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# A Simplification of Inclusion-Exclusion via Minimal Complexes

Andrew J. Brandt
abrandt@pugetsound.edu

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Andrew Brandt - Mathematics ’18, Professor Courtney Thatcher - Advisor

University of Puget Sound - Department of Mathematics and Computer Science

UNIVERSITY of PUGET SOUND

## Introduction

The goal of this project was to find a set of requirements for planar graphs that would simplify the inclusion-exclusion principle calculations for counting problems, and to explore the relationships between sets and complexes in various examples to expand the technique to a larger set of examples.

## Our Work

After researching planar graphs and working through many examples, we discovered a set of requirements for a planar graph associated to a family of sets that allows for the simplification of the inclusion-exclusion principle. We then proceeded to research simplicial complexes and intersection complexes associated to families of sets to expand and improve upon this application. After defining some new complexes and collapses, we were able to relate the weighted Euler characteristic to the order of the union of the family of sets, and hence simplify the inclusion-exclusion principle in a different way. While working with the new complexes, we discovered that we could use the generalized complexes from the second theorem to improve upon the conditions of the first. The revised version of Theorem 1 is what appears on this poster.

## Definitions

- Simplicial Complex - A simplicial complex $K$ is a set of simplices (generalized triangles) that satisfies the following conditions; any face of a simplex from $K$ is also in $K$ and the intersection of any two simplices $\sigma_1, \sigma_2 \in K$ is either $\emptyset$ or a face of both $\sigma_1$ and $\sigma_2$
- Weighted Intersection Complex - For a family of sets $F = \{F_1, \ldots, F_n\}$, the weighted intersection complex is a simplicial complex with vertices corresponding to each set and an $m$-simplex whenever $m$ sets have nonempty intersection. The simplices are labeled by the elements they contain.
- Weighted Collapse - A weighted collapse of $K$ is a complex $K'$ obtained from $K$ by the removal of a $n$-cell $\sigma \in K$ with labeling $\alpha$ along with the removal of an $(n-1)$-face of $\sigma$ with the same labeling.
- Minimal Intersection Complex - A minimal complex $M$ of a complex $K$ is the complex, derived from a series of weighted collapses, where no $n$-cell has the same labeling as an $(n-1)$-cell on the boundary of the $n$-cell.
- Weighted $\chi$ - The weighted Euler characteristic of a graph is equal to

$$\Sigma_{i=0}^{n}(-1)^i S_i$$

where $S_i$ is the sum of the order of the $i$-cells.

## Theorem 1

Suppose sets $F_1, F_2, \ldots, F_n$ can be represented as the vertices of a planar graph $G$. Let $C$ be the complex obtained from $G$ by labeling each nontrivial edge $e = \{F_i, F_j\}$ for some $i, j \in \{1, \ldots, n\}$ by the elements in the intersection $F_i \cap F_j$ and labeling each nontrivial interior face $f = \{F_{i_1}, \ldots, F_{i_k}\}$ for some $i_j \in \{1, \ldots, n\}$ whenever there is nontrivial intersection between the vertex sets $\bigcap_{i_j} F_{i_j} \neq \emptyset$. For each $x \in \bigcup_{i=1}^{n} F_i$, let $C_x$ be the subcomplex of vertices, edges, and faces that contain $x$. If $C_x$ is contractible $\forall x$, then

$$|\bigcup_{i=1}^{n} F_i| = \Sigma_{i=1}^{n}|v_i| - \Sigma_{i,j}|e_{i,j}| + \Sigma|f_l|$$

where $|v_i| = |F_i|, |e_{i,j}| = |F_i \cap F_j|$, and $|f_l| = \bigcap_{i_j} F_{i_j}$.

## Theorem 2

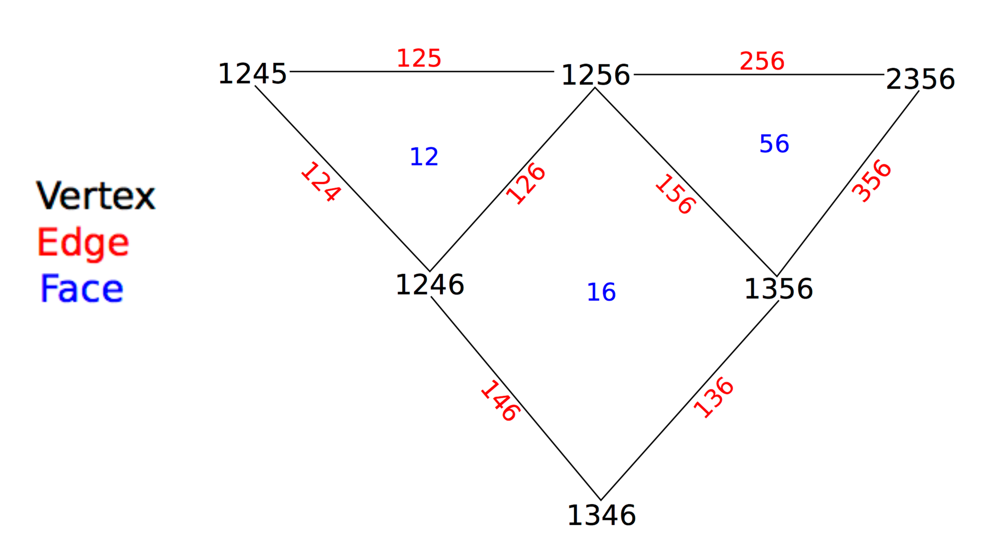
Let $F$ be a family of sets and let $M$ be a minimal complex from the weighted intersection complex $K$ created from $F$. The weighted Euler characteristic of $M$ is equal to the order of the union of the sets in $F$.

## Remark

While Theorem 1 can only be used for planar graphs, Theorem 2 involves the building and simplification of weighted intersection complexes to solve inclusion-exclusion problems. These two theorems involve different approaches with distinct requirements to solve the same problem.
Note that for a family of sets $F = F_1, \ldots, F_n$, if $F_1 \supseteq F_2$, the minimal complex will not contain any cells that have $F_2$ as a vertex. By extension, if $F_i \subseteq F_1 \forall i \neq 1$, then $|F_1| = |\cup F_i|$ and the minimal complex is simply a single vertex.

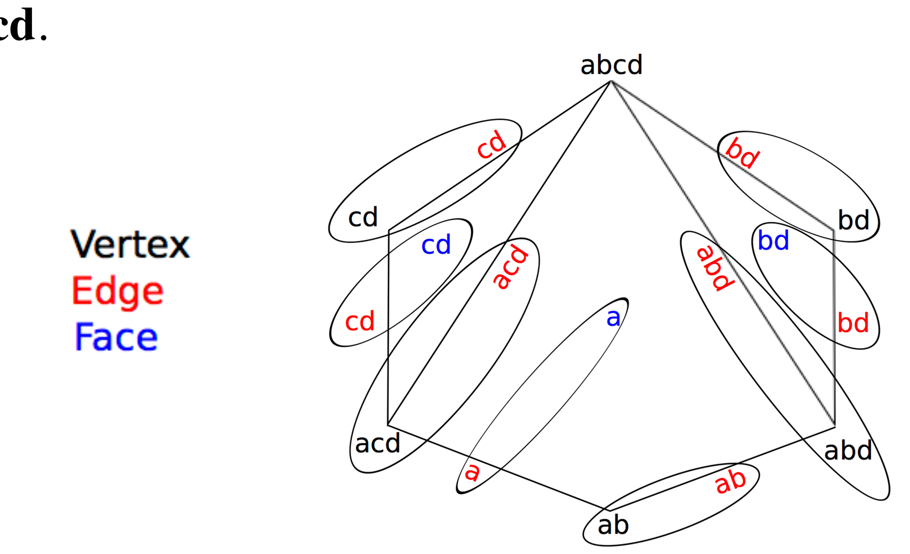
## Example 1

Question: If we toss $n$ six-sided dice, what is the probability that we obtain a three-length straight; i.e., at least one of 123, 234, 345, 456?
Begin with the formula $P = \frac{6^n - \bar{S}}{6^n}$. We calculate the size of the complement $\bar{S}$ (the number of ways to throw $n$ dice so that we do not obtain a three-length straight). There are 6 subsets of size 4 that do not result in a straight; 1245, 1246, 1256, 1346, 1356, 2356. Each of these subsets represents a combination of rolls that is comprised of only the four numbers in the subset. Using either theorem, we get $|\bar{S}| = 6 * 4^n - 8 * 3^n + 3 * 2^n$. The calculation for $P$ follows easily. Shown below is the minimal complex for the family of sets.




## Example 2

Shown below is a fully collapsible weighted intersection complex with ovals surrounding the components involved in each collapse. The minimal complex is the vertex **abcd**.

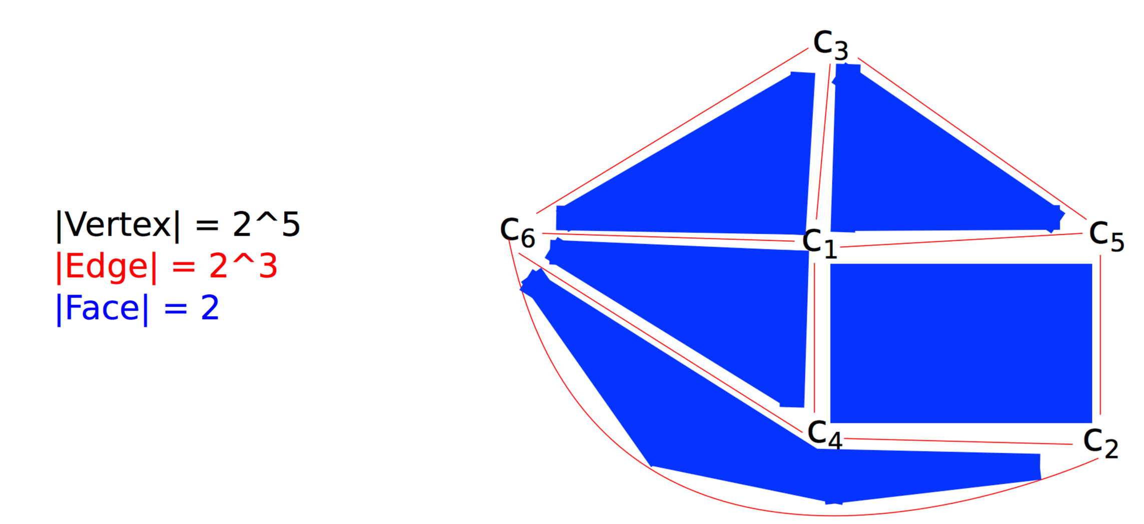



## Example 3

Question: How many ways can you tile a row of 7 black or white tiles with no black square preceded by a white square? Let $c_1$ be the set of all possible tilings with the first tile white and the second tile black. Repeat with $c_2$ through $c_6$. Since the order of each intersection is 2 raised to the number of tiles with undetermined value, each $c_i$ has weight of $2^5$. Pairwise intersections have weight of $2^3$ and threewise intersections have weight of 2.

$$\bar{N} = 2^7 - 6 * 2^5 + 10 * 2^3 - 4 * 2 = 8$$

Notice that there are no possible collapses, thus the weighted intersection complex is the same as the minimal complex




## Next Steps

Considering how well this topic relates well to other subject areas of mathematics, there are many possibilities for moving forward with further research. Possible topics to relate with our work include Betti numbers, homology theory, and the improved Bonferroni inequalities. We have done some preliminary investigation into using abstract tubes and believe they can be used as a different framework for Theorem 1.

## Acknowledgements

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