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THE CENTER OF SOME BRAID GROUPS AND THE FARRELL COHOMOLOGY OF CERTAIN PURE MAPPING CLASS GROUPS

YU QING CHEN, HENRY H. GLOVER AND CRAIG A. JENSEN

ABSTRACT. In this paper we first show that many braid groups of low genus surfaces have their centers as direct factors. We then give a description of centralizers and normalizers of prime order elements in pure mapping class groups of surfaces with spherical quotients using automorphism groups of fundamental groups of the quotient surfaces. As an application, we use these to show that the p-primary part of the Farrell cohomology groups of certain mapping class groups are elementary abelian groups. At the end we compute the p-primary part of the Farrell cohomology of a few pure mapping class groups.

1. Introduction

Braid groups were first introduced by Artin [1, 2, 3] for the purpose of studying knots and links. A geometric braid on n strings can be defined as follow. Let $E = \mathbb{E}^2 \times I$, where $\mathbb{E}^2$ is the Euclidean plane and I is the unit interval $[0, 1]$. Let $P_i = (i, 0, 1) \in \mathbb{E}^2 \times \{1\}$ and $P_i' = (i, 0, 0) \in \mathbb{E}^2 \times \{0\}$ for $i = 1, 2, \cdots, n$. An n-string $s = \{s_1, s_2, \cdots, s_n\}$ in E is a set of maps $s_i : [0, 1] \to E$, $i = 1, 2, \cdots, n$, satisfying

(I) $s_i(0) = P_i$ and $s_i(1) = P_{\sigma(i)}'$, $i = 1, 2, \cdots, n$, where σ is a permutation of the set $\{1, 2, \cdots, n\}$, and

(II) each plane $\mathbb{E}^2 \times \{t\}$ in E, $0 \leqslant t \leqslant 1$, intersects the strings $\bigcup_{i=1}^{n} s_i([0, 1])$ in exactly n points.

Two n-strings s and s' given by the same permutation σ of $\{1, 2, \cdots, n\}$ are said to be equivalent if there is a homotopy $H = (H_1, H_2, \cdots, H_n)$, where H_i, $i = 1, 2, \cdots, n$, are maps

$$H_i : [0, 1] \times [0, 1] \to E,$$

such that

$$\begin{cases} H_i(u, 0) = s_i(u), \\ H_i(u, 1) = s_i'(u), \\ H_i(0, v) = P_i, \\ H_i(1, v) = P_{\sigma(i)}' \end{cases}$$

and such that for every $s \in [0, 1]$, $H_v = \{H_1(\cdot, v), H_2(\cdot, v), \cdots, H_n(\cdot, v)\}$ is an n-string. Then a geometric braid on n strings is an equivalence class of n-strings. Geometric braids can be presented pictorially by two horizontal lines with the points P_i's on the top line and P_i''s on the bottom line and strings going from P_i's to P_i''s. The set of braids on n strings is denoted by B_n. We can define a product of two braids in B_n using these pictures. Let b_1 and b_2 be two geometric braids on n strings. The picture of $b = b_1 b_2$ is obtained as follow: placing the picture of b_1 above b_2 and identifying the bottom line of b_1 with the top line of b_2 in such a way that the points $P_1', P_2' \cdots, P_n'$ in b_1 match $P_1, P_2, \cdots, P_n$ in b_2. Then removing the common line between b_1 and b_2. It is easy to check that this product is well defined. Therefore B_n is a group which is called Artin braid group on n strings. A braid with the trivial permutation σ is called a pure braid. The set of pure braids forms a normal subgroup of B_n and is denoted by P_n. It is called the pure Artin braid group. There is a exact sequence

$$(1) \qquad\qquad 1 \to P_n \to B_n \to \Sigma_n \to 1,$$

where Σ_n is the symmetry group on n symbols. Given a subgroup G of Σ_n, we define G-Artin braid group B_n^G on n strings to be the subgroup of B_n such that for all braids in B_n^G, the permutations σ in (I) are from G. Clearly, $P_n \leqslant B_n^G \leqslant B_n$ and $B_n^{\Sigma_n} = B_n$ and $B_n^{\{1\}^n} = P_n$, where $\{1\}^k$ is the trivial subgroup of Σ_k. The group B_n^G is simply the pre-image of G in B_n of the exact sequence (1) and one has an exact sequence

$$1 \to P_n \to B_n^G \to G \to 1.$$

Another family of groups that we are interested in are the mapping class groups. They are closely related to braid groups. Let S_g^n denote the orientable surface of genus g with n punctures. The mapping class group Γ_g^n of S_g^n is defined to be the isotopy classes of orientation preserving homeomorphisms from S_g^n to itself. It is the discrete group of connected components of the group of orientation preserving

homeomorphisms of S_g^n. For other equivalent definitions of Γ_g^n and the properties of Γ_g^n with $n = 0$, we refer the reader to the survey paper of Mislin [16]. Each Γ_g^n contains a normal subgroup consisting of isotopy classes of orientation preserving homeomorphisms of S_g^n which point-wise fix the punctures. This subgroup is called pure mapping class group of S_g^n and is denoted by $P\Gamma_g^n$. Similar to the braid groups, we have the exact sequence

(2) $$1 \to P\Gamma_g^n \to \Gamma_g^n \to \Sigma_n \to 1.$$

For each subgroup G of Σ_n, we can also define the G-mapping class group of S_g^n, $\Gamma_g^{n,G}$, to be the pre-image of G in Γ_g^n of the exact sequence (2). Then one has $P\Gamma_g^n \leqslant \Gamma_g^{n,G} \leqslant \Gamma_g^n$, and $\Gamma_g^{n,\Sigma_n} = \Gamma_g^n$ and $\Gamma_g^{n,\{1\}^n} = P\Gamma_g^n$, and the exact sequence

$$1 \to P\Gamma_g^n \to \Gamma_g^{n,G} \to G \to 1.$$

In [13], Lu proved that $P\Gamma_g^n$ has periodic cohomology with period 2 for all $g \geqslant 1$ and $n \geqslant 1$, i.e. $\widehat{H}^{i+2}(P\Gamma_g^n, \mathbb{Z}) \cong \widehat{H}^i(P\Gamma_g^n, \mathbb{Z})$. She also computed the p-primary part of the Farrell cohomology of punctured pure mapping class group $\widehat{H}^*(P\Gamma_{n(p-1)/2}^i, \mathbb{Z})_{(p)}$ for some small values of n ($n \leqslant 3$) as well as $\widehat{H}^*(P\Gamma_g^n, \mathbb{Z})$ for small values of g ($g \leqslant 3$)[13, 14, 15].

In this paper, we first show in Theorems 3.3, 3.10 and 3.11 that for many braid groups of S_0^1, S_0^0 and S_1^0 (see the definition of these groups in Section 2), their centers are actually direct factors of these groups. We then give in Theorem 5.6 an algebraic description of centralizers and normalizers of elements of order p in $P\Gamma_g^m$ when the quotient surfaces are punctured 2-spheres. Combining the splitting of braid groups and the alternative description of normalizers, we are able to obtain structural information about the p-primary part of the Farrell cohomology of $P\Gamma_{n(p-1)/2}^m$ for certain values of n and m in Theorem 5.4. At the end of the of the paper in Theorem 5.8, we give an explicit computation of $\widehat{H}^i(P\Gamma_{n(p-1)/2}^{n+1}, \mathbb{Z})_{(p)}$, $\widehat{H}^i(P\Gamma_{n(p-1)/2}^{n+2}, \mathbb{Z})_{(p)}$, $\widehat{H}^i(P\Gamma_n^{2n+1}, \mathbb{Z})_{(2)}$ and $\widehat{H}^i(P\Gamma_n^{2n+2}, \mathbb{Z})_{(2)}$.

The rest of the paper is organized as follow. In section 2 we review some well known facts about braid groups and mapping class groups. In section 3 we prove that the braid groups of S_0^1, S_0^0 and S_1^0 are the direct product of their centers with other braid groups or mapping class groups. In section 4, we compute centralizers and normalizers of elements of prime order in pure mapping class groups, provided that the quotient surfaces are punctured 2-spheres. In the last section, we study the p-primary part of the Farrell cohomology of pure mapping class groups and compute the Farrell cohomology of the pure mapping class groups immediately above.

2. Braid groups and mapping class groups of surfaces

This section contains the descriptions of braid groups and mapping class groups of surfaces which are different from the ones given in the introduction. The relations between pure braid groups and pure mapping class groups of the punctured 2-spheres are discussed at the end of this section.

The use of configuration spaces to describe braid groups seems to be suggested by Fox. Let X be a topological space. Its n-fold configuration space $F_n(X)$ for

$n \geq 0$ is defined by

$$F_n(X) = \{(x_1, x_2, \cdots, x_n) \in \prod_{i=1}^{n} X \mid x_i \neq x_j \text{ for all } i \neq j\}$$

with the convention that when $n = 0$, the space $F_0(X)$ consists of a single point and when $n = 1$, the space $F_1(X) = X$ for any space X. Each subgroup G of Σ_n has a natural free action on $F_n(X)$ by permuting the coordinates. We denote by $F_n^G(X)$ the quotient space $F_n(X)/G$, i.e.

$$F_n^G(X) = \{(x_1, x_2, \cdots, x_n)^G \mid x_i \neq x_j \text{ for all } i \neq j\},$$

where $(x_1, x_2, \cdots, x_n)^G$ is the G-orbit of $(x_1, x_2, \cdots, x_n)$ in $F_n(X)$. Then the G-braid group of X on n strings is defined to be the fundamental group $\pi_1(F_n^G(X))$ of $F_n^G(X)$. When $G = \{1\}^n$, $F_n^{\{1\}^n}(X) = F_n(X)$ and $\pi_1(F_n(X))$ is called the pure braid group of X on n strings. When $G = \Sigma_n$, $\pi_1(F_n^{\Sigma_n}(X))$ is called full braid group of X on n strings. Obviously, the natural projection from $F_n(X)$ to $F_n^G(X) = F_n(X)/G$ is a regular covering with deck transformation group G. Thus we have an exact sequence of groups

$$1 \to \pi_1(F_n(X)) \to \pi_1(F_n^G(X)) \to G \to 1.$$

If the space X is the Euclidean plane, or equivalently, the 1-punctured 2-sphere S_0^1, then the braid group $\pi_1(F_n^G(S_0^1))$ of S_0^1 is precisely the G-Artin braid group B_n^G. The subgroup $\pi_1(F_n(S_0^1))$ is the pure Artin braid group P_n. For the connection between the two definitions of the Artin braid groups we refer the reader to [6]. For a fixed set of distinguished points $Q_m = \{x_1, x_2, \cdots, x_m\}$ of a topological space X, we define

$$F_{m,n}(X) = F_n(X - Q_m)$$

and for any subgroup G of Σ_n we define

$$F_{m,n}^G(X) = F_n^G(X - Q_m).$$

Note that the topological type of the configuration space $F_{m,n}(X)$ is independent of the choice of the particular points in the set Q_m, since one may always find an isotopy of X which deforms one such set to another. Let $n = s + t$, $G = G_s \times G_t$ for some subgroups G_s in Σ_s and G_t in Σ_t. The following theorem, which is a slight generalization of a version in [11], describes a link among various configuration spaces $F_{m,n}^G(X)$ for a given space X.

Theorem 2.1. [11, Fadell and Neuwirth] *Let* $\pi : F_{m,s+t}^{G_s \times F_t}(X) \to F_{m,s}^{G_s}(X)$ *be the projection defined by*

$$\pi((x_1, x_2, \cdots, x_{s+t})^{G_s \times G_t}) = (x_1, x_2, \cdots, x_s)^{G_s}.$$

Then π *is a locally trivial fibration with fibre* $F_{m+s,t}^{G_t}(X)$.

Remark 2.2. The fibration is usually not globally trivial.

We will use this theorem to establish a connection between $\pi_1(F_n^G(S_0^1))$ and $\pi_1(F_n^G(S_0^0))$.

In [1], Artin gave an algebraic description of the Artin braid groups which is now called the Artin representation.

Theorem 2.3. [1, Artin] *Let* $\mathrm{Aut}(F_n)$ *denote the group of automorphisms of the free group with generators* $x_1, x_2, \cdots x_n$. *The Artin braid group*

$$B_n \cong \{\gamma \in \mathrm{Aut}(F_n) \mid \gamma(x_i) \text{ is conjugate to } x_{\sigma(i)} \text{ for some permutation } \sigma \text{ of}$$
$$\{1, 2, \cdots, n\} \text{ and for all } i = 1, 2, \cdots, n \text{ and } \gamma(x_1 x_2 \cdots x_n) = x_1 x_2 \cdots x_n\}.$$

It follows easily from Theorem 2.3 that for any subgroup G of Σ_n, the G-Artin braid group

$$B_n^G \cong \{\gamma \in \mathrm{Aut}(F_n) \mid \gamma(x_i) \text{ is conjugate to } x_{\sigma(i)} \text{ for some } \sigma \in G$$
$$\text{and for all } i = 1, 2, \cdots, n \text{ and } \gamma(x_1 x_2 \cdots x_n) = x_1 x_2 \cdots x_n\}.$$

The centers of G-Artin braid groups are infinite cyclic groups. Geometrically, these groups are generated by the braids with a full twist. If we use the Artin representation to describe these centers, they are precisely the intersection of the braid groups with the inner automorphism groups of those free groups.

Mapping class groups $\Gamma_g^{n,G}$ introduced in the previous section can also be given an algebraic description using automorphisms of fundamental groups of the surfaces S_g^n as in [10, 19]. The fundamental group of S_g^n admits a presentation

$$\pi_1(S_g^n) = \langle x_1, x_2, \cdots, x_n, a_1, b_1, a_2, b_2, \cdots, a_g, b_g \mid x_1 x_2 \cdots x_n \prod_{i=1}^{n} [a_i, b_i] = 1 \rangle,$$

where $x_1, x_2, \cdots, x_n$ are the loops representing those punctures which are oriented in a consistent manner. For a subgroup G of Σ_n, we define

$$\mathrm{Aut}_+^G(\pi_1(S_g^n)) = \{\gamma \in \mathrm{Aut}(\pi_1(S_g^n)) \mid \gamma(x_i) \text{ is conjugate to } x_{\sigma(i)}$$
$$\text{for some } \sigma \in G \text{ and for all } i = 1, 2, \cdots, n\}.$$

Then the G-mapping class group of S_g^n

$$\Gamma_g^{n,G} \cong \mathrm{Aut}_+^G(\pi_1(S_g^n))/\mathrm{Inn}(\pi_1(S_g^n)),$$

where $\mathrm{Inn}(\pi_1(S_g^n))$ is the inner automorphism group of $\pi_1(S_g^n)$.

If we set $g = 0$, by comparing the algebraic descriptions of B_n^G and $\Gamma_0^{n,G}$, it is obvious that $B_n^G/Z(B_n^G) \cong \Gamma_0^{n+1, G \times \{1\}}$, where $Z(B_n^G)$ is the center of B_n^G.

We end this section with the following proposition that summarizes the known relation between the braid groups of S_0^1 and S_0^0 and the mapping class groups of the punctured 2-sphere. Some version of this can be found in [6, 11].

Proposition 2.4.   (1) *The center* $Z(B_n^G)$ *of* B_n^G *is an infinite cyclic group and* $B_n^G/Z(B_n^G) \cong \Gamma_0^{n+1, G \times \{1\}}$;
  (2) *the center* $Z(\pi_1(F_n^G(S_0)))$ *of* $\pi_1(F_n^G(S_0))$ *is a cyclic group of order* 2 *and* $\pi_1(F_n^G(S_0))/Z(\pi_1(F_n^G(S_0))) \cong \Gamma_0^{n,G}$.

3. The center of braid groups

In this section we will show that the center of braid groups of S_0^1, S_0^0, S_1^0 are direct factors of these groups. Let us recall a few more well known facts about the braid groups. The configuration space $F_n(S_0^1)$, which we used to define the pure Artin group P_n, is an Eilenberg-MacLane space $K(P_n, 1)$ of P_n since $\pi_i(F_n(S_0^1)) = 0$ for $i \geqslant 2$ (see [6]). Also since $F_n(S_0^1)$ is a finite cover of $F_n^G(S_0^1)$ for any subgroup G of Σ_n, one has $\pi_i(F_n^G(S_0^1)) = 0$ for $i \geqslant 2$. This implies that B_n^G is torsion free because it has finite cohomological dimension. In fact, If S is a surface with $\pi_2(S) = 1$, i.e.

$S \neq S_0^0$, the space $F_n^G(S)$ is an Eilenberg-MacLane space of $\pi_1(F_n^G(S))$ for any n and any subgroup G of Σ_n. Hence these braid groups are of finite cohomological dimension and are torsion free. Also from the geometric definition of P_n, there is an exact sequence

$$1 \to F_{n-1} \to P_n \to P_{n-1} \to 1,$$

where the projection $P_n \to P_{n-1}$ is given by removing a string from P_n and the kernel of this projection is isomorphic to the free group F_{n-1} on $n-1$ generators. The above exact sequence splits. Therefore

$$P_n \cong F_{n-1} \rtimes P_{n-1},$$

where the action of P_{n-1} on F_{n-1} is given by the Artin representation. For example, $P_3 \cong F_2 \rtimes \mathbb{Z}$ and the action of $\mathbb{Z}$ on F_2 is given by an inner automorphism of F_2.

We now prove a simple lemma concerning the configuration spaces of topological groups.

Lemma 3.1. *If K is a topological group and G is a subgroup of Σ_n such that $G = H \times \{1\}$, where H is a subgroup of Σ_{n-1}, then*

$$F_n^G(K) \cong F_{n-1}^H(K - \{\mathbf{1}\}) \times K,$$

where $\mathbf{1}$ is the identity element of K.

Proof. We define

$$u : F_n^G(K) \to F_{n-1}^H(K - \{\mathbf{1}\}) \times K,$$
$$u((k_1, k_2, \cdots, k_{n-1}, k_n)^G) = ((k_1 k_n^{-1}, k_2 k_n^{-1}, \cdots, k_{n-1} k_n^{-1})^H, k_n)$$

and

$$v : F_{n-1}^H(K - \{\mathbf{1}\}) \times K \to F_n^G(K),$$
$$v((k_1, k_2, \cdots, k_{n-1})^H, k_n) = (k_1 k_n, k_2 k_n, \cdots, k_{n-1} k_n, k_n)^G.$$

It is easy to verify that u and v are continuous and inverse to each other. $\square$

Remark 3.2. Comparing this Lemma with Theorem 2.1, we can see that the fibration in Theorem 2.1 with $m = 0$ and $s = 1$ is usually not a trivial fibration, while Lemma 3.1 provides a trivial one when the space has a group structure.

Theorem 3.3. *Let G be a subgroup of Σ_n such that $G = H \times \{1\}^2$, where H is a subgroup of Σ_{n-2}. The G-Artin braid group*

$$B_n^G \cong \pi_1(F_{n-2}^H(S_0^3)) \times \mathbb{Z} \cong \Gamma_0^{n+1, G \times \{1\}} \times \mathbb{Z}$$

for $n \geqslant 2$, where $G \times \{1\}$ is a subgroup of Σ_{n+1}.

Proof. We use the complex number plane $\mathbb{C}$ for the Euclidean plane $\mathbb{E}^2$ and denote $\mathbb{C} - \{0\}$ by $\mathbb{C}^*$, the non-zero complex numbers. Then we have

$$\begin{aligned}
B_n^G &= \pi_1(F_n^G(\mathbb{C})) \\
&= \pi_1(F_{n-1}^{H \times \{1\}}(\mathbb{C}^*) \times \mathbb{C}) \\
&= \pi_1(F_{n-1}^{H \times \{1\}}(\mathbb{C}^*)) \\
&= \pi_1(F_{n-2}^H(\mathbb{C}^* - \{1\}) \times \mathbb{C}^*) \\
&= \pi_1(F_{n-2}^H(\mathbb{C}^* - \{1\})) \times \pi_1(\mathbb{C}^*) \\
&= \pi_1(F_{n-2}^H(S_0^3)) \times \mathbb{Z}
\end{aligned}$$

Here we used Lemma 3.1 twice, once for the additive group of $\mathbb{C}$ and once for the multiplicative group $\mathbb{C}^*$. The factor $\pi_1(\mathbb{C}^*) \cong \mathbb{Z}$ is obviously contained in the center $Z(B_n^G)$ of B_n^G. In order to show that it is actually the full center, we only need that $Z(B_n^G) \cong \mathbb{Z}$. By statement (1) of Proposition 2.4, we have

$$B_n^G \cong \Gamma_0^{n+1, G \times \{1\}} \times \mathbb{Z}.$$

$\square$

Corollary 3.4.

$$P_n \cong P\Gamma_0^{n+1} \times \mathbb{Z}.$$

Remark 3.5. We can also use the fact that $P_n \cong F_{n-1} \rtimes P_{n-1}$ to prove Corollary 3.4 by induction.

Remark 3.6. For subgroups H of Σ_{n-1} which fixes no elements, the braid group $B_n^{H \times \{1\}}$ does not split.

Corollary 3.7. *The space $F_{n-3}^H(S_0^3)$ is an Eilenberg-MacLane space of $\Gamma_0^{n, H \times \{1\}^3}$ for any subgroup H of Σ_{n-3}. Therefore $\Gamma_0^{n, H \times \{1\}^3}$ has finite cohomological dimension and is torsion free.*

Remark 3.8. The corollary recovers the well known fact that $P\Gamma_0^4 \cong F_2$, the free group on two generators. Furthermore, combining this fact with Corollary 3.4, we obtain that $P_3 \cong F_2 \times \mathbb{Z}$. This description of P_3 is simpler than the one given before Lemma 3.1.

Remark 3.9. When H is a subgroup of Σ_{n-2}, the group $\Gamma_0^{n, H \times \{1\}^2}$ may contain torsion, and hence is not of finite cohomological dimension.

We now use Lemma 3.1 to give a similar decomposition of the G-braid group of the 2-sphere.

Theorem 3.10. *Let G be a subgroup of Σ_n such that $G = H \times \{1\}^3$ for some subgroup H of Σ_{n-3}. The G-braid group of the 2-sphere*

$$\pi_1(F_n^G(S_0^0)) \cong \Gamma_0^{n,G} \times \mathbb{Z}/2\mathbb{Z}$$

for $n \geqslant 3$.

Proof. If we apply Theorem 2.1 with $m = 0$, $s = 1$ and $X = S_0^0$, we obtain a fibration

$$\pi : F_n^G(S_0^0) \to S_0^0$$

with fibre $F_{1,n-1}^{H \times \{1\}^2}(S_0^0) = F_{n-1}^{H \times \{1\}^2}(S_0^1)$. Thus we have an exact sequence of the homotopy groups of the fibration

$$1 = \pi_2(F_{n-1}^{H \times \{1\}^2}(S_0^1)) \to \pi_2(F_n^G(S_0^0)) \to \pi_2(S_0^0) \to$$
$$\pi_1(F_{n-1}^{H \times \{1\}^2}(S_0^1)) \to \pi_1(F_n^G(S_0^0)) \to \pi_1(S_0^0) = 1 \quad .$$

Since $\pi_2(S_0^0) \cong \mathbb{Z}$ and, by Theorem 3.3, $\pi_1(F_{n-1}^{H \times \{1\}^2}(S_0^1)) \cong \Gamma_0^{n,G} \times \mathbb{Z}$, the above exact sequence is equivalent to

$$1 \to \pi_2(F_n^G(S_0^0)) \xrightarrow{i} \mathbb{Z} \xrightarrow{j} \Gamma_0^{n,G} \times \mathbb{Z} \xrightarrow{k} \pi_1(F_n^G(S_0^0)) \to 1.$$

The surjection k maps the center of $P\Gamma_0^n \times \mathbb{Z}$ into the center of $\pi_1(F_n(S_0^0))$ which is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Therefore we must have that $\pi_2(F_n^G(S_0^0)) = 1$ and $i(\mathbb{Z})$ is a subgroup of index 2 of the center of $\Gamma_0^{n,G} \times \mathbb{Z}$. From this one can easily see that

$$\pi_1(F_n^G(S_0^0)) \cong \Gamma_0^{n,G} \times \mathbb{Z}/2\mathbb{Z}.$$

$\square$

The last application of Lemma 3.1 is the following result whose proof is left to the reader.

Theorem 3.11. *Let G be a subgroup of Σ_n such that $G = H \times \{1\}$ for some subgroup H of Σ_{n-1}. The G-braid group of the torus*

$$\pi_1(F_n^G(S_1^0)) \cong \pi_1(F_{n-1}^H(S_1^1)) \times \mathbb{Z}^2$$

for $n \geqslant 1$.

4. The centralizer and normalizer of elements of prime order in mapping class groups

In this section we discuss the centralizer and normalizer of elements of prime order in mapping class groups Γ_g^n. Detailed discussion of conjugacy classes of prime order finite subgroup of $P\Gamma_g^n$ can be found in [13, 14]. Let us first recall two theorems from [7].

Theorem 4.1. *[7, Theorem 1] Let (p, S_g^m, S_h^n) be a regular covering space, either branched or unbranched, with a finite group of covering transformations and a finite number of branch points. Let the covering transformations leave each branch point fixed. In the case of a branched covering, assume that S_g^m is not homeomorphic to the closed sphere or closed torus. Let $g : S_g^m \to S_g^m$ be a fiber-preserving homeomorphism of S_g^m which is isotopic to the identity map. Then g is fiber-isotopic to the identity map.*

The term "fiber-isotopic" means the intermediate homeomorphisms between g and the identity map in the isotopy are all fiber-preserving.

Theorem 4.2. *[7, Theorem 3] Let $t, h \in \Gamma_g^0$ such that $t^k = 1$ and $hth^{-1} \in \langle t \rangle$. There are homeomorphisms $\tilde{t}$ and $\tilde{h}$ form S_g^0 to itself representing t and h respectively such that $\tilde{t}^k = 1$ and $\tilde{h}\tilde{t}\tilde{h}^{-1} \in \langle \tilde{t} \rangle$.*

The two theorems above imply that if $\alpha \in \Gamma_g^0$ is of finite order and it point-wise fixes all, say n, of its branch points, then the normalizer of α in Γ_g^0 is isomorphic to that of α in Γ_g^n with branch points being the punctures. The isomorphism between the two normalizers is established by selecting homeomorphisms of surfaces using Theorem 4.2 and map them to each other and Theorem 4.1 ensures that the maps are well defined at the mapping class level. More generally, if we remove, or mark, m branch points before hand, the same argument implies the following.

Proposition 4.3. *Let $m \geqslant 0$, $g \geqslant 0$ be integers such that $(m, g) \neq (0, 0), (0, 1)$. Let $\alpha \in P\Gamma_g^m$ be of finie order such that there are exactly n branch points in S_g^m which are point-wise fixed by α. By removing these n additional points from S_g^m, one has $\alpha \in \Gamma_g^{n+m, \Sigma_n \times \{1\}^m}$ and the normalizer of α in $P\Gamma_g^m$ is isomorphic to that of α in $\Gamma_g^{n+m, \Sigma_n \times \{1\}^m}$.*

Proposition 4.3 transfers the problem of finding the normalizer of a cyclic branched covering map to that of a cyclic regular covering map. Using this regular covering map, we can obtain a more detail description of these normalizers when the quotient surface is homeomorphic to a punctured 2-sphere. This was partially done in [7] and [17] by showing that there is an imbedding of the reduced normalizer of $\langle\alpha\rangle$, i.e. the normalizer of $\langle\alpha\rangle$ modulo $\langle\alpha\rangle$, into the mapping class group of the quotient surface. We will describe the normalizer of $\langle\alpha\rangle$ in terms of the automorphism group of the fundamental group of the quotient surface.

Assume that $\langle\alpha\rangle \cong \mathbb{Z}/p\mathbb{Z}$ is a subgroup of $\Gamma_g^{n+m,\Sigma_n\times\{1\}^m}$ with no additional fixed points and $S_g^{n+m}/\langle\alpha\rangle$ is homeomorphic to S_h^{n+m}, where the numbers g, h, $n+m$ and p must satisfy the Riemann-Hurwitz formula $2g-2 = p(2h-2) + (n+m)(p-1)$. Let $f : \pi_1(S_h^{n+m}) \to \langle\alpha\rangle$ be the group homomorphism obtained from the regular covering $S_g^{n+m} \to S_h^{n+m}$ and $x_1, x_2, \cdots, x_{n+m}$ be the elements in $\pi_1(S_h^{m+n})$ that represents the $n+m$ punctures. Then the values $f(x_i)$, $i = 1, 2, \cdots, n+m$, satisfy $f(x_1) + f(x_1) + \cdots + f(x_{n+m}) = 0$ and $f(x_i) \neq 0$ for all $1 \leqslant i \leqslant n+m$. We denote by $(\mathbb{Z}/p\mathbb{Z})^*$ the non-zero elements in $\mathbb{Z}/p\mathbb{Z}$ and call the vector $F_\alpha = (f(x_1), f(x_2), \cdots, f(x_{n+m})) \in ((\mathbb{Z}/p\mathbb{Z})^*)^{n+m}$ the fixed point data vector of α. Given a permutation $\sigma \in \Sigma_{n+m}$ and an element $k \in (\mathbb{Z}/p\mathbb{Z})^*$ define

$$\sigma(F_\alpha) = (f(x_{\sigma(1)}), f(x_{\sigma(2)}), \cdots, f(x_{\sigma(n+m)})) \in ((\mathbb{Z}/p\mathbb{Z})^*)^{n+m} \text{ and}$$
$$kF_\alpha = (kf(x_1), kf(x_2), \cdots, kf(x_{n+m})) \in ((\mathbb{Z}/p\mathbb{Z})^*)^{n+m}.$$

Let

$$\Sigma_\alpha^N = \{\gamma \in \Sigma_n \times \{1\}^m | \gamma(F_\alpha) = kF_\alpha \text{ for some } k \in (\mathbb{Z}/p\mathbb{Z})^*\},$$
$$\Sigma_\alpha^C = \{\gamma \in \Sigma_n \times \{1\}^m | \gamma(F_\alpha) = F_\alpha\}.$$

We call Σ_α^N the normalizing permutation group of α and Σ_α^C the centralizing permutation group of α. For each $t \in (\mathbb{Z}/p\mathbb{Z})^*$, let $l_t = |\{1 \leqslant i \leqslant n \mid f(x_i) = t\}|$. We observe from the definition that $\Sigma_\alpha^C = \Sigma_{l_1} \times \Sigma_{l_2} \times \cdots \times \Sigma_{l_{p-1}} \times \{1\}^m$. Consider the following commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(S_g^{n+m}) & \stackrel{i}{\longrightarrow} & \pi_1(S_h^{n+m}) & \stackrel{f}{\longrightarrow} & \alpha & \longrightarrow & 1 \\
& & \sigma\downarrow & & \sigma\downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\pi_1(S_g^{n+m})) & \stackrel{i}{\longrightarrow} & \mathrm{Aut}_+^{\Sigma_n\times\{1\}^m}(\pi_1(S_g^{n+m})) & \longrightarrow & \Gamma_g^{\Sigma_n\times\{1\}^m} & \longrightarrow & 1
\end{array}
$$

where the imbedding σ (as long as S_g^{n+m} and S_h^{n+m} are not S_0^1, S_0^0 or S_1^0) is given by the conjugation action on $\pi_1(S_g^{m+n})$, i.e., for every $u \in \pi_1(S_h^{n+m})$, $\sigma(u)(y) = i^{-1}(ui(y)u^{-1})$ for all $y \in \pi_1(S_g^{n+m})$. We know that the normalizer of $\langle\alpha\rangle$ in $\Gamma_g^{\Sigma_n\times\{1\}^m}$ is isomorphic to that of $\sigma(\pi_1(S_h^{n+m}))$ in $\mathrm{Aut}_+^{\Sigma_n\times\{1\}^m}(\pi_1(S_g^{n+m}))$ modulo $i(\mathrm{Inn}(\pi_1(S_g^{n+m})))$.

Lemma 4.4. *The normalizer of $\sigma(\pi_1(S_h^{n+m}))$ in $\mathrm{Aut}_+^{\Sigma_n\times\{1\}^m}(\pi_1(S_g^{n+m}))$ is isomorphic to $\mathrm{Aut}_+^{\Sigma_\alpha^N}(\pi_1(S_h^{n+m}))$ when $h = 0$.*

Proof. Let

$$K = \{\gamma \in \mathrm{Aut}_+^{\Sigma_n\times\{1\}^m}(\pi_1(S_h^{n+m})) \mid \gamma(i(\pi_1(S_g^{n+m}))) = i(\pi_1(S_g^{n+m}))\}$$

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and J be the normalizer of $\sigma(\pi_1(S_h^{n+m}))$ in $\text{Aut}_+^{\Sigma_n \times \{1\}^m}(\pi_1(S_g^{n+m}))$. We define two homomorphisms $\Phi : J \to K$ and $\Psi : K \to J$:

$$\text{for every } x \in J, \Phi(x)(u) = \sigma^{-1}(x\sigma(u)x^{-1}) \text{ for all } u \in \pi_1(S_h^{n+m});$$
$$\text{for every } y \in K, \Psi(y)(v) = i^{-1}(y(i(v))) \text{ for all } v \in \pi_1(S_g^{n+m}).$$

It can be routinely checked that Φ and Ψ are inverse to each other. Now we need to show that $K = \text{Aut}_+^{\Sigma_\alpha^N}(\pi_1(S_h^{n+m}))$ when $h = 0$. Recall that

$$\pi_1(S_0^{n+m}) = \langle x_1, x_2, \cdots, x_{n+m} \mid x_1 x_2 \cdots x_{n+m} = 1 \rangle,$$

Let H be the normal closure of $\{t^p \mid t \in \pi_1(S_0^{n+m})\}$ in $\pi_1(S_0^{n+m})$ and $M = H[\pi_1, \pi_1]$, where $[\pi_1, \pi_1]$ is the commutator subgroup of $\pi_1(S_0^{n+m})$. Then M is a characteristic subgroup of $\pi_1(S_0^{n+m})$ and the homomorphism f factors through

$$\pi_1(S_0^{n+m})/M = \bigoplus_{i=1}^{n+m} \mathbb{Z}/p\mathbb{Z}x_i/\langle x_1 + x_2 + \cdots + x_{n+m} \rangle.$$

An automorphism in $\text{Aut}_+^{\Sigma_n \times \{1\}^m}(\pi_1(S_0^{n+m}))$ fixes the subgroup $i(\pi_1(S_g^{n+m})) = \text{Ker}(f)$ if and only if its induced automorphism in $\text{Aut}(\pi_1(S_0^{n+m})/M)$ fixes the kernel of

$$\pi_1(S_0^{n+m})/M \to \langle \alpha \rangle,$$

which is given by

$$\{t_1 x_1 + t_2 x_2 + \cdots + t_{n+m} x_{n+m} \in \bigoplus_{i=1}^{n+m} \mathbb{Z}/p\mathbb{Z}x_i/\langle x_1 + x_2 + \cdots + x_{n+m} \rangle$$
$$\mid t_1 f(x_1) + t_2 f(x_2) + \cdots + t_{n+m} f(x_{n+m}) = 0\}.$$

In other words the kernel is the hyperplane $F_\alpha^\perp$ in $\pi_1(S_0^{n+m})/M$ that is orthogonal to the fixed point data vector of α. On the other hand, every automorphism in the group $\text{Aut}_+^{\Sigma_n \times \{1\}^m}(\pi_1(S_0^{n+m}))$ induces a permutation automorphism of $\pi_1(S_0^{n+m})/M$ which permutes the elements $x_1, x_2, \cdots, x_n$, and every such permutation automorphism in $\Sigma_n \times \{1\}^m$ is indeed induced by some automorphism in $\text{Aut}_+^{\Sigma_n \times \{1\}^m}(\pi_1(S_0^{n+m}))$. If $\gamma \in \Sigma_n \times \{1\}^m$, one has $\gamma(F_\alpha^\perp) = \gamma(F_\alpha)^\perp$. Hence $\gamma(F_\alpha^\perp) = F_\alpha^\perp$ if and only if $\gamma(F_\alpha)^\perp = F_\alpha^\perp$, or equivalently $\gamma \in \Sigma_\alpha^N$ and $K = \text{Aut}_+^{\Sigma_\alpha^N}(\pi_1(S_0^{n+m}))$, when $h = 0$. $\qquad \square$

Remark 4.5. The group K in the proof is the group that maps loops that lift to loops to loops that lift to loops as all such loops form the group $i(\pi_1(S_g^{n+m}))$. This property was used in [7, 17].

Combining Proposition 4.3 and Lemma 4.4, we have proved

Theorem 4.6. *Let $m \geqslant 0$, $g \geqslant 0$ be integers such that $(m, g) \neq (0, 0), (0, 1)$. Let $\alpha \in P\Gamma_g^m$ be an element of prime order p with exactly n branch points in S_g^m. Let Σ_α^N and Σ_α^C be the normalizing and centralizing permutation groups of α respectively. If the quotient surface $S_g^m/\langle \alpha \rangle$ is homeomorphic to S_0^m, then the normalizer of α in $P\Gamma_g^m$ is isomorphic to $\text{Aut}_+^{\Sigma_\alpha^N}(\pi_1(S_0^{n+m}))/\text{Inn}(\pi_1(S_g^{n+m}))$ and the centralizer of α in $P\Gamma_g^m$ is isomorphic to $\text{Aut}_+^{\Sigma_\alpha^C}(\pi_1(S_0^{n+m}))/\text{Inn}(\pi_1(S_g^{n+m}))$.*

5. The p-primary part of the Farrell cohomology of certain pure mapping class groups

The mapping class groups are known to have finite virtual cohomological dimensions (vcd)(see [14, 16]), i.e. they have subgroups of finite index which have finite cohomological dimensions. For group Γ of finite vcd, the ordinary cohomology and Farrell cohomology are isomorphic above the vcd, i.e.

$$H^k(\Gamma, M) \cong \widehat{H}^k(\Gamma, M)$$

for all Γ-module M and all $k > \mathrm{vcd}(\Gamma)$. Such a group Γ is said to have p-periodic cohomology for some prime p if there is a positive integer d such that $\widehat{H}^k(\Gamma, M)_{(p)} \cong \widehat{H}^{k+d}(\Gamma, M)_{(p)}$ for all integer k and all Γ-module M, where $\widehat{H}^k(\Gamma, M)_{(p)}$ is the p-primary part of $\widehat{H}^k(\Gamma, M)$. By [8, Theorem 6.7], a group of finite vcd is p-periodic for a prime p if and only if Γ is of p-rank 1, i.e. every finite elementary abelian p-subgroup is rank 1. The following theorem is needed in our study of the cohomology groups of certain pure mapping class groups.

Theorem 5.1. [8, Corollary 7.4] *Suppose that Γ is of finite vcd and every elementary abelian p-subgroup of Γ has rank $\leqslant 1$. Then*

$$\widehat{H}^n(\Gamma, \mathbb{Z})_{(p)} \cong \prod_{P \in \wp} \widehat{H}^n(N_\Gamma(P), \mathbb{Z})_{(p)},$$

where $\wp$ is a set of representatives for the conjugacy classes of subgroups of Γ of order p, $N_\Gamma(P)$ is the normalizer of P in Γ.

In this section we study the p-primary part of the Farrell cohomology of the pure mapping class groups. Lu [13, 14, 15] determined almost all of the p-primary part of Farrell cohomology of pure mapping class groups of low genus, as well as $P\Gamma^i_{n(p-1)/2}$ for small n. We obtain some general results on the p-primary part of the Farrell cohomology of $P\Gamma^k_{n(p-1)/2}$, and the 2-primary part of the Farrell cohomology of the pure mapping class group $P\Gamma^m_n$ for $m = 2n - 1, 2n, 2n + 1, 2n + 2$, where p is an odd prime and $n \geqslant 1$ is an integer. The explicit calculation is carried out for $\widehat{H}^m(P\Gamma^{n+1}_{n(p-1)/2}, \mathbb{Z})_{(p)}$, $\widehat{H}^m(P\Gamma^{n+2}_{n(p-1)/2}, \mathbb{Z})_{(p)}$, $\widehat{H}^m(P\Gamma^{2n+1}_n, \mathbb{Z})_{(2)}$ and $\widehat{H}^m(P\Gamma^{2n+2}_n, \mathbb{Z})_{(2)}$. The reason why we choose these groups is because the Riemann-Hurwitz formula implies

Lemma 5.2. *Let p be any odd prime, $m \geqslant 0$, and $n \geqslant 1$ be two integers.*

(1) *If $n - p + 3 \leqslant m \leqslant n + 2$, then every subgroup $\langle \alpha \rangle$ of order p in $P\Gamma^m_{n(p-1)/2}$ has $n + 2$ fixed points including the punctures and the quotient surface $S^m_{n(p-1)/2}/\langle \alpha \rangle$ is homeomorphic to S^m_0.*

(2) *If $2n - 1 \leqslant m \leqslant 2n + 2$, then every subgroup $\langle \alpha \rangle$ of order 2 in $P\Gamma^m_n$ has $2n + 2$ fixed points including the punctures and the quotient surface $S^m_n/\langle \alpha \rangle$ is homeomorphic to S^m_0.*

Remark 5.3. Lemma 5.2 is also a consequence of the main result of [12]: Every nonfree orientable action of $\mathbb{Z}/p\mathbb{Z}$ on S_g, p odd, is given by the connected sum at fixed points of the canonical actions of $\mathbb{Z}/p\mathbb{Z}$ on S_p and $S_{(p-1)/2}$. For $p = 2$ the canonical action is for $\mathbb{Z}/2\mathbb{Z}$ on S_1 and S_2.

In view of Theorem 5.1, we start our computation by obtaining some structural information about the cohomology groups of normalizers of elements of prime order in $P\Gamma^m_g$ such that the quotient surface is homeomorphic to S^m_0.

Lemma 5.4. *Let $\alpha \in P\Gamma_g^m$ be an element of prime order p such that $S_g^m/\langle\alpha\rangle$ is homeomorphic to S_0^m, and Σ_α^C its centralizing permutation group. Let $N(\alpha)$ be the normalizer of α in $P\Gamma_g^m$. If Σ_α^C contains a subgroup of the form $K = H \times \{1\}^l$ for some $l \geqslant 3$, i.e. K fixes at least three punctures, and the index $[\Sigma_\alpha^C : K]$ is relatively prime to p, then $\widehat{H}^*(N(\alpha), \mathbb{Z})_{(p)}$ is an elementary abelian group.*

Proof. By Theorem 4.6, the centralizer $C(\alpha)$ of α in $P\Gamma_g^m$ is isomorphic to the group $\mathrm{Aut}_+^{\Sigma_\alpha^C}(\pi_1(S_0^{n+m}))/\mathrm{Inn}(\pi_1(S_g^{n+m}))$, which will also be called $C(\alpha)$. Let $K(\alpha) = \mathrm{Aut}_+^K(\pi_1(S_0^{n+m}))/\mathrm{Inn}(\pi_1(S_g^{n+m}))$. Then $[C(\alpha) : K(\alpha)] = [\Sigma_\alpha^C : K]$ is prime to p. It follows that $[N(\alpha) : K(\alpha)] = [N(\alpha) : C(\alpha)][C(\alpha) : K(\alpha)]$ is also prime to p. Since the group $K = H \times \{1\}^l$ and $l \geqslant 3$, by the algebraic description of the Artin braid group and the mapping class group and Theorem 3.3,

$$\mathrm{Aut}_+^K(\pi_1(S_0^{n+m})) = \langle B_{n+m-1}^{H\times\{1\}^{l-1}}, \mathrm{Inn}(\pi_1(S_0^{m+n}))\rangle = \langle \Gamma_0^{n+m,K} \times \mathbb{Z}, \mathrm{Inn}(\pi_1(S_0^{m+n}))\rangle,$$

where the center $\mathbb{Z} = B_{n+m-1}^{H\times\{1\}^{l-1}} \cap \mathrm{Inn}(\pi_1(S_0^{m+n}))$, and $\langle A, B\rangle$ denotes the subgroup generated by A and B. Therefore,

$$K(\alpha) = \mathrm{Aut}_+^K(\pi_1(S_0^{n+m}))/\mathrm{Inn}(\pi_1(S_g^{m+n}))$$
$$= \langle B_{n+m-1}^{H\times\{1\}^{l-1}}, \mathrm{Inn}(\pi_1(S_0^{m+n}))\rangle/\mathrm{Inn}(\pi_1(S_g^{m+n}))$$

is generated by the group $B_{n+m-1}^{H\times\{1\}^{l-1}}/[B_{n+m-1}^{H\times\{1\}^{l-1}} \cap \mathrm{Inn}(\pi_1(S_g^{m+n}))]$ and the group $\mathrm{Inn}(\pi_1(S_0^{m+n}))/\mathrm{Inn}(\pi_1(S_g^{m+n})) = \langle\alpha\rangle$. Since

$$B_{n+m-1}^{H\times\{1\}^{l-1}} = \Gamma_0^{n+m,K} \times (B_{n+m-1}^{H\times\{1\}^{l-1}} \cap \mathrm{Inn}(\pi_1(S_0^{m+n}))),$$

one has

$$B_{n+m-1}^{H\times\{1\}^{l-1}}/[B_{n+m-1}^{H\times\{1\}^{l-1}} \cap \mathrm{Inn}(\pi_1(S_g^{m+n}))] =$$
$$\Gamma_0^{n+m,K} \times [(B_{n+m-1}^{H\times\{1\}^{l-1}} \cap \mathrm{Inn}(\pi_1(S_0^{m+n})))/(B_{n+m-1}^{H\times\{1\}^{l-1}} \cap \mathrm{Inn}(\pi_1(S_g^{m+n})))].$$

The group $(B_{n+m-1}^{H\times\{1\}^{l-1}} \cap \mathrm{Inn}(\pi_1(S_0^{m+n})))/(B_{n+m-1}^{H\times\{1\}^{l-1}} \cap \mathrm{Inn}(\pi_1(S_g^{m+n})))$ is a subgroup of $\langle\alpha\rangle = \mathrm{Inn}(\pi_1(S_0^{m+n}))/\mathrm{Inn}(\pi_1(S_g^{m+n}))\rangle$ and therefore

$$K(\alpha) = \Gamma_0^{n+m,K} \times \langle\alpha\rangle.$$

By the Künneth formula, $H^*(K(\alpha), \mathbb{Z})$ is an elementary abelian group in sufficiently high dimension as $\Gamma_0^{n+m,K}$ has finite cohomological dimension. Since $K(\alpha)$ is p-periodic as it has p-rank 1, $\widehat{H}^*(K(\alpha), \mathbb{Z})$ is elementary abelian. According to [8, Proposition 9.5], the composition of restriction map $\mathrm{res}_{K(\alpha)}^{N(\alpha)} : \widehat{H}^*(N(\alpha), \mathbb{Z}) \to \widehat{H}^*(K(\alpha), \mathbb{Z})$ with the transfer map $\mathrm{cor}_{N(\alpha)}^{K(\alpha)} : \widehat{H}^*(K(\alpha), \mathbb{Z}) \to \widehat{H}^*(N(\alpha), \mathbb{Z})$ yields the scalar multiplication map multiplying with $[N(\alpha) : K(\alpha)]$, i.e.

$$\mathrm{cor}_{N(\alpha)}^{K(\alpha)}\mathrm{res}_{K(\alpha)}^{N(\alpha)}(z) = [N(\alpha) : K(\alpha)]z$$

for all $z \in \widehat{H}^*(N(\alpha), \mathbb{Z})$. Hence $\widehat{H}^*(N(\alpha), \mathbb{Z})$ is annihilated by $p[N(\alpha) : N_K]$ as $\mathrm{res}_{K(\alpha)}^{N(\alpha)}(pz) = p(\mathrm{res}_{K(\alpha)}^{N(\alpha)}(z)) = 0$, and $\widehat{H}^*(N(\alpha), \mathbb{Z})_{(p)}$ is an elementary abelian group. $\square$

Corollary 5.5. *Let p be an odd prime and $\alpha \in P\Gamma^m_{n(p-1)/2}$ an element of order p with $n+2$ fixed points including punctures. Let $N(\alpha)$ be the normalizer of α in $P\Gamma^m_{n(p-1)/2}$. If m and n satisfy*

- (i) *$m \geqslant 3$; or*
- (ii) *$1 \leqslant m \leqslant 2$ and $n \not\equiv 0 \pmod{p}$; or*
- (iii) *$m = 0$ and $n \not\equiv 0$ or $-2 \pmod{p}$,*

then $\widehat{H}^*(N(\alpha), \mathbb{Z})_{(p)}$ is an elementary abelian group.

Proof. By the Riemann-Hurwitz formula, the quotient surface $S^m_{n(p-1)/2}/\langle\alpha\rangle$ is homeomorphic to S^m_0. The case $m \geqslant 3$ follows rather obviously from Lemma 5.4. If $m = 0$ and $n \not\equiv 0$ or $-2 \pmod{p}$, let Σ^C_α be the centralizing permutation group of α. Then $\Sigma^C_\alpha = \Sigma_{l_1} \times \Sigma_{l_2} \times \cdots \times \Sigma_{l_{p-1}}$ with

$$l_1 + l_2 + \cdots + l_{p-1} = n + 2 \not\equiv 0 \ (\bmod\ p)$$
$$l_1 + 2l_2 + \cdots + (p-1)l_{p-1} \equiv 0 \ (\bmod\ p).$$

If there are three l_i's, say l_1, l_2 and l_3, prime to p, we can choose $H = \Sigma_{l_1-1} \times \Sigma_{l_2-1} \times \Sigma_{l_3-1} \times \Sigma_{l_4} \cdots \times \Sigma_{l_{p-1}}$. If there are only two l_i's, say l_1 and l_2, prime to p, then $(l_1 - 1) + (l_2 - 1) \equiv n \not\equiv 0 \pmod{p}$ and there is at least one, say $l_1 - 1$, prime to p. Hence we can choose $H = \Sigma_{l_1-2} \times \Sigma_{l_2-1} \times \Sigma_{l_3} \times \cdots \times \Sigma_{l_{p-1}}$. By Lemma 5.4, $\widehat{H}^*(N(\alpha), \mathbb{Z})_{(p)}$ is elementary abelian. The case $1 \leqslant m \leqslant 2$ and $n \not\equiv 0 \pmod{p}$ can be similarly proved. $\square$

As we can see in most of the situations with quotient being the punctured 2-sphere, the p-primary part of Farrell cohomology has exponent p. By Lemma 5.2 and Corolary 5.5, we have

Theorem 5.6. *Let $m \geqslant 0$ and $n \geqslant 1$ be two integers.*

- (1) *If $n - p + 3 \leqslant m \leqslant n + 2$ then $\widehat{H}^*(P\Gamma^m_{n(p-1)/2}, \mathbb{Z})_{(p)}$ is an elementary abelian group.*
- (2) *If $2n - 1 \leqslant m \leqslant 2n + 2$, then $\widehat{H}^*(P\Gamma^m_n, \mathbb{Z})_{(2)}$ is an elementary abelian group when $P\Gamma^m_n \neq P\Gamma^1_1$ or $P\Gamma^2_1$.*

We are now ready to compute $\widehat{H}^m(P\Gamma^{n+1}_{n(p-1)/2}, \mathbb{Z})_{(p)}$, $\widehat{H}^m(P\Gamma^{n+2}_{n(p-1)/2}, \mathbb{Z})_{(p)}$, $\widehat{H}^m(P\Gamma^{2n+1}_n, \mathbb{Z})_{(2)}$ and $\widehat{H}^m(P\Gamma^{2n+2}_n, \mathbb{Z})_{(2)}$. By Theorem 5.1, we need to find all conjugacy classes of subgroups of order p, which was done in [13, 14]. The idea is that there is an one to one correspondence between the conjugacy classes of subgroups $\langle\alpha\rangle$ of order p in $P\Gamma^{n+1}_{n(p-1)/2}$, as well as in $P\Gamma^{n+2}_{n(p-1)/2}$, and the fixed point data vector up to scalars $\{kF_\alpha \mid k \in \mathbb{Z}/p\mathbb{Z}\}$. Hence the number of conjugacy classes of subgroups of order p in $P\Gamma^{n+1}_{n(p-1)/2}$ and in $P\Gamma^{n+2}_{n(p-1)/2}$ is equal to the number of solutions to the equation $f(x_1) + f(x_2) + \cdots + f(x_{n+1}) = 1$ in $\mathbb{Z}/p\mathbb{Z}$ and $f(x_i) \neq 0$ for all $i = 1, 2, \cdots, n+1$. If we denote the number of such solutions by $\nu(n+1)$, then we can get the following recursion formula based upon whether or not $f(x_n) = 1$,

$$(p - 1)\nu(n - 1) + (p - 2)\nu(n) = \nu(n + 1).$$

The initial values are obviously $\nu(1) = 1$ and $\nu(2) = p - 2$, By solving this recursion, one gets $\nu(n + 1) = [(p - 1)^{n+1} + (-1)^n]/p$. The groups $P\Gamma^{2n+1}_n$ and $P\Gamma^{2n+2}_n$ obviously have only one conjugacy class of subgroups of order 2 which fixes $2n + 2$ points. As a result,

Lemma 5.7. *The groups $P\Gamma^{n+1}_{n(p-1)/2}$ and $P\Gamma^{n+2}_{n(p-1)/2}$ have $[(p-1)^{n+1}+(-1)^n]/p$ conjugacy classes of subgroups of order p, and the groups $P\Gamma^{2n+1}_n$ and $P\Gamma^{2n+2}_n$ have 1 conjugacy class of subgroups of order 2.*

If one follows the same line of calculation as in Lemma 5.4, then for any $\alpha \in P\Gamma^{n+1}_{n(p-1)/2}$ or $\alpha \in P\Gamma^{n+2}_{n(p-1)/2}$ of order p, one can see that

$$N(\alpha) = C(\alpha) = K(\alpha) = P\Gamma^{n+2}_0 \times \mathbb{Z}/p\mathbb{Z}.$$

For any $\alpha \in P\Gamma^{2n+1}_n$ or $\alpha \in P\Gamma^{2n+2}_n$ of order 2, one has

$$N(\alpha) = C(\alpha) = K(\alpha) = P\Gamma^{2n+2}_0 \times \mathbb{Z}/2\mathbb{Z}.$$

The cohomology groups of $P\Gamma^m_0$ have been computed by Cohen [9]. These are free abelian groups. The Poincaré series of the cohomology is given by

$$P_m(t) = (1 + 2t)(1 + 3t) \cdots (1 + (m-2)t).$$

Using the Künneth formula, one finds that

$$
\begin{aligned}
H^i(N(\alpha), \mathbb{Z})_{(p)} &= (\mathbb{Z}/p\mathbb{Z})^{(P_{n+2}(1)+(-1)^i P_{n+2}(-1))/2} \\
&= (\mathbb{Z}/p\mathbb{Z})^{(\frac{1}{2}(n+1)!+(-1)^{i+n-1}(n-1)!)/2} \\
&= (\mathbb{Z}/p\mathbb{Z})^{(n-1)![n^2+n-(-1)^{n+i}2]/4}
\end{aligned}
$$

for i sufficiently large and $\alpha \in P\Gamma^{n+1}_{n(p-1)/2}$ or $P\Gamma^{n+2}_{n(p-1)/2}$, and

$$
\begin{aligned}
H^i(N(\alpha), \mathbb{Z})_{(2)} &= (\mathbb{Z}/p\mathbb{Z})^{(P_{2n+2}(1)+(-1)^i P_{2n+2}(-1))/2} \\
&= (\mathbb{Z}/p\mathbb{Z})^{(\frac{1}{2}(2n+1)!+(-1)^{i+2n-1}(2n-1)!)/2} \\
&= (\mathbb{Z}/p\mathbb{Z})^{(2n-1)![2n^2+n-(-1)^i]/2}
\end{aligned}
$$

for i sufficiently large and $\alpha \in P\Gamma^{2n+1}_n$ or $P\Gamma^{2n+2}_n$. Since pure mapping class groups are of finite vcd, we know that their Farrell cohomology groups coincide with their ordinary cohomology groups above the vcd. Since pure mapping class groups are periodic, by Theorem 5.1 and Lemma 5.7, one has

Theorem 5.8. *For any given odd prime p and positive integer n, the p-primary part of the Farrell cohomology of $P\Gamma^{n+1}_{n(p-1)/2}$ and $P\Gamma^{n+2}_{n(p-1)/2}$ is given by*

$$\widehat{H}^i(P\Gamma^{n+1}_{n(p-1)/2}, \mathbb{Z})_{(p)} \cong \widehat{H}^i(P\Gamma^{n+2}_{n(p-1)/2}, \mathbb{Z})_{(p)}$$
$$\cong (\mathbb{Z}/p\mathbb{Z})^{(n-1)![(p-1)^{n+1}+(-1)^n][n^2+n-(-1)^{n+i}2]/4}.$$

The 2-primary part of the Farrell cohomology of $P\Gamma^{2n+1}_n$ and $P\Gamma^{2n+2}_n$ is given by

$$\widehat{H}^i(P\Gamma^{2n+1}_n, \mathbb{Z})_{(2)} \cong \widehat{H}^i(P\Gamma^{2n+2}_n, \mathbb{Z})_{(2)} \cong (\mathbb{Z}/2\mathbb{Z})^{(2n-1)![2n^2+n-(-1)^i]/2}.$$

As illustrations of Theorem 5.8 we

Remark 5.9. cf.([14],[15])

$$
\begin{aligned}
\widehat{H}^0(P\Gamma^2_1, \mathbb{Z})_{(3)} &\cong (\mathbb{Z}/3\mathbb{Z})^3, \ \widehat{H}^1(P\Gamma^2_1, \mathbb{Z})_{(3)} \cong 0, \\
\widehat{H}^0(P\Gamma^2_2; \mathbb{Z})_{(5)} &\cong (\mathbb{Z}/5\mathbb{Z})^{15}, \ \widehat{H}^1(\Gamma^2_2, \mathbb{Z})_{(5)} \cong 0, \\
\widehat{H}^0(P\Gamma^3_2, \mathbb{Z})_{(3)} &\cong (\mathbb{Z}/3\mathbb{Z})^9, \ \widehat{H}^1(P\Gamma^3_2, \mathbb{Z})_{(3)} \cong (\mathbb{Z}/3\mathbb{Z})^{18}, \\
\widehat{H}^0(P\Gamma^3_1, \mathbb{Z})_{(2)} &\cong \mathbb{Z}/2\mathbb{Z}, \ \widehat{H}^1(P\Gamma^3_1, \mathbb{Z})_{(2)} \cong (\mathbb{Z}/2\mathbb{Z})^2.
\end{aligned}
$$

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