small, but considerablely larger than in neoclassical transport theory.

The spirit and method of this work is similar to that in [8] where a simplified two-fluid Braginski model was used to describe tokamak dynamics. There, equilibria were described and a procedure for the determination of the time evolution sketched. Small particle flows were included in a self-consistent manner. The work done by the fluid stresses was found to be an important mechanism for energy dissipation. Equilibria varying within flux surfaces were found. Determination of the energy dynamics required detailed information about the equilibrium flows. Electron and ion transport was seen on a time scale comparable with experiment. An objection to the analysis in [8] and a reason for using the thirteen moment model here is that the Braginski model is derived under the assumption of large collisionality and thus its validity for a weakly collisional fusion plasma is unclear.

The full two-fluid thirteen moment equations are quite complicated, so I extract from the two-fluid thirteen moments model a reduced set of equations tailored to match the operating parameters of a typical tokamak. When one examines the relative sizes of the terms in the equations, a number of characteristic dimensionless numbers appear. I restrict my interest to a specific range of operating parameters and fit these characteristic numbers using a single parameter. By neglecting small terms in the thirteen moment equations, I obtain a reduced model. Taking this reduced set of equations as a model for tokamak dynamics, one is led to a number of mathematics and physics questions, such as whether the system closed and on what time scale does the system evolve. The structure of the equations is not standard and such issues are not immediately clear. The reduced system still contains terms of very different sizes. A reasonable approach to understanding the structure of the equations and their solution is to expand the solution in a small parameter. I do so and then solve the system order by order. As is typical in asymptotic calculations, solvability conditions play an important role. Equilibria are first found on a fast time scale and then extended to longer times scales until sources are required to maintain a steady-state. The system evolves on a time scale comparable to that seen in experiment.

Guide to the Thesis

In Chapter 1 the complete two-fluid thirteen moment equations for a plasma are derived. Forms similar to this system are found in the literature [4,5]. I show that in a particular limit the thirteen moment equations reduce to a system like the standard collisional Braginski model [9]. In this collisional limit I sketch the usual classical transport results for the perpendicular fluxes of particles and energy. In Chapter 2 a scaling suitable for a tokamak plasma is introduced. A new reduced set of equations is obtained by neglecting small terms in the full thirteen moment equations. Differences between this system and standard collisional models are discussed. Axisymmetry is used to simplify the form of the equations. The equations are written using a coordinate system related to the magnetic field. In Chapter 3 the solution is expanded in an asymptotic series and the equations are solved through $O(\sqrt{m_e/m_i})$. In Chapter 4 the system is studied through $O(m_e/m_i)$ and the time evolution on the time scale $\epsilon^{-2}\tau_{ee}$ determined. Approximations for the moments of the Fokker-Plank collision terms are calculated in Appendix A.

Notation

I use the following conventions. The subscript a is a species subscript, here either e for electrons or i for ions. Fluid variables without species subscripts are total plasma variables. Vector notation is used where possible; vectors are written in boldface. When component notation is used, the summation convention is used; that is I sum over repeated indices. I use the following notation: $w_{er,z}$ is the partial derivative with respect to z of the component of $\mathbf{w}_e$ in the r direction.

Symbols

$\mathbf{E}$	electric field
$\mathbf{B}$	magnetic field
$\mathbf{x}$	3-vector in physical space
$\boldsymbol{\xi}$	3-vector in velocity space
f_a	distribution function for species a
C_{ab}	Fokker-Planck collision operator
$\mathbf{J}$	current density
e	electron charge
c	speed of light
n	number density
$\mathbf{u}_a$	fluid velocity
v_a	thermal velocity
$\mathbf{P}_a$	pressure tensor
p_a	scalar pressure
$\mathbf{p}_a$	stress tensor
T_a	temperature
$\mathbf{S}_a$	heat flow vector
$\mathbf{R}_a$	fourth order moment
$\mathbf{y}$	normalized 3-vector in velocity space
ω_H	Hermite polynomial weight function
$H^{(n)}$	Hermite polynomial of order n
δ_{ij}	Kronecker delta
$\mathbf{F}_a$	momentum transfer due to collisions
E_a	energy transfer due to collisions
$\mathbf{T}_a$	collisional term in stress equation
$\mathbf{Q}_a$	collisional term in heat flow equation
τ_{ab}	characteristic collision time
Ω_a	gyrofrequency
ρ_a	Lamor radius
ψ	poloidal flux function
χ	toroidal part of $\mathbf{B}$
$\mathbf{A}$	vector potential
$\boldsymbol{\Phi}$	scalar potential
$V(t)$	loop voltage
$\mathbf{U}$	flow across flux surfaces
λ_a	approximate poloidal stream function
ω_a	toroidal rotation frequency
ϕ	poloidal angle
J	Jacobian of (r, z) to (ψ, ϕ)

1. THE THIRTEEN MOMENT MODEL OF A PLASMA

In this chapter I apply the thirteen moment approximation to the kinetic Fokker-Planck model of a plasma. I compare the thirteen moment system to a standard collisional model, the Braginski equations. In the limit of large collisionality I calculate the fluxes of particles and energy across the magnetic field.

1.1. The Thirteen Moment Approximation

I consider a simple hydrogen plasma of electrons and singlely charged hydrogen ions. I assume the plasma to be accurately represented by the two Fokker-Planck equations

$$\frac{\partial f_a}{\partial t} + \boldsymbol{\xi} \cdot \frac{\partial f_a}{\partial \mathbf{x}} + \frac{e_a}{m_a}(\mathbf{E} + \frac{\boldsymbol{\xi}}{c} \times \mathbf{B}) \cdot \frac{\partial f_a}{\partial \boldsymbol{\xi}} = \sum_b C_{ab} \tag{1.1}$$

where a is the species index, e for electrons, i for hydrogen ions. The distribution function $f_a(\mathbf{x}, \boldsymbol{\xi}, t)$ is defined so that

$$\int_D f_a(\mathbf{x}, \boldsymbol{\xi}, t)\, d\mathbf{x}\, d\boldsymbol{\xi}$$

is the probable number of particles of species a, in the domain D, a subset of six dimensional $(\mathbf{x}, \boldsymbol{\xi})$ space, at time t. The particles are affected by the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ through the Lorentz force. The term C_{ab} represents the effects of collisions between species a and b on the distribution function f_a. I discuss the the form of the collision operator in Appendix A. Physical constants are: e_a the charge of a particle of species a, m_a its mass and c the speed of light. In this model the plasma is assumed to be neutrally charged, that is the number density of electrons is the same as that of the ions everywhere in physical space. Charge neutrality is equivalent to

$$\int f_e(\mathbf{x}, \boldsymbol{\xi}, t)\, d\boldsymbol{\xi} = \int f_i(\mathbf{x}, \boldsymbol{\xi}, t)\, d\boldsymbol{\xi}, \tag{1.2}$$

where the integrals are over all of velocity space. The electromagnetic field are governed by a quasi-magnetostatic form of Maxwell's equations:

$$\nabla \times \mathbf{B} = 4\pi \frac{\mu_0}{c} \mathbf{J}, \tag{1.3}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \tag{1.4}$$

and

$$\nabla \cdot \mathbf{B} = 0, \tag{1.5}$$

where the current density $\mathbf{J}$ is defined to be

$$\mathbf{J} = e \int \boldsymbol{\xi}(f_e - f_i)\, d\boldsymbol{\xi}. \tag{1.6}$$

In the above system, there are nine scalar differential equations (1.1) and (1.3) – (1.5), and the constraint (1.2) for eight scalar unknowns, suggesting the system may be over determined. However, equation (1.5) is an initial condition for equation (1.4); that is (1.4) implies that $\nabla \cdot \mathbf{B}$ is constant in time and (1.5) says that constant is zero. To determine the role of the constraint (1.2) it is convenient to use the fact that

$$\int C_{ab}\, d\boldsymbol{\xi} = 0. \tag{1.7}$$

Using (1.7), equations (1.3) and (1.1) imply that

$$\frac{\partial}{\partial t}\int (f_e - f_i)\, d\boldsymbol{\xi} = 0. \tag{1.8}$$

Hence, the constraint (1.2) is also an initial condition. The Fokker-Planck-Maxwell model consists of the two kinetic equations (1.1), Maxwell's equations (1.3)–(1.5), the definition (1.6) and the charge neutrality constraint (1.2).

The Fokker-Planck-Maxwell kinetic model above permits a detailed description of a plasma. Physical quantities such as those measured in experiment can be calculated by taking moments in velocity of f_a. Let us define some velocity moments of f_a. The zero order moment,

(1.9) $$n(\mathbf{x},t) = \int f_a(\mathbf{x},\boldsymbol{\xi},t)\, d\boldsymbol{\xi},$$

is number density of particles of species a, which according to (1.2) is the same for electrons and ions. The mean velocity of particles of species a is

(1.10) $$\mathbf{u}_a(\mathbf{x},t) = \frac{1}{n}\int \boldsymbol{\xi} f_a(\mathbf{x},\boldsymbol{\xi},t)\, d\boldsymbol{\xi}.$$

The pressure tensor $\mathbf{P}_a$ is the second order moment,

(1.11) $$\mathbf{P}_a = m\int(\boldsymbol{\xi}-\mathbf{u}_a)(\boldsymbol{\xi}-\mathbf{u}_a)f_a(\mathbf{x},\boldsymbol{\xi},t)\, d\boldsymbol{\xi}.$$

It is usual to define a scalar pressure, $p_a = \frac{1}{3}tr\mathbf{P}_a$ and a trace-free stress tensor $\mathbf{p}_a = \mathbf{P}_a - p_a\mathbf{I}$. I define the temperature T_a, so that $p_a = nT_a$; the units of T_a are such that the Boltzmann constant is unity. The heat flow vector $\mathbf{S}_a$ is the third order moment

(1.12) $$\mathbf{S}_a = m_a\int(\boldsymbol{\xi}-\mathbf{u}_a)^2(\boldsymbol{\xi}-\mathbf{u}_a)f_a(\mathbf{x},\boldsymbol{\xi},t)\, d\boldsymbol{\xi}.$$

Finally, I define the fourth order moment $\mathbf{R}_a$ to be

(1.13) $$\mathbf{R}_a = m_a\int(\boldsymbol{\xi}-\mathbf{u}_a)(\boldsymbol{\xi}-\mathbf{u}_a)(\boldsymbol{\xi}-\mathbf{u}_a)(\boldsymbol{\xi}-\mathbf{u}_a)f_a(\mathbf{x},\boldsymbol{\xi},t)\, d\boldsymbol{\xi}.$$

In the thirteen moment approximation the detailed velocity space structure found in f_a is replaced by the limited information given by the velocity moments defined above.

In Grad's moment description [1] the distribution function f (it is convenient to drop the species index) is expanded in Hermite polynomials about a locally Maxwellian distribution f_0 defined as

(1.14) $$f_0(\boldsymbol{\xi}) = \frac{n}{v^3\pi^{3/2}}exp(-y^2),$$

where

(1.15) $$\mathbf{y} = \frac{\boldsymbol{\xi}-\mathbf{u}}{v}.$$

The thermal velocity v is defined by

(1.16) $$T = \frac{1}{2}mv^2.$$

The moments n, $\mathbf{u}$ and T are functions of $\mathbf{x}$ and t. The Maxwellian distribution f_0 contains the moments n, $\mathbf{u}$ and T and is spherically symmetric in velocity space. Let $f(\boldsymbol{\xi}) = f_0(r)\hat{f}(\mathbf{y})$ and expand $\hat{f}(\mathbf{y})$ in a Hermite polynomial series in $\mathbf{y}$ using the weight function ω_H,

(1.17) $$\omega_H = \frac{v^3}{n} f_0(r).$$

That is,

(1.18) $$f(\boldsymbol{\xi}) = f_0(r) \sum_{n=1}^{\infty} a^{(n)} H^{(n)}(\mathbf{y}),$$

where

(1.19) $$H^{(n)}(\mathbf{y}) = \frac{(-1)^n}{2^n \omega_H} \nabla_{\mathbf{y}}^n \omega_H.$$

The coefficients $a^{(n)}$ are functions of $\mathbf{x}$ and t. Such a series expansion converges if

(1.20) $$\int f^2 / f_0 d\xi < \infty$$

where the integration is over all velocity. The infinite series (1.18) is equivalent to f. An infinite system of equations for the coefficients $a^{(n)}$ can be found by substituting the series into the Fokker-Planck equation (1.1). A level of approximation to the kinetic theory is chosen by truncating the Hermite series for f. One can then use the orthogonality of the expansion to obtain a finite set of equations for the truncated set of coefficients. By using a sufficient number of terms, f_a can be approximated arbitrarily closely. However, there is no guarantee that the solutions of the equations satisfied by the Hermite coefficients will approximate a solution of the Fokker-Planck equation.

The first few Hermite polynomials defined by (1.19) are:

(1.21) $$H^{(0)}(\mathbf{y}) = 1,$$

$$H_j^{(1)}(\mathbf{y}) = y_j, \tag{1.22}$$

$$H_{jk}^{(2)}(\mathbf{y}) = y_j y_k - \frac{1}{2}\delta_{jk}, \tag{1.23}$$

$$H_{jkl}^{(3)}(\mathbf{y}) = y_j y_k y_l - \frac{1}{2}(y_j\delta_{kl} + y_k\delta_{jl} + y_l\delta_{jk}). \tag{1.24}$$

The polynomials are orthogonal with respect to the weight ω_H.

The thirteen moment approximation uses the Hermite polynomials through second order completely and the contracted third order polynomial

$$H_j^{(3)}(\mathbf{y}) = \frac{2}{5}H_{jkk}^{(3)}(\mathbf{y}) = y_j(\frac{2}{5}y^2 - 1). \tag{1.25}$$

In the thirteen moment approximation f has the form

$$f = f_0(r)(a^{(0)}H^{(0)} + a_j^{(1)}H_j^{(1)} + a_{jk}^{(2)}H_{jk}^{(2)} + a_j^{(3)}H_j^{(3)}). \tag{1.26}$$

The Hermite coefficients $a^{(n)}$ in terms of the moments of f are,

$$a^{(0)} = 1, \tag{1.27}$$

$$a_i^{(1)} = 0, \tag{1.28}$$

$$a_{ij}^{(2)} = p_{ij}/p, \tag{1.29}$$

$$a_i^{(3)} = \frac{1}{6}S_i/pv. \tag{1.30}$$

Thus, in the thirteen moments approximation the distribution function f is

$$f(\boldsymbol{\xi}) = f_0(r)(1 + y_j y_k \frac{p_{jk}}{p} + y_j \frac{S_j}{6pv}(\frac{2}{5}y^2 - 1)). \tag{1.31}$$

The fourth order moment $\mathbf{R}$ can calculated to be

$$R_{ijkr} = T(p_{ij}\delta_{kr} + p_ik\delta_{jr} + p_{ir}\delta_{jk} + p_{jk}\delta_{ir} + p_{jr}\delta_{ik} + p_{kr}\delta_{ij})$$
$$+pT(\delta_{ij}\delta_{kr} + \delta_{ik}\delta_{jr} + \delta_{ir}\delta_{jk}). \tag{1.32}$$

The thirteen moment approximation contains the minimum number of moments needed to introduce anisotropy and skewness in the velocity dependence of f. The coefficients of the Hermite polynomials are linear combination of the moments of the distribution function. Since the moments of the distribution function have standard physical interpretations, it is convenient to use the moments as the unknowns instead of the Hermite coefficients. Hence, in the thirteen moment description the unknowns are the density n, the fluid velocity $\mathbf{u}$, the pressure p, the stress tensor $\mathbf{p}$ and the heat flow vector $\mathbf{S}$.

Taking moments of (1.1) gives the following equations:

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}_a) = 0 \tag{1.33}$$

$$m_a n\frac{\partial \mathbf{u}_a}{\partial t} + m_a n(\mathbf{u}_a \cdot \nabla)\mathbf{u}_a + \nabla \cdot \mathbf{P}_a - e_a n(\mathbf{E} + \frac{\mathbf{u}_a}{c} \times \mathbf{B}) = \mathbf{F}_a \tag{1.34}$$

$$\frac{\partial p_a}{\partial t} + \nabla \cdot (\mathbf{u}_a p_a + \frac{1}{3}\mathbf{S}_a) + \frac{2}{3}\mathbf{P}_a : \nabla \mathbf{u}_a = E_a \tag{1.35}$$

$$\frac{\partial \mathbf{p}_a}{\partial t} + \nabla_j \cdot (\mathbf{u}_{aj}\mathbf{p}_a) + \frac{1}{5}\{\nabla \mathbf{S}_a\} + \{\mathbf{p}_a \cdot \nabla \mathbf{u}_a\}$$
$$+p_a\{\nabla \mathbf{u}_a\} - \frac{e_a}{m_a c}\{\mathbf{p}_a \times \mathbf{B}\} = \mathbf{T}_a \tag{1.36}$$

$$\frac{\partial \mathbf{S}_a}{\partial t} + \nabla_j \cdot (\mathbf{u}_{aj}\mathbf{S}_a) + \frac{7}{5}(\mathbf{S}_a \cdot \nabla)\mathbf{u}_a + \frac{2}{5}\nabla \mathbf{u}_a \cdot \mathbf{S}_a + \frac{2}{5}\mathbf{S}_a \nabla \cdot \mathbf{u}_a$$
$$+\frac{1}{m_a n}(2\mathbf{p}_a + 5p_a\mathbf{I}) \cdot \mathbf{F}_a - \frac{2}{m_a n}\mathbf{p}_a \cdot \nabla \cdot (\mathbf{p}_a + p_a\mathbf{I}) + \frac{2T_a}{m_a}\nabla \cdot \mathbf{p}_a$$
$$+\frac{1}{m_a}\nabla T_a \cdot (7\mathbf{p}_a + 5p_a\mathbf{I}) - \frac{e_a}{m_a c}(\mathbf{S}_a \times \mathbf{B}) = \mathbf{Q}_a. \tag{1.37}$$

The relation (1.32) is used to calculate the fourth order moment that appears in (1.37). The moments of the collision term are

(1.38) $$\mathbf{F}_a = \sum_b \int \boldsymbol{\xi} C_{ab}\, d\boldsymbol{\xi},$$

(1.39) $$E_a = \sum_b \int (\boldsymbol{\xi} - \mathbf{u}_a)^2 C_{ab}\, d\boldsymbol{\xi},$$

(1.40) $$\mathbf{T}_a = trace\ free \sum_b \int (\boldsymbol{\xi} - \mathbf{u}_a)(\boldsymbol{\xi} - \mathbf{u}_a) C_{ab}\, d\boldsymbol{\xi},$$

and

(1.41) $$\mathbf{Q}_a = \sum_b \int (\boldsymbol{\xi} - \mathbf{u}_a)^2 (\boldsymbol{\xi} - \mathbf{u}_a) C_{ab}\, d\boldsymbol{\xi}.$$

In (1.36) I use the notation

(1.42) $$\{W_{jk}\} = W_{jk} + W_{jk} - \frac{2}{3} W_{ll}\delta_{jk}.$$

In (1.33) I use that the collision operator conserves particles, equivalent to (1.7); this result along with approximations for the moments of the collision term are found in Appendix A. I retain Maxwell's equations (1.3) – (1.5) with

(1.43) $$\mathbf{J} = en(\mathbf{u}_e - \mathbf{u}_i)$$

replacing (1.6).

Several of the equations in the thirteen moment system have physical interpretations familiar from fluid dynamics. Equation (1.33) expresses the conservation of particles. Equation (1.34) describes the momentum balance, including the Lorenz force. The collision term $\mathbf{F}_a$ allows momentum transfer from one species of particles to another but conserves the total momentum of the plasma; that is, $\mathbf{F}_e = -\mathbf{F}_i$. Equation (1.35) is the balance of energy including heat flows and work done by the pressure and the stresses. The collision term, E_a in this equation conserves the total energy of the plasma; $E_e = -E_i$. The less familiar equations (1.36) and (1.37) for the stress tensor $\mathbf{p}_a$ and the heat flow vector $\mathbf{S}_a$ complete the fluid equations.

The complete two-fluid model consists of the 26 scalar equations (1.33) – (1.37) for the fluid variables, coupled to the Maxwell equations, (1.3) – (1.5) and (1.43). Again, equation

(1.5) is to be interpreted as an initial condition since a consequence of (1.4) is that if $\nabla \cdot \mathbf{B}$ vanishes initially then it vanishes for all time. The relation between charge neutrality and the condition $\nabla \cdot \mathbf{J} = 0$ implied by (1.3) is perhaps clearer in the moment equations than in the Fokker-Planck-Maxwell system. If one assumes equation (1.33) for one species and equation (1.3), then equation (1.33) for the other species is a consequence.

The two-fluid thirteen moment description presented above appears quite complex. However, the thirteen moment model represents a considerable simplification of the kinetic theory. By assuming the distribution functions to have a simple structure in velocity space, the phase space has been reduced from six dimensions to three. Enough features have been retained in the model, such as a realistic treatment of the stresses and the heat flows, that one expects the model to provide a useful description of a tokamak plasma.

1.2. Classical Transport from the Thirteen Moment Model

In this section I examine the connection between the thirteen moment system and a widely used collisional model, the Braginski equations [9]. The Braginski model consists of equations for mass, momentum and energy balance along with relations for momentum and energy transfer due to collisions; the unknowns are the densities, the fluid velocities and the temperatures. The form of the equations for conservation of mass, momentum and energy is the same in both models; differences appear in the determination of the stress tensor and the heat flow vector. In the Braginski model, the stress tensor and the heat flow are expressed in terms of the density, the fluid velocity, the temperature, the magnetic field and their gradients. In the thirteen moment model the stress tensor and heat flow are solutions to differential equations. I show that by neglecting terms in the thirteen moment system a Braginski-like model can be obtained. The interest here is in identifying the assumptions underlying the Braginski model and examining their validity for a fusion plasma. The following are qualitative features of a fusion plasma: the system evolves on a time scale that is long compared to the collision time, the flow velocities are small relative to the thermal velocity, the pressure anisotropy is small, and the mean free path is long.

The differential equation for the stress tensor (1.36) can be reduced to an algebraic equation that can be solved explicitly, if terms containing derivatives of the stress tensor and derivatives of the heat flow vector are neglected and the stress tensor is taken to be small compared to the scalar presure. The resulting equation is:

$$p_a\{\nabla \mathbf{u}_a\} - \frac{e_a}{m_a c}\{\mathbf{p}_a \times \mathbf{B}\} = A\frac{\mathbf{p}_a}{\tau_{aa}} \tag{1.44}$$

where A is a numerical factor. Solving this equation for the components of the stress tensor p_{ij} one finds

$$p_{zz} = \frac{nT_a\tau_{aa}}{A}W_{zz} \tag{1.45}$$

$$(1 + \frac{A^2}{(\Omega_a\tau_{aa})^2})p_{yz} = -\frac{nT_aA}{\Omega_a^2\tau_{aa}}W_{yz} + \frac{nT_a}{\Omega_a}W_{xy} \tag{1.46}$$

$$(1 + \frac{A^2}{(\Omega_a\tau_{aa})^2})p_{xz} = \frac{nT_aA}{\Omega_a^2\tau_{aa}}W_{xz} + \frac{nT_a}{\Omega_a}W_{yz} \tag{1.47}$$

$$p_{xx} = nT_a\tau_{aa}((\frac{1}{A} - \frac{1}{2})W_{xx} + \frac{1}{2}W_{yy}) - \frac{A}{2\Omega_a\tau_{aa}}W_{xy} \tag{1.48}$$

$$p_{yy} = nT_a\tau_{aa}((\frac{1}{A} - \frac{1}{2})W_{yy} + \frac{1}{2}W_{xx}) + \frac{A}{2\Omega_a\tau_{aa}}W_{xy} \tag{1.49}$$

where **B** is taken to be in the z direction and $\mathbf{W} = \{\nabla \mathbf{u}_a\}$. This agrees with the Braginski results up to numerical factors of order one. The gyrofrequency Ω_a is

$$\Omega_a = \frac{eB}{m_a c}, \tag{1.50}$$

the like species collision time τ_{aa} is defined in Appendix A. The assumptions used to obtain (1.44) are:

$$\frac{\tau_{aa}}{\hat{t}} \ll 1, \tag{1.51}$$

the collision time is small compared to a characteristic time $\hat{t}$,

$$\frac{\mathbf{p}_a}{p_a} \ll 1, \tag{1.52}$$

the trace-free part of the stress tensor is small compared to the scalar pressure, and

$$\frac{\mathbf{S}_a}{p_a \mathbf{u}_a} \ll 1. \tag{1.53}$$

The assumptions (1.51) – (1.52) are consistent with the features of a fusion plasma mentioned before. The last assumption (1.53) used to neglect terms containing gradients of the heat flow, is less clearly appropriate. Later I will take $\mathbf{S}_a \sim p_a \mathbf{u}_a$ and retain the effects of the heat flow strain in the equation for the stress tensor.

The magnitude of the diagonal terms of the Braginski stress tensor is roughly

$$\frac{\mathbf{u}_a}{v_a} \frac{v_a \tau_{aa}}{L} p_a, \tag{1.54}$$

where L is the system length scale. The second factor in (1.54) $v_a \tau_{aa}/L$ the ratio of the mean free path to the system length, is a parameter that is large in a fusion plasma. If the scaling assumption (1.52) is to hold and the stress tensor be small relative to the pressure, the first factor in (1.54) $\mathbf{u}_a/v_a$ corresponding roughly to the Mach number of the flow, must be small. Hence, when the mean free path is long, particle flows with flow velocities small compared to the thermal velocity can lead to significant production of stresses.

I now consider the determination of the heat flow. By neglecting terms, the thirteen moment equation for the heat flow (1.37) can be reduced to the algebraic equation

$$\frac{5nT_a}{m_a} \nabla T_a - \frac{e_a}{m_a c} (\mathbf{S}_a \times \mathbf{B}) = A_1 \frac{\mathbf{S}_a}{\tau_{aa}} + A_2 \frac{nT_a}{\tau_{aa}} (\mathbf{u}_a - \mathbf{u}_b), \tag{1.55}$$

where A_1 and A_2 are numerical factors of order one. Equation (1.55) can be solved to obtain

$$\begin{aligned} (1 - \frac{A_1^2}{(\Omega_a \tau_{aa})^2}) \mathbf{S}_a = {} & \frac{5nT_a \tau_{aa}}{A_1 m_a} \nabla_{\|} T_a - \frac{5nT_a}{m_a \Omega_a} \mathbf{b} \times \nabla T_a + \frac{5 A_1 n T \tau_{aa}}{(m_a \Omega_a \tau_{aa})^2} \nabla_{\perp} T_a \\ & + A_2 n T_a \mathbf{b} \cdot (\mathbf{u}_a - \mathbf{u}_b) \mathbf{b} + A_2 \frac{nT_a}{\tau_{aa} \Omega_a} \mathbf{b} \times (\mathbf{u}_a - \mathbf{u}_b), \end{aligned} \tag{1.56}$$

where $\mathbf{b}$ is the unit vector in the direction of $\mathbf{B}$, $\nabla_{\|} = (\mathbf{b} \cdot \nabla)$ and $\nabla_{\perp} = -\mathbf{b} \times (\mathbf{b} \times \nabla)$. The representation (1.56) agrees with Braginski up to numerical factors of order one. The assumptions used to neglect terms in equation (1.37) are:

$$\frac{\mathbf{S}_a}{p_a \mathbf{u}_a} \frac{\mathbf{u}_a}{v_a} \frac{L}{v_a \tau} \frac{\tau}{\hat{t}} \ll 1 \tag{1.57}$$

(1.58)
$$\frac{\mathbf{S}_a}{p_a \mathbf{u}_a}(\frac{\mathbf{u}_a}{v_a})^2 \ll 1$$

(1.59)
$$\frac{\mathbf{u}_a}{v_a}\frac{L}{v_a \tau} \ll 1,$$

and

(1.60)
$$\frac{\mathbf{P}_a}{p_a} \ll 1.$$

These assumptions are reasonable for a fusion plasma at least in the sense of a lowest order approximation. The usual conclusion drawn from equation (1.55) is that the temperature gradient in the direction of $\mathbf{B}$ must be small since it is balanced only by the very small collisional terms on the left hand side. However, consider the size of the collisional terms that are retained in (1.55). The sizes of the two terms on the right hand side of (1.55) (relative to the term containing the gradient of the temperature on the left hand side) are respectively

(1.61)
$$\frac{\mathbf{S}_a}{p_a \mathbf{u}_a}\frac{\mathbf{u}_a}{v_a}\frac{L}{v_a \tau},$$

and

(1.62)
$$\frac{\mathbf{u}_a}{v_a}\frac{L}{v_a \tau}.$$

The two collisional terms in (1.55) are of comparable size to those that have been neglected. If instead of retaining the collisional terms in (1.55) the terms involving the stress tensor are kept, terms that I have previously argued may be significant, then the temperature variation along field lines may be balanced by the effects of the stresses.

The assumptions needed to extract a collisional Braginski-like system from the thirteen moments have been stated quite explicitly. Assuming that the thirteen moment approximation is more accurate, one can comment on the validity of the Braginski model in weakly collisional regimes. Most of the assumptions used here hold even in the case of long mean free path. The Braginski form of the stress tensor except for not including the heat strain does seem reasonable in lowest order. The Braginski form of the heat flow is more questionable in the weakly collisional regime. The Braginski form of the heat flow in the direction perpendicular to **B** seems reasonable but there are difficulties with the heat flow parallel to **B**.

I now sketch how "classical" transport results are derived from this collisional model. First, I describe how to calculate the flux of particles across the magnetic field due to collisions. Take the momentum equation (1.34) and neglect the derivatives of $\mathbf{u}_a$. Take only the scalar pressure p_a and neglect the electric field **E**. For simplicity take just two species of charged particles denoted by a and b. The momentum balance is then

$$\nabla p_a - \frac{e_a}{c} n\mathbf{u}_a \times \mathbf{B} = \frac{-m_a n}{\tau_{ab}}(\mathbf{u}_a - \mathbf{u}_b) + \frac{3}{5}\frac{1}{v_a^2 + v_b^2}\left(\frac{\mathbf{S}_a}{\tau_{ab}} - \frac{\mathbf{S}_b}{\tau_{ba}}\right), \tag{1.63}$$

where I have used the expression for $\mathbf{F}_a$ given by (A.25). Then one can solve for $\mathbf{u}_{a\perp} = \mathbf{b} \times (\mathbf{u}_a \times \mathbf{b})$

$$n\mathbf{u}_{a\perp} = -\frac{c}{e_a B}\mathbf{b} \times \left(\frac{-m_a n}{\tau_{ab}}(\mathbf{u}_a - \mathbf{u}_b) + \frac{3}{5}\frac{n}{v_a^2 + v_b^2}\left(\frac{\mathbf{S}_a}{\tau_{ab}} - \frac{\mathbf{S}_b}{\tau_{ba}}\right)\right) + \frac{c}{e_a B}\mathbf{b} \times \nabla p_a. \tag{1.64}$$

Using the expression (1.56) for the heat flow and keeping terms only through order $(\Omega_a \tau_{ab})^{-2}$ gives

$$\begin{aligned} n\mathbf{u}_{a\perp} &= -\frac{1}{m_a \Omega_a^2}\frac{1}{\tau_{ab}}\frac{1}{c}(\mathbf{J} \times \mathbf{B}) + \frac{c}{e_a B}\mathbf{b} \times \nabla p_a \\ &+ \frac{3}{2}\frac{1}{m_a \Omega_a^2}\frac{n}{(1 + \frac{m_a T_b}{m_b T_a})}\left(\frac{1}{\tau_{ab}}\nabla_\perp T_a - \frac{1}{\tau_{ba}}\frac{T_b}{T_a}\nabla_\perp T_b\right). \end{aligned} \tag{1.65}$$

Summing equation (1.63) over species gives

$$\frac{1}{c}\mathbf{J} \times \mathbf{B} = \nabla p, \tag{1.66}$$

where $p = p_e + p_i$ is the total pressure. It is reasonable to assume that the surfaces $p = const$ form nested surfaces. From (1.66 the pressure gradient is perpendicular to **B**, and $\nabla p = \nabla_\perp p$. One then has

$$\begin{aligned} n\mathbf{u}_{a\perp} \cdot \nabla_\perp p &= -\frac{1}{m_a \Omega_a^2}\frac{1}{\tau_{ab}}|\nabla_\perp p|^2 \\ &+ \frac{3}{2}\frac{1}{m_a \Omega_a^2}\frac{n}{(1 + \frac{m_a T_b}{m_b T_a})}\left(\frac{1}{\tau_{ab}}\nabla T_a - \frac{1}{\tau_{ba}}\frac{T_b}{T_a}\nabla T_b\right) \cdot \nabla_\perp p, \end{aligned} \tag{1.67}$$

for the particle fluxes across magnetic field lines. Note that $n\mathbf{u}_{a\perp} \cdot \nabla_\perp p = n\mathbf{u}_{b\perp} \cdot \nabla_\perp p$; the particle flux across the magnetic field is the same for both species. From (1.67) one can write

$$n\mathbf{u}_{a\perp} \sim D_a \nabla_\perp n \tag{1.68}$$

where $D_a = \rho_a^2/\tau_{ab}$, and $\rho_a = v_a/\Omega_a$. Note that $D_e \sim D_i \sim D$. Using equation (1.33) and only considering perpendicular derivatives,

$$n_{,t} + D\nabla_\perp^2 n = 0. \tag{1.69}$$

This suggests that the density evolves on the time scale $\tau_{ab}L^2/\rho_a^2$. This result can be interpreted as diffusion generated by a random walk with step size ρ_a taken at time intervals τ_{ab}.

The similar relation for the heat flow across the magnetic field

$$\mathbf{S}_a \sim \chi_{\perp a} \nabla_\perp T_a \tag{1.70}$$

with $\chi_{\perp a} = n\rho_a^2/\tau_{aa}$ is obtained from (1.56). This along with equation (1.35) gives

$$T_{a,t} + \chi_{\perp a} \nabla_\perp^2 T_a = 0, \tag{1.71}$$

suggesting that the temperature evolves on the time scale $\tau_{aa}L^2/\rho_a^2$. This is equivalent to diffusion generated by a random walk with step size ρ_a taken at time intervals of τ_{aa}. Like-particle collisions lead to heat transport, in contrast to particle transport that is driven only by unlike-particle collisions. Note that $\tau_{ee} \sim \sqrt{m_e/m_i}\,\tau_{ii}$. Thus, the electron temperature evolves on a time scale that is longer by a factor of $\sqrt{m_i/m_e}$ than that of the ions.

Typical tokamak parameters are $n = 10^{20}m^{-3}$, $T_e \sim T_i = 2.5KeV$, $B = 4T$. The collisional transport calculations above suggest that the particle confinement time is on the order $\sim 8000sec$ for both electrons and ions. The energy confinement time for the electrons is predicted to be $\sim 16000sec$ and for the ions to be $200sec$. In experiment, typical energy confinement times are roughly the same for electrons and ions and are on the order of a hundred miliseconds [10]. Particle confinement times are more difficult to measure. Hence, classical collisional transport theory calculations do not at all agree with experiment.

2. Reduced Model

In this chapter I extract a reduced set of equations from the full two-fluid thirteen moment model presented in Chapter 1. For a choice of physical parameters relevant to a fusion plasma, I estimate the order of magnitude of the various terms in the full system and neglect the terms that are small. The axisymmetry of the tokamak is used to simplify the form of the equations. Finally, I write the equations using coordinates related to the magnetic field geometry.

2.1. Scaling

The two-fluid thirteen moment equation set is quite complicated. A reasonable method of reducing the complexity of the system is to estimate the sizes of the various terms in the equations and neglect those that are small. In section (1.2) it was shown that with a particular collisional scaling the thirteen moment model reduces to a Braginski-like description. Here, I will use a scaling more appropriate for a fusion plasma. A systematic way to examine the relative sizes of the terms in the equations is to non-dimensionalize the equations. Let u_a be a characteristic velocity, a a characteristic length (here the minor radius of the tokamak), $\hat{t}$ a characteristic time and let the heat flow $S \sim O(pu)$ and assume smooth laminar flow. As seen in section (1.2), several dimensionless numbers arise. Familiar from gas dynamics are the following: u_a/v_a a quantity that is roughly the Mach number of the fluid flow, $v_a\tau/a$ the ratio of the mean free path to the minor radius, and $\hat{t}/\tau_a$ the Knudsen number. The magnetic field introduces another scale into the problem. It is convenient to use the for this scale the quantity $\Omega_a\tau_a$, roughly the number of rotations a particle makes about a field line between collisions. In addition a drift velocity $\bar{v}$ is defined by

$$\frac{B}{a} = \frac{4\pi}{c}en\bar{v}. \tag{2.1}$$

One of the interesting features of this problem is that for typical tokamak parameters, the dimensionless numbers introduced above are not order one quantities. The flow velocity is quite small compared to the thermal velocity; the mean free path is long; the Knudsen number is large; $\Omega_a \tau_a$ is large. In particular consider a "typical" tokamak with the following characteristics: number density $n = 10^{20} m^{-3}$, $\mathbf{B} = 4T(tesla)$, $T_i = T_e = 2.5KeV$, minor radius $a = 0.5m$, coulomb logarithm $= 15$. Using the above parameters one can calculate for each species in a hydrogen plasma the self-collision time τ, the thermal velocity v, the mean free path $v\tau/a$ and the parameter $\Omega\tau$:

	τ	v	$v\tau/a$	$\bar{v}/v_{th}$	$\Omega\tau$
e	$2.9 \times 10^{-5} s$	$2.1 \times 10^{7}\ m/s$	1.3×10^{3}	1.9×10^{-2}	7.8×10^{5}
i	$1.7 \times 10^{-3} s$	$4.9 \times 10^{5}\ m/s$	1.8×10^{3}	8×10^{-1}	2.2×10^{7}

The usual measure of the collisionality of a plasma is the smallness of the mean free path and thus as mentioned before a fusion plasma is only weakly collisional.

To compare the relative magnitude of terms it is convenient, though somewhat artificial to introduce a single small parameter into the problem. I take as a small parameter $\epsilon^2 = m_e/m_i = 1/1836$. I write the dimensionless quantities in terms of the small parameter ϵ

	$v_{th}\tau/a$	$\bar{v}/v_{th}$	u_T/v_{th}	u_P/v_{th}	$\Omega_a\tau$
e	ϵ^{-2}	ϵ	$\epsilon^{3/2}$	ϵ^{2}	ϵ^{-4}
i	ϵ^{-2}	1	$\epsilon^{1/2}$	ϵ	ϵ^{-3}

where the P and T subscripts refer to poloidal and toroidal components. Scaling all these quantities in terms of ϵ is not completely natural and in the case of $\Omega\tau$, ϵ could be replaced by 2ϵ to better fit the data. The flow velocities are such that the Lorentz force balances the pressure gradient, that is

$$\frac{e}{c} n u_P B \sim \frac{p}{a}. \tag{2.2}$$

A priori I assume that quantities vary on the time scale $\epsilon^{-2}\tau_{ee}$. That is I take,

$$\frac{\partial}{\partial t} \sim \frac{\epsilon}{\tau_i} \sim \frac{\epsilon^2}{\tau_e}. \tag{2.3}$$

For the parameters above $\epsilon^{-2}\tau_{ee} \sim 50ms$. Actually, I will show that (2.3) is not an assumption but a property of the system.

In this scaling β, the ratio of thermal energy to magnetic energy is

$$\beta \sim \frac{p}{B^2} \sim \epsilon, \tag{2.4}$$

that is on the order of a few percent. I take the aspect ratio, the ratio of the major radius R, to the minor radius a to be

$$\frac{R}{a} \sim \epsilon^{-1/2}. \tag{2.5}$$

I take the size of the heat flow to be

$$\mathbf{S}_a \sim p\mathbf{u}_a. \tag{2.6}$$

Hence, the order of magnitude of the deviation of the distribution function from a Maxwellian due to the heat flows is $S_a/pv_a \sim u_a/v_a$ which is $O(\epsilon^{1/2})$ for the ions and $O(\epsilon^{3/2})$ for the electrons. I assume that

$$\mathrm{p}_a/p_a \sim O(\epsilon^{1/2}). \tag{2.7}$$

This implies that the distribution function f_a is locally Maxwellian to order $\epsilon^{1/2}$. This choice of scaling is quite different from the usual neo-classical scaling one which takes f_a to be a Maxwellian to order $\delta_a = \rho_a/a$, the ratio of the Larmor radius to minor radius. For the parameters presented here, $\delta_e = 6 \times 10^{-5}$ and $\delta_i = 3 \times 10^{-3}$. I present a simple argument for a choice of scaling with larger anisotropy than in the neoclassical theory. The non-Maxwellian part of the distribution depends on the particle flows and temperature gradients. This dependence is seen in the following simple calculation. Take $f = f_M(1+g)$ where f_M is a locally Maxwellian distribution. Then g satisfies the equation

$$\frac{\partial g}{\partial t} + (\boldsymbol{\xi}\cdot\nabla)g - e(E + \boldsymbol{\xi}/c \times \mathbf{B})\frac{\partial g}{\partial \boldsymbol{\xi}} + (1+g)F_s = C'(g, f_M) \tag{2.8}$$

where the source term F_s is

$$F_s = \frac{2}{v}\mathbf{y}(\mathbf{u}_{,t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \frac{1}{mn}\nabla p - e(\mathbf{E} + \frac{\mathbf{u}}{c}\times\mathbf{B}) - \frac{\mathbf{F}}{mn})$$
$$+\frac{1}{nT}(y^2 - \frac{5}{2})(nT_{,t} + n(\mathbf{u}\cdot\nabla)T + \frac{2}{3}nT\nabla\cdot\mathbf{u} - \frac{1}{3}tr\mathrm{T})$$

$$+\frac{v}{T}(y^2 - \frac{5}{2})(\mathbf{y}\cdot\nabla)T + (\mathbf{yy} - \mathbf{I}) : (\nabla\mathbf{u} + (\nabla\mathbf{u})^T - \frac{2}{3}y^2\nabla\cdot\mathbf{u}\mathbf{I}), \tag{2.9}$$

and $C'(g, f_M)$ is a collision operator with magnitude of order g/τ. The strain and temperature gradient act as sources for the non-Maxwellian part of the distribution function. The magnitude of g is then approximately

$$\frac{\mathbf{u}}{v}\frac{v\tau}{a}. \tag{2.10}$$

So that in the case of small flows but long mean free path, g may not be extremely small. The same estimate for the the size of the anisotropy can be obtained using the stress tensor equation directly and examining the terms that produce the classical stress tensor as was done in Section 1.2.

Another reason for taking the non-Maxwellian part of the distribution functions to be relatively large is that tokamaks are driven by external sources. Thus, the system is not in the thermodynamic sense a closed system. The effect of sources on the system will be included in the model. This choice of scaling is a key element in model presented here. By considering this scaling I select a different set of solutions than is considered in neoclassical calculations. The scaling used in neoclassical theory results in a system with quite different properties.

I apply the scaling described above to the thirteen moment system introduced in Chapter 1 and keep terms through $O(\epsilon^2)$. A reason for keeping terms through $O(\epsilon^2)$ is that this is the size of the temperature and density time derivatives that appear in the system. A consequence of this scaling is that in the equation for the stress tensor (1.36), the term $\mathbf{p}_a \times \mathbf{B}$ has relative size ϵ^{-4}. Thus to the order needed for this calculation the trace free tensor $\mathbf{p}_a$ has the form

$$\mathbf{p}_a = (p_{\|a} - p_a)\mathbf{bb} + (p_{\perp a} - p_a)(\mathbf{I} - \mathbf{bb}) \tag{2.11}$$

where $\mathbf{b}$ is the unit vector in the direction of $\mathbf{B}$. The condition that the stress tensor be trace free is:

$$p_{\|a} + 2p_{\perp a} = 3p_a. \tag{2.12}$$

Of the three scalar pressures that appear in (2.11) one may choose any two independent linear combinations as our variables; I take as unknowns p_a, and $(p_a - p_{\perp a})$. Thus

$$\mathbf{p}_a = 2(p_a - p_{\perp a})\mathbf{bb} - (p_a - p_{\perp a})(\mathbf{I} - \mathbf{bb}). \tag{2.13}$$

It is convenient to introduce the temperature anisotropy $(T_a - T_{\perp a}) = (p_a - p_{\perp a})/n$.

I now examine the remaining equations. The equations (1.33) for the conservation of particles are retained completely. In the momentum equations (1.34) all terms are scaled relative to the size of the pressure gradient term, p/a. The order of magnitude of the collisional terms $\mathbf{F}_a$ is

$$\mathbf{F}_e \sim \mathbf{F}_i \sim \frac{m_e n}{\tau_e}\mathbf{u}_e \sim \frac{p}{a}\epsilon^{7/2}. \tag{2.14}$$

and hence are dropped. The term $m_a n \mathbf{u}_{a,t}$ is $O(\epsilon^{11/2})$ for the electrons and $O(\epsilon^{7/2})$ for the ions and are neglected. The inertial term in the ion momentum equation is

$$m_i n(\mathbf{u}_i \cdot \nabla)\mathbf{u}_i \sim \epsilon^{3/2}\frac{p}{a} \tag{2.15}$$

and is retained. The inertial term in the electron momentum equation is $\sim \epsilon^{7/2}p/a$ and is dropped. The pressure equations (1.35) are kept completely. The equation for the stress tensor (1.36) is replaced by its $\mathbf{bb}$ component and the time derivative terms dropped. The scaling of the heat flow equation (1.37) parallels that of the momentum equation with collisional terms and time derivatives of the heat flow being neglected for both electrons and ions and with convective terms being kept in the ion heat flow equation and neglected in the electron heat flow equation.

The reduced equations are: mass balance for both species,

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{u}_a) = N_a, \tag{2.16}$$

electron momentum balance,

$$3(\mathbf{B} \cdot \nabla)(\frac{\mathbf{b}}{|B|}(p_e - p_{\perp e})) - \nabla(p_e - p_{\perp e}) + \nabla p_e - en(\mathbf{E} + \frac{\mathbf{u}_e}{c} \times \mathbf{B}) = \mathbf{P}_{se}, \tag{2.17}$$

ion momentum balance,

$$m_i n(\mathbf{u}_a \cdot \nabla)\mathbf{u}_i + 3\mathbf{B}\cdot\nabla(\frac{\mathbf{b}}{|B|}(p_i - p_{\perp i})) - \nabla(p_i - p_{\perp i}) + \nabla p_i$$

$$+en(\mathbf{E} + \frac{\mathbf{u}_i}{c} \times \mathbf{B}) = \mathbf{P}_{si}, \tag{2.18}$$

pressure balance for both species,

$$\frac{\partial p_a}{\partial t} + \nabla\cdot(\mathbf{u}_a p_a + \frac{1}{3}\mathbf{S}_a) + p_a \nabla\cdot\mathbf{u}_a + 2(p_a - p_{\perp a})(\frac{\mathbf{B}\cdot(\mathbf{B}\cdot\nabla)}{B^2}\mathbf{u}_a - \frac{1}{3}\nabla\cdot\mathbf{u}_a)$$

$$= -\frac{2^{4/3}}{3}\frac{m_e}{m_i}\frac{n(T_a - T_b)}{\tau_{ee}} + E_{as}, \tag{2.19}$$

pressure anisotropy equations for both species,

$$2\nabla\cdot(\mathbf{u}_a(p_a - p_{\perp a})) + \frac{2}{5}(\frac{\mathbf{B}\cdot(\mathbf{B}\cdot\nabla)}{B^2}\mathbf{S}_a - \frac{1}{3}\nabla\cdot\mathbf{S}_a)$$

$$+(p_a - 2(p_a - p_{\perp a}))(\frac{\mathbf{B}\cdot(\mathbf{B}\cdot\nabla)}{B^2}\mathbf{u}_a - \frac{1}{3}\nabla\cdot\mathbf{u}_a) = \mathbf{b}\cdot\mathbf{T}'_a, \tag{2.20}$$

electron heat flow equation,

$$\frac{\mathbf{b}}{|B|}(-8n(T_e - T_{\perp e})(\mathbf{B}\cdot\nabla)(T_e - T_{\perp e}) - 8(T_e - T_{\perp e})^2(\mathbf{B}\cdot\nabla)n$$

$$+10n(T_e - T_{\perp e})(\mathbf{B}\cdot\nabla)T_e + 4nT_e(\mathbf{B}\cdot\nabla)(T_e - T_{\perp e})$$

$$+5nT_e(\mathbf{B}\cdot\nabla)T_e + 6(-2n(T_e - T_{\perp e})^2 + nT_e(T_e - T_{\perp e}))|B|\nabla\cdot\mathbf{b})$$

$$-2n(T_e - T_{\perp e})\nabla_\perp(T_e - T_{\perp e}) - 2n(T_e - T_{\perp e})^2\nabla_\perp n$$

$$-5n(T_e - T_{\perp e})\nabla_\perp T_e + 2T(T_e - T_{\perp e})\nabla_\perp n + 5nT_e\nabla_\perp T_e$$

$$+6n((T_e - T_{\perp e})^2 + T_e(T_e - T_{\perp e}))(\mathbf{b}\cdot\nabla)\mathbf{b} - \frac{e}{c}\mathbf{S}_e \times \mathbf{B} = 0, \tag{2.21}$$

and the ion heat flow equation,

$$m_i\nabla\cdot(\mathbf{u}_i\mathbf{S}_i) + m_i\frac{7}{5}(\mathbf{S}_i\cdot\nabla)\mathbf{u}_i + m_i\frac{2}{5}\nabla\mathbf{u}_i\cdot\mathbf{S}_i + m_i\frac{2}{5}\mathbf{S}_i\nabla\cdot\mathbf{u}_i$$

$$+\frac{\mathbf{b}}{|B|}(-8n(T_i - T_{\perp i})(\mathbf{B}\cdot\nabla)(T_i - T_{\perp i}) - 8(T_i - T_{\perp i})^2(\mathbf{B}\cdot\nabla)n$$

$$+10n(T_i - T_{\perp i})(\mathbf{B}\cdot\nabla)T_i + 4nT_i(\mathbf{B}\cdot\nabla)(T_i - T_{\perp i})$$

$$+5nT_e(\mathbf{B}\cdot\nabla)T_e + 6(-2n(T_i - T_{\perp i})^2 + nT_i(T_i - T_{\perp i}))|B|\nabla\cdot\mathbf{b})$$
$$-2n(T_i - T_{\perp i})\nabla_\perp(T_i - T_{\perp i}) - 2n(T_i - T_{\perp i})^2\nabla_\perp n$$
$$-5n(T_i - T_{\perp i})\nabla_\perp T_i + 2T(T_i - T_{\perp i})\nabla_\perp n + 5nT_i\nabla_\perp T_i$$
$$+6n((T_i - T_{\perp i})^2 + T_i(T_i - T_{\perp i}))(\mathbf{b}\cdot\nabla)\mathbf{b} + \frac{e}{c}\mathbf{S}_i \times \mathbf{B} = 0. \tag{2.22}$$

The only collision terms retained in the system are those appearing in the pressure equations and those appearing in the stress equations,

$$\mathbf{b}\cdot\mathbf{T}'_e = \frac{-3\cdot(1+2^{1/3})}{5}\frac{(p_e - p_{\perp e})}{\tau_{ee}}, \tag{2.23}$$

$$\mathbf{b}\cdot\mathbf{T}'_i = -\frac{3}{5}(\frac{T_e}{T_i})^{3/2}\sqrt{\frac{m_e}{m_i}}\frac{(p_i - p_{\perp i})}{\tau_{ee}}. \tag{2.24}$$

The terms N_a, $\mathbf{P}_{sa}$ and E_{as} are particle, momentum, and energy sources respectively, all $O(\epsilon^2)$. These source terms are taken to be known. I keep the Maxwell equations (1.3) – (1.5) and (1.43). A consequence of (1.3) is that the particle sources must be the same for each species, that is $N_e = N_i = N$. At this point one can only hypothesize that this reduced system is a closed one. At least, the number of unknowns is equal to the number of equations. The question of what data is appropriate for the system is complicated and requires more careful examination of the system; the system is not of standard type. These questions of what is a well-posed problem are one motivation for the asymptotic expansion of the solution in Chapters 3 and 4.

In [8] a similar scaling was used to extract a reduced model from the two-fluid Braginski model. I comment on some of the similarities and differences of the reduced thirteen moment system with the reduced Braginski model used in [8]. The mass and momentum balance is essentially the same in both models. In [8] the stress tensor appearing in the momentum and energy equations is generated by the component of the fluid strain parallel to the magnetic field. Here, the stress tensor includes in addition to the parallel strain, terms coming from the gradients of the heat flow. From the results in [8] I expect the work done by the stresses to be important in determining energy transport.

In [8] the heat flow is given by an expression like (1.56). Hence, the temperature is forced to be constant on magnetic surfaces. If the temperature is constant on magnetic surfaces, then the Braginski heat flow across those surfaces is $O((\Omega_a\tau_a)^2)$ and in this scaling negligible. Hence in [8] there is no energy transport to heat flows. Here, the parallel component of the heat flow is not determined from the heat flow equation. Instead, the parallel component of the heat flow equation provides a relation between variation of the temperature and of the temperature anisotropy along field lines. By allowing the temperature to vary on magnetic surfaces, I have included realistic perpendicular heat flows in this model.

2.2. Axisymmetry

I now use the axisymmetry of the system to simplify the form of the equations. I use usual polar coordinates (r, z, θ) with unit vectors $\hat{\mathbf{r}}$, $\hat{\mathbf{z}}$ and $\hat{\boldsymbol{\theta}}$ and assume that no quantities depend on $\boldsymbol{\theta}$. I first examine the electromagnetic relations. From (1.5) the magnetic field has the form

$$\begin{aligned}\mathbf{B} &= \nabla\psi \times \nabla\theta + \chi\nabla\theta \\ &= -\hat{\mathbf{r}}\frac{\psi_{,r}}{r} + \hat{\mathbf{z}}\frac{\psi_{,r}}{r} + \hat{\boldsymbol{\theta}}\frac{\chi}{r}.\end{aligned} \tag{2.25}$$

The toroidal and poloidal components of the magnetic field are $\mathbf{B}_T = \chi\nabla\theta$ and $\mathbf{B}_P = \nabla\psi \times \nabla\boldsymbol{\theta}$. The surfaces $\psi = const.$ are assumed to form a family of nested flux surfaces. Using (1.43) and (2.25) the components of (1.3) are

$$\chi_{,z} = -4\pi\frac{e}{c}\mu_0 nr(u_{ir} - u_{er}), \tag{2.26}$$

$$\chi_{,r} = -4\pi\frac{e}{c}\mu_0 nr(u_{iz} - u_{ez}), \tag{2.27}$$

and

$$\Delta^*\psi = \psi_{,zz} + r(\frac{\psi_{,r}}{r})_{,r} = -4\pi\frac{e}{c}\mu_0 nr(u_{i\theta} - u_{e\theta}). \tag{2.28}$$

I introduce a vector potential $\mathbf{A}$ such that $\nabla \times \mathbf{A} = \mathbf{B}$, with choice of gauge $\nabla \cdot \mathbf{A} = 0$. Then

$$\mathbf{A} = \psi\nabla\theta + \nabla\Lambda \times \nabla\theta \tag{2.29}$$

where

$$\Delta^* \Lambda = -\chi \tag{2.30}$$

with the boundary condition $\Lambda = 0$ on the boundary. Solving (1.4) for $\mathbf{E}$,

$$\mathbf{E} = -\nabla\Phi - \mathbf{A}_{,t} - V(t)\nabla\theta \tag{2.31}$$

where Φ is the scalar potential and $V(t)$ the loop voltage taken to be $O(\epsilon^2)$.

I reexamine the scaling of the mass balance equation (2.16). The time derivative of the density $n_{,t}$ is $\sim \epsilon^2 nu/a$, as is the particle source N. It is convenient to introduce a flow $\mathbf{U}$ in the direction of $\nabla\psi$, that is across flux surfaces with magnitude $\epsilon^2 u$ that contains the effects of the density time variation and of the particle source N. That is let $\mathbf{U}$ satisfy

$$n_{,t} + \nabla \cdot (n\mathbf{U}) = N. \tag{2.32}$$

Then, it follows that $\nabla \cdot (n\mathbf{u}_a - n\mathbf{U}) = 0$. Thus, the fluid flow can be written as

$$n\mathbf{u}_a = \nabla\lambda_a \times \nabla\theta + r^2 n\omega_a \nabla\theta + n\mathbf{U}. \tag{2.33}$$

To $O(\epsilon^2)$, the functions λ_a are streamfunctions for the flow in the poloidal plane; the approximate streamlines of the flow are $\lambda_a = constant$. The functions ω_a are toroidal rotation frequencies. If the flux of particles through a flux surface $\psi = \overline{\psi}$ is calculated, one finds that

$$\int_{\psi=\overline{\psi}} \mathbf{n} \cdot (n\mathbf{u}_a)\, dS = \int_{\psi=\overline{\psi}} \mathbf{n} \cdot (n\mathbf{U})\, dS, \tag{2.34}$$

where $\mathbf{n}$ is the unit normal to the surface $\psi = \overline{\psi}$ and dS the surface element. That is, the flux of particles through a flux surface (or through any closed surface for that matter) is the same for both species. This result is a direct consequence of the assumption of charge neutrality. The equations for conservation of mass (2.16) are now replaced by the relation (2.32) for $\mathbf{U}$ and the definitions (2.33).

The definition (2.33) implies that

$$\chi_{,r} = 4\pi \frac{e}{c} \mu_0 (\lambda_{i,r} - \lambda_{e,r}), \tag{2.35}$$

and

$$\chi_{,z} = 4\pi\frac{e}{c}\mu_0(\lambda_{i,z} - \lambda_{e,z}), \tag{2.36}$$

or that

$$\begin{aligned} \chi &= \chi_0(t) + 4\pi\mu_0\frac{e}{c}(\lambda_i - \lambda_e) \\ &= \chi_0(t) + \chi_2. \end{aligned} \tag{2.37}$$

The part of the toroidal field given by χ_0 can be identified as the vacuum magnetic field and that given by χ_2 as the contribution from the poloidal plasma currents. The generalized Grad-Shafranov [11] equation (2.28) has the form

$$\Delta^*\psi = -4\pi\frac{e}{c}\mu_0 n r^2(\omega_i - \omega_e). \tag{2.38}$$

In summary, the electromagnetic equations (1.3) – (1.4) have been replaced by representations for the magnetic field (2.25) and the electric field (2.31), along with definitions for the vector potential (2.29) and (2.30), the toroidal field (2.37), and a generalized Grad-Shafranov equation for the poloidal flux (2.38).

I now examine the reduced momentum equations (2.17) and (2.18). The $\hat{\theta}$ components of (2.17) and (2.18) are respectively

$$(\mathbf{B}\cdot\nabla)[\frac{e}{c}\lambda_e - \frac{3\chi}{B^2}(p_e - p_{\perp e})] = en(\psi_{,t} + V(t) + \frac{1}{c}\mathbf{U}\cdot\nabla\psi) + rP_{se\theta}, \tag{2.39}$$

and

$$\begin{aligned} (\mathbf{B}\cdot\nabla)[\frac{e}{c}\lambda_i + \frac{3\chi}{B^2}(p_i - p_{\perp i})] &= -en(\psi_{,t} + V(t) \\ &+ \frac{1}{c}\mathbf{U}\cdot\nabla\psi) - \frac{1}{r}(\lambda_{i,r}(\omega_i r^2)_{,z} - \lambda_{i,z}(\omega_i r^2)_{,r}) + rP_{si\theta}. \end{aligned} \tag{2.40}$$

In order that solutions to (2.39) and (2.40) exist, the following conditions must be satisfied

$$\int_{\psi\leq\overline{\psi}} en(\psi_{,t} + V(t) + \frac{1}{c}\mathbf{U}\cdot\nabla\psi + rP_{se\theta})\,dV = 0 \tag{2.41}$$

$$\int_{\psi\le\overline{\psi}} \frac{1}{r}[\lambda_{i,r}(\omega_i r^2)_{,z} - \lambda_{i,z}(\omega_i r^2)_{,r}]\, dV = \int_{\psi\le\overline{\psi}} r(P_{se\theta} + P_{si\theta})\, dV \tag{2.42}$$

for all values of $\overline{\psi}$, where the integrals are calculated over the volume bounded by the surface $\psi = \overline{\psi}$. If the conditions (2.41) and (2.42) are satisfied then equations (2.39) and (2.40) are equivalent to

$$\frac{e}{c}\lambda_e - F_e(\psi) - \frac{3\chi}{B^2}(p_e - p_{\perp e}) = f_4 \tag{2.43}$$

$$\frac{e}{c}\lambda_i + F_i(\psi) + \frac{3\chi}{B^2}(p_i - p_{\perp i}) = -f_4 - f_{i4} \tag{2.44}$$

where $F_e(\psi)$ and $F_e(\psi)$ are arbitrary functions of ψ and the order ϵ^2 quantities f_4 and f_{i4} are given by

$$(\mathbf{B}\cdot\nabla)f_4 = en(\psi_{,t} + V(t) + \frac{1}{c}\mathbf{U}\cdot\nabla\psi) + rP_{se\theta} \tag{2.45}$$

$$(\mathbf{B}\cdot\nabla)f_{i4} = \frac{1}{r}[\lambda_{i,r}(\omega_i r^2)_{,z} - \lambda_{i,z}(\omega_i r^2)_{,r}] - r(P_{se\theta} + P_{si\theta}). \tag{2.46}$$

Solving (2.43) and (2.44) for $(T_e - T_{\perp e})$ and $(T_i - T_{\perp i})$, one finds

$$T_e - T_{\perp e} = \frac{B^2}{3n\chi}(\frac{e}{c}\lambda_e - F_e(\psi) - f_4), \tag{2.47}$$

and

$$T_i - T_{\perp i} = \frac{B^2}{3n\chi}(-\frac{e}{c}\lambda_i + F_i(\psi) + f_4 - f_{i4}). \tag{2.48}$$

These can be rewritten accurate through order ϵ^2 as

$$\begin{aligned} T_e - T_{\perp e} &= \frac{\chi_0}{3nr^2}(\frac{e}{c}\lambda_e - F_e(\psi)) \\ &+\frac{1}{3nr^2}(\frac{|\nabla\psi|^2}{\chi} + \chi_2)(\frac{e}{c}\lambda_e - F_e(\psi)) - \frac{\chi_0}{3nr^2}f_4 \end{aligned} \tag{2.49}$$

and

$$T_i - T_{\perp i} = \frac{\chi_0}{3nr^2}(F_i(\psi) - \frac{e}{c}\lambda_i)$$

$$(2.50) \qquad + \frac{1}{3nr^2}(\frac{|\nabla\psi|^2}{\chi} + \chi_2)(F_i(\psi) - \frac{e}{c}\lambda_i) - \frac{\chi_0}{3nr^2}(f_{i4} - f_4).$$

The toroidal components of the two momentum balance equations have been replaced by the above relations for $(T_e - T_{\perp e})$ and $(T_i - T_{\perp i})$ along with the definitions (2.45) and (2.46) for f_4 and f_{i4} and the constraints (2.41) and (2.42).

I now turn to the poloidal components of the momentum equation. The poloidal component of the electron momentum equation is

$$(\mathbf{B}_p \cdot \nabla)[\frac{\mathbf{b}_p}{|B|}(p_e - p_{\perp e})] + n\nabla T_e + T_e \nabla n + \nabla(p_e - p_{\perp e})$$

$$(2.51) \qquad + \hat{\mathbf{r}}\frac{3\chi^2}{B^2 r^3}(p_e - p_{\perp e}) = -en\nabla\Phi - \frac{e\chi}{cr^2}\nabla\lambda_e + \frac{e}{c}n\omega_e\nabla\psi + P'_e,$$

and for the ions

$$(\mathbf{B}_p \cdot \nabla)[\frac{\mathbf{b}_p}{|B|}(p_i - p_{\perp i})] + n\nabla T_i + T_i \nabla n + \nabla(p_i - p_{\perp i})$$

$$(2.52) \qquad + \hat{\mathbf{r}}\frac{3\chi^2}{B^2 r^3}(p_i - p_{\perp i}) = en\nabla\Phi + \frac{e\chi}{cr^2}\nabla\lambda_i - \frac{e}{c}n\omega_i\nabla\psi + P'_i,$$

where

$$(2.53) \qquad P'_e = P_{epol} + \frac{e}{c}n\chi\mathbf{U} \times \nabla\theta - en(\nabla\Lambda_{,t} \times \nabla\theta),$$

and

$$(2.54) \qquad P'_i = P_{ipol} - \frac{e}{c}n\chi\mathbf{U} \times \nabla\theta + en(\nabla\Lambda_{,t} \times \nabla\theta);$$

P_{epol} and P_{ipol} are the poloidal components of the the momentum sources; $\mathbf{B}_p$ is the poloidal part of $\mathbf{B}$; $\mathbf{b}_p$ is the poloidal part of $\mathbf{b}$. P'_e and P'_i are order $O(\epsilon^2)$ momentum sources generated by the time varying parts of the density and electromagnetic fields and the external momentum sources.

Equations (2.51) and (2.52) have the form

$$\mathbf{M}_e = -e\nabla\Phi, \tag{2.55}$$

and

$$\mathbf{M}_i = e\nabla\Phi., \tag{2.56}$$

where the vectors $\mathbf{M}_e$ and $\mathbf{M}_i$ have only poloidal components. This form is not particularly convenient for the analysis that will follow. I seek a system where Φ does not appear and that with appropriate data is equivalent to (2.51) and (2.52). The equations (2.55) and (2.56) imply

$$\mathbf{M}_{er,z} - \mathbf{M}_{ez,r} = 0 \tag{2.57}$$

and

$$\mathbf{M}_{ir,z} - \mathbf{M}_{iz,r} = 0. \tag{2.58}$$

The equations (2.57) and (2.58) only require that $\mathbf{M}_e$ and $\mathbf{M}_i$ be gradients; a condition relating these two gradients is needed, for example the condition

$$\mathbf{M}_e + \mathbf{M}_i = 0, \tag{2.59}$$

is sufficient. The condition (2.59) along with the equations (2.57) and (2.58) is equivalent to

$$\mathbf{B} \cdot (\mathbf{M}_e + \mathbf{M}_i) = 0 \tag{2.60}$$

applied at all points and the constraint

$$\mathbf{a} \cdot (\mathbf{M}_e + \mathbf{M}_i) = 0 \tag{2.61}$$

applied on a curve from the magnetic axis to the plasma edge with the vector $\mathbf{a}$ nowhere parallel to $\mathbf{B}$. Thus, I replace (2.55) and (2.56) with the equations (2.57), (2.58) and (2.60) and the constraint (2.61). Explicitly (2.57) and (2.58) are

$$
\begin{aligned}
&-\frac{1}{n}(n_{,r}T_{e,z}-n_{,z}T_{e,r})-\frac{4\chi_0 e}{3r^2n^2c}(n_{,r}\lambda_{e,z}-n_{,z}\lambda_{e,r})\\
&\quad+\frac{\chi_0}{3rn^2}F_e'(\psi)(\mathbf{B}\cdot\nabla)n+\frac{e\chi}{ncr^3}\lambda_{e,z}+\frac{e}{c}r(\mathbf{B}\cdot\nabla)\omega_e\\
&\quad-\frac{2\chi_0}{3r^3n^2}[\frac{e}{c}\lambda_e-F_e(\psi)]n_{,z}+\frac{\chi_0}{nr^3}rB_rF_e'(\psi)\\
&\quad-\frac{e\chi_2 e}{cr^2n^2c}(n_{,r}\lambda_{e,z}-n_{,z}\lambda_{e,r})+\frac{e}{nr^2}(\chi_{2,r}\lambda_{e,z}-\chi_{2,z}\lambda_{e,r})\\
&\quad-\frac{1}{3n^2}n_{,z}[\frac{1}{r^2}(\frac{|\nabla\psi|^2}{\chi}+\chi_2)(\frac{e}{c}\lambda_e-F_e(\psi))]_{,r}\\
&\quad+\frac{1}{3n^2}n_{,r}[\frac{1}{r^2}(\frac{|\nabla\psi|^2}{\chi}+\chi_2)(\frac{e}{c}\lambda_e-F_e(\psi))]_{,z}\\
&\quad-(\frac{e}{c}\lambda_e-F_e(\psi)-f_4)(\frac{\chi_{2,z}}{nr^3}-\frac{2\chi n_{,z}}{n^2r^3})+[(\frac{1}{n}\mathbf{B}\cdot\nabla)(\frac{B_r}{n\chi}(\frac{e}{c}\lambda_e-F_e(\psi)-f_4))]_{,z}\\
&\quad-[(\frac{1}{n}\mathbf{B}\cdot\nabla)(\frac{B_z}{n\chi}(\frac{e}{c}\lambda_e-F_e(\psi)-f_4))]_{,r}+\frac{\chi_0}{nr^3}f_{4,z}+(\frac{P'_{ez}}{n})_{,r}-(\frac{P'_{er}}{n})_{,z}=0,
\end{aligned}
\tag{2.62}
$$

and

$$
\begin{aligned}
&-\frac{1}{n}(n_{,r}T_{i,z}-n_{,z}T_{i,r})+\frac{4\chi_0 e}{3r^2n^2c}(n_{,r}\lambda_{i,z}-n_{,z}\lambda_{i,r})-\frac{\chi_0}{3rn^2}F_i'(\psi)(\mathbf{B}\cdot\nabla)n\\
&\quad-\frac{e\chi}{ncr^3}\lambda_{i,z}-\frac{e}{c}r(\mathbf{B}\cdot\nabla)\omega_i+\frac{2\chi_0}{3r^3n^2}[-\frac{e}{c}\lambda_i+F_i(\psi)]n_{,z}-\frac{\chi_0}{nr^3}rB_rF_i'(\psi)\\
&\quad+\frac{e\chi_2 e}{r^2n^2c}(n_{,r}\lambda_{i,z}-n_{,z}\lambda_{i,r})-\frac{e}{ncr^2}(\chi_{2,r}\lambda_{i,z}-\chi_{2,z}\lambda_{i,r})\\
&\quad-\frac{1}{3n^2}n_{,z}[\frac{1}{r^2}(\frac{|\nabla\psi|^2}{\chi}+\chi_2)(-\frac{e}{c}\lambda_i+F_i(\psi))]_{,r}\\
&\quad+\frac{1}{3n^2}n_{,r}[\frac{1}{r^2}(\frac{|\nabla\psi|^2}{\chi}+\chi_2)(-\frac{e}{c}\lambda_i+F_i(\psi))]_{,z}\\
&\quad-(-\frac{e}{c}\lambda_i+F_i(\psi)-f_4)(\frac{\chi_{2,z}}{nr^3}-\frac{2\chi n_{,z}}{n^2r^3})+[(\frac{1}{n}\mathbf{B}\cdot\nabla)(\frac{B_r}{n\chi}(-\frac{e}{c}\lambda_i+F_i(\psi)-f_4))]_{,z}\\
&\quad-[(\frac{1}{n}\mathbf{B}\cdot\nabla)(\frac{B_z}{n\chi}(-\frac{e}{c}\lambda_i+F_i(\psi)-f_4))]_{,r}+\frac{\chi_0}{nr^3}f_{4,z}+(\frac{P'_{iz}}{n})_{,r}-(\frac{P'_{ir}}{n})_{,z}\\
&\quad+m_i\frac{\partial}{\partial\psi}(\frac{\lambda'_{i0}(\psi)}{n_0})(\mathbf{B}\cdot\nabla)|\nabla\psi|^2+\frac{m_i}{n_0}\lambda'^2_{i0}(\psi)[(\mathbf{B}\cdot\nabla B_r)_{,z}-(\mathbf{B}\cdot\nabla B_z)_{,r}]=0.
\end{aligned}
\tag{2.63}
$$

Equations (2.60) and (2.61) are

$$
\begin{aligned}
&(\mathbf{B}\cdot\nabla)p-(\mathbf{B}\cdot\nabla)[\frac{\chi_0}{3r^2}(F(\psi)-\frac{e}{c}\lambda]-\frac{B_r\chi}{r^3}(F(\psi)-\frac{e}{c}\lambda)\\
&\quad+\frac{e\chi}{r^2}(\mathbf{B}\cdot\nabla)\lambda=-\mathbf{B}\cdot[\mathbf{B}_p\cdot\nabla(\frac{\mathbf{B}_p}{\chi}(F(\psi)-\frac{e}{c}\lambda))]
\end{aligned}
$$

$$+(\mathbf{B}\cdot\nabla)[(\frac{|\nabla\psi|^2}{\chi}+\chi_2)(\frac{(F(\psi)-\frac{e}{c}\lambda)}{3r^2})]+B_r r m_i n\omega_i^2$$

$$+\mathbf{B}\cdot(\mathbf{B}_p\cdot\nabla)(\mathbf{B}_p\frac{f_{i4}}{\chi})-(\mathbf{B}\cdot\nabla)(\frac{\chi_0}{3r^2}f_{i4})+B_r\frac{\chi_0}{r^3}f_{i4}+P'\cdot\mathbf{B}, \tag{2.64}$$

applied at all points and

$$\mathbf{a}\cdot(\mathbf{B}_p\cdot\nabla)(\mathbf{B}_p\frac{(F(\psi)-\frac{e}{c}\lambda)}{\chi})+\mathbf{a}\cdot\nabla p-\mathbf{a}\cdot\nabla[\frac{\chi_0}{3r^2}(F(\psi)-\frac{e}{c}\lambda)]$$

$$-\mathbf{a}\cdot\nabla[\frac{1}{3r^2}(\frac{|\nabla\psi|^2}{\chi}+\chi_2)(F(\psi)-\frac{e}{c}\lambda)]-a_r\frac{\chi}{r^3}(F(\psi)-\frac{e}{c}\lambda)$$

$$+\mathbf{a}\cdot\frac{e\chi}{cr^2}\nabla\lambda-en\omega\mathbf{a}\cdot\nabla\psi=-a_r r m_i n\omega_i^2-$$

$$\mathbf{a}\cdot\frac{m_i}{n_0}\lambda_{i0}'^2(\psi))(\mathbf{B}\cdot\nabla)\mathbf{B}_p+\mathbf{a}\cdot\nabla[(\mathbf{B}_p\cdot\nabla)(\mathbf{B}_p\frac{f_{i4}}{\chi_0}(\frac{\chi_0 f_{i4}}{3r^2}))], \tag{2.65}$$

applied on a curve from the magnetic axis to the plasma edge, where the total plasma variables are

$$\lambda=\lambda_i-\lambda_e \tag{2.66}$$

$$\omega=\omega_i-\omega_e \tag{2.67}$$

$$F(\psi)=F_i(\psi)-F_e(\psi) \tag{2.68}$$

$$p=p_i+p_e. \tag{2.69}$$

In (2.64) and (2.65) I have anticipated that in lowest order λ_i and n are functions of ψ alone. The electrostatic potential Φ has been eliminated from the system and the four poloidal momentum balance equations replaced by the three equations (2.62), (2.63) and (2.64) and the constraint (2.65).

The pressure and stress equations are kept in their original form. The pressure equations are

$$\frac{1}{r}(T_{a,z}\lambda_{a,r} - T_{a,r}\lambda_{a,z}) + \frac{1}{3}\nabla\cdot\mathbf{S}_a - \frac{T_a}{rn}(n_{,z}\lambda_{,r} - n_{,r}\lambda_{,z})$$
$$+\frac{2(T_a - T_{\perp a})}{3nr}(n_{,z}\lambda_{a,r} - n_{,r}\lambda_{a,z}) + \frac{2n(T_a - T_{\perp a})}{B^2}[\chi(\mathbf{B}\cdot\nabla)\omega_a$$
$$-\frac{\chi^2}{r^4 n}\lambda_{a,z} - B_r(\mathbf{B}\cdot\nabla)(\frac{\lambda_{a,r}}{rn}) + B_z(\mathbf{B}\cdot\nabla)(\frac{\lambda_{a,r}}{rn})] = \frac{2^{4/3}}{3}\frac{m_e}{m_i}\frac{n(T_a - T_b)}{\tau_{ee}}$$
$$-nT_{a,t} - T_a N - n(\mathbf{U}\cdot\nabla)T_a - T_a\nabla\cdot(n\mathbf{U}) + E_{as}. \tag{2.70}$$

The equations for the stresses are

$$\frac{2}{r}[(T_a - T_{\perp a})_{,z}\lambda_{a,r} - (T_a - T_{\perp a})_{,r}\lambda_{a,z}] + \frac{1}{3rn}T_a(n_{,z}\lambda_{a,r} - n_{,r}\lambda_{a,z})$$
$$-\frac{2}{3rn}(T_a - T_{\perp a})(n_{,z}\lambda_{a,r} - n_{,r}\lambda_{a,z}) + \frac{2}{5}[\frac{1}{B^2}\mathbf{B}\cdot(\mathbf{B}\cdot\nabla)\mathbf{S}_a - \frac{1}{3}\nabla\cdot\mathbf{S}_a]$$
$$+n[T_a - 2(T_a - T_{\perp a})]\frac{1}{B^2}[\chi(\mathbf{B}\cdot\nabla)\omega_a - \frac{\chi^2}{r^4 n}\lambda_{a,z} - B_r(\mathbf{B}\cdot\nabla)(\frac{\lambda_{a,r}}{rn})$$
$$+B_z(\mathbf{B}\cdot\nabla)(\frac{\lambda_{a,r}}{rn})] = -\frac{1}{3}T_a n_{,t} + \mathbf{b}\cdot\mathbf{T}'_a. \tag{2.71}$$

I now consider the reduced heat flow equations (2.21) and (2.22). It is convenient to separate these equations into components perpendicular and parallel to the magnetic field. Consider first the reduced electron heat flow equation (2.21). The scalar product of this equation with $\mathbf{B}$ is

$$-8n(T_e - T_{\perp e})(\mathbf{B}\cdot\nabla)(T_e - T_{\perp e}) - 8(T_e - T_{\perp e})^2(\mathbf{B}\cdot\nabla)n$$
$$+10n(T_e - T_{\perp e})(\mathbf{B}\cdot\nabla)T_e + 4nT_e(\mathbf{B}\cdot\nabla)(T_e - T_{\perp e}) + 5nT_e(\mathbf{B}\cdot\nabla)T_e$$
$$+6[-2n(T_e - T_{\perp e})^2 + nT_e(T_e - T_{\perp e})]|B|\nabla\cdot\mathbf{b} = 0. \tag{2.72}$$

This equation relates the variation of the temperature, density and anisotropy along field lines. The scalar product of the electron heat flow equation (2.21) with $\nabla\psi\times\mathbf{B}$ is

$$2n(T_e - T_{\perp e})(\mathbf{B}\cdot\nabla)(T_e - T_{\perp e}) + 2(T_e - T_{\perp e})^2(\mathbf{B}\cdot\nabla)n$$
$$-5n(T_e - T_{\perp e})(\mathbf{B}\cdot\nabla)T_e - 2T_e(T_e - T_{\perp e})(\mathbf{B}\cdot\nabla)n - 5nT_e(\mathbf{B}\cdot\nabla)T_e$$
$$+6n[(T_e - T_{\perp e})^2 + T_e(T_e - T_{\perp e})][((\mathbf{b}\cdot\nabla)\mathbf{b})\cdot(\nabla\psi\times\mathbf{B})]/\chi - \frac{e}{c\chi}B^2(\mathbf{S}_e\cdot\nabla\psi), \tag{2.73}$$

an algebraic equation for the component of the electron heat flow in the direction of $\nabla\psi$. Finally, the scalar product of the electron heat flow equation with $\nabla\psi$ is

$$\begin{aligned}
&\nabla\psi\cdot[2n(T_e-T_{\perp e})\nabla(T_e-T_{\perp e})+2(T_e-T_{\perp e})^2\nabla n-5n(T_e-T_{\perp e})\nabla T_e \\
&-2T_e(T_e-T_{\perp e})\nabla n-5nT_e\nabla T_e]+6n[(T_e-T_{\perp e})^2 \\
&+T_e(T_e-T_{\perp e})][((\mathbf{b}\cdot\nabla)\mathbf{b})\cdot\nabla\psi]-\frac{e}{c}\mathbf{S}_e\cdot(\nabla\psi\times\mathbf{B})=0,
\end{aligned}\tag{2.74}$$

an algebraic equation for the electron heat flow in the direction of $(\nabla\psi\times\mathbf{B})$. The parallel component of the electron heat flow equation (2.107) will be one of the primary equations in the system and the perpendicular components (2.73) and (2.74) will be used to define the perpendicular electron heat flow.

The reduced ion heat flow equation dotted with $\mathbf{B}$ is

$$\begin{aligned}
&-\frac{12}{5}m_iB_r\frac{u_{i\theta}S_{i\theta}}{r}+m_iB_\theta(\mathbf{u}_i\cdot\nabla)S_{i\theta}+\frac{7}{5}m_iB_\theta(\mathbf{S}_i\cdot\nabla)u_{i\theta} \\
&+\frac{2}{5}m_iS_{i\theta}(\mathbf{B}\cdot\nabla)u_{i\theta}+m_iB_\theta\frac{u_{i\theta}S_{i\theta}}{r}+\frac{9}{5}m_iB_\theta\frac{S_{i\theta}u_{ir}}{r} \\
&+m_iB_r(\mathbf{u}_i\cdot\nabla)S_{ir}+m_iB_z(\mathbf{u}_i\cdot\nabla)S_{iz}+\frac{9}{5}m_iB_r(\mathbf{S}_i\cdot\nabla)u_{ir} \\
&+\frac{9}{5}B_z(\mathbf{S}_i\cdot\nabla)u_{iz}-\frac{2}{5}m_iB_rS_{iz}(u_{ir,z}-u_{iz,r})+\frac{2}{5}m_iB_zS_{ir}(u_{ir,z}-u_{iz,r}) \\
&-\frac{7}{5}m_i\frac{\mathbf{S}_i\cdot\mathbf{B}}{rn^2}(\lambda_{i,r}n_{,z}-\lambda_{i,z}n_{,r})-8n(T_i-T_{\perp i})(\mathbf{B}\cdot\nabla)(T_i-T_{\perp i}) \\
&-8(T_i-T_{\perp i})^2(\mathbf{B}\cdot\nabla)n+10n(T_i-T_{\perp i})(\mathbf{B}\cdot\nabla)T_i \\
&+4nT_i(\mathbf{B}\cdot\nabla)(T_i-T_{\perp i})+5nT_i(\mathbf{B}\cdot\nabla)T_i \\
&+6[-2n(T_i-T_{\perp i})^2+nT_i(T_i-T_{\perp i})]|B|\nabla\cdot\mathbf{b}=0,
\end{aligned}\tag{2.75}$$

This equation is like the electron one (2.73) except that derivatives of the ion fluid and heat flows are included. The reduced ion heat flow equation dotted with $\nabla\psi\times\mathbf{B}$ is

$$\begin{aligned}
&-\frac{12}{5}m_iB_r\frac{u_{i\theta}S_{i\theta}}{r}-m_i\frac{|\nabla\psi|^2}{r\chi_0}(\mathbf{u}_i\cdot\nabla)S_{i\theta}-\frac{7}{5}m_i\frac{|\nabla\psi|^2}{\chi_0 r}(\mathbf{S}_i\cdot\nabla)u_{i\theta} \\
&+\frac{2}{5}m_iS_{i\theta}(\mathbf{B}\cdot\nabla)u_{i\theta}-m_i\frac{|\nabla\psi|^2}{\chi_0 r}\frac{u_{i\theta}S_{i\theta}}{r}-\frac{9}{5}m_i\frac{|\nabla\psi|^2}{\chi_0 r}\frac{S_{i\theta}u_{ir}}{r} \\
&+m_iB_r(\mathbf{u}_i\cdot\nabla)S_{ir}+m_iB_z(\mathbf{u}_i\cdot\nabla)S_{iz}+\frac{9}{5}m_iB_r(\mathbf{S}_i\cdot\nabla)u_{ir}
\end{aligned}$$

$$+\frac{9}{5}B_z(\mathbf{S}_i\cdot\nabla)u_{iz}-\frac{2}{5}m_iB_rS_{iz}(u_{ir,z}-u_{iz,r})+\frac{2}{5}m_iB_zS_{ir}(u_{ir,z}-u_{iz,r})$$
$$-\frac{7}{5}m_i\frac{1}{rn^2}(\lambda_{i,r}n_{,z}-\lambda_{i,z}n_{,r})(S_{i\theta}\frac{|\nabla\psi|^2}{\chi_0 r}-\mathbf{S}_i\cdot\mathbf{B})$$
$$+2n(T_i-T_{\perp i})(\mathbf{B}\cdot\nabla)(T_i-T_{\perp i})+2(T_i-T_{\perp i})^2(\mathbf{B}\cdot\nabla)n$$
$$-5n(T_i-T_{\perp i})(\mathbf{B}\cdot\nabla)T_i-2T_i(T_i-T_{\perp i})(\mathbf{B}\cdot\nabla)n-5nT_i(\mathbf{B}\cdot\nabla)T_i$$
$$+6n[(T_i-T_{\perp i})^2+T_i(T_i-T_{\perp i})][((\mathbf{b}\cdot\nabla)\mathbf{b})\cdot(\nabla\psi\times\mathbf{B})]/\chi$$
$$+\frac{e}{c\chi}B^2(\mathbf{S}_i\cdot\nabla\psi)=0. \tag{2.76}$$

The scalar product of reduced ion heat flow equation with $\nabla\psi$ is

$$\frac{12}{5}m_iB_zS_{i\theta}u_\theta+\frac{2}{5}m_iS_{i\theta}\nabla\psi\cdot\nabla u_{i\theta}+m_irB_z(\mathbf{u}_i\cdot\nabla)S_{ir}-m_irB_r(\mathbf{u}_i\cdot\nabla)S_{iz}$$
$$+\frac{9}{5}m_irB_z(\mathbf{S}_i\cdot\nabla)u_{ir}-\frac{9}{5}m_irB_r(\mathbf{S}_i\cdot\nabla)u_{iz}-2m_ir(\mathbf{S}_i\cdot\mathbf{B})(u_{ir,z}-u_{iz,r})$$
$$-\frac{7}{5}m_i\frac{1}{rn^2}\mathbf{S}_i\cdot\nabla\psi(\lambda_{i,r}n_{,z}-\lambda_{i,z}n_{,r})+\nabla\psi\cdot[2n(T_i-T_{\perp i})\nabla(T_i-T_{\perp i})$$
$$+2(T_i-T_{\perp i})^2\nabla n-5n(T_i-T_{\perp i})\nabla T_i-2T_i(T_i-T_{\perp i})\nabla n-5nT_i\nabla T_i]$$
$$+6n[(T_i-T_{\perp i})^2+T_i(T_i-T_{\perp i})][((\mathbf{b}\cdot\nabla)\mathbf{b})\cdot\nabla\psi]+\frac{e}{c}\mathbf{S}_i\cdot(\nabla\psi\times\mathbf{B})=0. \tag{2.77}$$

One would like to be able to use equations (2.76) and (2.77) as algebraic definitions of the perpendicular ion heat flow. However, these equations contain derivatives of the ion heat flow. Let us examine the terms that contain derivatives of the heat flow. In equation (2.76), there is the term

$$-m_i\frac{|\nabla\psi|^2}{r\chi_0}(\mathbf{u}_i\cdot\nabla)S_{i\theta}. \tag{2.78}$$

Since, this term is $O(\epsilon^2)$ it only requires knowing $S_{i\theta}$ in lowest order and thus does not contain perpendicular components of the ion heat flow. Also in equation (2.76) are the $O(\epsilon^2)$ terms

$$m_iB_r(\mathbf{u}_i\cdot\nabla)S_{ir}+m_iB_z(\mathbf{u}_i\cdot\nabla)S_{iz}. \tag{2.79}$$

However, the terms above can be eliminated from (2.76) using equation (2.75). The terms

$$m_irB_z(\mathbf{u}_i\cdot\nabla)S_{ir}-m_irB_r(\mathbf{u}_i\cdot\nabla)S_{iz} \tag{2.80}$$

in equation (2.77) are nominally $O(\epsilon^2)$ but it is convenient to anticipate that in lowest order $\mathbf{u}_i$ is parallel to $\mathbf{B}$ and that the heat flow is in lowest order a function of ψ alone. Hence, the above terms are $O(\epsilon^{5/2})$ relative to the largest terms in equation (2.77), $\nabla\psi \cdot \nabla T_i$ and may be dropped.

Let us now review our equation set. The primary unknowns are ψ, λ_a, ω_a, n, T_a, and the components of the electron and ion heat flow parallel to the magnetic field, a total of ten scalar variables. The primary equations are a generalized Grad-Shafranov equation (2.38), electron poloidal momentum balance (2.62), ion poloidal momentum balance (2.63), the parallel component of the sum of poloidal momentum balance (2.64), electron and ion pressure equations (2.70), electron and ion stress equations (2.71), parallel component of the electron heat flow equation (2.72), and parallel component of ion heat flow equation (2.75). The secondary variables $(T_e - T_{\perp e})$, $(T_i - T_{\perp i})$, $\mathbf{U}$, f_4, f_{i4}, $\mathbf{S}_e \cdot \nabla\psi$, $\mathbf{S}_e \cdot (\nabla\psi \times \mathbf{B})$, $\mathbf{S}_i \cdot \nabla\psi$, $\mathbf{S}_i \cdot (\nabla\psi \times \mathbf{B})$ are defined by (2.49), (2.50), (2.33), (2.45), (2.46), (2.73), (2.74), (2.76), (2.77). In addition there are the constraints (2.41), (2.42), and (2.65). The system I study consists of ten equations for the ten primary unknowns along with three constraints and a number of side relations.

Clearly, the form of the system is complicated. However, the reduced system is a considerable simplification of the full thirteen moment system. The complete thirteen moment equation set has twenty-five scalar fluid variables and six scalar electromagnetic variables. Just in the number of unknowns, the reduced system is much simpler. In addition, by neglecting small effects I have gone from a system that described a wide range of phenomena on a variety of time and length scales, to system that describes a much narrower range of plasma behavior. In the reduction, care has been taken to retain essential physics of the problem. In particular, effects such as particle flows, anisotropy, variation of the temperature along field lines, and a realistic treatment of the heat flow have been kept in the model. Hence, an investigation of the reduced model should give valuable information about tokamak transport.

2.3. Flux Coordinates

The equation set presented in the previous section has a simpler form when variables related to the magnetic field are used. Issues of solvability conditions are more easily addressed in such a coordinate system. I now replace the coordinates (r, z) with the flux coordinates (ψ, ϕ) (see Fig. 2). The coordinate ψ labels flux surfaces and the coordinate ϕ is a poloidal angle variable in a flux surface going from 0 to 2π. The choice of poloidal angle is not specified; the calculations are carried out for a general system of flux coordinates. In order to rewrite the equation set using these flux variables some simple calculations are useful. The Jacobian, J of the transformation from (ψ, ϕ) to (r, z) is

(2.81) $$J = \psi_{,r}\phi_{,z} - \psi_{,z}\phi_{,r} = r(\mathbf{B}\cdot\nabla)\phi.$$

Thus the volume element is

(2.82) $$dV = 2\pi r dr\, dz = 2\pi d\psi\, d\phi/J.$$

From the relations

(2.83) $$r_{,\phi}\phi_{,r} + r_{,\psi}\psi_{,r} = 1,$$

(2.84) $$z_{,\phi}\phi_{,z} + z_{,\psi}\psi_{,z} = 1,$$

(2.85) $$r_{,\phi}\phi_{,z} + r_{,\psi}\psi_{,z} = 0,$$

and

(2.86) $$z_{,\phi}\phi_{,r} + z_{,\psi}\psi_{,r} = 0,$$

the poloidal magnetic field is found, $rB_r = Jr_{,\phi}$ and $rB_z = Jz_{,\phi}$. The above implies

(2.87) $$r(\mathbf{B}\cdot\nabla) = J\frac{\partial}{\partial\phi}.$$

Simple calculations give the following relations for other derivatives appearing in the equations

(2.88) $$\frac{\partial}{\partial z} = J(r_{,\psi}\frac{\partial}{\partial\phi} - r_{,\phi}\frac{\partial}{\partial\psi}),$$

(2.89)
$$\frac{\partial}{\partial r} = J(-z_{,\psi}\frac{\partial}{\partial \phi} + z_{,\phi}\frac{\partial}{\partial \psi}),$$

and

(2.90)
$$(f_{,r}g_{,z} - f_{,z}g_{,r}) = J(f_{,\psi}g_{,\phi} - f_{,\phi}g_{,\psi}).$$

The heat flow vector $\mathbf{S}$ can be written

(2.91)
$$\mathbf{S}_a = S_{\|a}\mathbf{B} + S_{\perp a}\frac{\nabla\psi}{r} + \gamma_a\frac{\mathbf{B}\times\nabla\psi}{rB_\theta}.$$

As in the case of the particle flows, the toroidal heat flow is taken to be larger than the poloidal heat flow by a factor of $\epsilon^{-1/2}$. Thus, initially I assume

(2.92)
$$S_\| \sim S_{\perp a} \sim \gamma_a \sim \frac{pu}{B}.$$

In order to write the equation set in flux coordinates a few vector calculations are useful. The component of the strain parallel to the magnetic field is

(2.93)
$$\begin{aligned}\frac{r}{J}(\frac{1}{B^2}(\mathbf{B}\cdot(\mathbf{B}\cdot\nabla)\mathbf{u}) - \frac{1}{3}\nabla\cdot\mathbf{u}) &= \frac{1}{3n}(n_{,\phi}\lambda_{,\psi} - n_{,\psi}\lambda_{,\phi})\\ &+\frac{\chi}{B^2}(\omega_{,\phi} + \frac{\chi}{nr^3}(\lambda_{,\phi}r_{,\psi} - \lambda_{,\psi}r_{,\phi})) + \frac{B_p^2}{B^2}(\frac{\lambda_{,\psi}}{n})_{,\phi}\\ &+\frac{\lambda_{,\psi}}{2nB^2}(B_p^2)_{,\phi} + (\frac{\lambda_{,\phi}}{rn})_{,\phi}(\frac{\nabla\psi\cdot\nabla\phi}{rB^2})\\ &-\frac{J\lambda_{,\phi}}{nr^2B^2}(z_{,\phi}(Jr_{,\psi})_{,\phi} + r_{,\phi}(Jz_{,\psi})_{,\phi}) - \frac{J}{3r}\nabla\cdot(n\mathbf{U}).\end{aligned}$$

The divergence of the heat flow is

(2.94)
$$\nabla\cdot\mathbf{S}_a = \frac{J}{r}((S_{\|a} - \gamma + S_{\perp a}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi} + (S_{\perp a}\frac{|\nabla\psi|^2}{J})_{,\psi}).$$

The parallel component of the heat flow strain, appearing the pressure anisotropy equations is

$$\frac{1}{B^2}\mathbf{B}\cdot(\mathbf{B}\cdot\nabla)\mathbf{S}_a = (\mathbf{B}\cdot\nabla)S_{\|a} - \frac{S_{\|a}}{|\mathbf{B}|}(\mathbf{B}\cdot\nabla)|\mathbf{B}| - \mathbf{S}_a\cdot(\mathbf{b}\cdot\nabla)\mathbf{b}$$

$$= \frac{J}{r}(S_{\|a,\phi} - S_{\|a}(-\frac{r_{,\phi}}{r} + \frac{1}{r^2B^2}(2\chi_0\chi_{2,\phi} + (|\nabla\psi|^2)_{,\phi}))$$

(2.95)
$$+\frac{b_\theta^2}{r}(S_{\perp a}z_{,\phi} - \gamma_a r_{,\phi})).$$

The following geometric expressions are useful in the heat flow equations

(2.96)
$$\frac{r}{J}|\mathbf{B}|\nabla\cdot\mathbf{b} = \frac{r_{,\phi}}{r}\frac{\chi}{r^2B^2}\chi_{2,\phi} + \frac{1}{2r^2B^2}(|\nabla\psi|^2)_{,\phi},$$

(2.97)
$$\frac{r}{J}((\mathbf{b}\cdot\nabla)\mathbf{b})\cdot(\nabla\psi\times\mathbf{B}) = \frac{\chi}{r}r_{,\phi} - \frac{\chi}{2r^2B^2}(|\nabla\psi|^2)_{,\phi} + (1-b_\theta^2)\chi_{2,\phi},$$

(2.98)
$$\frac{r}{J}((\mathbf{b}\cdot\nabla)\mathbf{b})\cdot\nabla\psi = -b_\theta z_{,\phi} + r\frac{B_z^2}{B^2}(\frac{B_r}{B_z})_{,\phi},$$

(2.99)
$$\mathbf{S}_a\cdot(\nabla\psi\times\mathbf{B}) = \gamma_a\frac{\chi_0|\nabla\psi|^2}{R^2}(1 - 2\frac{r}{R} + (\frac{|\nabla\psi|^2}{\chi_0} + \chi_2 + 3\frac{r^2\chi_0}{R^2})) + O(\epsilon^{3/2}).$$

With the above calculations, the equation set can now be easily written using flux variables. Equations (2.62) and (2.63), poloidal momentum balance for the electrons and the ions respectively have the following form in flux coordinates

$$-(n_{,\psi}T_{e,\phi} - n_{,\phi}T_{e,\psi}) - \frac{4\chi_0 e}{3nr^2c}(n_{,\psi}\lambda_{e,\phi} - n_{,\phi}\lambda_{e,\psi}) + \frac{\chi_0}{3r^2}F_e'(\psi)n_{,\phi}$$
$$-\frac{e\chi_0}{cr^3}(r_{,\phi}\lambda_{e,\psi} - r_{,\psi}\lambda_{e,\phi}) + \frac{e}{c}n\omega_{e,\phi} - \frac{2\chi_0}{3nr^3}(\frac{e}{c}\lambda_e - F_e(\psi))r_{,\phi}n_{,\psi}$$
$$-\frac{\chi_0}{nr^3}F_e'(\psi)r_{,\phi} = -\frac{2\chi_0}{3r^3}(\frac{e}{c}\lambda_e - F_e(\psi))n_{,\phi}r_{,\psi}$$
$$+\frac{e\chi_2}{ncr^2}(n_{,\psi}\lambda_{e,\psi} - n_{,\psi}\lambda_{e,\phi}) + \frac{e}{cr^2}(\chi_{2,\phi}\lambda_{e,\psi} - \chi_{2,\psi}\lambda_{e,\phi})$$

(2.100)
$$-\frac{n}{J}(\frac{P_{ez}'}{n})_{,r} + \frac{n}{J}(\frac{P_{er}'}{n})_{,z} + O(\epsilon^2)$$

$$-(n_{,\psi}T_{i,\phi} - n_{,\phi}T_{i,\psi}) + \frac{4\chi_0 e}{3nr^2c}(n_{,\psi}\lambda_{i,\phi} - n_{,\phi}\lambda_{i,\psi}) - \frac{\chi_0}{3r^2}F_i'(\psi)n_{,\phi}$$
$$+\frac{e\chi_0}{cr^3}(r_{,\phi}\lambda_{,\psi} - r_{,\psi}\lambda_{,\phi}) - \frac{e}{c}n\omega_{i,\phi} + \frac{2\chi_0}{3nr^3}(-\frac{e}{c}\lambda_i + F_i(\psi))r_{,\phi}n_{,\psi}$$
$$+\frac{\chi_0}{nr^3}F_i'(\psi)r_{,\phi} = \frac{2\chi_0}{3r^3}(\frac{e}{c}\lambda_i - F_i(\psi))n_{,\phi}r_{,\psi}$$
$$-\frac{e\chi_2}{ncr^2}(n_{,\psi}\lambda_{i,\psi} - n_{,\psi}\lambda_{i,\phi}) - \frac{e}{cr^2}(\chi_{2,\phi}\lambda_{i,\psi} - \chi_{2,\psi}\lambda_{i,\phi})$$

(2.101)
$$-\frac{n}{J}(\frac{P_{iz}'}{n})_{,r} + \frac{n}{J}(\frac{P_{ir}'}{n})_{,z} + O(\epsilon^2)$$

Equation (2.64), the parallel component of the sum of the electron and ion poloidal momentum equations is

$$p_{,\phi} - \frac{\chi_0}{3r^3}(F(\psi) - \frac{e}{c}\lambda)r_{,\phi} + \frac{4e\chi_0}{3cr^2}\lambda_{,\phi} = O(\epsilon^{3/2}). \tag{2.102}$$

Pressure and stress equations are written using flux coordinates. The electron pressure equation is

$$\begin{aligned}
&(T_{e,\phi}\lambda_{e,\psi} - T_{e,\psi}\lambda_{e,\phi}) - \frac{T_e}{n}(n_{,\phi}\lambda_{e,\psi} - n_{,\psi}\lambda_{e,\phi}) \\
&+\frac{2}{3n}(T_e - T_{\perp e})(n_{,\phi}\lambda_{e,\psi} - n_{,\psi}\lambda_{e,\phi}) + \frac{1}{3}(S_{\| e} - \gamma_e + S_{\perp e}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi} \\
&+\frac{1}{3}(S_{\perp e}\frac{|\nabla\psi|^2}{J})_{,\psi} + \frac{3}{2}[\frac{e}{c}\lambda_e - F_e(\psi)][\omega_{e,\phi} - \frac{\chi}{r^3 n}(r_{,\psi}\lambda_{e,\phi} - r_{,\phi}\lambda_{e,\psi})] \\
&= -\frac{2}{3}[\frac{e}{c}\lambda_e - F_e(\psi)][\frac{B_p^2}{B^2}(\frac{\lambda_{e,\psi}}{n})_{,\phi} + \frac{\lambda_{e,\psi}}{2nB^2}(B_p^2)_{,\phi} \\
&+(\frac{\lambda_{e,\phi}}{rn})_{,\phi}(\frac{\nabla\psi\cdot\nabla\phi}{rB^2}) - \frac{\lambda_{e,\phi}}{r^2nB^2}(Jz_{,\phi}(Jr_{,\psi})_{,\phi} + Jr_{,\phi}(Jz_{,\psi})_{,\phi})] \\
&-\frac{r}{J}(-nT_{e,t} - T_eN - n(\mathbf{U}\cdot\nabla)T_e - \frac{2^{4/3}}{3}\frac{m_e}{m_i}\frac{n(T_e - T_i)}{\tau_{ee}} + E_{es}).
\end{aligned} \tag{2.103}$$

The ion pressure equation is

$$\begin{aligned}
&(T_{i,\phi}\lambda_{i,\psi} - T_{i,\psi}\lambda_{i,\phi}) - \frac{T_i}{n}(n_{,\phi}\lambda_{i,\psi} - n_{,\psi}\lambda_{i,\phi}) + \frac{2}{3n}(T_i - T_{\perp i})(n_{,\phi}\lambda_{i,\psi} - n_{,\psi}\lambda_{i,\phi}) \\
&+\frac{1}{3}((S_{\| i} - \gamma_i + S_{\perp i}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi} + (S_{\perp i}\frac{|\nabla\psi|^2}{J})_{,\psi}) \\
&+\frac{2}{3}[-\frac{e}{c}\lambda_i - F_i(\psi)][\omega_{i,\phi} - \frac{\chi}{r^3 n}(r_{,\psi}\lambda_{i,\phi} - r_{,\phi}\lambda_{i,\psi})] \\
&= -\frac{2}{3}[-\frac{e}{c}\lambda_i + F_i(\psi)][\frac{B_p^2}{B^2}(\frac{\lambda_{i,\psi}}{n})_{,\phi} + \frac{\lambda_{i,\psi}}{2nB^2}(B_p^2)_{,\phi} \\
&+(\frac{\lambda_{i,\phi}}{rn})_{,\phi}(\frac{\nabla\psi\cdot\nabla\phi}{rB^2}) - \frac{\lambda_{i,\phi}}{r^2nB^2}(Jz_{,\phi}(Jr_{,\psi})_{,\phi} + Jr_{,\phi}(Jz_{,\psi})_{,\phi})] \\
&-\frac{r}{J}(nT_{i,t} - T_iN - n(\mathbf{U}\cdot\nabla)T_i + \frac{2^{4/3}}{3}\frac{m_e}{m_i}\frac{n(T_e - T_i)}{\tau_{ee}} + E_{is}).
\end{aligned} \tag{2.104}$$

The electron stress equation is

$$\frac{2\chi_0}{3nr^2}F_e'(\psi)\lambda_{e,\phi}+\frac{2\chi_0}{3r^2}[\frac{e}{c}\lambda_e-F_e(\psi)][-\frac{2}{r}(r_{,\phi}\lambda_{e,\psi}-r_{,\psi}\lambda_{e,\phi})$$
$$-\frac{1}{n}(n_{,\phi}\lambda_{e,\psi}-n_{,\psi}\lambda_{e,\phi})]+\frac{2}{5}(S_{\|e,\phi}+S_{\|e}\frac{r_{,\phi}}{r}+\frac{b_\theta^2}{r}(S_{\perp e}z_{,\phi}-\gamma_e r_{,\phi}))$$
$$-\frac{2}{15}((S_{\|e}-\gamma_e+S_{\perp e}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi}+(S_{\perp e}\frac{|\nabla\psi|^2}{J})_{,\psi})$$
$$+[\frac{nT_e\chi}{B^2}-(\frac{e}{c}\lambda_e-F_e(\psi))][\omega_{e,\phi}-\frac{\chi}{r^3n}(r_{,\psi}\lambda_{e,\phi}-r_{,\phi}\lambda_{e,\psi})]$$
$$+\frac{1}{3n}[T_e-2(T_e-T_{\perp e})](\lambda_{e,\psi}n_{,\phi}-\lambda_{,\phi}n_{,\psi})=\frac{r}{J}(\frac{1}{3}T_e\nabla\cdot(n\mathbf{U})$$
$$+\frac{-3\cdot 2^{1/3}}{5}\frac{nT_e-T_{\perp e}}{\tau_{ee}})-\frac{2}{3nr^2}(\frac{|\nabla\psi|^2}{\chi}+\chi_2)F_e'(\psi)\lambda_{e,\phi}$$
$$-n(\frac{e}{c}\lambda_e-F_e(\psi))((\frac{2}{3nr^2}(\frac{|\nabla\psi|^2}{\chi}+\chi_2))_{,\phi}\lambda_{,\psi}-(\frac{2}{3nr^2}(\frac{|\nabla\psi|^2}{\chi}+\chi_2))_{,\psi}\lambda_{e,\phi})$$
$$-[\frac{nT_e\chi}{B^2}-\frac{2}{3}(\frac{e}{c}\lambda_e-F_e'(\psi))][\frac{B_p^2}{B^2}(\frac{\lambda_{e,\psi}}{n})_{,\phi}+\frac{\lambda_{,\psi}}{2nB^2}(B_p^2)_{,\phi}$$
$$+(\frac{\lambda_{e,\phi}}{rn})_{,\phi}(\frac{\nabla\psi\cdot\nabla\phi}{rB^2})-\frac{\lambda_{e,\phi}}{r^2nB^2}(Jz_{,\phi}(Jr_{,\psi})_{,\phi}+Jr_{,\phi}(Jz_{,\psi})_{,\phi})]. \tag{2.105}$$

The ion stress equation is

$$-\frac{2\chi_0}{3nr^2}F_i'(\psi)\lambda_{i,\phi}+\frac{2\chi_0}{3r^2}(-\frac{e}{c}\lambda_i+F_i(\psi))(-\frac{2}{r}(r_{,\phi}\lambda_{i,\psi}-r_{,\psi}\lambda_{i,\phi})$$
$$-\frac{1}{n}(n_{,\phi}\lambda_{i,\psi}-n_{,\psi}\lambda_{i,\phi}))+\frac{2}{5}(S_{\|i,\phi}+S_{\|i}\frac{r_{,\phi}}{r}+\frac{b_\theta^2}{r}(S_{\perp i}z_{,\phi}-\gamma_i r_{,\phi}))$$
$$-\frac{2}{15}((S_{\|i}-\gamma_i+S_{\perp i}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi}+(S_{\perp i}\frac{|\nabla\psi|^2}{J})_{,\psi})$$
$$+[\frac{nT_i\chi}{B^2}-(-\frac{e}{c}\lambda_i+F_i(\psi))](\omega_{i,\phi}-\frac{\chi}{r^3n}(r_{,\psi}\lambda_{i,\phi}-r_{,\phi}\lambda_{i,\psi}))$$
$$+\frac{1}{3n}[T_i-2(T_i-T_{\perp i})](\lambda_{i,\psi}n_{,\phi}-\lambda_{i,\phi}n_{,\psi})=\frac{r}{J}(-2(T_i-T_{\perp i})N$$
$$-2n(\mathbf{U}\cdot\nabla)(T_i-T_{\perp i})+\frac{1}{3}(T_i-2(T_i-T_{\perp i}))\nabla\cdot(n\mathbf{U})$$
$$-2n(T_i-T_{\perp i})_{,t}+\frac{1}{5}(\frac{T_{e0}}{T_{i0}})^{3/2}\sqrt{\frac{m_e}{m_i}}\frac{rn(T_i-T_{\perp i})}{J\tau_{ee}}+\frac{2}{3nr^2}(\frac{|\nabla\psi|^2}{\chi}+\chi_2)F_i'(\psi)\lambda_{i,\phi}$$
$$-n(-\frac{e}{c}\lambda_i+F_i(\psi))((\frac{2}{3nr^2}(\frac{|\nabla\psi|^2}{\chi}+\chi_2))_{,\phi}\lambda_{i,\psi}-(\frac{2}{3nr^2}(\frac{|\nabla\psi|^2}{\chi}+\chi_2))_{,\psi}\lambda_{i,\phi})$$
$$-[\frac{nT_i\chi}{B^2}-\frac{2}{3}(-\frac{e}{c}\lambda_i+F_i'(\psi))][\frac{B_p^2}{B^2}(\frac{\lambda_{e,\psi}}{n})_{,\phi}+\frac{\lambda_{,\psi}}{2nB^2}(B_p^2)_{,\phi}$$
$$+(\frac{\lambda_{e,\phi}}{rn})_{,\phi}(\frac{\nabla\psi\cdot\nabla\phi}{rB^2})-\frac{\lambda_{e,\phi}}{r^2nB^2}(Jz_{,\phi}(Jr_{,\psi})_{,\phi}+Jr_{,\phi}(Jz_{,\psi})_{,\phi})]. \tag{2.106}$$

The parallel components of the heat flow equations (2.72) and (2.75) are

$$\begin{aligned}&-8n(T_e - T_{\perp e})(T_e - T_{\perp e})_{,\phi} - 8(T_e - T_{\perp e})^2 n_{,\phi} + 10n(T_e - T_{\perp e})T_{e,\phi}\\ &+4nT_e(T_e - T_{\perp e})_{,\phi} + 5nT_eT_{e,\phi}\\ &+6\frac{r}{J}[-2n(T_e - T_{\perp e})^2 + nT_e(T_e - T_{\perp e})]|\mathbf{B}|\nabla\cdot\mathbf{b} = 0,\end{aligned} \tag{2.107}$$

and

$$\begin{aligned}&-\frac{12}{5}u_{i\theta}S_{i\theta}\frac{r_{,\phi}}{r} + B_\theta\frac{r}{J}(\mathbf{u}_i\cdot\nabla)u_{i\theta} + \frac{7}{5}B_\theta S_{\|i}u_{i\theta,\phi} + \frac{2}{5}S_{i\theta}u_{i\theta,\phi}\\ &+B_\theta u_{i\theta}S_{\|i}\frac{r_{,\phi}}{r} + \frac{9}{5}B_\theta S_{i\theta}u_{ir}\frac{1}{J} - 8n(T_i - T_{\perp i})(T_i - T_{\perp i})_{,\phi}\\ &-8(T_i - T_{\perp i})^2 n_{,\phi} + 10n(T_i - T_{\perp i})T_{i,\phi} + 4nT_i(T_i - T_{\perp i})_{,\phi} + 5nT_iT_{i,\phi}\\ &+6\frac{r}{J}[-2n(T_i - T_{\perp i})^2 + nT_i(T_i - T_{\perp i})]|\mathbf{B}|\nabla\cdot\mathbf{b} = O(\epsilon^2).\end{aligned} \tag{2.108}$$

Of the remaining side relations, it is convenient to write those for the perpendicular electron heat flow (2.73) and (2.74). The equation for $S_{\perp e}$ is

$$\begin{aligned}&2n(T_e - T_{\perp e})(T_e - T_{\perp e})_{,\phi} + 2(T_e - T_{\perp e})^2 n_{,\phi}\\ &-5n(T_e - T_{\perp e})T_{e,\phi} - 2T_e(T_e - T_{\perp e})n_{,\phi} - 5nT_eT_{e,\phi}\\ &+6\frac{rn}{J}[(T_e - T_{\perp e})^2 + T_e(T_e - T_{\perp e})][(\mathbf{b}\cdot\nabla)\mathbf{b}]\cdot(\nabla\psi\times\mathbf{B})\frac{1}{\chi}\\ &-\frac{e}{c\chi}B^2(S_{\perp e}\frac{|\nabla\psi|^2}{J}) = 0.\end{aligned} \tag{2.109}$$

The equation for γ_e is

$$\begin{aligned}&\nabla\psi\cdot(2n(T_e - T_{\perp e})\nabla(T_e - T_{\perp e}) + 2(T_e - T_{\perp e})^2\nabla n\\ &-5n(T_e - T_{\perp e})\nabla T_e - 2T_e(T_e - T_{\perp e})\nabla n - 5nT_e\nabla T_e)\\ &+6n[(T_e - T_{\perp e})^2 + T_e(T_e - T_{\perp e})][(\mathbf{b}\cdot\nabla)\mathbf{b}]\cdot\nabla\psi\\ &-\frac{e}{c}\frac{\gamma_e\chi|\nabla\psi|^2}{r^2}(\frac{|\nabla\psi|^2}{\chi^2} + 1) = 0.\end{aligned} \tag{2.110}$$

The perpendicular ion heat flow equations (2.76) and (2.77) are

$$-\frac{12}{5}\frac{r_{,\phi}}{r}u_{i\theta}S_{i\theta} + \frac{|\nabla\psi|^2}{\chi_0 J}(\mathbf{u}_i\cdot\nabla)S_{i\theta} + \frac{7}{5}\frac{|\nabla\psi|^2}{\chi_0 J}(\mathbf{S}_i\cdot\nabla)u_{i\theta}$$
$$+2n(T_i - T_{\perp i})(T_i - T_{\perp i})_{,\phi} + 2(T_i - T_{\perp i})^2 n_{,\phi}$$
$$-5n(T_i - T_{\perp i})T_{i,\phi} - 2T_i(T_i - T_{\perp i})n_{,\phi} - 5nT_iT_{i,\phi}$$
$$+6\frac{rn}{J}[(T_i - T_{\perp i})^2 + T_i(T_i - T_{\perp i})][(\mathbf{b}\cdot\nabla)\mathbf{b}]\cdot(\nabla\psi\times\mathbf{B})\frac{1}{\chi}$$
$$+\frac{e}{c\chi}B^2(S_{\perp i}\frac{|\nabla\psi|^2}{J}) = O(\epsilon^2), \tag{2.111}$$

and

$$-\frac{12}{5}\frac{z_{,\phi}}{r}JS_{i\theta}u_{i\theta} + \frac{2}{5}S_{i\theta}\nabla\psi\cdot\nabla u_{i\theta}$$
$$+\nabla\psi\cdot(2n(T_i - T_{\perp i})\nabla(T_i - T_{\perp i}) + 2(T_i - T_{\perp i})^2\nabla n$$
$$-5n(T_i - T_{\perp i})\nabla T_i - 2T_i(T_i - T_{\perp i})\nabla n - 5nT_i\nabla T_i)$$
$$+6n[(T_i - T_{\perp i})^2 + T_i(T_i - T_{\perp i})][(\mathbf{b}\cdot\nabla)\mathbf{b}]\cdot\nabla\psi$$
$$+\frac{e}{c}\frac{\gamma_e\chi|\nabla\psi|^2}{r^2}(\frac{|\nabla\psi|^2}{\chi^2} + 1) = O(\epsilon^2). \tag{2.112}$$

For the constraint (2.65) arising from poloidal momentum equations, I choose $\mathbf{a} = \mathbf{r}_{,\psi} = (r_{,\psi}, z_{,\psi})$ and impose the constraint on the curve $\phi = const.$ from the magnetic axis to the plasma boundary. The constraint (2.65) is:

$$p_{,\psi} - \frac{e}{c}n\omega + \frac{\chi_0}{r^2}F'(\psi) - \frac{\chi_0}{3r^3}(F(\psi) - \frac{e}{c}\lambda)r_{,\psi} - \frac{\chi_0}{3r^2}(F(\psi) - \frac{e}{c}\lambda)_{,\psi} = O(\epsilon^{3/2}). \tag{2.113}$$

I now discuss the role of the constraint (2.113). In the equation set the unknown ω_e appears only in the form $\omega_{e,\phi}$. Hence, one may add an arbitrary function of ψ to ω_e. I claim that the constraint (2.113) can be satisfied by the appropriate choice of this flux function. To verify this claim, I show that $\omega_{,\phi}$ calculated by taking the sum of (2.100) and (2.101) is the same as $\omega_{,\phi}$ calculated by taking the derivative of (2.113) with respect to ϕ. The sum of the poloidal momentum equations (2.51) and (2.52) has the symbolic form

$$n\omega\nabla\psi = \mathbf{M}' \tag{2.114}$$

where $\mathbf{M}'$ has only poloidal components. The constraint (2.113) is then just

$$n\omega = \mathbf{M}' \cdot \mathbf{r}_{,\psi}. \tag{2.115}$$

Taking the derivative of (2.115) with respect to ϕ gives

$$\omega_{,\phi} = (\frac{1}{n}\mathbf{M}' \cdot \mathbf{r}_{,\psi})_{,\phi} = \frac{1}{J}\hat{\theta} \cdot \nabla \times (\frac{1}{n}\mathbf{M}'), \tag{2.116}$$

where I use that $\mathbf{M}' \cdot \mathbf{B} = 0$. The sum of equations (2.100) and (2.101) is just

$$\omega_{,\phi} = \frac{1}{J}\hat{\theta} \cdot \nabla \times (\frac{1}{n}\mathbf{M}'). \tag{2.117}$$

Thus (2.113) may always be satisfied by the appropriate choice of the arbitrary flux function part of ω.

Let us now review the equation set. The primary unknowns are ψ, λ_a, ω_a, n, T_a, $S_{\|a}$, a total of ten scalar variables. The equations in flux coordinates are a generalized Grad-Shafranov equation (2.38), electron poloidal momentum balance (2.100), ion poloidal momentum balance (2.101), the parallel component of the sum of poloidal momentum balance (2.102), the electron pressure equation (2.103), the ion pressure equation (2.104), the electron stress equation (2.105), the ion stress equation (2.106), the parallel component of the electron heat flow equation (2.107), the parallel component of ion heat flow equation (2.108). The secondary variables $(T_e - T_{\perp e})$, $(T_i - T_{\perp i})$, $\mathbf{U}$, f_4, f_{i4}, $S_{\perp e}$, γ_e, $S_{\perp i}$, and γ_i are defined by (2.49), (2.50), (2.33), (2.45), (2.46), (2.109), (2.110), (2.111), and (2.112). In addition there are the constraints (2.41), (2.42), and (2.65). Later I will show that the constraint (2.42) reduces to a condition only on the momentum sources. I have shown that the constraint (2.65) can always be satisfied by choosing the part of ω_e constant on flux surfaces appropriately. The remaining constraint (2.41) will be used to determine the time evolution of ψ. Note that the time evolution appears explicitly only in the definition of $\mathbf{U}$, the pressure equations, and in the constraint (2.41).

3. Formal Expansion

To understand better the physics and mathematics of the model, the solution is expanded in an asymptotic series in powers of $\epsilon^{1/2}$. Equilibria are found first on the electron-electron collision time scale τ_{ee}, and are then extended to longer time scales. In lowest order the solutions are functions of ψ alone and the poloidal magnetic field is given by a Grad-Shafranov type equation. Corrections to the lowest order solution give the poloidal variation.

3.1. Expansion Procedure

The reduced two-fluid thirteen moment system is still quite complicated. The structure of the equations is not standard. Even after neglecting small quantities, the largest and smallest terms in the equations differ by a factor of $\epsilon^2 \sim 2000$. The appearance of small quantities in the system suggests that a reasonable method of investigating the properties of the solution, is to expand the solution in a formal series. The largest of the small parameters that appear is $O(\epsilon^{1/2})$, so the solution is expanded in powers of $\epsilon^{1/2}$. That is, all variables are written in the form

$$w = w_0 + w_1 + w_2 + w_3 + w_4 + \dots \tag{3.1}$$

where $w_n/w_0 \sim O(\epsilon^{n/2})$. I define

$$r = R + r_1(\psi, \phi), \tag{3.2}$$

and

$$z = z_1(\psi, \phi). \tag{3.3}$$

I substitute these expansions in the equation set and find the asymptotic solution order by order. At each order, the system can be arranged in a almost "triangular" manner. The solution scheme is the following. First, I know the temperature anisotropy $(T_a - T_{\perp a})$, a secondary variable, in terms of the poloidal stream functions λ_a from equations (2.49) and (2.50). Then I solve equations (2.107) and (2.108), coming from the parallel component of the heat flow equations to find the temperature T_a in terms of the anisotropy $(T_a - T_{\perp a})$. I then solve equation (2.102) to find the density n in terms of the temperature and the poloidal stream functions. At this point the temperatures T_a, and the density n are known in terms of the poloidal stream functions λ_a. The perpendicular components of the heat flows $S_{\perp a}$ and γ_a, secondary variables, are given by the algebraic definitions (2.109), (2.110), (2.111) and (2.112). Then I find the parallel heat flow $S_{\|a}$ from the pressure equations (2.103) and (2.104) in terms of density, temperature, and poloidal flow. I find the toroidal flow ω_a using (2.100) and (2.101), equations derived from poloidal momentum balance. At this point, all quantities are known in term of the poloidal stream functions λ_e and λ_i, which are then determined using the two stress equations (2.105) and (2.106). Finally, there is the generalized Grad-Shafranov equation for ψ. The ten primary unknowns in the system have been at least partially determined. There remain the three constraints, (2.41), (2.42), (2.65). The constraint (2.41) contains only fourth order quantities and will only be used in fourth order. The constraint (2.42) will be shown to be satisfied by appropriate choice of momentum sources. It was shown in the previous chapter that the constraint (2.65) is satisfied by choosing correctly the part of ω_e that is constant on flux surfaces, a quantity that is not determined by the equation set.

I now describe in more detail the structure of the equations that will be solved. Important issues are the existence and uniqueness of solutions. The structure of the generalized Grad-Shafranov equation is that of a nonlinear elliptic differential equation and is fairly standard. The other nine equations in the ordered system have two distinct forms. The eight ordered equations (2.107), (2.108), (2.102), (2.103), (2.104), (2.100), (2.101) and (2.106) have the general form

$$w_{,\phi} = G(\phi, \psi), \tag{3.4}$$

where w is an unknown, and G is periodic in ϕ; equation (3.4) can be solved by integration. In first and second order the right hand side of (3.4) is an exact derivative with respect to ϕ and the solutions are quite explicit. The unknown w is periodic in ϕ if and only if the condition

$$\langle G\rangle = \frac{1}{2\pi}\int_0^{2\pi} G\, d\phi = 0 \tag{3.5}$$

holds. If, as in the first and second order systems, G is an exact derivative with respect to ϕ then (3.5) holds trivially. The solution w is not determined uniquely since an arbitrary flux function can be added to w. The other form of equation encountered is that of (2.105) the electron temperature anisotropy equation which has the form

$$w_{,\phi} + G_1(\phi,\psi)w = G_2(\phi,\psi) \tag{3.6}$$

with G_1 and G_2 periodic in ϕ. The undifferentiated term G_1 comes from collisional terms. The homogeneous equation ($G_2 = 0$) has a non-zero periodic solution only if $\langle G_1\rangle = 0$. Thus, if $\langle G_1\rangle \neq 0$ equation (3.6) has a unique periodic solution.

3.2. Zero Order

I now consider the lowest order system. I follow the procedure sketched in the previous section. The scaling assumption

$$\frac{(T_a - T_{\perp a})}{T_a} \sim O(\epsilon^{1/2}). \tag{3.7}$$

along with the relations (2.49) and (2.50) imply that $\lambda_{e0} = F_e(\psi)$ and $\lambda_{i0} = F_i(\psi)$. From the parallel components of the heat equations (2.107) and (2.108), I find that $T_{e0} = T_{e0}(\psi)$ and $T_{i0} = T_{i0}(\psi)$ with T_{e0} and T_{i0} undetermined. All these flux functions also have an explicit dependence on time that will be suppressed until required. From equation (2.102) one finds that $n_0 = n_0(\psi)$. From (2.73), (2.74), (2.76) and (2.77) the perpendicular components of the heat flow in lowest order are

$$\mathbf{S}_{e0}\cdot\nabla\psi = \mathbf{S}_{i0}\cdot\nabla\psi = 0 \tag{3.8}$$

and

$$\gamma_{e0} = -\frac{5 n_0 T_{e0} c R^2}{e\chi_0} T_{e0,\psi} \tag{3.9}$$

$$\gamma_{i0} = \frac{5 n_0 T_{i0} c R^2}{e \chi_0} T_{i0,\psi}. \tag{3.10}$$

The pressure equations (2.103) and (2.104) in lowest order are $\nabla \cdot \mathbf{S}_{a0} = 0$, implying $S_{\|a0} = S_{\|a0}(\psi)$. From equations (2.100) and (2.101), I find that $\omega_{a0} = \omega_{a0}(\psi)$. Thus, the smallness of the anisotropy forces all the fluid variables to be in lowest order functions of ψ alone.

The remainder of the system is the generalized Grad-Shafranov equation (2.38) and the constraints (2.113), (2.41), and (2.42). The integral constraint (2.42) reduces to a constraint on the toroidal components of the momentum sources:

$$\int_{\psi \le \overline{\psi}} r(P_{se\theta} + P_{si\theta}) = 0, \tag{3.11}$$

and once satisfied can be dropped from the system. The constraint (2.41) involves only fourth order quantities. The constraint (2.113) is in lowest order the pressure balance

$$p_{0,\psi} - \frac{e}{c} n_0 \omega_0 + \frac{\chi_0}{R^2} F'(\psi) = 0. \tag{3.12}$$

Combining this with equation (2.38) gives

$$\Delta^* \psi = 4\pi\mu_0 (R^2 p_0'(\psi) + \chi_0 F'(\psi)). \tag{3.13}$$

The equilibrium magnetic field depends on the total fluid pressure profile $p_0(\psi)$, the poloidal current $\lambda_0(\psi)$ and the vacuum toroidal field $\chi_0(t)$. The flux functions $n_0(\psi)$, $T_{a0}(\psi)$, $\lambda_{a0}(\psi)$, $\omega_{a0}(\psi)$ and $S_{\|a0}(\psi)$ are arbitrary except for satisfying the relation (3.12). The lowest order solution contains eight arbitrary flux functions. At this point in the calculation there is no information about the time evolution of the lowest order solution described here. The system must be solved to higher order to determine the evolution of the lowest order solution and to determine additional constraints on the zero order solution.

3.3. First Order

The first order system introduces the poloidal dependence of the unknowns. The general structure of the system was described in section (3.1). From (2.43) and (2.44), the anisotropy in first order is

(3.14) $$(T_e - T_{\perp e})_1 = \frac{e\chi_0}{3n_0cR^2}\lambda_{e1}$$

and

(3.15) $$(T_i - T_{\perp i})_1 = -\frac{e\chi_0}{3n_0cR^2}\lambda_{i1}.$$

From the parallel component of the heat flow equations (2.107) and (2.108), the poloidal variation of the temperature is balanced by that of the anisotropy so that,

(3.16) $$T_{e1,\phi} = -\frac{4}{5}(T_e - T_{\perp e})_{,\phi}$$

and

(3.17) $$T_{i1,\phi} = -\frac{4}{5}(T_i - T_{\perp i})_{,\phi}.$$

Thus,

(3.18) $$T_{e1} = -\frac{4e\chi_0}{15n_0R^2c}\lambda_{e1} + \tilde{T}_{e1}(\psi)$$

and

(3.19) $$T_{i1} = \frac{4e\chi_0}{15n_0R^2c}\lambda_{i1} + \tilde{T}_{i1}(\psi)$$

where $\tilde{T}_{e1}(\psi)$ and $\tilde{T}_{i1}(\psi)$ are arbitrary flux functions of order $O(\epsilon^{1/2})$. From equation (2.102) one finds that

(3.20) $$n_{1,\phi} = \frac{1}{T_0}(-n_0T_{1,\phi} - \frac{4e\chi_0}{3cR^2}\lambda_{1,\phi})$$

or

(3.21) $$n_1 = -\frac{8e\chi_0}{5cR^2(T_{e0} + T_{i0})}(\lambda_{i1} - \lambda_{e1}) + \tilde{n}_1(\psi),$$

where $\tilde{n}_1(\psi)$ is an $O(\epsilon^{1/2})$ arbitrary flux function.

The components of the heat flow in the direction of $\nabla\psi$ are found from the algebraic equations (2.109) and (2.111) to be:

$$S_{\perp e1}\frac{|\nabla\psi|^2}{J} = -\frac{5cR^2 n_0 T_{e0}}{e\chi_0}T_{e1,\phi}, \tag{3.22}$$

and

$$S_{\perp i1}\frac{|\nabla\psi|^2}{J} = \frac{5cR^2 n_0 T_{i0}}{\chi_0 e}T_{i1,\phi}. \tag{3.23}$$

The net heat flow across a flux surface for electrons and ions respectively is

$$\langle S_{\perp e1}\frac{|\nabla\psi|^2}{J}\rangle = 0, \tag{3.24}$$

and

$$\langle S_{\perp i1}\frac{|\nabla\psi|^2}{J}\rangle = 0. \tag{3.25}$$

The components of the heat flow perpendicular to **B** but within the flux surface are:

$$\begin{aligned}\gamma_{e1} = {} & \frac{cR^2}{e\chi_0}(-2T_{e0}(T_e - T_{\perp e})_1 n_{0,\psi} - 5n_0(T_e - T_{\perp e})_1 T_{e0,\psi} \\ & -5(n_1 T_{e0} + n_0 T_{e1})T_{e0,\psi} - 5n_0 T_{e0}(T_{e1,\psi} + T_{e1,\phi}\frac{\nabla\psi\cdot\nabla\phi}{|\nabla\psi|^2})) + 2\frac{r_1}{R}\gamma_{e0}\end{aligned} \tag{3.26}$$

$$\begin{aligned}\gamma_{i1} = {} & \frac{cR^2}{e\chi_0}(2T_{i0}(T_i - T_{\perp i})_1 n_{0,\psi} + 5n_0(T_i - T_{\perp i})_1 T_{i0,\psi} \\ & +5(n_1 T_{i0} + n_0 T_{i1})T_{i0,\psi} + 5n_0 T_{i0}(T_{i1,\psi} + T_{i1,\phi}\frac{\nabla\psi\cdot\nabla\phi}{|\nabla\psi|^2})) + 2\frac{r_1}{R}\gamma_{i0}.\end{aligned} \tag{3.27}$$

The divergence of the perpendicular part of the heat flow is then for the electrons

$$\begin{aligned}(\gamma_{e1,\phi} - (S_{\perp e1}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi} - (S_{\perp e1}\frac{|\nabla\psi|^2}{J})_{,\psi}) = {} & \frac{cR^2}{e\chi_0}(-2T_{e0}n_{0,\psi}(T_e - T_{\perp e})_1 \\ & -5n_0 T_{e0,\psi}(T_e - T_{\perp e})_1 - 5T_{e0}T_{e0,\psi}n_1 + 5n_{0,\psi}T_{e0}T_{e1})_{,\phi} + 2\gamma_{e0}\frac{r_{1,\phi}}{R},\end{aligned} \tag{3.28}$$

and for the ions

$$\begin{aligned}(\gamma_{i1,\phi} - (S_{\perp i1}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi} - (S_{\perp i1}\frac{|\nabla\psi|^2}{J})_{,\psi}) = {} & \frac{cR^2}{e\chi_0}(2T_{i0}n_{0,\psi}(T_i - T_{\perp i})_1 \\ & +5n_0 T_{i0,\psi}(T_i - T_{\perp i})_1 + 5T_{i0}T_{i0,\psi}n_1 - 5n_{0,\psi}T_{i0}T_{i1})_{,\phi} + 2\gamma_{i0}\frac{r_{1,\phi}}{R}.\end{aligned} \tag{3.29}$$

The pressure equations (2.103) and (2.104) are used to find the parallel heat flows $S_{\|e1}$ and $S_{\|i1}$ that give an equilibrium energy balance

$$
\begin{aligned}
\frac{1}{3}S_{\|e1,\phi} &= -(T_{e1,\phi}\lambda_{e0,\psi} - T_{e0,\psi}\lambda_{e1,\phi}) + \frac{T_{e0}}{n_0}(n_{1,\phi}\lambda_{e0,\psi} - n_{0,\psi}\lambda_{e1,\phi}) \\
&+\frac{1}{3}(\gamma_{e1,\phi} - (S_{\perp e1}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi} - (S_{\perp e1}\frac{|\nabla\psi|^2}{J})_{,\psi})
\end{aligned} \tag{3.30}
$$

and

$$
\begin{aligned}
\frac{1}{3}S_{\|i1,\phi} &= -(T_{i1,\phi}\lambda_{i0,\psi} - T_{i0,\psi}\lambda_{i1,\phi}) + \frac{T_{i0}}{n_0}(n_{1,\phi}\lambda_{i0,\psi} - n_{0,\psi}\lambda_{i1,\phi}) \\
&+\frac{1}{3}(\gamma_{i1,\phi} - (S_{\perp i1}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi} - (S_{\perp i1}\frac{|\nabla\psi|^2}{J})_{,\psi}).
\end{aligned} \tag{3.31}
$$

Using the previous relations the parallel heat flow can be written in terms of r_1, λ_{e1} and λ_{i1}

$$
\begin{aligned}
\frac{1}{3}S_{\|e1} &= -\frac{10cRn_0T_{e0}T_{e0,\psi}}{e\chi_0}r_1 + \lambda_1(\frac{8T_{e0}T_{e0,\psi}}{3(T_{e0}+T_{i0})} - \frac{8e\chi_0T_{e0}\lambda_{e0,\psi}}{5cR^2n_0(T_{e0}+T_{i0})}) \\
&+\lambda_{e1}(\frac{4e\chi_0\lambda_{e0,\psi}}{15cR^2} + \frac{4}{9}T_{e0,\psi} - \frac{5T_{e0}n_{0,\psi}}{3n_0}) + \frac{1}{3}\tilde{S}_{\|e1}(\psi)
\end{aligned} \tag{3.32}
$$

and

$$
\begin{aligned}
\frac{1}{3}S_{\|i1} &= \frac{10cRn_0T_{i0}T_{i0,\psi}}{e\chi_0}r_1 + \lambda_1(\frac{8T_{i0}T_{i0,\psi}}{3(T_{e0}+T_{i0})} - \frac{8e\chi_0T_{i0}\lambda_{i0,\psi}}{5cR^2n_0(T_{e0}+T_{i0})}) \\
&+\lambda_{i1}(-\frac{4e\chi_0\lambda_{i0,\psi}}{15cR^2} + \frac{4}{9}T_{i0,\psi} - \frac{5T_{i0}n_{0,\psi}}{3n_0}) + \frac{1}{3}\tilde{S}_{\|i1}(\psi),
\end{aligned} \tag{3.33}
$$

where $\tilde{S}_{\|e1}(\psi)$ and $\tilde{S}_{\|i1}(\psi)$ are arbitrary flux functions of $O(\epsilon^{1/2})$.

The poloidal momentum balance equations (2.100) and (2.101) are used to find the toroidal flows ω_{e1} and ω_{i1}

$$
\begin{aligned}
\frac{e}{c}\omega_{e1} &= T_{e1}\frac{n_{0,\psi}}{n_0} - n_1(\frac{T_{e0,\psi}}{n_0} - \lambda_{e0,\psi}\frac{e\chi_0}{cn_0^2R^2}) \\
&+\lambda_{e1}(\frac{4e\chi_0}{3cn_0^2R^2}n_{0,\psi}) + r_1(\frac{e\chi_0}{cR^3n_0}\lambda_{e0,\psi}) + \frac{e}{c}\tilde{\omega}_{e1}(\psi)
\end{aligned} \tag{3.34}
$$

$$\frac{e}{c}\omega_{i1} = -T_{i1}\frac{n_{0,\psi}}{n_0} + n_1(\frac{T_{i0,\psi}}{n_0} - \lambda_{i0,\psi}\frac{x\chi_0}{cn_0^2R^2})$$

$$+\lambda_{i1}(\frac{4e\chi_0}{3cn_0^2R^2}n_{0,\psi}) + r_1(\frac{e\chi_0}{cR^3n_0}\lambda_{i0,\psi}) + \frac{e}{c}\tilde{\omega}_{i1}(\psi) \qquad (3.35)$$

Using the previous relations

$$\frac{e}{c}\omega_{e1} = r_1\frac{e\chi_0\lambda_{e0,\psi}}{cR^3n_0} + \lambda_{e1}\frac{16e\chi_0n_{0,\psi}}{15cR^2n_0^2}$$

$$+\lambda_1(\frac{8e\chi_0}{5cR^2n_0(T_{e0}+T_{i0})})(\frac{e\chi_0\lambda_{e0,\psi}}{n_0cR^2} + T_{e0,\psi}) + \frac{e}{c}\tilde{\omega}_{e1}(\psi) \qquad (3.36)$$

$$\frac{e}{c}\omega_{i1} = r_1\frac{e\chi_0\lambda_{i0,\psi}}{cR^3n_0} + \lambda_{i1}\frac{16e\chi_0n_{0,\psi}}{15cR^2n_0^2}$$

$$+\lambda_1(\frac{8e\chi_0}{5cR^2n_0(T_{e0}+T_{i0})})(\frac{e\chi_0\lambda_{i0,\psi}}{n_0cR^2} - T_{i0,\psi}) + \frac{e}{c}\tilde{\omega}_{i1}(\psi) \qquad (3.37)$$

I have now expressed the unknowns n_1, T_{e1}, T_{i1}, $S_{||e1}$, $S_{||i1}$, ω_{e1}, ω_{i1} in terms of λ_{e1}, λ_{i1}, r_1 and seven undetermined $O(\epsilon^{1/2})$ flux functions. The structure of the system has been such that all solvability conditions have been trivially satisfied. Finally λ_{e1} and λ_{i1} are found using the two stress equations. Note the difference between the ion and electron stress equations; the electron stress equation contains the undifferentiated term λ_{e1} coming from the collision term. There is no corresponding collisional term in the ion equation in first order. The electron and ion stress equations in first order are respectively:

$$\frac{2\chi_0}{3n_0R^2}F_e'(\psi)\lambda_{e1,\phi} + \frac{2}{5}(S_{||e1,\phi} + \frac{r_{1,\phi}}{R}(S_{||e0} - \gamma_{e0})) + \frac{2}{5}(T_{e1,\phi}\lambda_{e0,\psi} - T_{e0,\psi}\lambda_{e1,\phi})$$

$$+\frac{T_{e0}}{n_0}(n_{1,\phi}\lambda_{e0,\psi} - n_{0,\psi}\lambda_{e1,\phi}) + \frac{n_0T_{e0}R^2}{\chi_0}(\omega_{e1,\phi} + \frac{\chi_0}{n_0R^3}\lambda_{e0,\psi}r_{1,\phi}) = -\frac{2^{1/3}e\chi_0}{5cR^2}\frac{\lambda_{e1}}{J\tau_{ee}} \qquad (3.38)$$

and

$$-\frac{2\chi_0}{3n_0R^2}F_i'(\psi)\lambda_{i1,\phi} + \frac{2}{5}(S_{||i1,\phi} + \frac{r_{1,\phi}}{R}(S_{||i0} - \gamma_{i0})) + \frac{2}{5}(T_{i1,\phi}\lambda_{i0,\psi} - T_{i0,\psi}\lambda_{i1,\phi})$$

$$+\frac{T_{i0}}{n_0}(n_{1,\phi}\lambda_{i0,\psi} - n_{0,\psi}\lambda_{i1,\phi}) + \frac{n_0T_{i0}R^2}{\chi_0}(\omega_{i1,\phi} + \frac{\chi_0}{n_0R^3}\lambda_{i0,\psi}r_{1,\phi}) = 0. \qquad (3.39)$$

From these two equations one can obtain an expression for λ_{i1} in terms of λ_{e1}, r_1 and an arbitrary flux function $\tilde{\lambda}_{i1}(\psi)$. Then one can write an ordinary differential equation for λ_{e1} of the form

$$A(\psi)\lambda_{e1,\phi} + \frac{B(\psi)}{J}\lambda_{e1} = C(\psi)\frac{r_{1,\phi}}{R} \tag{3.40}$$

where $A(\psi)$, $B(\psi)$ and $C(\psi)$ depend on the zero order solution. In general the above equation has a unique periodic solution if $B(\psi)$ is not zero. Thus one can solve (3.40) for λ_{e1} and the first order solution would be known up to the eight arbitrary first order flux functions already introduced.

Before calculating λ_{e1} more explicitly, it is convenient at this point to leave the sequence of the solution scheme and look ahead to the second order system. The reason for doing so is that solvability conditions encountered in the second order system have a striking effect on the nature of the first order solution. Consider equation (2.102) in second order,

$$n_0T_{2,\phi} + n_{2,\phi}T_0 + (n_1T_1)_{,\phi} - \frac{\chi_0}{3R^2}\lambda_1 r_{1,\phi} + \frac{4e\chi_0}{3cR^2}\lambda_{2,\phi} + \frac{4e\chi_0}{3cR^3}r_1\lambda_{1,\phi} = 0. \tag{3.41}$$

In order that equation (3.41) have a periodic solution the first order solution must satisfy the condition

$$\langle r_{1,\phi}\lambda_{e1}\rangle = 0. \tag{3.42}$$

However, multiplying (3.40) by λ_{e1} and applying $\langle\cdot\rangle$ gives that

$$\langle\frac{\lambda_{e1}^2}{J}\rangle = 0 \tag{3.43}$$

which implies that $\lambda_{e1} = 0$. Thus to $O(\epsilon)$ the electron poloidal flow is in the flux surface. Setting $\lambda_{e1} = 0$ imposes the condition on the zero order solution that

$$C(\psi) = 0, \tag{3.44}$$

reducing the number of arbitrary flux functions in the zero order solution to seven. Thus, the collisional term in the electron anisotropy equation forces the electron distribution function to be Maxwellian to $O(\epsilon)$ rather then $O(\epsilon^{1/2})$ as was assumed.

I now summarize the changes in the solution resulting from setting λ_{e1} to zero. From the electron stress equation λ_{i1} is

$$\lambda_{i1} = \frac{r_1}{R}(2T_{e0}\lambda_{e0,\psi} + \frac{2T_{e0}T_{e0,\psi}cR^2n_0}{3e\chi_0} + \frac{2}{5}S_{\|e0}) \cdot(\frac{8T_{e0}T_{e0,\psi}}{3(T_{e0}+T_{i0})} - \frac{25e\chi_0T_{e0}\lambda_{e0,\psi}}{15cR^2n_0(T_{e0}+T_{i0})})^{-1} + \tilde{\lambda}_{i1}(\psi). \tag{3.45}$$

Other effects of setting $\lambda_{e1} = 0$ are

$$T_{e1} = S_{\perp e1} = (T_e - T_{\perp e})_1 = 0, \tag{3.46}$$

the electron temperature is a flux function to $O(\epsilon)$ and the perpendicular electron heat flow and the electron anisotropy are $O(\epsilon)$. Also

$$\gamma_{e1} = -\frac{8T_{e0}T_{e0,\psi}}{T_0}\lambda_{i1} - \frac{10cR^2n_0T_{e0}T_{e0,\psi}}{e\chi_0}\frac{r_1}{R}, \tag{3.47}$$

and

$$\frac{1}{3}S_{\|e1} = -\frac{10cRn_0T_{e0}T_{e0,\psi}}{e\chi_0}r_1 + \lambda_1(\frac{8T_{e0}T_{e0,\psi}}{3(T_{e0}+T_{i0})} - \frac{8e\chi_0T_{e0}\lambda_{e0,\psi}}{5cR^2n_0(T_{e0}+T_{i0})}) + \frac{1}{3}\tilde{S}_{\|e1}(\psi). \tag{3.48}$$

Note that $\nabla \cdot \mathbf{S}_{e1} = 0$. Also

$$n_1 = -\frac{8e\chi_0}{5cR^2(T_{e0}+T_{i0})}\lambda_{i1} + \tilde{n}_1(\psi), \tag{3.49}$$

and

$$\frac{e}{c}\omega_{e1} = r_1\frac{e\chi_0\lambda_{e0,\psi}}{n_0cR^3} + \lambda_{i1}(\frac{8e\chi_0}{5n_0cR^2(T_{e0}+T_{i0})})(\frac{e\chi_0\lambda_{e0,\psi}}{n_0cR^2} + T_{e0,\psi}) + \frac{e}{c}\tilde{\omega}_{e1}(\psi). \tag{3.50}$$

The final part of the first order solution is the constraint (2.113) which determines the arbitrary flux function part of $(n\omega)_1$. The constraint (2.113) in first order is

$$\frac{e}{c}(n_0\omega_1 + n_1\omega_0) = \frac{\partial}{\partial\psi}(p_1 + \frac{4e\chi_0}{3cR^2}\lambda_{i1}) - \frac{2e\chi_0}{cR^2}F'(\psi)\frac{r_1}{R}. \tag{3.51}$$

From the above solutions one can find that

$$\frac{e}{c}(n_0\omega_1 + n_1\omega_0) = -\frac{2e\chi_0}{cR^2}F'(\psi)\frac{r_1}{R} + \frac{e}{c}(n_0\tilde{\omega}_1 + \tilde{n}_1\omega_0). \tag{3.52}$$

From the constraint (3.51) one finds

$$\begin{aligned} \frac{e}{c}(n_0\tilde{\omega}_1 + \tilde{n}_1\omega_0) &= \frac{\partial}{\partial\psi}(p_1 + \frac{4e\chi_0}{3cR^2}\lambda_{i1}) \\ &= \frac{\partial}{\partial\psi}(n_0\tilde{T}_{i1} + \tilde{n}_1 T_0 + \frac{4e\chi_0}{3cR^2}\tilde{\lambda}_{i1}). \end{aligned} \tag{3.53}$$

This constraint reduces the number of arbitrary $O(\epsilon^{1/2})$ flux functions in the first order solution from eight to seven. Assuming the zero order solution given, the first order solutions are determined up to the seven first order arbitrary flux functions appearing in the solution.

A simple consequence of the structure of the first order solutions is that for any k

$$\begin{aligned} \langle r_1^k\lambda_{i1,\phi}\rangle = \langle r_1^k\omega_{i1,\phi}\rangle = \langle r_1^k\omega_{e1,\phi}\rangle = \langle r_1^k n_{1,\phi}\rangle \\ = \langle r_1^k T_{i1,\phi}\rangle = \langle r_1^k S_{\|i1,\phi}\rangle = \langle r_1^k S_{\|e1,\phi}\rangle = 0. \end{aligned} \tag{3.54}$$

These properties will be used repeatedly in the evaluation of solvability conditions in higher order.

It is convenient to absorb the seven arbitrary $O(\epsilon^{1/2})$ flux functions that were introduced in the first order solution into the seven arbitrary flux functions in the zero order solution. For example, λ_i through first order is

$$\lambda_i = \lambda_{i0}(\psi) + L_{i1}(\psi)\frac{r_1}{R} + \tilde{\lambda}_{i1}(\psi), \tag{3.55}$$

where L_{i1} is determined from the zero order solution accurate to $O(\epsilon^{1/2})$. I now redefine $\lambda_{i0}(\psi)$ so that through first order λ_i is

$$\lambda_i = \lambda_{i0}(\psi) + L_{i1}(\psi)\frac{r_1}{R}. \tag{3.56}$$

The following representation is convenient for the first order solution:

$$\lambda_{i1} = L_{i1}(\psi)\frac{r_1}{R}, \tag{3.57}$$

(3.58)
$$n_1 = N_1(\psi)\frac{r_1}{R},$$

(3.59)
$$T_{i1} = \hat{T}_{i1}(\psi)\frac{r_1}{R},$$

where

(3.60)
$$L_{i1}(\psi) = (2T_{e0}\lambda_{e0,\psi} + \frac{2T_{e0}T_{e0,\psi}cR^2n_0}{3e\chi_0} + \frac{2}{5}S_{\|e0}) \cdot (\frac{8T_{e0}T_{e0,\psi}}{3(T_{e0}+T_{i0})} - \frac{25e\chi_0T_{e0}\lambda_{e0,\psi}}{15cR^2n_0(T_{e0}+T_{i0})})^{-1},$$

(3.61)
$$N_1(\psi) = -\frac{8e\chi_0}{5cR^2}\frac{L_{i1}(\psi)}{T_{e0}+T_{i0}},$$

and

(3.62)
$$\hat{T}_{i1}(\psi) = \frac{4e\chi_0}{15n_0cR^2}L_{i1}(\psi).$$

3.4. Second Order

The same solution scheme used in first order will be applied to the second order system. As in first order, most of the second order solutions are not unique; they are known up to the addition of arbitrary flux functions. It will be seen later that the arbitrary flux functions appearing in the second order solution do not affect transport. First, from equations (2.43) and (2.44) the second order correction to the temperature anisotropy is

(3.63)
$$(T_e - T_{\perp e})_2 = \frac{e\chi_0}{3n_0cR^2}\lambda_{e2}$$

and

(3.64)
$$(T_i - T_{\perp i})_2 = -\frac{e\chi_0}{3n_0cR^2}\lambda_{i2} + \frac{e\chi_0}{3n_0cR^2}\lambda_{i1}(\frac{n_1}{n_0} + \frac{2r_1}{R_0}).$$

The parallel electron heat flow equation (2.107) in second order is

(3.65)
$$T_{e2,\phi} = -\frac{4}{5}(T_e - T_{\perp e})_{2,\phi}.$$

The parallel ion heat flow equation (2.108) in second order is

$$T_{i2,\phi} = -\frac{2}{T_{i0}}(T_i - T_{\perp i})_1 T_{i1,\phi} + \frac{8}{5T_{i0}}(T_i - T_{\perp i})_1 (T_i - T_{\perp i})_{1,\phi}$$
$$+\frac{4}{5}(\frac{T_{i1}}{T_{i0}} + \frac{n_1}{n_0})(T_i - T_{\perp i})_{1,\phi} + (\frac{T_{i1}}{T_{i0}} + \frac{n_1}{n_0})T_{i1,\phi} + \frac{4}{5}(T_i - T_{\perp i})_{2,\phi}. \tag{3.66}$$

Using the first order solution equation (3.66) can be written

$$T_{i2,\phi} = \frac{2}{5T_{i0}}(T_i - T_{\perp i})_1 (T_i - T_{\perp i})_{1,\phi} + \frac{4}{5}(T_i - T_{\perp i})_{2,\phi}. \tag{3.67}$$

Equations (3.65) and (3.66) can be integrated to find T_{e2} and T_{i2} up to the addition of $O(\epsilon)$ flux functions. These $O(\epsilon)$ arbitrary flux function will not be needed to calculate transport; only $T_{i2,\phi}$ and $T_{i2,\phi}$ are needed. The solvability conditions associated with equations (2.107) and (2.108) are clearly satisfied.

From equation (2.102) one finds the second order correction to the density to be

$$n_{2,\phi} = -\frac{1}{T_0}(n_0 T_{2,\phi} + (n_1 T_{i1})_{,\phi} + \frac{e\chi_0}{3cR^2}\lambda_{i1} r_{1,\phi} + \frac{4e\chi_0}{3cR^2}(\lambda_{2,\phi} - 2\lambda_{i1,\phi}\frac{r_1}{R})) \tag{3.68}$$

Equation (3.68) can be integrated to find n_2. The solvability condition is automatically satisfied. From the perpendicular heat flow equations, (2.109) and (2.111) one finds

$$S_{\perp e2}\frac{|\nabla\psi|^2}{J} = -5\frac{cR^2}{e\chi_0} n_0 T_{e0} T_{e2,\phi}, \tag{3.69}$$

and

$$S_{\perp i2}\frac{|\nabla\psi|^2}{J} = \frac{cR^2}{e\chi_0}(-2n_0(T_i - T_{\perp i})_1(T_i - T_{\perp i})_{1,\phi} + 5n_0(T_i - T_{\perp i})_1 T_{i1,\phi}$$
$$+2T_{i0}(T_i - T_{\perp i})_1 n_{1,\phi} + 5(n_0 T_{i0} T_{i2,\phi} + (n_0 T_{i1} + n_1 T_{i0})T_{i1,\phi})$$
$$+6n_0 T_{i0}(T_i - T_{\perp i})_1 \frac{r_{1,\phi}}{R}) + 2\frac{r_1}{R} S_{\perp i1}\frac{|\nabla\psi|^2}{J}. \tag{3.70}$$

There is no net heat flow through a flux surface, that is

$$\langle S_{\perp e2}\frac{|\nabla\psi|^2}{J}\rangle = \langle S_{\perp i2}\frac{|\nabla\psi|^2}{J}\rangle = 0. \tag{3.71}$$

To calculate the parallel heat flow in second order I need to calculate γ_{e2} and γ_{i2}. From the equations (2.110) and (2.112) one finds

$$\gamma_{e2} = \frac{cR^2}{e\chi_0}(-5n_0(T_e - T_{\perp e})_2 T_{e0,\psi} - 2T_{e0}(T_e - T_{\perp e})_2 n_{0,\psi}$$
$$-5n_0 T_{e0}(T_{e2,\psi} + T_{e2,\phi}\frac{\nabla\phi\cdot\nabla\psi}{|\nabla\psi|^2} - 5n_0 T_{e2} T_{e0,\psi}$$
$$+\frac{cr_1 R}{e\chi_0}(-10n_1 T_{e0} T_{e0,\psi} - 5n_0 T_{e0} T_{i0,\psi}\frac{r_1}{R})$$
$$+\frac{cR^2}{e\chi_0}5n_0 T_{e0} T_{e0,\psi}(\frac{|\nabla\psi|^2}{\chi_0^2} + \frac{\chi_2}{\chi_0} + \frac{3r_1^2}{R^2}), \tag{3.72}$$

and

$$\gamma_{i2} = 2\frac{r_1}{R}\gamma_{i1} - (3\frac{r_1^2}{R^2} + \frac{\chi_2}{\chi_0} + \frac{|\nabla\psi|^2}{\chi_0^2})\gamma_{i0}$$
$$-\frac{cR^2}{e\chi_0}(2n_0(T_i - T_{\perp i})_1(T_i - T_{\perp i})_{1,\psi} + 2(T_i - T_{\perp i})_1^2 n_{0,\psi} -$$
$$5n_0(T_i - T_{\perp i})_1 T_{i1,\psi} - 5n_1(T_i - T_{\perp i})_1 T_{i0,\psi}$$
$$-5n_0(T_i - T_{\perp i})_2 T_{i0,\psi} - 2T_{i0}(T_i - T_{\perp i})_1 n_{1,\psi}$$
$$-2T_{i1}(T_i - T_{\perp i})_1 n_{0,\psi} - 2T_i(T_i - T_{\perp i})_2 n_{0,\psi}$$
$$-5n_0 T_{i0} T_{i2,\psi} - 5(n_0 T_{i1} + n_1 T_{i0})T_{i1,\psi} - 5n_1 T_{i1} T_{i0,\psi}$$
$$-6n_0 T_{i0}(T_i - T_{\perp i})_1\frac{Jz_{1,\phi}}{R} + \frac{\nabla\phi\cdot\nabla\psi}{|\nabla\psi|^2}(2n_0(T_i - T_{\perp i})_1(T_i - T_{\perp i})_{1,\phi}$$
$$-5n_0(T_i - T_{\perp i})_1 T_{i1,\phi} - 2T_{i0}(T_i - T_{\perp i})_1 n_{1,\phi}$$
$$-5n_0 T_{i0} T_{i2,\phi} + 5(n_0 T_{i0} + n_1 T_{i0})T_{i1,\phi})). \tag{3.73}$$

The parallel heat flows are found from the pressure equations to be

$$\frac{1}{3}S_{\|e2,\phi} = -(T_{e2,\phi}\lambda_{e0,\psi} - T_{e0,\psi}\lambda_{e2,\phi})$$
$$-\frac{cR^2}{e\chi_0}(5n_0 T_{e0,\psi}(T_e - T_{\perp e})_{2,\phi} - 2T_{e0} n_{0,\psi}(T_e - T_{\perp e})_{2,\phi}$$
$$-5n_{0,\psi} T_{e0} T_{e2,\phi} + 10T_{e0} T_{e0,\psi}(\frac{r_1}{R}n_1)_{,\phi} + 5n_0 T_{e0,\phi}\frac{r_1 r_{1,\phi}}{R^2}$$
$$-5n_0 T_{e0} T_{e0,\psi}(\frac{|\nabla\psi|^2}{\chi_0^2} + \frac{\chi_2}{\chi_0} + \frac{3r_1^2}{R^2})_{,\phi}), \tag{3.74}$$

and

$$\frac{1}{3}S_{\|i2,\phi} = -(T_{i2,\phi}\lambda_{i0,\psi} - T_{i0,\psi}\lambda_{i2,\phi}) - (T_{i1,\phi}\lambda_{i1,\psi} - T_{i1,\psi}\lambda_{i1,\phi})$$
$$+\frac{T_{i0}}{n_0}(n_{2,\phi}\lambda_{i0,\psi} - n_{0,\psi}\lambda_{i2,\phi} + n_{1,\phi}\lambda_{i1,\psi} - n_{1,\psi}\lambda_{i1,\phi})$$
$$+\frac{T_{i0}}{n_0}(\frac{T_{i1}}{T_{i0}} - \frac{n_1}{n_0})(n_{1,\phi}\lambda_{i0,\psi} - n_{0,\psi}\lambda_{i1,\phi})$$
$$-\frac{2}{3n_0}(T_i - T_{\perp i})_1(n_{1,\phi}\lambda_{i0,\psi} - n_{0,\psi}\lambda_{i1,\phi}) + \frac{2}{3}\lambda_{i1}(\omega_{i1,\phi} + \frac{\chi_0}{n_0 R^3}r_{1,\phi}\lambda_{i0,\psi})$$
(3.75)
$$+\frac{1}{3}\gamma_{i2,\phi} - \frac{1}{3}(S_{\perp i2}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi} - \frac{1}{3}(S_{\perp i2}\frac{|\nabla\psi|^2}{J})_{,\psi}.$$

These equations have solutions that are determined up to the addition of $O(\epsilon)$ flux functions. All the solvability conditions associated with these equations are satisfied.

From the momentum balance I find the toroidal flow

$$\omega_{e2,\phi} = -\frac{n_{0,\phi}}{n_0}T_{e2,\phi} + n_{2,\phi}(\frac{T_{e0,\psi}}{n_0} + \frac{5e\chi_0}{3cR^2n_0^2}\lambda_{e0,\psi}) - \frac{4e\chi_0}{3cR^2n_0^2}n_{0,\psi}\lambda_{e2,\phi}$$
(3.76)
$$-\frac{n_1 n_{1,\phi}}{n_0^2}\frac{5e\chi_0}{cR^2n_0}\lambda_{e0,\psi} + \frac{r_1}{R}(\frac{10e\chi_0}{3cR^2n_0^2}n_{1,\phi}\lambda_{e0,\psi} + \frac{6e\chi_0}{cR^3n_0}\lambda_{e0,\psi}r_{1,\phi}),$$

and

$$\omega_{i2,\phi} = -\frac{n_{0,\phi}}{n_0}T_{i2,\phi} + n_{2,\phi}(\frac{T_{i0,\psi}}{n_0} + \frac{5e\chi_0}{3cR^2n_0^2}\lambda_{i0,\psi}) - \frac{4e\chi_0}{3cR^2n_0^2}n_{0,\psi}\lambda_{i2,\phi}$$
$$-\frac{1}{n_0}(n_{1,\psi}T_{i1,\phi} - n_{1,\phi}T_{i1,\psi}) - \frac{4e\chi_0}{3cR^2n_0^2}(n_{1,\psi}\lambda_{i1,\phi} - n_{1,\phi}\lambda_{i1,\psi})$$
$$-(\frac{n_1}{n_0} + \frac{2r_1}{R})\frac{e\chi_0}{cR^2n_0^2}(\frac{5}{3}n_{1,\phi}\lambda_{i0,\psi} + \frac{4}{3}n_{0,\psi}\lambda_{i1,\phi}) - \frac{6\chi_0}{n_0R^4}\lambda_{i0,\psi}r_1r_{1,\phi}$$
(3.77)
$$+\frac{e\chi_0}{cR^3n_0}(r_{1,\phi}\lambda_{1,\psi} - r_{1,\psi}\lambda_{i1,\phi}) + \frac{3e\chi_0}{2cR^2n_0}n_{0,\psi}\lambda_{i1}r_{1,\phi}.$$

The equations above determine ω_{e2} and ω_{i2} up to the addition of $O(\epsilon)$ flux functions. I have expressed the unknowns, T_{e2}, T_{i2}, n_2 $S_{\|e2}$, $S_{\|i2}$, ω_{e2} and ω_{i2} in terms of λ_{e2}, λ_{i2}, r_1 and z_1. Finally the two stress equations are used to determine λ_{e2} and λ_{i2}. The ion stress equation in second order is

$$-(\frac{2\chi_0}{3n_0R^2}F_i'(\psi) + \frac{2}{5}T_{i0,\psi} + \frac{2T_{i0}n_{0,\psi}}{n_0})\lambda_{i2,\phi} + \frac{2}{5}S_{\|i2,\phi}$$
$$+\frac{2}{5}\lambda_{i0,\psi}T_{i2,\phi} + \frac{2T_{i0}}{n_0}\lambda_{i0,\psi}n_{2,\phi} + \frac{2\chi_0}{3n_0R^2}F_i'(\psi)(\frac{n_1}{n_0} + \frac{2r_1}{R})\lambda_{i1,\phi}$$

$$-2\lambda_{i0,\phi}\lambda_{i1}r_{1,\phi}+\frac{2}{5}(S_{\|i1}\frac{r_{1,\phi}}{R}-S_{\|i0}\frac{r_1 r_{1,\phi}}{R^2}+S_{\perp 1}\frac{z_{1,\phi}}{R}$$
$$-\gamma_{i1}\frac{r_{1,\phi}}{R}+\gamma_{i0}\frac{r_1 r_{1,\phi}}{R^2})+\frac{2}{5}(T_{i1,\phi}\lambda_{i1,\psi}-T_{i1,\psi}\lambda_{i1,\phi})$$
$$+\frac{2T_{i0}}{n_0}(n_{1,\phi}\lambda_{i1,\psi}-n_{1,\psi}\lambda_{i1,\phi})+(\frac{T_{i1}}{T_{i0}}-\frac{n_1}{n_0})(n_{1,\phi}\lambda_{i0,\psi}-n_{0,\psi}\lambda_{i1,\phi})$$
$$+\frac{3}{2}(-\lambda_{i1}+\frac{2n_0 T_{i0}R^2}{3\chi_0}(\frac{n_1}{n_0}+\frac{T_{i1}}{T_{i0}}+\frac{2r_1}{R}))(\omega_{i1,\phi}+\frac{\chi_0}{n_0 R^2}\lambda_{i0,\psi}r_{1,\phi})=0. \tag{3.78}$$

This equation can be solved for λ_{i2} in terms of λ_{e2} and other known quantities if the following solvability condition holds

$$\langle S_{\perp i1}z_{1,\phi}\rangle = 0, \tag{3.79}$$

which implies

$$\langle \frac{J}{|\nabla\psi|^2}r_{1,\phi}z_{1,\phi}\rangle = 0. \tag{3.80}$$

I now make a final assumption on the system. I assume that $\psi(r,z)$, correct through order ϵ is an even function of z. To the same order, $r_1(\psi,\phi)$ is then an even function of ϕ as are $|\nabla\psi|$ and J; $z_1(\psi,\phi)$ is an odd function of ϕ. Under this symmetry assumption the solvability condition (3.80) is satisfied. I will show later that this assumption is indeed reasonable and consistent with the structure of the generalized Grad-Shafranov equation for ψ.

The electron stress equation is

$$\frac{2\chi_0}{3n_0R^2}F_e'(\psi)\lambda_{e2,\phi}+\frac{2}{5}S_{\|e2,\phi}+\frac{2n_0T_{e0}R^2}{\chi_0}\omega_{e2,\phi}$$
$$+\frac{2}{5}(S_{\|e1}\frac{r_{1,\phi}}{R}-\gamma_{e1}\frac{r_{1,\phi}}{R}+\gamma_{e0}\frac{r_1 r_{1,\phi}}{R^2})$$
$$+\frac{2n_0T_{e0}R^2}{\chi_0}(\frac{n_1}{n_0}+\frac{2r_1}{R})\omega_{e1,\phi}=\frac{-2^{1/3}e\chi_0}{5J\tau_{ee}cR}\lambda_{e2}. \tag{3.81}$$

Using the two stress equations one obtains a single equation for λ_{e2} of the form

$$\tilde{A}(\psi)\lambda_{e2,\phi}+\frac{\tilde{B}(\psi)}{J}\lambda_{e2}=\tilde{C}(\psi)\frac{r_1 r_{1,\phi}}{R^2}+D(\psi)\frac{r_{1,\phi}z_{1,\phi}}{R^2}. \tag{3.82}$$

This equation determines λ_{e2} uniquely in terms of the coefficients in (3.82). The functions $\tilde{A}(\psi)$, $\tilde{B}(\psi)$, $\tilde{C}(\psi)$ and $D(\psi)$ are determined by the zero order solution. Thus, λ_{e2}

is undetermined to the extent that the zero order solution contains seven undetermined functions.

The final part of the first order system is the generalized Grad-Shafranov equation for ψ. The solutions above give that $n\omega$ through $O(\epsilon)$ is

$$\frac{e}{c}n\omega = \frac{\chi_0}{r^2}F'(\psi) - \frac{e\chi_0}{2cR^4}L'_{i1}(\psi)r_1^2 + \nu(\psi) \tag{3.83}$$

where $\nu(\psi)$ is an arbitrary flux function. Imposing the constraint (2.113) determines $\nu(\psi)$ to be

$$\nu(\psi) = \frac{\partial}{\partial\psi}(p - \frac{\chi_0}{3r^2}(F(\psi) - \frac{e}{c}\lambda) + \frac{e\chi_0}{2cR^2}\frac{r_1^2}{R^2}L_{i1} + \frac{e\chi_0}{cR^2}\frac{r_1}{R}\tilde{\lambda}_{i1}). \tag{3.84}$$

Recall the discussion of section (2.3) where it was shown that the right hand side of (3.84) is indeed a function of ψ alone.

I now comment on the assumption that ψ to $O(\epsilon)$ is an even function of z. Assume that that the domain in which the Grad-Shafranov equation is solved has reflection symmetry in the plane $z = 0$. Since the source terms to $O(\epsilon)$ in the generalized Grad-Shafranov equation depend only on ψ and r, the equation for ψ is to $O(\epsilon)$ symmetric with respect to z. Thus, it is reasonable to hypothesize that $\psi(r,z)$, correct through order ϵ is an even function of z. This is a real assumption since solutions to approximately symmetric nonlinear elliptic differential equations need not have approximately symmetric solutions. If the assumption of the up-down symmetry of ψ does not hold, then the equilibria still must satisfy (3.80).

Unlike the first order solutions that have up-down symmetry given symmetry assumptions on ψ, λ_{e2} and hence the second order solution does not have a given partity. This breaking of symmetry is due to the collisional term that appears in the electron stress equation. Later in fourth order it will be shown that the odd part of λ_{e2} is a source of transport.

I now summarize the characterization of $O(\epsilon)$ steady solution. The lowest order solution contains nine flux functions, of which seven may be prescribed independently. The poloidal flux function ψ is then given by a Grad-Shafranov type equation. This solution is steady on the fast time scale τ_e. To extend this solution to the time scale $\epsilon^{-1/2}\tau_e$, the self-consistent first order corrections to the solution must be calculated. These corrections include the poloidal variation of the solution. To extend the zero order solution to the time scale $\epsilon^{-1}\tau_e$ requires calculating the second order corrections. In order that the second order solutions exist, there are certain restrictions placed on the first order corrections; the poloidal variation of the electron temperature $T_{e,\phi}$, the electron temperature anisotropy $(T_e - T_{\perp e})$, and the poloidal variation of the electron stream function $\lambda_{e,\phi}$, are all $O(\epsilon)$ rather than $O(\epsilon^{1/2})$ as they are for the ions.

If the poloidal flux function ψ is approximately up-down symmetric then some comments can be made about the symmetry of the solutions; the first order solution has up-down symmetry. Collisional effects lead to a loss of symmetry in the second order solution. The structure of the equation for λ_{e2} is such that if the zero order solution is given then λ_{e2} is completely determined. In third and fourth order constraints on λ_{e2} will be interpreted as constraints on the zero order solution.

The following simple results from the second order solution are useful in later calculations in fourth order. The first set of relations show that averages of r_1^2 and second order quantities can be expressed in terms of $\langle r_1^2\lambda_{e2,\phi}\rangle$. I present the relations in the same sequence as was used to solve the system.

$$\langle r_1^2 T_{e2,\phi}\rangle = -\frac{4e\chi_0}{15n_0cR^2}\langle r_1^2\lambda_{e2,\phi}\rangle, \tag{3.85}$$

$$\langle r_1^2 T_{i2,\phi}\rangle = -\frac{4e\chi_0}{15n_0cR^2}\langle r_1^2\lambda_{i2,\phi}\rangle, \tag{3.86}$$

$$\langle r_1^2(T_e - T_{\perp e})_2\rangle = \frac{e\chi_0}{3n_0cR^2}\langle r_1^2\lambda_{e2,\phi}\rangle, \tag{3.87}$$

$$\langle r_1^2(T_i - T_{\perp i})_{2,\phi}\rangle = -\frac{e\chi_0}{3n_0cR^2}\langle r_1^2\lambda_{i2,\phi}\rangle, \tag{3.88}$$

$$\langle r_1^2 n_{2,\phi}\rangle = -\frac{8e\chi_0}{5T_0cR^2}\langle r_1^2(\lambda_{i2,\phi} - \lambda_{e2,\phi})\rangle, \tag{3.89}$$

$$\begin{aligned}\langle r_1^2 S_{\|e2,\phi}\rangle = &-3\lambda_{e0,\psi}\langle r_1^2 T_{e2,\phi}\rangle + 3T_{e0,\psi}\langle r_1^2\lambda_{e2,\phi}\rangle \\ &-\frac{3cR^2}{e\chi_0}(5n_0T_{e0,\psi}\langle r_1^2(T_e - T_{\perp e})_{2,\phi}\rangle - 2T_{e0}n_{0,\psi}\langle r_1^2(T_e - T_{\perp e})_{2,\phi}\rangle\end{aligned}$$

$$-5n_{0,\psi}T_{e0}\langle r_1^2 T_{e2,\phi}\rangle), \tag{3.90}$$

$$\begin{aligned}\langle r_1^2 S_{\|i2,\phi}\rangle &= -\lambda_{i0,\psi}\langle r_1^2 T_{i2,\phi}\rangle + T_{i0,\psi}\rangle r_1^2\lambda_{i2,\phi}\rangle + \frac{T_{i0}}{n_0}\lambda_{i0,\psi}\langle r_1^2 n_{2,\phi}\rangle \\ &-\frac{T_{i0}}{n_0}n_{0,\psi}\langle\lambda_{i2,\phi}\rangle + \frac{1}{3}\langle r_1^2\gamma_{i2,\phi}\rangle - \frac{1}{3}\langle r_1^2(S_{\perp i2}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi}\rangle - \frac{1}{3}\langle r_1^2(S_{\perp i2}\frac{|\nabla\psi|^2}{J})_{,\psi}\rangle,\end{aligned} \tag{3.91}$$

$$\begin{aligned}\langle\gamma_{i2}r_1r_{1,\phi}\rangle &= -5n_0T_{i0,\psi}\langle r_1r_{1,\phi}(T_i - T_{\perp i})_2\rangle - 2T_{i0}n_{0,\psi}\langle r_1r_{1,\phi}(T_i - T_{\perp i})_2\rangle \\ &-5n_0T_{i0}\langle r_1^2 T_{i2,\psi}\rangle - 5n_0T_{i0}\langle r_1r_{1,\phi}T_{i2,\phi}\frac{\nabla\phi\cdot\nabla\psi}{|\nabla\psi|^2}\rangle,\end{aligned} \tag{3.92}$$

$$\begin{aligned}&(\frac{2\chi_0}{3n_0R^2}F_i'(\psi) + \frac{2}{5}T_{i0,\psi} + \frac{2T_{i0}}{n_0})\langle r_1^2\lambda_{i2,\phi}\rangle \\ &= \frac{2}{5}\langle r_1^2 S_{\|i2,\phi}\rangle + \frac{2}{5}\lambda_{i0,\psi}\langle r_1^2 T_{i2,\phi}\rangle + \frac{2T_{i0}}{n_0}\lambda_{i0,\psi}\langle r_1^2 n_{2,\phi}\rangle.\end{aligned} \tag{3.93}$$

The following relations for the averages of products of second order quantities will be used in fourth order. The point of the calculation is to show that the averages of product of second order quantities that appear in fourth order solvability conditions can be expressed in terms of averages of λ_{e2} alone.

$$\langle n_{2,\phi}(T_e - T_{\perp e})_2\rangle = \frac{e\chi_0}{3n_0cR^2}\langle n_{2,\phi}\lambda_{e2}\rangle \tag{3.94}$$

$$\langle n_{2,\phi}(T_i - T_{\perp i})_2\rangle = -\frac{e\chi_0}{3n_0cR^2}\langle n_{2,\phi}\lambda_{i2}\rangle + \frac{e\chi_0}{3n_0cR^2}L_{i1}(\frac{N_1}{n_0} + 2)\langle n_{2,\phi}\frac{r_1^2}{R^2}\rangle \tag{3.95}$$

$$\langle n_2T_{e2,\phi}\rangle = -\frac{4}{5}\langle n_2(T_e - T_{\perp e})_{2,\phi}\rangle \tag{3.96}$$

$$\begin{aligned}\langle n_2T_{i2,\phi}\rangle &= \langle n_2\frac{r_1r_{1,\phi}}{R^2}\rangle(\frac{2}{T_{i0}}\frac{e\chi_0}{3n_0cR^2}L_{i1}\hat{T}_{i1} + \frac{8}{5T_{i0}}(L_{i1}\frac{e\chi_0}{3n_0cR^2})^2 \\ &+(\frac{4e\chi_0}{15n_0cR^2}L_{i1} + \hat{T}_{i1})(\frac{\hat{T}_{i1}}{T_{i0}} + \frac{N_1}{n_0})) + \frac{4}{5}\langle n_2(T_i - T_{\perp i})_{2,\phi}\rangle\end{aligned} \tag{3.97}$$

$$(3.98)\, T_0\langle n_{2,\phi}\lambda_{e2}\rangle = -n_0\langle\lambda_{e2}T_{i2,\phi}\rangle - \frac{4e\chi_0}{3cR^2}\langle\lambda_{e2}\lambda_{i2,\phi}\rangle - (N_1\hat{T}_{i1} - \frac{7e\chi_0}{3cR^2}L_{i1})\langle\frac{r_1 r_{1,\phi}}{R^2}\lambda_{e2}\rangle$$

$$\begin{aligned} T_0\langle n_{2,\phi}\lambda_{i2}\rangle &= -n_0\langle\lambda_{i2}T_{i2,\phi}\rangle - n_0\langle\lambda_{i2}T_{e2,\phi}\rangle \\ &+(\frac{7e\chi_0}{3cR^2}L_{i1} - N_1\hat{T}_{i1})\langle\lambda_{i2}\frac{r_1 r_{1,\phi}}{R^2}\rangle + \frac{4e\chi_0}{3cR^2}\langle\lambda_{i2}\lambda_{e2,\phi}\rangle \end{aligned} \tag{3.99}$$

$$\begin{aligned} \langle\lambda_{e2}T_{i2,\phi}\rangle &= (\frac{2}{T_{i0}}\frac{e\chi_0}{3n_0cR^2}L_{i1}\hat{T}_{i1} + \frac{8}{5T_{i0}}(\frac{e\chi_0}{3n_0cR^2}L_{i1})^2 \\ &-(\frac{\hat{T}_{i1}}{T_{i0}} + \frac{N_1}{n_0})(\frac{4}{5}\frac{e\chi_0}{3n_0cR^2}L_{i1} - \hat{T}_{i1})\langle\lambda_{e2}\frac{r_1 r_{1,\phi}}{R^2}\rangle + \frac{4}{5}\langle\lambda_{e2}(T_i - T_{\perp i})_{2,\phi}\rangle \end{aligned} \tag{3.100}$$

$$\lambda_{e2}(T_i - T_{\perp i})_{2,\phi}\rangle = -\frac{e\chi_0}{3n_0cR^2}\langle\lambda_{e2}\lambda_{i2,\phi}\rangle + \frac{e\chi_0}{3n_0cR^2}L_{i1}(\frac{N_1}{n_0} + 2)\langle\lambda_{e2}\frac{r_1 r_{1,\phi}}{R^2}\rangle \tag{3.101}$$

$$\langle\lambda_{i2}T_{e2,\phi}\rangle = -\frac{4}{5}\langle\lambda_{i2}(T_e - T_{\perp e})_{2,\phi}\rangle \tag{3.102}$$

$$\langle\lambda_{i2}(T_e - T_{\perp e})_{2,\phi}\rangle = \frac{e\chi_0}{3n_0cR^2}\langle\lambda_{i2}\lambda_{e2,\phi}\rangle \tag{3.103}$$

At this point all the averages of products of second order quantities above can be expressed in terms of $\langle\lambda_{e2,\phi}r_1^2\rangle$ and $\langle\lambda_{e2,\phi}\lambda_{i2}\rangle$. The following relations shows that $\langle\lambda_{e2,\phi}\lambda_{i2,\phi}\rangle$ can be expressed in terms of $\langle r_1^2\lambda_{e2,\phi}\rangle$ and other averages of λ_{e2}.

$$\begin{aligned} &-(\frac{2\chi_0}{3n_0R^2}F_i'(\psi) + \frac{2}{5}T_{i0,\phi} + 2\frac{T_{i0}n_{0,\psi}}{n_0}\langle\lambda_{i2,\phi}\lambda_{e2}\rangle + \frac{2}{5}\langle S_{\|i2,\phi}\lambda_{e2}\rangle \\ &+\frac{2}{5}\lambda_{i0,\psi}\langle T_{i2,\phi}\lambda_{e2}\rangle + \frac{T_{i0}}{n_0}\lambda_{i0,\psi}\langle n_{2,\phi}\lambda_{e2}\rangle + \frac{2\chi_0}{3n_0R^2}F_i'(\psi)(\frac{N_1}{n_0} + 2)L_{i1}\langle\lambda_{e2}\frac{r_1 r_{1,\phi}}{R^2}\rangle \\ &-2\lambda_{i0,\psi}L_{i1}\langle\lambda_{e2}\frac{r_1 r_{1,\phi}}{R^2} + \frac{2}{5}\langle S_{\|i1}\frac{r_{1,\phi}}{R}\lambda_{e2}\rangle - \frac{2}{5}S_{\|i0}\langle\frac{r_1 r_{1,\phi}}{R^2}\lambda_{e2}\rangle + \frac{2}{5}\langle S_{\perp i1}\frac{z_{1,\phi}}{R}\lambda_{e2}\rangle \\ &-\frac{2}{5}\langle\gamma_{i1}\frac{r_{1,\phi}}{R}\lambda_{e2} + \gamma_{i0}\langle\frac{r_1 r_{1,\phi}}{r^2}\lambda_{e2}\rangle + \frac{2}{5}(\hat{T}_{i1}L_{i1}' - \hat{T}_{i1}'L_{i1})\langle\frac{r_1 r_{1,\phi}}{R^2}\lambda_{e2}\rangle \\ &+2\frac{T_{i0}}{n_0}(N_1L_{i1}' - N_1'L_{i1})\langle\frac{r_1 r_{1,\phi}}{R^2}\lambda_{e2}\rangle + (\frac{\hat{T}_{i1}}{T_{i0}} - \frac{N_1}{n_0})(N_1\lambda_{i0,\psi} - n_{0,\psi}L_{i1})\langle\frac{r_1 r_{1,\phi}}{R^2}\lambda_{e2}\rangle \\ &+\frac{2}{3}(-L_{i1} + \frac{2n_0T_{i0}R^2}{3\chi_0}(\frac{N_1}{n_0} + \frac{\hat{T}_{i1}}{T_{i0}} + 2))(W_{i1} + \frac{\chi_0}{n_0R^3}\lambda_{i0,\psi})\langle\frac{r_1 r_{1,\phi}}{R^2}\lambda_{e2}\rangle = 0, \end{aligned} \tag{3.104}$$

$$\frac{1}{3}\langle S_{\| i2,\phi}\lambda_{e2}\rangle = -\lambda_{i0,\psi}\langle T_{i2,\phi}\lambda_{e2}\rangle + T_{i0,\psi}\langle\lambda_{i2,\phi}\lambda_{e2}\rangle$$
$$-(\hat{T}_{i1}L'_{i1} - \hat{T}'_{i1}L_{i1})\langle\frac{r_1 r_{1,\phi}}{R^2}\rangle + \frac{T_{i0}}{n_0}(\lambda_{i0,\psi}\langle n_{2,\phi}\lambda_{e2}\rangle - n_{0,\psi}\langle\lambda_{i2,\phi}\lambda_{e2}\rangle)$$
$$+\frac{T_{i0}}{n_0}((N_1 L'_{i1} - N'_1 L_{i1})\langle\frac{r_1 r_{1,\phi}}{R^2}\lambda_{e2}\rangle + (\frac{\hat{T}_{i1}}{T_{i0}} - \frac{N_1}{n_0})(N_1\lambda_{i0,\psi} - n_{0,\psi}L_{i1})\langle\frac{r_1 r_{1,\phi}}{R^2}\lambda_{e2}\rangle)$$
$$-\frac{2}{3n_0}(N_1\lambda_{i0,\psi} - n_{0,\psi}L_{i1})\langle(T_i - T_{\perp i})_1\frac{r_{1,\phi}}{R}\lambda_{e2}\rangle$$
$$+\frac{2}{3}L_{i1}\langle\frac{r_1}{R}(\omega_{i1,\phi} + \frac{\chi_0}{n_0 R^3}r_{1,\phi}\lambda_{i0,\psi})\lambda_{e2}\rangle + \frac{1}{3}\langle\gamma_{i2,\phi}\lambda_{e2}\rangle - \frac{1}{3}\langle(S_{\perp i2}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi}\lambda_{e2}\rangle$$
$$-\frac{1}{3}\langle(S_{\perp i2}\frac{|\nabla\psi|^2}{J})_{,\psi}\lambda_{e2}\rangle = 0, \tag{3.105}$$

$$\langle(\gamma_{i2,\phi} - (S_{\perp i2}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi} - (S_{\perp i2}\frac{|\nabla\psi|^2}{J})_{,\psi})\lambda_{e2}\rangle$$
$$= 2\langle\frac{r_1}{R}(\gamma_{i1,\phi} - (S_{\perp i1}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi} - (S_{\perp i1}\frac{|\nabla\psi|^2}{J})_{,\psi})\lambda_{e2}\rangle$$
$$+2n_{0,\psi}\langle(T_i - T_{\perp i})_1(T_i - T_{\perp i})_{1,\phi}\lambda_{e2}\rangle + 5n_{0,\psi}\langle(T_i - T_{\perp i})_1 T_{i1,\phi}\lambda_{e2}\rangle$$
$$+5T_{i0,\psi}\langle n_1(T_i - T_{\perp i})_1\lambda_{e2,\phi}\rangle - 5n_0 T_{i0,\psi}\langle(T_i - T_{\perp i})_{2,\phi}\lambda_{e2}\rangle$$
$$+2n_{0,\psi}\langle T_{i1}(T_i - T_{\perp i})_1\lambda_{e2,\phi}\rangle - 2n_{0,\psi}T_{i0}\langle(T_i - T_{\perp i})_{2,\phi}\lambda_{e2}\rangle$$
$$-5(n_0 T_{io})_{,\psi}\langle T_{i2,\phi}\lambda_{e2}\rangle + 5T_{i0,\psi}\langle n_1 T_{i1}\lambda_{e2,\phi}\rangle - 6\langle(n_0 T_{i0}(T_i - T_{\perp i})_1\frac{r_{1,\phi}}{R})_{,\psi}\lambda_{e2}\rangle$$
$$-6n_0 T_{i0}\langle(T_i - T_{\perp i})_1\frac{z_{1,\phi}}{R}J)_{,\phi}\lambda_{e2}\rangle - \gamma_{i0}\langle(\frac{3r_1^2}{R^2} + \frac{|\nabla\psi|^2}{\chi_0^2}\lambda_{e2,\phi}\rangle. \tag{3.106}$$

4. Third and Fourth Order Systems

In this chapter I examine the third and fourth order systems. Only limited information is needed from the third and fourth order systems. In third order I check that all solvability conditions are satisfied and calculate some averages of the third order solution. The information needed to calculated the explicit time evolution of T_{e0} and T_{i0} in the fourth order system is determined. The time evolution of the lowest order states is presented.

4.1. Third Order

The third order equations will be examined in the same sequence used in Chapter 3. In first and second order it was possible to calculate the solutions explicitly. The third order solutions are more difficult to calculate explicitly because solving the third order equations requires integrating expressions of the form $r_1\lambda_{e2,\phi}$. Fortunately, only limited information from the third order solution will be required to compute the time evolution of the zero order solutions. It is necessary that any solvability conditions associated with the third order equations be satisfied.

The third order corrections to the temperature anisotropy found from (2.43) and (2.44) are

$$(T_e - T_{\perp e})_3 = \frac{e\chi_0}{3n_0cR^2}(\lambda_{e3} - \frac{n_1}{n_0}\lambda_{e2} - \frac{2r_1}{R}\lambda_{e2}), \tag{4.1}$$

and

$$\begin{aligned}(T_i - T_{\perp i})_3 = {} & \frac{e\chi_0}{3n_0cR^2}(-\lambda_{i3} + \lambda_{i2}(\frac{n_1}{n_0} + \frac{2r_1}{R})) \\ & - \frac{e\chi_0}{3n_0cR^2}\lambda_{i1}(\frac{n_1^2}{n_0^2} + \frac{n_2}{n_0} + \frac{3r_1^2}{R^2} + \frac{|\nabla\psi|^2}{\chi_0^2} + \frac{\chi_2}{\chi_0}).\end{aligned} \tag{4.2}$$

Equation (2.107) in third order is

(4.3) $$4n_0T_{e0}(T_e - T_{\perp e})_{3,\phi} + 4n_1T_{e0}(T_e - T_{\perp e})_{2,\phi} + 5n_0T_{e0}T_{e3,\phi} + 5n_1T_{e0}T_{e2,\phi} = 0.$$

This equation can be integrated to find T_{e3} up to the addition of a flux function. The condition that a solution to (4.3) exist is

(4.4) $$4T_{e0}\langle n_1(T_e - T_{\perp e})_{2,\phi}\rangle + 5T_{e0}\langle n_1T_{e2,\phi}\rangle = 0.$$

Equation (3.65) implies that condition (4.4) is satisfied.

The third order quantity T_{i3} is found from equation (2.108) in third order,

(4.5) $$\begin{aligned}&-8n_1(T_i - T_{\perp i})_1(T_i - T_{\perp i})_{1,\phi} - 8n_0(T_i - T_{\perp i})_1(T_i - T_{\perp i})_{2,\phi}\\ &\quad -8n_0(T_i - T_{\perp i})_2(T_i - T_{\perp i})_{1,\phi} - 8(T_i - T_{\perp i})_1^2 n_{1,\phi}\\ &\quad +10n_0(T_i - T_{\perp i})_1T_{i2,\phi} + 10n_0(T_i - T_{\perp i})_2T_{i1,\phi}\\ &\quad +10n_0(T_i - T_{\perp i})_2T_{i1,\phi} + 10n_1(T_i - T_{\perp i})T_{i1,\phi} + 4n_0T_{i0}(T_i - T_{\perp i})_{3,\phi}\\ &\quad +4(n_1T_{i0} + n_0T_{i1})(T_i - T_{\perp i})_{2,\phi} + 4n_1T_{i1}(T_i - T_{\perp i})_{1,\phi}\\ &\quad +(4n_0T_{i2} + n_2T_{i0})(T_i - T_{\perp i})_{1,\phi} + 5n_0T_{i0}T_{i3,\phi}\\ &\quad +5(n_0T_{i1} + n_1T_{i0})T_{i2,\phi} + 5(n_1T_{i1}n_0T_{i2} + n_2T_{i0})T_{i1,\phi}\\ &\quad +6n_0T_{i0}(T_i - T_{\perp i})_1\frac{(|\nabla\psi|^2)_{,\phi}}{2\chi_0^2} = 0.\end{aligned}$$

The solvability condition for this equation reduces to

(4.6) $$\langle r_1((|\nabla\psi|^2)_{,\phi})\rangle = 0.$$

which is satisfied with the symmetry assumptions that r_1 and ψ are even functions of ϕ to $O(\epsilon)$. I have identified the equations that determine T_{e3} and T_{i3} up to the addition of flux functions and have verified that the solvability conditions associated with these equations hold.

Equation (2.102) in third order is

$$n_{3,\phi}T_0 + n_0T_{3,\phi} + (n_2T_1)_{,\phi} + (n_1T_2)_{,\phi} + \frac{e\chi_0}{cR^3}(r_{1,\phi}\lambda_2 - \frac{r_1r_{1,\phi}}{R}\lambda_{i1})$$

$$+\frac{4e\chi_0}{3cR^2}(\lambda_{3,\phi} + \frac{\chi_2}{\chi_0}\lambda_{i1,\phi} - \frac{2r_1}{R}\lambda_{2,\phi} + \frac{3r_1^2}{R^2}\lambda_{i1,\phi}) - r_{1,\phi}\omega_{i0}^2 n_0 R = 0. \tag{4.7}$$

This equation can be solved for n_3 if

$$\langle r_{1,\phi}\lambda_2\rangle = 0. \tag{4.8}$$

Examining the second order solution one finds that (4.8) is equivalent to

$$\begin{aligned}\langle r_1\lambda_{e2,\phi}\rangle &= \langle r_{1,\phi}\lambda_{i2}\rangle = \langle r_{1,\phi}T_{e2}\rangle = \langle r_{1,\phi}T_{i2}\rangle = \langle r_{1,\phi}n_2\rangle \\ &= \langle r_{1,\phi}(T_i - T_{\perp i})_2\rangle = \langle r_{1,\phi}(T_e - T_{\perp e})_2\rangle = \langle r_{1,\phi}S_{\parallel e2}\rangle = 0.\end{aligned} \tag{4.9}$$

In particular, to satisfy (4.9) one need only insure that $\langle r_1\lambda_{e2,\phi}\rangle = 0$. The condition (4.9) provides an additional constraint on the zero order solution, reducing the number of independent functions in the zero order solution from seven to six.

As in the previous calculations the parallel heat flows are found using the pressure balance equations. In third order the electron pressure equation is

$$\begin{aligned}&(T_{e3,\phi}\lambda_{e0,\psi} - T_{e0,\psi}\lambda_{e3,\phi}) + \frac{2}{3n_0}(T_e - T_{\perp e})_2 n_{1,\phi}\lambda_{e0,\psi} \\ &+\frac{1}{3}(S_{\parallel e3,\phi} - \gamma_{e3,\phi} + (S_{\perp e3}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi} + (S_{\perp e3}\frac{|\nabla\psi|^2}{J})_{,\psi} \\ &+\frac{3e}{2c}\lambda_{e2}(\omega_{e1,\phi} + \frac{\chi_0}{n_0R^3}\lambda_{e0,\psi}r_{1,\phi}) = 0.\end{aligned} \tag{4.10}$$

This equation need not be solved for $S_{\parallel e3}$ but for a solution to exist the following condition must hold:

$$\begin{aligned}&\frac{2\lambda_{e0,\psi}}{3n_0}\langle n_{1,\phi}(T_e - T_{\perp e})_2\rangle + \frac{1}{3}\langle(S_{\perp e3}\frac{|\nabla\psi|^2}{J})_{,\psi}\rangle \\ &+\frac{3e}{2c}\langle\lambda_{e2}\omega_{e1,\phi}\rangle + \frac{3e\chi_0\lambda_{e0,\psi}}{2n_0cR^3}\langle\lambda_{e2}r_{1,\phi}\rangle = 0.\end{aligned} \tag{4.11}$$

All the terms containing products of first and second order solutions are clearly proportional to $\langle r_{1,\phi}\lambda_{e2}\rangle$ and vanish. The remaining term involving $S_{\perp e3}$ can be calculated easily from (2.109) to be

$$(4.12) \qquad \langle (S_{\perp e3}\frac{|\nabla\psi|^2}{J})_{,\psi}\rangle = -\frac{2T_{e0}}{3n_{e0}}\langle n_{1,\phi}\lambda_{e2}\rangle - \frac{14T_{e0}}{3}\langle \frac{r_{1,\phi}}{R}\lambda_{e2}\rangle = 0.$$

Hence, the condition (4.11) is satisfied and no additional information is gained.

The next equation is the ion pressure equation (2.104)

$$\begin{aligned}
&(T_{i3,\phi}\lambda_{i0,\psi} - T_{i0,\psi}\lambda_{i3,\phi}) + (T_{i2,\phi}\lambda_{i1,\psi} + T_{i1,\phi}\lambda_{i2,\psi} - T_{i2,\phi}\lambda_{i1,\phi} - T_{i1,\psi}\lambda_{i2,\phi})\\
&+\frac{2}{3n_0}[(T_i - T_{\perp i})_2(n_{1,\phi}\lambda_{i0,\psi} - n_{0,\psi}\lambda_{i1,\phi})\\
&+(T_i - T_{\perp i})_1(n_{2,\phi}\lambda_{i0,\psi} - n_{0,\psi}\lambda_{i2,\phi} - \frac{n_1}{n_0}(n_{1,\phi}\lambda_{i0,\psi} - n_{0,\psi}\lambda_{i1,\phi})\\
&+(n_{1,\phi}\lambda_{i1,\psi} - n_{1,\psi}\lambda_{i1,\phi}))] + \frac{1}{3}(S_{\| i3} - \gamma_{i3} + S_{\perp i3})_{,\phi} + \frac{1}{3}(S_{\perp i3}\frac{|\nabla\psi|^2}{J})_{,\psi}\\
&-\frac{2e}{3c}\lambda_{i2}(\omega_{i1,\phi} + \frac{\chi_0}{n_0R^2}r_{1,\phi}\lambda_{i0,\psi})\\
(4.13) \qquad &-\frac{2e}{3c}\lambda_{i1}[\omega_{i2,\phi} - \frac{\chi_0}{n_0R^2}(r_{1,\psi}\lambda_{i1,\phi} - r_{1,\phi}\lambda_{i1,\psi} - (\frac{2r_1}{R} + \frac{n_1}{n_0})r_{1,\phi}\lambda_{i0,\psi})] = 0.
\end{aligned}$$

This equation does not need to be solved explicitly. The condition that a solution exist is satisfied and gives no new information. The conditions that the poloidal momentum equations (2.100) and (2.101) can be solved for ω_{e3} and ω_{i3} are satisfied.

The ion stress equation in third order has the solvability condition

$$(4.14) \qquad \frac{2}{R}(\langle S_{\perp i2}z_{1,\phi}\rangle - \langle\gamma_{i2}r_{1,\phi}\rangle) + \frac{3e\chi_0}{R\tau_{ee}}\sqrt{\frac{m_e}{m_i}}(\frac{T_{e0}}{T_{i0}})^{3/2}\langle\frac{\lambda_{i1}}{J}\rangle = 0.$$

Let us examine the terms in this condition:

$$(4.15) \qquad \langle S_{\perp i2}z_{1,\phi}\rangle = 5\frac{cR^2}{e\chi_0}n_0T_{i0}\langle\frac{J}{|\nabla\psi|^2}z_{1,\phi}T_{e2,\phi}\rangle,$$

$$(4.16) \qquad \langle\gamma_{i2}r_{1,\phi}\rangle = -5n_0T_{i0}\langle r_{1,\phi}T_{i2,\psi}\rangle - 5n_0T_{i0}\langle\frac{\nabla\phi\cdot\nabla\psi}{|\nabla\psi|^2}r_{1,\phi}T_{i2,\phi}\rangle,$$

and

$$(4.17) \qquad \langle\frac{\lambda_{i1}}{J}\rangle = \frac{L_{i1}(\psi)}{R}\langle\frac{r_1}{J}\rangle.$$

While the condition (4.14) contains second order solutions, it does not depend on the arbitrary flux functions that enter in second order. It does depend on λ_{e2} and the zero order profiles in the equation for λ_{e2}. Hence, the condition (4.14) is another relation between the zero order profiles. At this point, having used the constraints (4.9) and (4.14) there are five independent zero order functions. The final equation in the system is (2.105) for λ_{e3} which in general has a unique periodic solution.

In third order the solutions have not been calculated explicitly. Rather, I have determined what conditions are necessary for solutions to exist and have shown that they can be satisfied. I have determined additional constraints on the zero order solution, (4.9) and (4.14. In the next section the fourth order system containing the explicit time evolution of T_{e0} and T_{i0} will be analyzed. It will be seen that only partial information about the third order solution is needed to calculate transport. In particular, one only needs some averages of the product of third order quantities with r_1. I show below that these averages can be expressed in terms of $\langle \lambda_{e3,\phi} r_1 \rangle$. The sequence of calculating these relations below parallels that used in solving the ordered system. Just as the unknowns, $(T_e - T_{\perp e})$, $(T_i - T_{\perp i})$, T_e, T_i, n, $S_{\|e}$, $S_{\|i}$, ω_e, ω_i and λ_i were expressed in terms of λ_e, here the averages $\langle r_{1,\phi}(T_e - T_{\perp e})_3 \rangle$, $\langle r_{1,\phi}(T_i - T_{\perp i})_3 \rangle$, $\langle r_{1,\phi} T_{e3} \rangle$, $\langle r_{1,\phi} T_{i3} \rangle$, $\langle r_{1,\phi} n_3 \rangle$, $\langle r_{1,\phi} S_{\|e3} \rangle$, $\langle r_{1,\phi} S_{\|i3} \rangle$, $\langle r_{1,\phi} \omega_{e3} \rangle$, $\langle r_{1,\phi} \omega_{i3} \rangle$ and $\langle r_1 \lambda_{i3} \rangle$ will be expressed in terms of $\langle r_1 \lambda_{e3} \rangle$.

I begin with the relations (2.43) and (2.44). I multiply the relations by $r_{1,\phi}$ and take the average with respect to ϕ to obtain

$$\langle (T_e - T_{\perp e})_3 r_{1,\phi} \rangle = \frac{e\chi_0}{3cn_0R^2}[\langle \lambda_{e3} r_{1,\phi} \rangle - (\frac{N_1}{n_0} + 2)\langle \lambda_{e2} \frac{r_1 r_{1,\phi}}{R} \rangle] \tag{4.18}$$

and

$$\langle (T_i - T_{\perp i})_3 r_{1,\phi} \rangle = -\frac{e\chi_0}{3cn_0R^2}[\langle \lambda_{i3} r_{1,\phi} \rangle - (\frac{N_1}{n_0} + 2)\langle \lambda_{i2} \frac{r_1 r_{1,\phi}}{R} \rangle]. \tag{4.19}$$

I calculate from equations (2.107) and (2.108) that

$$\langle r_1 T_{e3,\phi} \rangle = -\frac{4e\chi_0}{15cn_0R^2}[\langle \lambda_{e3} r_{1,\phi} \rangle - (\frac{N_1}{n_0} + 2)\langle \lambda_{e2} \frac{2r_1 r_{1,\phi}}{R} \rangle] \tag{4.20}$$

and

$$\begin{aligned} \langle r_1 T_{i3,\phi} \rangle = & -\frac{4}{5}\langle r_1 (T_i - T_{\perp i})_{3,\phi} \rangle \\ & +20\langle r_1 (T_i - T_{\perp i})_1 T_{i2,\phi} \rangle + 20\langle r_1 (T_i - T_{\perp i})_2 T_{i1,\phi} \rangle. \end{aligned} \tag{4.21}$$

Using equation (2.102) one finds

$$\begin{aligned} T_0\langle r_1 n_{3,\phi}\rangle &= -n_0\langle T_{3,\phi} r_1\rangle - \langle r_1(n_1T_2)_{,\phi}\rangle \\ &-\langle r_1(n_2T_1)_{,\phi}\rangle + \frac{\chi_0}{3R^3}\langle r_1^2\lambda_{2,\phi}\rangle - \frac{4e\chi_0}{3cR^2}[\langle r_1\lambda_{3,\phi}\rangle + 2\langle\lambda_{2,\phi}\frac{r_1^2}{R}\rangle]. \end{aligned} \tag{4.22}$$

From the relations (2.109) and (2.111) one can calculate

$$\begin{aligned} \frac{\chi_0}{R^2}\langle r_1\frac{|\nabla\psi|^2}{J}S_{\perp e3}\rangle &= -2\frac{\chi_0}{R^2}\langle\frac{r_1^2}{R}\frac{|\nabla\psi|^2}{J}S_{\perp e2}\rangle - 2T_{e0}\langle r_1 n_{1,\phi}(T_e - T_{\perp e})_2\rangle \\ &-5n_0T_{e0}\langle r_1T_{e3,\phi}\rangle - 5T_{e0}\langle r_1n_1T_{e2,\phi}\rangle + 6n_0T_{e0}\langle(T_e - T_{\perp e})_2\frac{r_1 r_{1,\phi}}{R}\rangle \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} \frac{e\chi_0}{cR^2}\langle r_1S_{\perp i3}\frac{|\nabla\psi|^2}{J}\rangle &= \frac{e\chi_0}{cR^2}\langle\frac{r_1^2}{R}S_{\perp i2}\frac{|\nabla\psi|^2}{J}\rangle - 5n_0\langle r_1T_{i1,\phi}(T_i - T_{\perp i})_2\rangle \\ &-5n_0\langle r_1(T_i - T_{\perp i})_1(T_i - T_{\perp i})_2\rangle - 2T_{i0}\langle r_1(T_i - T_{\perp i})_2 n_{1,\phi}\rangle \\ &-2T_{i0}\langle r_1(T_i - T_{\perp i})1n_{2,\phi}\rangle - 5n_0T_{i0}\langle r_1T_{13,\phi}\rangle - 5T_{i0}\langle r_1n_1T_{i2,\phi}\rangle - 5T_{i0}\langle r_1n_2T_{i1,\phi}\rangle, \end{aligned} \tag{4.24}$$

$$\begin{aligned} \langle r_1S_{\|i3,\phi}\rangle &= -3\lambda_{i0,\psi}\langle r_1T_{i3,\phi}\rangle + 3T_{i0,\psi}\langle r_1\lambda_{i3,\phi}\rangle - 3L_{i1}'\langle T_{i2,\phi}\frac{r_1^2}{R}\rangle \\ &-3L_{i1}\frac{\partial}{\partial\psi}\langle T_{i2,\phi}\frac{r_1^2}{2R}\rangle - 3\hat{T}_{i1}'\langle\lambda_{i2,\phi}\frac{r_1^2}{R}\rangle - 3\hat{T}_{i1}\frac{\partial}{\partial\psi}\langle\lambda_{i2,\phi}\frac{r_1^2}{2R}\rangle \\ &-\frac{2}{n_0}[(\lambda_{i0,\psi}N_1 - n_{0,\psi}L_{i1})\langle r_1r_{1,\phi}(T_i - T_{\perp i})_2\rangle + \lambda_{i0,\psi}\langle r_1(T_i - T_{\perp i})_1n_{2,\phi}\rangle \\ &-n_{0,\psi}\langle r_1(T_i - T_{\perp i})_1\lambda_{i2,\phi}\rangle] - \langle r_1\gamma_{i3,\phi}\rangle - \langle r_1(S_{\perp i3}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi}\rangle \\ &-\langle r_1(S_{\perp i3}\frac{|\nabla\psi|^2}{J})_{,\psi}\rangle + 2\frac{e}{c}\langle r_1\omega_{i1,\phi}\lambda_{i2,\phi}\rangle - \frac{2e\chi_0}{n_0cR^2}\lambda_{i0,\psi}\langle r_1r_{1,\phi}\lambda_{i2}\rangle - 2\frac{e}{c}\langle\lambda_{i1}r_1\omega_{i1,\phi}\rangle \end{aligned} \tag{4.25}$$

From the pressure equations (2.103) and (2.104)

$$\begin{aligned} \frac{1}{3}\langle r_1\frac{r}{J}\nabla\cdot\mathbf{S}_{e3}\rangle &= -\lambda_{e0,\psi}\langle r_1T_{e3,\phi}\rangle + T_{e0,\psi}\langle r_1\lambda_{e3,\phi}\rangle + \frac{T_{e0}}{n_0}[\lambda_{e0,\psi}\langle r_1n_{3,\phi}\rangle \\ &-n_{0,\psi}\langle r_1\lambda_{e3,\phi}\rangle + \langle r_1n_{1,\phi}\lambda_{e2,\psi} - r_1n_{1,\psi}\lambda_{e2,\phi}\rangle \\ &+\langle r_1(\frac{T_{e1}}{T_{e0}} - \frac{n_1}{n_0})(n_{2,\phi}\lambda_{e0,\psi} - n_{0,\psi}\lambda_{e2,\phi})\rangle + \langle r_1(\frac{T_{e2}}{T_{e0}} - \frac{n_2}{n_0})n_{1,\phi}\lambda_{e0,\psi}\rangle] \\ &-\frac{3e}{2c}\langle r_1\lambda_{e2}\omega_{e1,\phi}\rangle - \frac{3e\chi_0\lambda_{e0,\psi}}{2cR^3n_0}\langle r_1r_{1,\phi}\lambda_{e2}\rangle, \end{aligned} \tag{4.26}$$

and

$$\frac{1}{3}\langle r_1\frac{r}{J}\nabla\cdot\mathbf{S}_{i3}\rangle = -\lambda_{i0,\psi}\langle r_1 T_{i3,\phi}\rangle + T_{i0,\psi}\langle r_1\lambda_{i3,\phi}\rangle + \frac{L_{e1}}{R}\frac{\partial}{\partial\psi}\langle T_{i2,\phi}r_1^2\rangle$$
$$+\frac{T_{i1}}{R}\langle r_1 r_{1,\phi}\lambda_{i2}\rangle + \frac{2\lambda_{i0,\psi}}{3n_0}\langle(T_i - T_{\perp i})_1 r_1 n_{2,\phi}\rangle$$
$$+\frac{2n_{0,\psi}}{n_0}\langle(T_i - T_{\perp i})_1 r_1\lambda_{i2,\phi}\rangle - \lambda_{i0,\psi}\langle r_1(T_i - T_{\perp i})_1 n_{2,\phi}\rangle$$
$$+n_{0,\psi}\langle r_1\lambda_{i2,\phi}(T_i - T_{\perp i})_1\rangle + \frac{2e}{3c}\langle\lambda_{i2}r_1\omega_{i1,\phi}\rangle$$
$$+\frac{2e\chi_0\lambda_{i0,\psi}}{3n_0 cR^2}\langle\lambda_{i2}r_1 r_{1,\phi}\rangle + \frac{2e}{3c}\langle r_1\lambda_{i1}\omega_{i2,\phi}\rangle. \tag{4.27}$$

In calculating the transport the quantities $\langle r_1\omega_{e3,\phi}\rangle$ and $\langle r_1\omega_{i3,\phi}\rangle$. The final relation needed from third order is obtained from equation (2.106) and is

$$\frac{2\chi_0}{3n_0R^2}[\langle r_1\lambda_{i3,\phi}\rangle - \langle r_1(\frac{n_1}{n_0} + \frac{2r_1}{R})\lambda_{i2,\phi}\rangle] + \frac{4e\chi_0}{3cR^3}\lambda_{i0,\psi}\langle r_1 r_{1,\phi}\lambda_{i2}\rangle$$
$$+\frac{2e\chi_0}{3n_0cR^2}\langle r_1\lambda_{i1}(n_{2,\phi}\lambda_{i0,\psi} - n_{0,\psi}\lambda_{i2,\phi})\rangle + \frac{2}{5}[\langle r_1 S_{\|i3,\phi}\rangle + \langle\frac{r_1 r_{1,\phi}}{R}S_{\|i2}\rangle$$
$$+\frac{1}{R}\langle S_{\perp i2}z_{1,\phi}r_1\rangle - \frac{1}{R}\langle r_1 r_{1,\phi}\gamma_{i2}\rangle] - \frac{2}{15}\langle\frac{r_1 R}{J}\nabla\cdot\mathbf{S}_{i3}\rangle$$
$$+\frac{R^2 nT_{i0}}{\chi_0}[\langle r_1\omega_{i3,\phi}\rangle + \langle r_1(\frac{2r_1}{R} + \frac{n_1}{n_0} + \frac{T_{i1}}{T_{i0}})\omega_{i2,\phi}\rangle]$$
$$+\frac{T_{i0}}{n_0}[\langle -r_1(r_{1,\psi}\lambda_{i2,\phi} + r_{1,\phi}\lambda_{i2,\psi})\rangle + \langle r_1 r_{1,\phi}\frac{T_{i2}}{T_{i0}}\rangle]$$
$$+\frac{T_{i0}}{3n_0}[\lambda_{i0,\psi}\langle r_1 n_{3,\phi}\rangle - n_{0,\psi}\langle r_1\lambda_{i3,\phi}\rangle + \langle(\frac{T_{i1}}{T_{i0}} - \frac{n_1}{n_0})r_1(\lambda_{i0,\psi}n_{2,\phi} - \lambda_{i2,\phi}n_{0,\psi})\rangle]$$
$$+\frac{T_{i0}}{3n_0}\langle r_1\frac{T_{i2}}{T_{i0}}(\lambda_{i0,\psi}n_{1,\phi} - \lambda_{i1,\phi}n_{0,\psi})\rangle - \frac{2}{3n_0}[\langle(T_i - T_{\perp i})_1 r_1(\lambda_{i0,\psi}n_{2,\phi} - \lambda_{i2,\phi}n_{0,\psi})\rangle$$
$$+\langle r_1(T_i - T_{\perp i})_2(\lambda_{i0,\psi}n_{1,\phi} - n_{0,\psi}\lambda_{i1,\phi})\rangle] + \frac{1}{5}\sqrt{\frac{m_e}{m_i}}(\frac{T_{e0}}{T_{i0}})^{3/2}\frac{cR}{e\chi_0}\langle r_1\frac{\lambda_{i1}}{J}\rangle = 0. \tag{4.28}$$

Thus, the quantities $\langle r_1(T_e - T_{\perp e})_{3,\phi}\rangle$, $\langle r_1(T_i - T_{\perp i})_{3,\phi}\rangle$, $\langle r_1 T_{e3,\phi}\rangle$, $\langle r_1 T_{i3,\phi}\rangle$, $\langle r_1 n_{3,\phi}\rangle$, $\langle r_1 S_{\perp e3}|\nabla\psi|^2/J\rangle$, $\langle r_1 S_{\perp i3}|\nabla\psi|^2/J\rangle$, $\langle r_1 r/J\nabla\cdot\mathbf{S}_{e3}\rangle$, $\langle r_1 r/J\nabla\cdot\mathbf{S}_{i3}\rangle$ and $\langle r_1\lambda_{i3,\phi}\rangle$ have been expressed in terms of $\langle r_1\lambda_{e3,\phi}\rangle$. The quantity $\langle r_1\lambda_{e3,\phi}\rangle$ has not yet been determined. It will be seen that fourth order solvability conditions give sufficient information to find $\langle r_1\lambda_{e3,\phi}\rangle$. This process by which solvability conditions give information about lower order solutions was seen earlier when the second order solvability conditions gave that $\lambda_{e1} = 0$ and when the third order solvability conditions implied that $\langle r_1\lambda_{e2,\phi}\rangle = 0$.

4.2. Fourth order

The analysis of the fourth order system, as in third order, focuses on determining and satisfying the solvability conditions rather than calculating the explicit solutions. A goal here is the calculation of the time evolution of T_{e0} and T_{i0} appearing in the pressure equations and subsequently the calculation of the time evolution of the complete zero order solution. It will not be necessary to calculate the fourth order solution in order to determine the transport. Instead the time evolution of the temperature will be found by satisfying the solvability conditions associated with the fourth order pressure equations. Terms of the form $\langle(\textit{third order})r_{1,\phi}\rangle$ appear in these conditions; they are determined using the relations from the previous section. Additional relations between the zero profiles will be determined.

I now examine the fourth order equations in the same sequence used in lower order, not computing the solutions but verifying that the solvability conditions are satisfied. Equation (2.107) in fourth order can be used to determine T_{e4}. The condition that one can solve for T_{e4} is

$$\begin{aligned}&\langle(T_e - T_{\perp e})_2\frac{(|\nabla\psi|^2)_{,\phi}}{\chi_0^2}\rangle - \langle(T_e - T_{\perp e})_2\frac{r_1 r_{1,\phi}}{R^2}\rangle\\ &+\langle(T_e - T_{\perp e})_2\frac{n_1 r_{1,\phi}}{n_0 R}\rangle + \langle(T_e - T_{\perp e})_3\frac{r_{1,\phi}}{R}\rangle = 0.\end{aligned} \tag{4.29}$$

This condition can be rewritten as

$$-\langle\lambda_{e2,\phi}\frac{|\nabla\psi|^2}{\chi_0^2}\rangle + \frac{1}{2R^2}(\frac{N_1}{n_0} + 1)\langle r_1^2\lambda_{e2,\phi}\rangle + \langle\lambda_{e3}\frac{r_{1,\phi}}{R}\rangle = 0. \tag{4.30}$$

The condition (4.30) expresses $\langle\lambda_{e3,\phi}r_1\rangle$ in terms of $\langle\lambda_{e2,\phi}r_1^2\rangle$, and $\langle\lambda_{e2,\phi}|\nabla\psi|^2\rangle$. Using the relations at the end of the previous section, a number of other averages of third quantities are also known in terms of averages of λ_{e2}.

Equation (2.108) is used to determine T_{i4}. The solvability condition for this equation gives the following relation between $\langle r_1\lambda_{i3,\phi}\rangle$, $\langle r_1^2\lambda_{e2,\phi}\rangle$, $\langle\lambda_{e2}S_{\perp i1}z_{1,\phi}\rangle$and $\langle\lambda_{e2,\phi}|\nabla\psi|^2\rangle$,

$$\begin{aligned}&-8n_0\langle(T_i - T_{\perp i})_1(T_i - T_{\perp i})_{3,\phi}\rangle + 8\langle n_{1,\phi}(T_i - T_{\perp i})_1(T_i - T_{\perp i})_2\rangle\\ &-8\langle(T_i - T_{\perp i})_1^2 n_{2,\phi}\rangle - 16\langle(T_i - T_{\perp i})_1(T_i - T_{\perp i})_2 n_{1,\phi}\rangle\end{aligned}$$

$$+10n_0\langle (T_i - T_{\perp i})_1 T_{i3,\phi}\rangle + 10n_0\langle (T_i - T_{\perp i})_2 T_{i2,\phi}\rangle$$
$$+10n_0\langle T_{i1,\phi}(T_i - T_{\perp i})_3\rangle + 10\langle n_1 (T_i - T_{\perp i})_1 T_{i2,\phi}\rangle$$
$$+10\langle n_1 T_{i1,\phi}(T_i - T_{\perp i})_2\rangle + 10\langle n_2 (T_i - T_{\perp i})_1 T_{i1}\rangle$$
$$+4n_0\langle T_{i1}(T_i - T_{\perp i})_{3,\phi}\rangle + 4n_0\langle T_{i2}(T_i - T_{\perp i})_{2,\phi}\rangle$$
$$+4T_{i0}\langle n_1(T_i - T_{\perp i})_{3,\phi}\rangle + 4T_{i0}\langle n_2(T_i - T_{\perp i})_{2,\phi}\rangle + 5n_0\langle T_{i1}T_{i3,\phi}\rangle$$
$$+5T_{i0}\langle n_1 T_{i3,\phi}\rangle + 5T_{i0}\langle n_2 T_{i2,\phi}\rangle - 24n_0\langle \frac{r_{1,\phi}}{R}(T_i - T_{\perp i})_1 (T_i - T_{\perp i})_2\rangle$$
$$+6n_0T_{i0}\langle (T_i - T_{\perp i})_2 (\frac{r_1 r_{1,\phi}}{R^2} + \frac{(|\nabla\psi|^2)_{,\phi}}{\chi_0^2}\rangle$$
$$+6n_0\langle \frac{r_{1,\phi}}{R} T_{i2}(T_i - T_{\perp i})_1\rangle + 6T_{i0}\langle n_2 (T_i - T_{\perp i})_1 \frac{r_{1,\phi}}{R}\rangle$$
$$+6n_0T_{i0}\langle \frac{r_{1,\phi}}{R}(T_i - T_{\perp i})_3\rangle = 0. \tag{4.31}$$

Since the average $\langle r_1\lambda_{e3,\phi}\rangle$ is expressed in terms of the second order averages $\langle r_1^2\lambda_{e2}\rangle$, and $\langle \lambda_{e2,\phi}|\nabla\psi|^2\rangle$, the above condition (4.31) gives a constraint on λ_{e2}. This constraint reduces the number of independent functions in the zero order solution from five to four.

Continuing with the remaining fourth order solvability conditions, equation (2.102) in fourth order is used to find n_4. The solvability condition is

$$\frac{e\chi_0}{cR^3}\langle r_{1,\phi}\lambda_3\rangle - 3\frac{e\chi_0}{cR^4}\langle r_1 r_{1,\phi}\lambda_2\rangle + \frac{e\chi_0}{2cR^2}\langle \frac{|\nabla\psi|^2}{\chi_0^2}\lambda_{2,\phi}\rangle - \langle \mathbf{P}'\cdot \mathbf{r}_{,\phi}\rangle. \tag{4.32}$$

The condition (4.32) is an additional constraint on λ_{e2}, since the term $\langle r_{1,\phi}\lambda_3\rangle$ is known in terms of averages of λ_{e2}. Hence, the number of independent functions in the zero order solution is reduced from four to three.

The pressure equations are used to determine the fourth order parallel heat flow. These equations have solvability conditions. Appearing in these conditions are the time derivatives of the zero order temperatures, T_{e0} and T_{i0}. Hence, to determine the slow time evolution of the temperature it is not necessary to solve the fourth order equations but only to evaluate the solvability conditions. Evaluating the solvability conditions is relatively simple since no information is required from the fourth order solution and only limited information is needed about the third order solution.

The solvability conditions are complicated. It is convenient to write the solvability conditions not in flux coordinates but in the original polar coordinates,

$$\langle \frac{r}{J}\frac{\partial p_{a0}}{\partial t}\rangle + \langle \frac{r}{J}\nabla\cdot(\mathbf{u}_a p_a + \frac{1}{3}\mathbf{S}_a)\rangle + \langle \frac{r}{J}p_a\nabla\cdot\mathbf{u}_a\rangle$$
$$+2\langle \frac{r}{J}(p_a - p_{\perp a})(\frac{\mathbf{B}\cdot(\mathbf{B}\cdot\nabla)}{B^2}\mathbf{u}_a - \frac{1}{3}\nabla\cdot\mathbf{u}_a)\rangle$$
$$= -\frac{2^{4/3}}{3}\frac{m_e}{m_i}\frac{n_0(T_{a0}-T_{b0})}{\tau_{ee}} + \langle \frac{r}{J}E_{as}\rangle. \tag{4.33}$$

I will show that in general it is not possible to set the time derivative in the above expression to zero. That is, the lowest order solution does evolve on the time scale hypothesized initially, $\epsilon^{-2}\tau_{ee}$. To determine the slow time evolution one must calculate the various terms in this expression. First consider terms explicitly involving the time variation of the lowest order solution are,

$$\langle \frac{r}{J}\frac{\partial p_{a0}}{\partial t}\rangle + \langle \frac{r}{J}\nabla\cdot(\mathbf{u}_a p_a)\rangle = n_0\langle \frac{r}{J}\frac{\partial T_{a0}}{\partial t}\rangle + T_{a0}\langle \frac{r}{J}N\rangle + \langle \frac{r}{J}n_0(\mathbf{U}\cdot\nabla)T_{a0}\rangle. \tag{4.34}$$

The time derivative of a flux function is no longer a flux function. The work done by the pressure is calculated to be

$$\langle \frac{r}{J}p_a\nabla\cdot\mathbf{u}_a\rangle = T_{a0}\langle \frac{r}{J}(N - \frac{\partial n_0}{\partial t} - (\mathbf{U}\cdot\nabla)n_0)\rangle - \langle \frac{T_a}{n}(n_{,\phi}\lambda_{a,\psi} - n_{,\psi}\lambda_{a,\phi})\rangle \tag{4.35}$$

where

$$\langle \frac{T_e}{n}(n_{,\phi}\lambda_{e,\psi} - n_{,\psi}\lambda_{e,\phi})\rangle = \frac{T_{e0}}{n_0^2}\frac{\partial}{\partial\psi}(n_0\langle n_1\lambda_{e3,\phi}\rangle) + \frac{T_{e0}}{n_0}\frac{\partial}{\partial\psi}(n_0\langle n_2\lambda_{e2,\phi}\rangle)$$
$$-\frac{T_{e0}}{n_0}\frac{\partial}{\partial\psi}(n_0\langle n_1^2\lambda_{e2,\phi}\rangle) + \frac{\lambda_{e0,\psi}}{n_0}\langle T_{e2}n_{2,\phi}\rangle + \frac{\lambda_{e0,\psi}}{n_0}\langle T_{e3}n_{1,\phi}\rangle$$
$$-\frac{T_{e0}\lambda_{e0,\psi}}{n_0^3}\langle n_1 n_{1,\phi} n_2\rangle - \frac{T_{e0}\lambda_{e0,\psi}}{n_0^2}\langle n_{1,\phi}n_3\rangle \tag{4.36}$$

and

$$\langle \frac{T_i}{n}(n_{,\phi}\lambda_{i,\psi} - n_{,\psi}\lambda_{i,\phi})\rangle = \frac{T_{i0}}{n_0}\frac{\partial}{\partial\psi}(\langle n_{3,\phi}\lambda_{i1}\rangle + \langle n_{1,\phi}\lambda_{i3}\rangle + \langle n_{2,\phi}\lambda_{i2}\rangle)$$
$$+\frac{T_{i0}}{n_0}\langle(\frac{T_{i1}}{T_{i0}} - \frac{n_1}{n_0})(n_{3,\phi}\lambda_{i0,\psi} - n_{0,\psi}\lambda_{i3,\phi} + n_{2,\phi}\lambda_{i1,\psi} - n_{2,\psi}\lambda_{i1,\phi}$$
$$+n_{1,\phi}\lambda_{i2,\psi} - n_{1,\psi}\lambda_{i2,\phi})\rangle$$

$$+\frac{T_{i0}}{n_0}\langle(\frac{T_{i2}}{T_{i0}}+\frac{n_1^2}{n_0^2}-\frac{T_{i1}n_1}{T_{i0}n_0})(n_{2,\phi}\lambda_{i0,\psi}-n_{0,\psi}\lambda_{i2,\phi})\rangle$$
$$+\frac{T_{i0}}{n_0}\langle\frac{n_2}{n_0}n_{0,\psi}\lambda_{i2,\phi}\rangle+\frac{T_{i0}}{n_0}\langle(\frac{T_{i2}}{T_{i0}}-\frac{n_2}{n_0})(n_{1,\phi}\lambda_{i1,\psi}-n_{1,\psi}\lambda_{i1,\phi})\rangle$$
$$+\frac{T_{i0}}{n_0}\langle(\frac{T_{i3}}{T_{i0}}+\frac{n_1n_2}{n_0^2}-\frac{T_{i2}n_1}{T_{i0}n_0}-\frac{T_{i1}n_2}{T_{i0}n_0})(n_{1,\phi}\lambda_{i0,\psi}-n_{0,\psi}\lambda_{i1,\phi})\rangle. \tag{4.37}$$

Consider the next term in (4.33) containing the divergence of the heat flow,

$$\frac{1}{3}\langle\frac{r}{J}\nabla\cdot\mathbf{S}_a\rangle=\frac{\partial}{\partial\psi}\langle S_{\perp a}\frac{|\nabla\psi|^2}{J}\rangle. \tag{4.38}$$

This requires calculating the flux surface average of the fourth order perpendicular heat flow. For the electrons, this is

$$\langle S_{\perp e4}\frac{|\nabla\psi|^2}{J}\rangle=\langle\frac{|\nabla\psi|^2}{J}\frac{r_1}{R}S_{\perp e3}\rangle-\langle\frac{|\nabla\psi|^2}{J}(\frac{3r_1^2}{R^2}+\frac{|\nabla\psi|^2}{\chi_0^2})S_{\perp e2}\rangle$$
$$-2\frac{cR^2}{e\chi_0}\langle T_{e1}(T_e-T_{\perp e})_2n_{1,\phi}\rangle+3\frac{cR^2}{e\chi_0}\langle T_{e0}(T_e-T_{\perp e})_2n_{2,\phi}\rangle$$
$$-2T_{e0}\frac{cR^2}{e\chi_0}\langle(T_e-T_{\perp e})_3n_{1,\phi}\rangle-5T_{eo}\frac{cR^2}{e\chi_0}\langle n_1T_{e3,\phi}\rangle$$
$$+6T_{e0}\frac{cR^2}{e}\langle\frac{r_{1,\phi}}{R}n_1(T_e-T_{\perp e})_2\rangle-6n_0T_{e0}\frac{cR^2}{e}\langle\frac{r_1r_{1,\phi}}{R^2}(T_e-T_{\perp e})_2\rangle$$
$$-3n_0T_{e0}\frac{cR^2}{e\chi_0^2}\langle(T_e-T_{\perp e})_2(|\nabla\psi|^2)_{,\phi}\rangle. \tag{4.39}$$

The flux surface average of the fourth order ion perpendicular heat flow is

$$\langle S_{\perp i4}\frac{|\nabla\psi|^2}{J}\rangle=2\langle S_{\perp i3}\frac{|\nabla\psi|^2}{J}\frac{r_1}{R}\rangle+\langle S_{\perp i2}\frac{|\nabla\psi|^2}{J}(\frac{3r_1^2}{R^2}+\frac{|\nabla\psi|^2}{\chi_0^2})\rangle$$
$$-2\langle n_{1,\phi}(T_i-T_{\perp i})_1(T_i-T_{\perp i})_2\rangle+2\langle n_2(T_i-T_{\perp i})_1(T_i-T_{\perp i})_{1,\phi}\rangle$$
$$+2\langle(T_i-T_{\perp i})_1^2n_{2,\phi}\rangle+4\langle(T_i-T_{\perp i})_1(T_i-T_{\perp i})_2n_{1,\phi}\rangle$$
$$-5n_0\langle(T_i-T_{\perp i})_1T_{i3,\phi}\rangle-5n_0\langle(T_i-T_{\perp i})_2T_{i2,\phi}\rangle$$
$$-5n_0\langle(T_i-T_{\perp i})_3T_{i1,\phi}\rangle-5\langle n_1(T_i-T_{\perp i})_1T_{i2,\phi}\rangle$$
$$-5\langle n_1(T_i-T_{\perp i})_2T_{i1,\phi}\rangle-5\langle n_2(T_i-T_{\perp i})_1T_{i1,\phi}\rangle$$
$$-2T_{i0}\langle(T_i-T_{\perp i})_1n_{3,\phi}\rangle-2T_{i0}\langle(T_i-T_{\perp i})_2n_{2,\phi}\rangle$$
$$-2T_{i0}\langle(T_i-T_{\perp i})_3n_{1,\phi}\rangle-2\langle T_{i1}(T_i-T_{\perp i})_1n_{2,\phi}\rangle$$

$$-2\langle T_{i1}(T_i - T_{\perp i})_2 n_{1,\phi}\rangle + \langle T_{i2}(T_i - T_{\perp i})_1 n_{2,\phi}\rangle$$
$$-5T_{i0}\langle n_1 T_{i3,\phi}\rangle - 5T_{i0}\langle n_2 T_{i2,\phi}\rangle - 5\langle n_2 T_{i1} T_{i1,\phi}\rangle$$
$$-5T_{i0}\langle n_3 T_{i1,\phi}\rangle + \frac{6\chi_0}{R}(n_0 T_{i0}\langle (T_i - T_{\perp i})_3 r_{1,\phi}\rangle$$
$$+T_{i0}\langle n_1(T_i - T_{\perp i})_2 r_{1,\phi}\rangle + \langle n_2(T_i - T_{\perp i})_1 r_{1,\phi}\rangle). \tag{4.40}$$

I now calculate the flux surface average of the work done by the stresses,

$$\langle \frac{r}{J}(p_a - p_{\perp a})(\frac{\mathbf{B}\cdot(\mathbf{B}\cdot\nabla)}{B^2}\mathbf{u}_a - \frac{1}{3}\nabla\cdot\mathbf{u}_a)\rangle. \tag{4.41}$$

This calculation is straightforward but lengthy. I first calculate the work done by the electron stresses. It is convenient to return for a moment to the original form of the reduced electron stress equation (2.20). An exact consequence of the reduced equation is

$$(p_e - p_{\perp e})(\frac{1}{B^2}\mathbf{B}\cdot(\mathbf{B}\cdot\nabla)\mathbf{u}_e - \frac{1}{3}\nabla\cdot\mathbf{u}_e) = \frac{-3\cdot 2^{1/3}}{5}\frac{(p_e - p_{\perp e})^2}{\tau_{ee}}$$
$$-2\frac{(p_e - p_{\perp e})}{p_e}\nabla\cdot(\mathbf{u}_e(p_e - p_{\perp e})) - \frac{2(p_e - p_{\perp e})}{5p_e}(\frac{1}{B^2}\mathbf{B}\cdot(\mathbf{B}\cdot\nabla)\mathbf{S}_e - \frac{1}{3}\nabla\cdot\mathbf{S}_e)$$
$$+\frac{2(p_e - p_{\perp e})^2}{p_e}(\frac{1}{B^2}\mathbf{B}\cdot(\mathbf{B}\cdot\nabla)\mathbf{u}_e - \frac{1}{3}\nabla\cdot\mathbf{u}_e). \tag{4.42}$$

If terms through order ϵ^2 only are retained then the last term on the right hand side may be dropped. Now calculate the flux surface average of the terms in the above expression (4.42). The flux surface average of the first term is

$$\langle \frac{R}{J}\frac{-3\cdot 2^{1/3}}{5}\frac{(p_e - p_{\perp e})^2}{\tau_{ee}}\rangle = n_0^2\frac{-3\cdot 2^{1/3}R}{5\tau_{ee}}\langle\frac{(T_e - T_{\perp e})_2^2}{J}\rangle. \tag{4.43}$$

The flux surface average of the second term is

$$\langle -\frac{R}{J}2\frac{(p_e - p_{\perp e})}{p_e}\nabla\cdot(\mathbf{u}_e(p_e - p_{\perp e}))\rangle$$
$$= -2\frac{n_0\lambda_{e0,\psi}}{T_{e0}}\langle (T_e - T_{\perp e})_2 (T_e - T_{\perp e})_{2,\phi}\rangle = 0. \tag{4.44}$$

The flux surface average of the remaining term is

$$\langle -\frac{R}{J}\frac{2(p_e - p_{\perp e})}{5p_e}(\frac{1}{B^2}\mathbf{B}\cdot(\mathbf{B}\cdot\nabla)\mathbf{S}_e - \frac{1}{3}\nabla\cdot\mathbf{S}_e)\rangle$$

$$= -\frac{2}{5T_{e0}}\langle (T_e - T_{\perp e})_2 (S_{\|e2,\phi} + S_{\|e1}\frac{r_{1,\phi}}{R} - S_{\|e0}\frac{(|\nabla\psi|^2)_{,\phi}}{\chi_0^2} -$$

$$\gamma_{e1}\frac{r_{1,\phi}}{R} - \frac{1}{3}(S_{\perp e2}\frac{|\nabla\psi|^2}{J})_{,\psi})\rangle. \tag{4.45}$$

The flux surface average of the work done by the electron stresses has been specified.

Now, I calculate the work done by the ion stresses. First one finds that

$$(p_i - p_{\perp i})(\frac{1}{B^2}\mathbf{B}\cdot(\mathbf{B}\cdot\nabla)\mathbf{u}_i - \frac{1}{3}\nabla\cdot\mathbf{u}_i) = -\frac{3}{5}(\frac{T_{e0}}{T_{i0}})^{3/2} n_0^2 \sqrt{\frac{m_e}{m_i}}\frac{(T_i - T_{\perp i})^2}{\tau_{ee}}$$

$$-2(1 - 2\frac{(p_i - p_{\perp i})}{p_i})^{-1}\frac{(p_i - p_{\perp i})}{p_i}\nabla\cdot(\mathbf{u}_i(p_i - p_{\perp i})) -$$

$$(1 - 2\frac{(p_i - p_{\perp i})}{p_i})^{-1}\frac{2(p_i - p_{\perp i})}{5p_i}(\frac{1}{B^2}\mathbf{B}\cdot(\mathbf{B}\cdot\nabla)\mathbf{S}_i - \frac{1}{3}\nabla\cdot\mathbf{S}_i). \tag{4.46}$$

Taking the average of this expression, the first term is

$$-\langle\frac{R}{J}\frac{3}{5}(\frac{T_{e0}}{T_{i0}})^{3/2} n_0^2\sqrt{\frac{m_e}{m_i}}\frac{(T_i - T_{\perp i})^2}{\tau_{ee}}\rangle = -\frac{3}{5}(\frac{T_{e0}}{T_{i0}})^{3/2}\frac{R}{\tau_{ee}}\sqrt{\frac{m_e}{m_i}}\langle\frac{(T_i - T_{\perp i})_1^2}{J}\rangle. \tag{4.47}$$

The average of the next term is

$$-\langle\frac{R}{J}2(1 - 2\frac{(p_i - p_{\perp i})}{p_i})^{-1}\frac{(p_i - p_{\perp i})}{p_i}\nabla\cdot(\mathbf{u}_i(p_i - p_{\perp i}))\rangle$$

$$= -\frac{1}{2T_{i0}}\frac{\partial}{\partial\psi}\langle\lambda_{i2,\phi}(T_i - T_{\perp i})_1^2\rangle - \frac{1}{T_{i0}}\frac{\partial}{\partial\psi}\langle\lambda_{i1,\phi}(T_i - T_{\perp i})_1(T_i - T_{\perp i})_2\rangle$$

$$-\frac{\lambda_{i0,\psi}}{T_{i0}^2}\langle T_{i1,\phi}(T_i - T_{\perp i})_1(T_i - T_{\perp i})_2\rangle$$

$$-\frac{\lambda_{i0,\psi}}{T_{i0}^2}\langle T_{i2}(T_i - T_{\perp i})_1(T_i - T_{\perp i})_{1,\phi}\rangle. \tag{4.48}$$

The average of the final term is

$$\langle\frac{R}{J}(1 - 2\frac{(p_i - p_{\perp i})}{p_i})^{-1}\frac{2(p_i - p_{\perp i})}{5p_i}(\frac{1}{B^2}\mathbf{B}\cdot(\mathbf{B}\cdot\nabla)\mathbf{S}_i - \frac{1}{3}\nabla\cdot\mathbf{S}_i)\rangle$$

$$= -\frac{2}{5T_{i0}}[\langle(T_i - T_{\perp i})_1 S_{\|i3,\phi}\rangle + \langle(T_i - T_{\perp i})_3 S_{\|i1,\phi}\rangle] + \frac{1}{T_{i0}}\langle T_{i1}(T_i - T_{\perp i})_2 S_{\|i1,\phi}\rangle$$

$$+\frac{1}{T_{i0}}\langle T_{i2}(T_i - T_{\perp i})_1 S_{\|i1,\phi}\rangle + \frac{2}{T_{i0}}\langle(T_i - T_{\perp i})_1^2 S_{\|i2,\phi}\rangle$$

$$-\frac{S_{\|i0}}{\chi_0^2}\langle(T_i - T_{\perp i})_2(|\nabla\psi|^2)_{,\phi}\rangle - \frac{1}{3}\langle(T_i - T_{\perp i})_1(\frac{r}{J}\nabla\cdot\mathbf{S}_{i3})\rangle$$

$$-\frac{1}{3}\langle((T_i - T_{\perp i})_2 - \frac{(T_i - T_{\perp i})_1 T_{i1}}{T_{i0}} + 2\frac{(T_i - T_{\perp i})_1^2}{T_{i0}})(S_{\|i2,\phi} - \gamma_{i2,\phi} + (S_{\perp i2}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi}$$

$$-(S_{\perp i2}\frac{|\nabla\psi|^2}{J})_{,\psi})\rangle - \frac{1}{3}\ langle(T_i - T_{\perp i})_3 - \frac{(T_i - T_{\perp i})_1 T_{i2}}{T_{i0}})(S_{\|i1,\phi}$$

$$-\gamma_{i1,\phi} + (S_{\perp i1}\frac{\nabla\psi\cdot\nabla\phi}{J})_{,\phi} - (S_{\perp i1}\frac{|\nabla\psi|^2}{J})_{,\psi})\rangle. \tag{4.49}$$

The flux surface average of the work done by the stresses has been calculated.

I summarize the analysis of the fourth order system up to this point. Using the condition (4.30), $\langle r_1\lambda_{e3,\phi}\rangle$ is known completely if λ_{e2} is known. The relations (4.31) and (4.32) are satisfied by the appropriate choice of two of the arbitrary first order flux functions in λ_{e2}. The calculation of the solvability condition associated with the pressure equations gives expressions for the time evolution of the lowest order temperature. All the terms in this solvability can be expressed in terms of averages of λ_{e2}. Thus, when λ_{e2} is completely specified the time evolution of the temperature will be known. I proceed with the remainder of the fourth order system.

The solvability conditions for ω_{e4}, ω_{i4} and n_4 are not independent; there are only two independent conditions. This can be shown by examining the structure of these three equations. While three equations are derived from the poloidal momentum balance, there are only two independent solvability constraints. Recall that these equations have the form

$$\frac{1}{J}(\mathbf{M}_{er,z} - \mathbf{M}_{ez,r}) = 0, \tag{4.50}$$

$$\frac{1}{J}(\mathbf{M}_{ir,z} - \mathbf{M}_{iz,r}) = 0, \tag{4.51}$$

and

$$\frac{1}{J}\mathbf{B}\cdot(\mathbf{M}_e + \mathbf{M}_i) = 0. \tag{4.52}$$

The associated solvability conditions are obtained by taking the average of the above equations with respect to ϕ. A simple calculation shows that

$$\int\frac{d\phi}{J}(v_{r,z} - v_{z,r}) = -\frac{\partial}{\partial\psi}\int\frac{d\phi}{J}r\mathbf{B}\cdot\mathbf{v} \tag{4.53}$$

where $\mathbf{v}$ is any vector with no $\hat{\theta}$ component. Hence, if the average of any two of the above equations vanishes so does the third.

The solvability condition for ω_{e4} is:

$$\begin{aligned}
&\frac{1}{n_0}\frac{\partial}{\partial\psi}\langle n_{2,\phi}T_{e2}\rangle + \frac{1}{n_0}\frac{\partial}{\partial\psi}\langle n_{1,\phi}T_{e3}\rangle - \frac{T_{e0,\psi}}{n_0^2}\langle n_1 n_{3,\phi}\rangle + \frac{n_{0,\psi}}{n_0^2}\langle n_1 T_{e3,\phi}\rangle \\
&+\frac{1}{2n_0}\frac{\partial}{\partial\psi}\langle (n_1^2)_{,\phi}T_{e2}\rangle + \frac{T_{e0,\psi}}{n_0^3}\langle n_1^2 n_{2,\phi}\rangle - \frac{n_{0,\psi}}{n_0^3}\langle n_1^2 T_{e2,\phi}\rangle + \frac{n_{0,\psi}}{n_0^2}\langle n_2 T_{e2,\phi}\rangle \\
&+\frac{T_{e0,\psi}}{n_0^3}\langle n_1 n_{1,\phi} n_2\rangle - \frac{T_{e0,\psi}}{n_0^2} + \frac{4e\chi_0}{3n_0 c R^2}[\frac{\partial}{\partial\psi}\langle n_{2,\phi}\lambda_{e2}\rangle + \frac{\partial}{\partial\psi}\langle n_{1,\phi}\lambda_{e3,\phi}\rangle \\
&-2\lambda_{e0,\psi}\langle(\frac{n_1}{n_0}+\frac{r_1}{R})n_{3,\phi}\rangle - 2n_0\langle(\frac{n_1}{n_0}+\frac{r_1}{R})\lambda_{e3,\phi}\rangle - \frac{1}{n_0}\frac{\partial}{\partial\psi}\langle (n_1^2)_{,\phi}\lambda_{e2}\rangle \\
&-\frac{N_1}{R}\frac{\partial}{\partial\psi}\langle r_1 r_{1,\phi}\lambda_{e2}\rangle + 3(\frac{N_1^2}{n_0^2}+\frac{1}{R^2})\langle r_1^2 n_{2,\phi}\lambda_{e0,\psi} - n_{0,\psi}\lambda_{e2,\phi} r_1^2\rangle \\
&+\frac{2n_{0,\psi}}{n_0}\langle n_2\lambda_{e2,\phi}\rangle] + \frac{e\chi_0}{3n_0^2 cR^2}\lambda_{e0,\psi}[-2\langle(\frac{n_1}{n_0}+\frac{r_1}{R})n_{3,\phi}\rangle + 3\langle(\frac{n_1^2}{n_0^2}+\frac{r_1^2}{R^2})n_{2,\phi}\rangle \\
&+\frac{6}{n_0^2}\langle n_1 n_{1,\phi} n_2\rangle + \frac{6}{n_0 R}\langle r_1 n_2 n_{1,\phi}\rangle] + \frac{e\chi_0}{n_0 c R^3}[\frac{\partial}{\partial\psi}\langle r_1\lambda_{e3,\phi}\rangle \\
&-(\frac{N_1}{n_0}+3)\frac{1}{2R}\frac{\partial}{\partial\psi}\langle r_1^2\lambda_{e2,\phi}\rangle + \lambda_{e0,\psi}\langle\frac{n_3}{n_0}r_{1,\phi}\rangle - \frac{2\lambda_{e0,\psi}}{n_0^2}\langle n_1 n_2 r_{1,\phi}\rangle] \\
&-\frac{2\chi_0}{3n_0^2 R^3}[\frac{L_{e1}}{2R}\frac{\partial}{\partial\psi}\langle r_1^2 n_{2,\phi}\rangle + \langle\frac{n_2}{n_0}\lambda_{e1} r_{1,\phi}\rangle - \frac{N_1'}{R}\langle\lambda_{e2} r_1 r_{1,\phi}\rangle \\
&+(3+\frac{2N_1}{n_0})\frac{n_{0,\psi}}{R}\langle r_1 r_{1,\phi}\lambda_{e2}\rangle - n_{0,\psi}\langle\lambda_{e3} r_{1,\phi}\rangle] + \frac{e\chi_0}{n_0 c R^2}\lambda_{e0,\psi}[\frac{2}{n_0^2}\langle n_1 n_2 r_{1,\phi}\rangle \\
&-\frac{1}{n_0}\langle r_{1,\phi} n_3\rangle + \frac{3}{n_0 R}\langle r_{1,\phi} r_1 n_2\rangle] + \frac{1}{n_0}\langle\frac{1}{J}(P'_{ez,r} - P'_{er,z})\rangle.
\end{aligned} \tag{4.54}$$

The constraint (4.54) reduces the number of independent function in the zero order solution from three to two. With all the quantities in the energy balance solvability conditions known, the time evolution of the lowest order temperature is determined.

The remaining equations in the system are the ion and electron stress equations. The solvability condition for the ion stress equation includes the term $\langle\frac{\lambda_{i2}}{J}\rangle$. The second order solution λ_{i2} has an undetermined flux function part that may be chosen such that the solvability condition is satisfied. There is no solvability condition associated with the electron stress equation. The constraint (2.41) must now be satisfied. This constraint determines the time evolution of ψ.

I summerize the zero order system. There zero order system initially consists of nine arbitrary functions of ψ and a Grad-Shafranov type equation for ψ. The seven constraints (3.12), (3.44), (4.9), (4.14), (4.31), (4.32), and (4.54) serve to reduce the number of independent functions to two. Hence, with the time evolution of T_{e0} and T_{i0} given by (4.33) the time evolution of the nine zero order functions is known. The remaining part of the zero order solution is ψ whose time evolution is determined by the constraint (2.41) and the system is closed.

5. Summary and Discussion

In this thesis, I have presented and analyzed a mathematical model describing transport in a tokamak. The model is derived from a thirteen moment two-fluid description. The thirteen moment system was chosen as an initial description because one expects it to provide a reasonable description of a weakly collisional plasma while still being sufficiently simple that a detailed analysis is possible. The full thirteen moment system includes a wide variety of phenomena. A reduced model is obtained by neglecting small effects. The question of which effects are important to transport and which may be neglected is not easily answered. I have taken typical values of the temperature, the density and the magnetic field and assuming the plasma to be stable and quiescent, used these to estimated the order of magnitude of various effects. I include small smooth laminar flows. I introduce a single scaling parameter, $\epsilon^2 = m_e/m_i$.

It is in the matter of scaling that this work differs most significantly from neoclassical calculations. In particular, I take the velocity space anisotropy of the distribution function to be considerablely larger than in standard neoclassical calculations. One of the results of this work is that the assumed size of the anisotropy directly determines the magnitude of the flux surface variation of the other quantities in the system and plays an important role in transport. It would appear that by assuming different sizes of the anisotropy one can obtain self-consistent systems with dramatically different properties. The question is then what is appropriate in a tokamak system.

There are plausible reasons for taking the anisotropy to be relatively large. One reason is related to the low collisionality of the system. An estimate of the size of the anisotropy is given by the product of the Mach number and the mean free path. While the flow velocities are small, the mean free path is long and thus the possibility of significant anisotropy in the system exists. Another perhaps more fundamental reason is that tokamak devices are not in the thermodynamic sense closed systems. They are driven by external sources such as the transformer loop voltage and various heating schemes. This model includes particle, momentum and energy source terms. It is the order of magnitude of these sources that determine the character of the system. Hence, I take the anisotropy to be initially $O(\epsilon^{1/2})$; collisional effects force the electron anisotropy to be $O(\epsilon)$.

Once the scaling is decided it is straightforward to extract from the full thirteen moment system a reduced model that includes terms through $O(\epsilon^2)$. Using the toroidal symmetry of the physical system some additional reduction in the form of the equations is possible. The system finally consists of ten primary unknowns and ten equations along with three constraints. The reduced system though considerablely simpler than the full thirteen moment system still retains particle flows and pressure anisotropy. The qualitative features of the solutions of such a system are not at all apparent.

This reduced model is quite complex and non-standard. The equations still contain terms of very different size, varying from $O(1)$ to $O(\epsilon^2)$. There are a number of fundamental questions about the model that one would like to address. There are questions about the mathematical structure of the system, such as what data can be specified, is the system closed and the time evolution determined. Also, there are questions about the physics of the model, what are the effects of particle flows, what is the role of the anisotropy, how does collisionality affect the solution, and on what time scale do solutions evolve. A reasonable method of exploring these questions is to expand the solution in an asymptotic series using the scaling already introduced. The calculation of the asymptotic solution is straightforward but lengthy.

The asymptotic solution provides detailed information about the model. I find that with two flux function profiles initially specified and the external sources known, the lowest order solution and its self-consistent time evolution is determined. I find that the lowest order solution evolves on the time scale $\epsilon^{-2}\tau_e \sim 50ms$. This time scale is comparable to energy confinement times seen in experiment.

I now describe the asymptotic solution in detail. The lowest order system gives that the poloidal flux function ψ is given by a Grad-Shafranov type equation and the other nine unknowns are undetermined functions of ψ. A first order constraint reduces the number of independent undetermined functions from nine to eight. This solution is steady on the time scale τ_e. It is typical of asymptotic solutions that the zero order solution is not completely determined by the zero order system. From the first order system I calculate corrections to the solution so that the lowest order solution is steady on the time scale $\epsilon^{-1/2}\tau_e$. I next calculate the second order corrections and extend the zero order solution to the time scale $\epsilon^{-1}\tau_e$. In order that solutions to the second order system exist, restrictions are placed on the first order solution. These restriction imply that $\lambda_{e1} = 0$. By setting λ_{e1} to zero, I impose a constraint on the zero order solution, reducing the number of undetermined functions in the zero order solution from eight to seven. Another solvability condition in the second order system is satisfied by assuming that ψ is approximately symmetric with respect to z. It is seen that with this assumption of symmetry, the solution to $O(\epsilon^{1/2})$ is up-down symmetric but that collisional effects lead to a loss of this symmetry in second order. Later, this up-down asymmetry is found to be a mechanism for transport.

The analysis of the third order system concentrates on identifying and satisfying all the solvability conditions associated with the third order equations. The third order system provides two additional conditions on the lower order solution, which are interpreted as additional constraints on the zero order solution. Hence, the number of undetermined function in the zero order solution is reduced from seven to five. The analysis of the fourth order system is similar. From solvability conditions, I find that the time derivatives of the zero order temperature profiles can be set to zero only if the energy sources are carefully chosen. Hence, in general the system evolves on the time scale $\epsilon^{-2}\tau_e$. By satisfying the remaining solvability conditions for the fourth order system, I obtain three additional constraints on the zero order solution. Thus, with the time evolution of the temperature profiles known and the time evolution of the magnetic field given by another constraint, the self-consistent time evolution of the zero order solution is determined.

I have presented a model for tokamak transport derived based on a thirteen moment model of a plasma. The model contains a careful treatment of particle flows, anisotropy and heat flow. A key element in this model is the assumed size of the anisotropy. While the model is somewhat complicated it is sufficiently simple that a detailed study of the behavior of asymptotic solutions is possible. The work done by the stresses is a significant mechanism for energy dissipation. Up-down asymmetry is another source of transport. This model also finds energy transport due to heat flows. The system evolves on the time scale $\sim 50ms$, a time scale comparable to experimental energy confinement times.

Acknowledgments

I would like to thank my advisor Harold Weitzner for his patient support and advice. This work was supported by U.S. Department of Energy Grant No. DE-FG02-86ER53223.

A. The Fokker-Planck Collision Operator

I now make explicit the form of the collision operator C_{ab} in (1.1) and calculate its moments. In this section it is convenient to supress the spatial and time dependance of various quantities. The Fokker-Planck collision operator has the form (see for example [2])

$$C_{ab} = -\nabla_\xi \cdot (\mathbf{A}_{ab} f_a - \frac{1}{2}\nabla_\xi \cdot (\mathbf{D}_{ab} f_a)), \tag{A.1}$$

where the frictional force vector $\mathbf{A}_{ab}$ is

$$\mathbf{A}_{ab} = z_b^2 \Gamma_a (1 + \frac{m_a}{m_b}) \nabla_\xi h_b, \tag{A.2}$$

the diffusion tensor $\mathbf{D}_{ab}$ is

$$\mathbf{D}_{ab} = z_b^2 \Gamma_a \nabla_\xi \nabla_\xi g_b, \tag{A.3}$$

and

$$\Gamma_a = \frac{4\pi z_a e^4 \ln \Lambda_{coul}}{m_a^2}. \tag{A.4}$$

Here $\ln \Lambda_{coul}$ is the Coulomb logarithm, an approximate quantity related to the introduction of a cut-off of the Coulomb potential. The operator ∇_ξ is the gradient operator in velocity space. The functions g_b and h_b are the Rosenbluth potentials defined by

$$\Delta_\xi^2 g_b = 2h_b \tag{A.5}$$

$$\Delta_\xi^2 h_b = -4\pi f_b, \tag{A.6}$$

with the boundary conditions $h(\boldsymbol{\xi})$ goes to zero and $g(\boldsymbol{\xi})/\xi n$ goes to one as ξ goes to infinity. I first show that this collision operator has the necessary properties of conserving mass, momentum and energy of the plasma. I assume that the distribution function f_a is sufficiently smooth and vanishes as ξ goes to infinity. Integration by parts gives zero order moment

$$\int C_{ab}\, d\xi = 0; \tag{A.7}$$

hence, particles are conserved in collisions. A simple calculation gives

$$m_a \int \xi C_{ab}\, d\xi = -m_b \int \xi C_{ba}\, d\xi. \tag{A.8}$$

Thus

$$\sum_b m_a \int \xi C_{ab}\, d\xi = 0, \tag{A.9}$$

and the total momentum of the plasma is conserved by collisions. Likewise the total energy of the plasma is conserved by collisions since

$$m_a \int (\xi - \mathbf{u}_a)^2 C_{ab}\, d\xi = -m_b \int (\xi - \mathbf{u}_b)^2 C_{ba}\, d\xi \tag{A.10}$$

and

$$\sum_b m_a \int (\xi - \mathbf{u}_a)^2 C_{ab}\, d\xi = 0. \tag{A.11}$$

There are no similar conservation relations for higher order moments. Another important property of this collision operator is that $\sum C_{ab} = 0$ if and only if f_a and f_b are uniform Maxwellians with common velocity and temperature.

In order to complete the thirteen moment equations I need to calculate moments of the collision terms. I first calculate the thirteen moment approximations for the Rosenbluth potentials g and h. Using (1.31), a simple calculation shows that

$$f = (1 + \frac{1}{4}\frac{p_{jk}}{p}\frac{\partial^2}{\partial y_j \partial y_k} - \frac{1}{120}\frac{\mathrm{S}_j}{pv}\frac{\partial^3}{\partial y_j \partial y_k \partial y_k}) f_0. \tag{A.12}$$

Since f_0 is a function of $r = |\mathbf{y}|$ alone this can be written

$$f = (1 + \frac{1}{4}\frac{p_{jk}}{p}\hat{y}_j\hat{y}_k r(\frac{f_0'}{r})' - \frac{1}{120}\frac{\mathrm{S}_j}{pv}\hat{y}_j(\frac{1}{r^2}(r^2 f_0')')'), \tag{A.13}$$

where $\hat{\mathbf{y}} = \mathbf{y}/r$ and prime denotes differentiation with respect to r. Symbolically, (A.13) can be written, $f = L[f_0]$. This form allows the thirteen moments approximations for g and h to be calculated as $h = L[h_0]$ and $g = L[g_0]$ where

$$\Delta_\xi^2 g_0 = 2h_0 \tag{A.14}$$

and

$$\Delta_\xi^2 h_0 = -4\pi f_0. \tag{A.15}$$

The functions g_0 and h_0 are found to be

$$g_0 = nv((\frac{1}{2r} + r)\Phi(r) + \pi^{1/2}exp(-r^2)) \tag{A.16}$$

and

$$h_0 = \frac{n}{v}\frac{\Phi(r)}{r} \tag{A.17}$$

where $\Phi(r)$ is the error function.

In order to make the calculation as explicit as possible I use the following standard approximation, valid for fluid velocities much less than the thermal velocity (see for example [2])

$$\begin{aligned} f_{a0} &= \frac{n_a\pi^{3/2}}{va^3}exp(-\mu^2 r_b^2)(1 - 2\mu^2\hat{\mathbf{y}}_b\cdot(\mathbf{u}_b - \mathbf{u}_a)/v_b) \\ f_{a0} &= f_{a0}^{(0)} + f_{a0}^{(1)} \end{aligned} \tag{A.18}$$

where $\mu = v_b/v_a$. Now to minimize notation let $\mathbf{y} = \mathbf{y}_b$.

The integrals over velocity space are conveniently calculated using spherical coordinates. The following identities are used for the radial integrals,

$$\int_{-\infty}^{\infty} dr\ r^n e^{-sr^2} = \frac{1}{2}\Gamma(\frac{n+1}{2})s^{-\frac{n+1}{2}} \tag{A.19}$$

$$\int_{-\infty}^{\infty} dr\ r\Phi(r)e^{-sr^2} = \frac{\sqrt{1+s}}{2s}, \tag{A.20}$$

where $\Gamma(x)$ is the usual gamma function. The calculation of the integrals with respect to the angular variables uses the following simple identities for the solid angle integral of various tensor products of the unit vector $\hat{\mathbf{y}}$ with itself

$$\int d\Omega = 4\pi, \tag{A.21}$$

$$\int d\Omega\ \hat{y}_i \hat{y}_j = \frac{4}{3}\pi\delta_{ij}, \tag{A.22}$$

$$\int d\Omega\ \hat{y}_i \hat{y}_j \hat{y}_k \hat{y}_l = \frac{4}{15}\pi(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{A.23}$$

where the integration is over the surface of a sphere. The integral of odd tensor products of $\hat{y}$ vanishes. With above identities the calculation of the moments of the collision term while complicated is straightforward. As an additional simplification, I will present only the collision terms linear in the moments $\mathbf{u}_a$, $\mathbf{p}_a$ and $\mathbf{S}_a$; quadratic terms are neglected.

I can now simply calculate the first order moment

$$\mathbf{F}_a = z_b^2 \Gamma_a m_a (1 + \frac{m_a}{m_b}) v_b^2 \int f_a \frac{\partial h_b}{\partial \xi}\, d\mathbf{y} \tag{A.24}$$

where f_a is given by (1.31) and (A.18). Keeping only linear terms one has

$$\mathbf{F}_a = \frac{-m_a n}{\tau_{ab}}(\mathbf{u}_a - \mathbf{u}_b) + \frac{3}{5}\frac{1}{v_a^2 + v_b^2}(\frac{\mathbf{S}_a}{\tau_{ab}} - \frac{\mathbf{S}_b}{\tau_{ba}}) \tag{A.25}$$

where

$$\tau_{ab}^{-1} = \frac{4}{3\pi^{1/2}} \frac{n z_b^2 \Gamma_a (1 + m_a/m_b)}{(v_a^2 + v_b^2)^{3/2}}. \tag{A.26}$$

The time τ_{ab} gives the characteristic time scale for momentum transfer from species b to species a. There is no momentum transfer in like-species collisions.

The integral for the second order moments is

$$\mathbf{T}_a = z_b^2 \Gamma_a m_a v_b^3 \int (2(1 + \frac{m_a}{m_b})\mathbf{y}\frac{\partial h_b}{\partial \mathbf{y}} + \frac{1}{v_b^2}\frac{\partial^2 g_b}{\partial \mathbf{y} \partial \mathbf{y}}) f_a\, d\mathbf{y}. \tag{A.27}$$

The resulting linear terms are

$$\mathbf{T}_a = \frac{m_a n}{\tau_{ab}}\frac{(m_a v_a^2 - m_b v_b^2)\mathbf{I}}{m_a + m_b} - \frac{\mathbf{p}_a}{\tau_{ab}}\frac{(2m_a v_a^2 + 3m_b v_a^2 + 5m_a v_b^2 + 6m_b v_b^2)}{(m_a + m_b)(v_a^2 + v_b^2)}$$

$$+\frac{1}{5}\frac{\mathbf{p}_b}{\tau_{ba}}\frac{3m_a v_a^2 + 2m_b v_a^2 - m_b v_b^2}{(m_a + m_b)(v_a^2 + v_b^2)}. \tag{A.28}$$

The integral for the third order moments is

$$\mathbf{Q}_a = z_b^2 \Gamma_a m_a v_b^3 \int (r^2(1 + \frac{m_a}{m_b})(2\hat{\mathbf{y}}\hat{y}_k \frac{\partial h_b}{\partial y_k} + \frac{\partial h_b}{\partial \mathbf{y}})$$

$$+2 v_b \hat{\mathbf{y}} r h_b + \frac{2r}{v_b}\hat{y}_k \frac{\partial^2 g_b}{\partial y_k \partial \mathbf{y}}) f_a \, d\mathbf{y}. \tag{A.29}$$

Integration gives the linear terms

$$\mathbf{Q}_a = \frac{m_a n}{\tau_{ab}}\frac{(\mathbf{u}_a - \mathbf{u}_b)}{m_a + m_b}(3m_a v_a^2 + 3m_b v_a^2 - 2m_b v_b^2)$$

$$+\frac{3}{10}\frac{\mathbf{S}_a}{\tau_{ab}}\frac{1}{(m_a + m_b)}\frac{1}{v_a^2(v_a^2 + v_b^2)^2}(18m_a v_a^6 + 6m_b v_a^6 + 63m_a v_a^4 v_b^2 + 17m_b v_a^4 v_b^2$$

$$+60m_a v_a^2 v_b^4 + 6m_b v_a^2 v_b^4 + 30m_a v_b^6 + 10m_b v_b^6)$$

$$-\frac{3\mathbf{S}_b}{\tau_{ba}}\frac{v_a^2}{(v_a^2 + v_b^2)^2}\frac{1}{(m_a + m_b)}(10m_a v_a^2 + 4m_b v_a^2 - 5m_a v_b^2 - 11m_b v_b^2). \tag{A.30}$$

These results can be simplified using that m_e/m_i is small. It will be shown that for our purposes the only the leading order terms are needed. They are:

$$\mathbf{F}_e = -\mathbf{F}_i = -2^{1/3}\frac{n m_e}{\tau_{ee}}(\mathbf{u}_e - \mathbf{u}_i) + \frac{2^{1/3}\cdot 9}{10}\frac{m_e}{T_e \tau_{ee}}\mathbf{S}_e + O(\frac{m_e}{m_i}), \tag{A.31}$$

$$\mathbf{T}_e = -\frac{2\cdot 2^{1/3}}{3}\frac{m_e}{m_i}\frac{n(T_e - T_i)}{\tau_{ee}}\mathbf{I} - \frac{3\cdot(1 + 2^{1/3})}{5}\frac{\mathbf{p}_e}{\tau_{ee}} + O(\frac{m_e}{m_i}), \tag{A.32}$$

$$\mathbf{T}_i = \frac{2\cdot 2^{1/3}}{3}\frac{m_e}{m_i}\frac{n(T_e - T_i)}{\tau_{ee}}\mathbf{I} - \frac{3}{5}\frac{\mathbf{p}_i}{\tau_{ii}} + O(\frac{m_e}{m_i}), \tag{A.33}$$

$$\mathbf{Q}_e = 1.70\frac{\mathbf{S}_e}{\tau_{ee}} - 0.84\frac{nT_e}{\tau_{ee}}(\mathbf{u}_e - \mathbf{u}_i) + O(\frac{m_e}{m_i}), \tag{A.34}$$

$$\mathbf{Q}_i = 1.89\frac{T_e}{T_i}\frac{\mathbf{S}_i}{\tau_{ee}} + 7.95\frac{\mathbf{S}_i}{\tau_{ii}} + O(\frac{m_e}{m_i}). \tag{A.35}$$

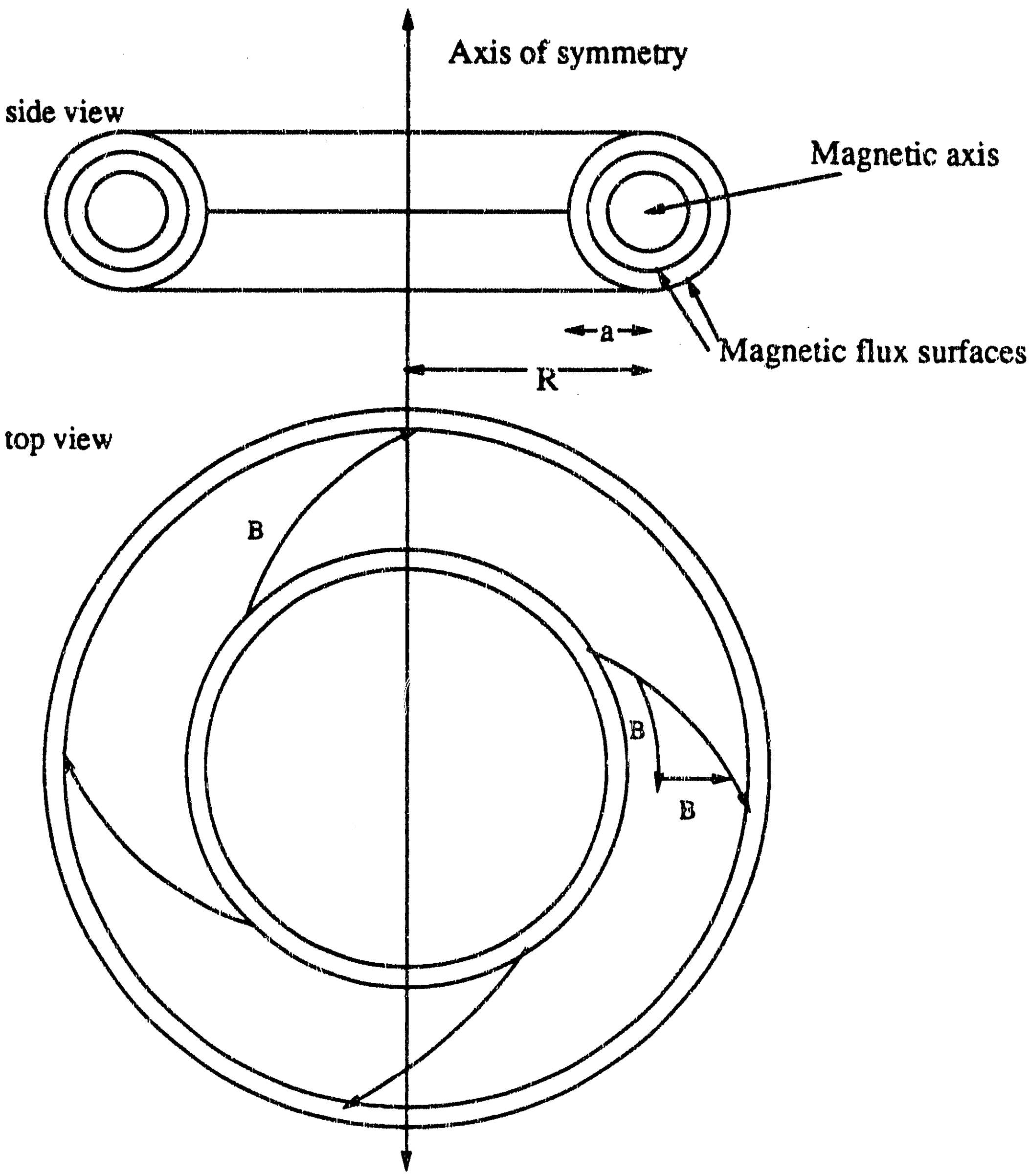

Figure 1. Tokamak Geometry

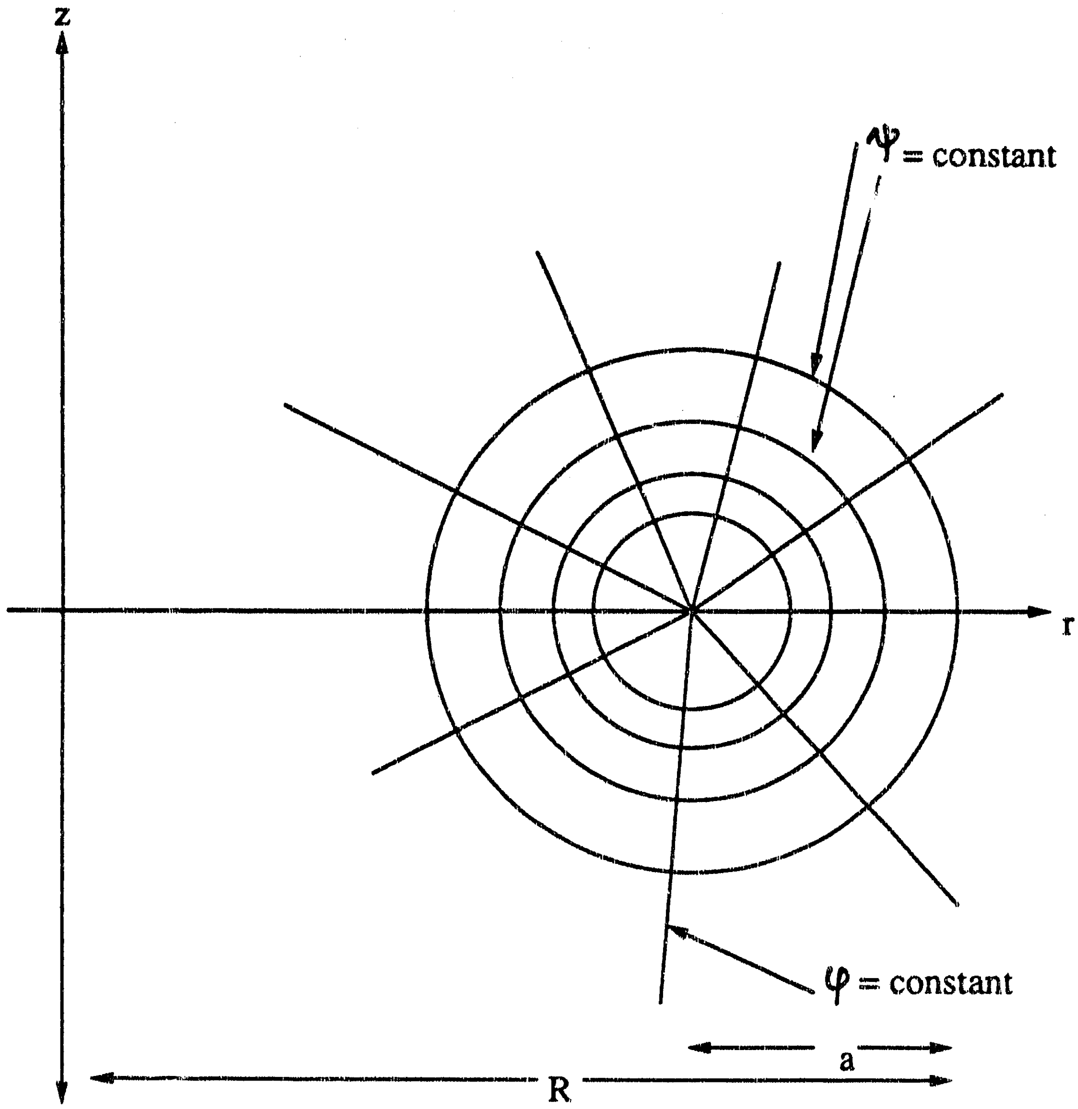

Figure 2. Flux Coordinates

BIBLIOGRAPHY

[1] H. Grad, Comm. Pure Appl. Math. **2**, 331 (1949).

[2] F. Hinton, in *Plasma Physics Handbook*, Vol. 1, eds. A.A. Galeev and R.N. Sudan, North-Holland Publishing Company, New York, 1983

[3] H. Grad, *Principles of the Kinetic Theory of Gases. Handbuch der Physik*, Vol. **12**, Springer-Verlag , 1958

[4] I. I. Kolodner, Moment Descriptions of Gas Mixtures-I, Courant Institute of Mathematical Sciences, Report No. NYU-7980,September 1957, New York.

[5] B. Herdan and B. S. Lily, Rev. Mod. Phys. **32**, 731, (1960)

[6] F. L. Hinton and R. D. Hazeltine, Phys. Fluids **16** 1883 (1973)

[7] S. P. Hirshman and D. J. Sigmar, Nuclear Fusion **21** 1079 (1981)

[8] H. Weitzner and W. Kerner, Z. Naturforsch. **42 a** 1101 (1987)

[9] S. I. Braginski, in *Reviews of Plasma Physics*, Vol. 1, ed. by M. A. Leontovich, Consultants Bureau, New York 1965, 205.

[10] K. S. Riedel, Nuclear Fusion, **5**, 927 (1991)

[11] V. D. Shafranov, Sov. Phys. JETP, **8**, 494 (1958)

END

DATE FILMED

10 / 14 / 92