10-1-1988

The Classical Limit Of Quantum-mechanics From Fermat Principle And The Debroglie Relation

Mark Rosenquist
Shell Oil Company, Rocky Mountain Division, P. O. Box 831, Houston, Texas 77001

James C. Evans
University of Puget Sound, jcevans@pugetsound.edu

Citation

Rosenquist, M., and James C. Evans. 1988. "The Classical Limit of Quantum-Mechanics from Fermat Principle and the Debroglie Relation." American Journal Of Physics 56(10): 881-882.

AAPT PHYSICS EDUCATION
AMERICAN JOURNAL of PHYSICS

The classical limit of quantum mechanics from Fermat's principle and the de Broglie relation

Mark Rosenquist and James Evans

Citation: American Journal of Physics **56**, 881 (1988); doi: 10.1119/1.15763
View online: http://dx.doi.org/10.1119/1.15763
View Table of Contents: http://scitation.aip.org/content/aapt/journal/ajp/56/10?ver=pdfcov
Published by the American Association of Physics Teachers

The classical limit of quantum mechanics from Fermat's principle and the de Broglie relation

Mark Rosenquist
Shell Oil Company, Rocky Mountain Division, P. O. Box 831, Houston, Texas 77001

James Evans
Department of Physics, University of Puget Sound, Tacoma, Washington 98416

(Received 2 September 1987; accepted for publication 5 December 1987)

The classical limit of quantum mechanics may be obtained, in a much simpler fashion than usual, by applying the de Broglie relation to Fermat's principle.

Consider a wave disturbance in an isotropic medium. Assume a time-independent situation, so that the wave-number k for a particular frequency depends only upon the position $\mathbf{r}$. In the "geometrical optics limit," when the wavelength is much shorter than any other significant distance, the disturbance may be considered to propagate along well-defined rays. In this situation, the principle of stationary phase may be used to derive a differential equation governing the rays.

Refer to Fig. 1. A wave disturbance originates at a source and propagates eventually to a receiver along a path we shall call the "true path." The phase difference between the source and the receiver is

$$\int k(\mathbf{r})|d\mathbf{r}|,$$

where $|d\mathbf{r}|$ is an element of length along the path and where the integration is carried out from the source to the receiver. The principle of stationary phase then requires that this integral be stationary against small variations in the shape of the path,

$$\delta \int k(\mathbf{r})|d\mathbf{r}| = 0. \qquad (1)$$

Physically, this condition originates from the requirement that there be constructive interference among a bundle of virtual paths closely neighboring the true path. In accordance with contemporary usage, we shall refer to Eq. (1) as Fermat's principle, although Fermat certainly would not have recognized it in this form.[1]

Equation (1) applies quite generally to all wave phenomena obeying the superposition principle, and thus must apply to the matter waves of quantum mechanics. That is, if we pass to the short-wavelength limit of quantum mechanics, the trajectory of a material particle (analogous to an optical ray) is determined by Eq. (1). The general equation can be applied specifically to matter waves by the use of the de Broglie relation,

$$p = mv = \hbar k. \qquad (2)$$

As usual, p denotes the momentum of the classical particle, m the mass, and v the speed, while $\hbar$ is Planck's quantum of action divided by 2π. Thus we have, by substitution,

$$\delta \int v(\mathbf{r})|d\mathbf{r}| = 0. \qquad (3)$$

The variational principle of Eq. (3) is known as Maupertuis' principle and represents the first historical statement of a least-action principle.[2] Maupertuis' principle can be used in classical mechanics for determining the trajectory of a particle whose speed is a function of position only. In other words, Eq. (3) applies to time-independent situations in which the force is derivable from a potential and in which the principle of conservation of energy therefore holds. Maupertuis' principle is, of course, neither the most familiar nor the most useful expression of the laws of mechanics. However, the transition to this formulation of mechanics is accomplished in a single step. Moreover, it is immediately clear that this variational principle of classical mechanics rests upon the underlying wavelike nature of material particles.

A more familiar formulation of mechanics can be obtained by carrying out the variational calculation of Eq. (1). To do this, we imagine the position of the wave disturbance along the true path to be a function of the time t. As t increases, the point specified by $\mathbf{r}(t)$ moves smoothly along the path. (See Fig. 1.) The integral is varied by integrating along a slightly different path. For each value of t, the varied path differs from the true path by an infinitesimal variation $\mathbf{e}(t)$. Thus Eq. (1) becomes

$$\delta \int k|\mathbf{r}'|dt = 0,$$

where the prime denotes differentiation with respect to t. Carrying out the variation on both factors of the integrand, we have

$$\int [(\delta k)|\mathbf{r}'| + k(\delta|\mathbf{r}'|)]dt = 0. \qquad (4)$$

Now, to the first order in the variation,

$$\delta k = \nabla k \cdot \mathbf{e}, \qquad (5)$$

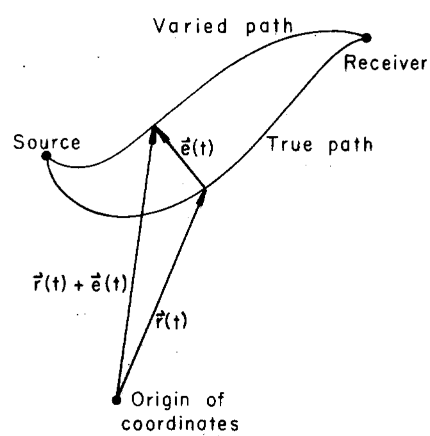

Fig. 1. Variation of the path of integration.

and

$$\delta|\mathbf{r}'| = |\mathbf{r}' + \mathbf{e}'| - |\mathbf{r}'| \approx \mathbf{r}'\cdot\mathbf{e}'/|\mathbf{r}'|. \quad (6)$$

Upon substituting (5) and (6) into (4), we obtain

$$\int [\,|\mathbf{r}'|\nabla k\cdot\mathbf{e} + (k\mathbf{r}'/|\mathbf{r}'|)\cdot\mathbf{e}'\,]dt = 0.$$

When the expression involving $\mathbf{e}'$ is integrated by parts, the integrated term vanishes because the variation $\mathbf{e}$ is zero at the source and the receiver. We have then

$$\int \left[|\mathbf{r}'|\nabla k - \frac{d}{dt}\left(\frac{k\mathbf{r}'}{|\mathbf{r}'|}\right)\right]\cdot\mathbf{e}\, dt = 0.$$

Since this equation holds for any infinitesimal $\mathbf{e}$, the expression in square brackets must itself be identically zero. This gives us a differential equation governing the motion of the disturbance along the ray[3]:

$$|\mathbf{r}'|\nabla k = \frac{d}{dt}\left(\frac{k\mathbf{r}'}{|\mathbf{r}'|}\right). \quad (7)$$

As before, the transition from a general statement about wave phenomena in the short wavelength limit to a statement about matter waves is accomplished by applying the de Broglie relation (2). Thus if in Eq. (7) we replace k by $mv/\hbar$, we obtain

$$\nabla\left(\frac{1}{2}mv^2\right) = m\frac{d\mathbf{v}}{dt}. \quad (8)$$

Since we have supposed k to be a function of position alone, the speed v is similarly a function of position alone. We may then write $\frac{1}{2}mv^2 = E - U(\mathbf{r})$, where E is the total energy and U is the potential energy. Equation (8) then assumes the form

$$m\frac{d\mathbf{v}}{dt} = -\nabla U. \quad (9)$$

Thus Newton's law of motion results from the application of the de Broglie relation to the equation (7) governing the ray in the short wavelength limit.

It is worth pointing out that the relativistic form of Newton's law of motion is obtained if, instead of Eq. (2), we use the relativistic form of the de Broglie relation,

$$\hbar k = mv\gamma, \quad (10)$$

where m is the rest mass and $\gamma = (1 - v^2/c^2)^{-1/2}$, with c being the speed of light in vacuum. Use of this expression for k in Eq. (7) gives

$$mv\,\nabla(v\gamma) = \frac{d}{dt}(m\mathbf{v}\gamma). \quad (11)$$

Now,

$$v\,\nabla(v\gamma) = v^2\,\nabla\gamma + v\gamma\,\nabla v = c^2\,\nabla\gamma + (v^2 - c^2)\nabla\gamma + v\gamma\,\nabla v,$$

but the last two terms on the right-hand side of this equation cancel identically, so we have

$$v\,\nabla(v\gamma) = c^2\,\nabla\gamma.$$

Use of this result in Eq. (11) gives

$$\nabla(mc^2\gamma) = \frac{d}{dt}(m\mathbf{v}\gamma), \quad (12)$$

the relativistic form of Eq. (8). The expression in parentheses on the left side of Eq. (12) is the kinetic energy plus rest mass energy of the particle. As the kinetic energy is supposed to be a function of position alone, the total energy E is conserved and the expression in parentheses may be replaced by $E - U(\mathbf{r})$. Thus Eq. (12) becomes

$$\frac{d}{dt}(m\mathbf{v}\gamma) = -\nabla U, \quad (13)$$

the relativistic form of Newton's law of motion.

This approach to the short-wavelength limit of quantum mechanics has several advantages for undergraduate instruction. It does not involve a rarely used formulation of classical mechanics such as the Hamiltonian–Jacobi formulation, nor does it involve higher level quantum concepts such as the probability current density.[4] Instead, it relies only on the short-wavelength behavior of waves in general and the simple physics of the de Broglie relation. Because Fermat's principle does not depend upon the particular form of the wave equation, it yields the relativistic form of classical mechanics as readily as the Newtonian. This approach has, moreover, a certain historical legitimacy. The analogy between the principles of Fermat and Maupertuis played an important part in de Broglie's thought as he worked toward the momentum–wavelength relation.[5]

[1]If we suppose that the speed of light depends only upon position and restrict ourselves to a single frequency (or to a nondispersive medium), a simple manipulation reduces Eq. (1) to $\delta\int c^{-1}n(\mathbf{r})|d\mathbf{r}| = 0$, where c is the speed of light in vacuum and n is the index of refraction; i.e., a statement that the time of travel is stationary against small variations in the shape of the path. In this form, the variational principle is somewhat closer to the principle of least time enunciated by Fermat.

[2]For a discussion of Maupertuis' principle and its elaboration by Euler and Lagrange, see W. Yourgrau and S. Mandelstam, *Variation Principles in Dynamics and Quantum Theory* (Dover, New York, 1979), 3rd ed., pp. 19–32.

[3]Equation (7) may be generalized somewhat: The four differentiations with respect to t may be replaced by differentiations with respect to any other parameter (e.g., the arc length) that increases smoothly as one moves along the path of integration. The more general form of (7) results from the adoption of some variable other than t as the variable of integration in Eq. (4).

As will be shown, the application of Eq. (7) to the matter waves of quantum mechanics results in Newton's law of motion. But Eq. (7) applies quite generally to any linear wave phenomenon in the short-wavelength limit: Eq. (7) may also be applied, for instance, to ordinary light optics to obtain the shape of the optical ray in a region of varying index of refraction. If the independent variable is chosen appropriately, the differential equation governing the shape of the light ray may also be cast into the form of Newton's law of motion. See J. Evans and M. Rosenquist, Am. J. Phys. **54**, 876 (1986).

[4]For a typical demonstration of the classical limit of quantum mechanics making use of the Hamilton–Jacobi equation, see L. I. Schiff, *Quantum Mechanics* (McGraw–Hill, New York, 1968), 3rd ed., pp. 269–270. For a typical demonstration based on Ehrenfest's theorem and using the probability current density, see E. Merzbacher, *Quantum Mechanics* (Wiley, New York, 1962), pp. 41–42.

[5]L. de Broglie, *Recherches sur la théorie des quanta* [The thesis of 1924] (Masson, Paris, 1963), pp. 32–49. Summaries of de Broglie's use of the principles of Fermat and Maupertuis are given in the following two works: Ref. 2, pp. 116–118 and T.-Y. Wu, *Quantum Mechanics* (World Scientific, Singapore, 1986), pp. 131–135.