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Lisa A. Pflug

George E. Ioup
University of New Orleans

Juliette W. Ioup
University of New Orleans

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Known source detection predictions for higher order correlators

Lisa A. Pflug
Naval Research Laboratory, Code 7173, Stennis Space Center, Mississippi 39529-5004

George E. Ioup and Juliette W. Ioup
Department of Physics, University of New Orleans, New Orleans, Louisiana 70148

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The problem addressed in this paper is whether higher order correlation detectors can perform better in white noise than the cross correlation detector for the detection of a known transient source signal, if additional receiver information is included in the higher order correlations. While the cross correlation is the optimal linear detector for white noise, additional receiver information in the higher order correlations makes them nonlinear. In this paper, formulas that predict the performance of higher order correlation detectors of energy signals are derived for a known source signal. Given the first through fourth order signal moments and the noise variance, the formulas predict the SNR for which the detectors achieve a probability of detection of 0.5 for any level of false alarm, when noise at each receiver is independent and identically distributed. Results show that the performance of the cross correlation, bicorrelation, and tricorrelation detectors are proportional to the second, fourth, and sixth roots of the sampling interval, respectively, but do not depend on the observation time. Also, the SNR gains of the higher order correlation detectors relative to the cross correlation detector improve with decreasing probability of false alarm. The source signal may be repeated in higher order correlations, and gain formulas are derived for these cases as well. Computer simulations with several test signals are compared to the performance predictions of the formulas. The breakdown of the assumptions for signals with too few sample points is discussed, as are limitations on the design of signals for improved higher order gain. Results indicate that in white noise it is difficult for the higher order correlation detectors in a straightforward application to achieve better performance than the cross correlation. [S0001-4966(98)01805-0]

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INTRODUCTION

The potential advantages over conventional detection methods that may be obtained using higher order moment and related spectral techniques have received much attention in recent years. Higher order techniques show promise in applications for stationary signals and also for short-time transients where only a single occurrence of a signal may be available for detection (Dwyer, 1984; Hinich, 1990; Hinich and Wilson, 1990; Kletter and Messer, 1990; Sangfelt and Persson, 1993; Delaney, 1994; Tague *et al.*, 1994; Baugh and Hardwicke, 1994; Nuttall, 1994). The latter case, for correlation detectors, has been investigated in previous papers by the authors using both computer simulations (Pflug *et al.*, 1992b, 1994b) and more recently for unknown source detection, using theoretical performance predictions for the case of uncorrelated noise (Pflug *et al.*, 1995b). The theoretical performance predictions are extended in this paper to include predictions for known source detection, of which active detection is the most common application. The formulas can be used to determine under what conditions higher order correlations perform better than the cross correlation detector in uncorrelated noise, if the higher order correlations include more than one hydrophone, or channel, of data. Although the cross correlation is the optimal linear detector in white noise, the inclusion of additional channels of data in the higher order correlations makes them nonlinear detectors, which makes possible improvement over the cross correlation. Preliminary results using these formulas have been included in two abstracts (Pflug *et al.*, 1994a; Ioup *et al.*, 1995) and a proceedings article (Pflug *et al.*, 1995a).

After a background discussion given in Sec. I, the formulas for known source detection are derived in Sec. II. Comparison of the formula predictions with simulations using a set of various test signals is given in Sec. III. Section IV presents a discussion of signal design limitations for higher order gain and the applicability of the prediction formulas. Repeating the source signal in higher order correlation detectors is addressed in Sec. V. Finally, a summary of the findings appears in Sec. VI.

I. DEFINITIONS AND ASSUMPTIONS

The detection criteria used in this paper are based on the second through fourth order moments of a deterministic transient signal $s(t)$, defined by

$$m_p^s=\Delta t\sum_{k=0}^{N_s-1} s^p(t), \tag{1}$$

in which p is the moment order, $t=k\Delta t$, and N_s is the number of points in the source signal. The second through fourth order moments correspond to central ordinate, or zero time lag, values for the cross correlation, bicorrelation, and tricorrelation, respectively. Although the definition of m_p^s includes the factor Δt, changing the sampling rate of a signal results in a corresponding change in the summation of Eq. (1), leaving the value of m_p^s the same, if the signal is adequately sampled.

The underlying noise process is assumed to be zero-mean stationary, independent, and identically distributed (i.i.d.), and therefore higher order white and uncorrelated. Less restrictive assumptions concerning the noise can be found in Appendices A and B of Pflug *et al.* (1995b). These assumptions can be modified in a straightforward manner to substitute the known source signal for one of the noise sequences. No averaging is used in the detection process, since the transients are nonstationary over their domains. Thus the noise is treated as an energy signal in the process of detecting an energy transient and has moments defined as in Eq. (1).

It is assumed for all but Sec. V that the source signal is simultaneously recorded on each of p-1 spatially separated sensors for the pth order moment. When the source signal is present, the second, third, and fourth order correlation detectors, also called the cross correlation (CC), bicorrelation (BC), and tricorrelation (TC) detectors, have detection statistics

$$\mathrm{CC}=\sum_{k=0}^{N_s-1} s(t)[s(t)+n_1(t)]\Delta t, \tag{2a}$$

$$\mathrm{BC}=\sum_{k=0}^{N_s-1} s(t)[s(t)+n_1(t)][s(t)+n_2(t)]\Delta t, \tag{2b}$$

and

$$\mathrm{TC}=\sum_{k=0}^{N_s-1} s(t)[s(t)+n_1(t)][s(t)+n_2(t)] \times[s(t)+n_3(t)]\Delta t. \tag{2c}$$

In these equations $n_i(t)$ represents the noise at one of the p-1 spatially separated sensors. The first terms in the extended sums of Eqs. (2a), (2b), and (2c) are simply the moments of the signal, as given in Eq. (1). When the source signal is absent, the correlation detectors are given by

$$\mathrm{CC}=\sum_{k=0}^{N_s-1} s(t)n_1(t)\Delta t, \tag{3a}$$

$$\mathrm{BC}=\sum_{k=0}^{N_s-1} s(t)n_1(t)n_2(t)\Delta t, \tag{3b}$$

and

$$\mathrm{TC}=\sum_{k=0}^{N_s-1} s(t)n_1(t)n_2(t)n_3(t)\Delta t. \tag{3c}$$

The amplitude signal-to-noise ratio (SNR) for energy signals is defined by

$$\mathrm{SNR}=\frac{\sigma_s}{\sigma_n}. \tag{4}$$

Conversion of the amplitude SNR in Eq. (4) to power SNR in decibels is accomplished using $20\log_{10}(\sigma_s/\sigma_n)$. The variance of a deterministic signal $s(t)$ is

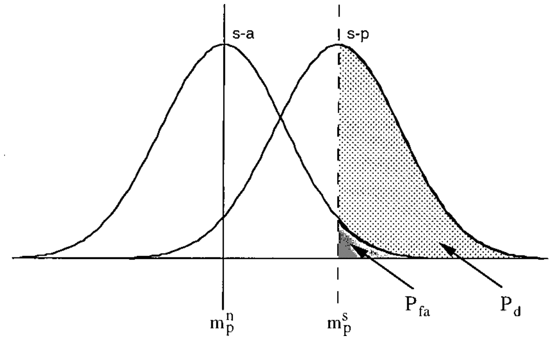


FIG. 1. The signal-absent (s-a) and signal-present (s-p) PDFs that determine the P_d=0.5 point of a ROC curve.

$$\sigma_s^2=\frac{\Delta t}{T}\sum_{k=0}^{N_s-1}[s(t)-\overline{s}]^2=\frac{1}{T}\left[m_2^s-\frac{(m_1^s)^2}{T}\right], \tag{5}$$

where the mean is $\overline{s}=m_1^s/T$. For known source detection, the processing window duration, $T=\Delta t N_s$, is by definition equal to the transient signal duration, denoted by T_s. The noise variance, σ_n^2, is defined in the same way as σ_s^2 because the correlations are defined for energy transients. The noise ensemble is ergodic and differences between the sample and population noise means and variances are assumed to be small for comparison of theoretical and simulated results.

II. KNOWN SOURCE DETECTION PREDICTION FORMULAS

Derivations of formulas that predict the SNR at which a passive correlation detector achieves the minimum detectable level (MDL) are given by Pflug *et al.* (1995b). The MDL is the SNR at which a detector achieves a probability of detection (P_d) equal to 0.5 for a selected probability of false alarm (P_{fa}). The derivations are based on the areas beneath the signal-absent (s-a) and signal-present (s-p) probability density functions (PDFs) of the zero lag correlation values that define a receiver operating characteristic (ROC) curve (see Fig. 1). For the pth order correlation, the mean of the s-p PDF is at m_p^s (the pth order signal correlation), and the mean of the s-a PDF is at m_p^n (the pth order noise correlation), which is zero under the assumptions in this paper. The s-a PDF moments are consistent with Gaussian moments, as shown in the Appendix.

Since the s-p PDF is symmetric, the mean is equal to the median of the PDF and the $P_d=0.5$ threshold occurs at the mean, m_p^s (see Fig. 1). For a fixed SNR, this threshold also defines the P_{fa}. The P_{fa}, or area to the right of the threshold, is related to a standardized score, called z_n, by

$$z_n=\frac{m_p^s-m_p^n}{\sqrt{\alpha_p^n}}, \tag{6}$$

where α_p^n represents the variance of the s-a PDF. The standardized score is the abscissa value which defines the tail area for a normalized Gaussian distributed PDF having zero-mean and unit variance. This makes it possible to use one formula or table of areas to find the P_{fa} for any Gaussian distribution, regardless of its defining parameters. A table of these areas can be found in most basic statistics books or computed numerically. Here, z_n and $P_{\rm fa}$ are inversely related. When α_p^n in Eq. (6) and the corresponding SNR are smaller, z_n is larger, and the two PDFs in Fig. 1 are narrower and have less overlap, resulting in a smaller $P_{\rm fa}$. For more details on the relationship between z scores and ROC curves, see Egan (1975).

For $P_d=0.5$, Eq. (6) is related to the remaining ROC curve parameters through α_p^n, SNR, and $P_{\rm fa}$. Since m_p^s and m_p^n are known, if α_p^n can be expressed theoretically as a function of signal and noise parameters, then Eq. (6) can ultimately be used as a prediction formula for detection performance. The derivations of α_p^n for $p=2$, 3, and 4 follow.

The PDF variance, α_2^n, of cross correlation realizations of a finite-length transient signal, $s(t)$, and members of an infinite ensemble of finite-length noise sequences, $n_i(t)$, is given by

$$\alpha_2^n=E\left\{\left[\sum_{k=0}^{N_s-1} s(t)n_{i_1}(t)\Delta t\right]^2\right\} -E^2\left\{\sum_{k=0}^{N_s-1} s(t)n_{i_1}(t)\Delta t\right\}. \tag{7}$$

The expectation operator represents averaging the large number of realizations generated in performing Monte Carlo simulations over the noise ensemble to evaluate detector performance, and is not related to any averaging in the detector itself. Since the signal and noise are approximately uncorrelated across the finite-time interval of interest and the noise is assumed to be zero mean, the second term can be neglected. Then,

$$\alpha_2^n=E\left\{(\Delta t)^2\sum_{k_1=0}^{N_s-1}\sum_{k_2=0}^{N_s-1} s(t_1)s(t_2)n_{i_1}(t_1)n_{i_1}(t_2)\right\} \tag{8a}$$

$$=E\left\{(\Delta t)^2\sum_{k_1=0}^{N_s-1} s^2(t_1)n_{i_1}^2(t_1)\right\} \tag{8b}$$

$$=(\Delta t)m_2^s E\{n_{i_1}^2(t_1)\} \tag{8c}$$

$$=\Delta t m_2^s\sigma_n^2. \tag{8d}$$

The summation over k_2 in the derivation of Eq. (8b) from Eq. (8a) is a result of the assumption that the noise is i.i.d. $E\{n^2(t)\}$ represents the infinite ensemble variance of the noise, which is equivalent to σ_n^2 in Eq. (8d) for an infinite time sum due to the ergodicity of the noise. This derivation also follows from the result given in the Appendix and is based on the assumption that finite-time averages are equal to infinite-time averages, which is approximately true for large enough N_s (see the discussion in Sec. IV for practical restrictions on the signal). For the bicorrelation, noise received from two sensors is used in the s-a PDF variance.

$$\alpha_3^n=E\left\{\left[\sum_{k=0}^{N_s-1} s(t)n_{i_1}(t)n_{i_2}(t)\Delta t\right]^2\right\} -E^2\left\{\sum_{k=0}^{N_s-1} s(t)n_{i_1}(t)n_{i_2}(t)\Delta t\right\} \tag{9a}$$

$$=E\left\{(\Delta t)^2\sum_{k_1=0}^{N_s-1}\sum_{k_2=0}^{N_s-1} s(t_1)s(t_2)n_{i_1}(t_1) \times n_{i_1}(t_2)n_{i_2}(t_1)n_{i_2}(t_2)\right\} \tag{9b}$$

$$=\Delta t m_2^s\sigma_n^4. \tag{9c}$$

Again, N_s must be large. For the tricorrelation, noise received from three sensors is used in the s-a PDF variance.

$$\alpha_4^n=E\left\{\left[\sum_{t=0}^{N_s-1} s(t)n_{i_1}(t)n_{i_2}(t)n_{i_3}(t)\Delta t\right]^2\right\} -E^2\left\{\sum_{t=0}^{N_s-1} s(t)n_{i_1}(t)n_{i_2}(t)n_{i_3}(t)\Delta t\right\} \tag{10a}$$

$$=\Delta t m_2^s\sigma_n^6. \tag{10b}$$

These s-a PDF variances can be substituted in Eq. (6). As mentioned previously, the cross correlation s-p PDF has mean equal to m_2^s, and the expression for z_n becomes

$$z_n=\frac{m_2^s-m_2^n}{\sqrt{\alpha_2^n}}=\frac{\sqrt{m_2^s}}{\sigma_n\sqrt{\Delta t}}. \tag{11}$$

Using Eq. (4) gives

$${\rm SNR}_{\rm CC}=\frac{\sigma_s z_n\sqrt{\Delta t}}{\sqrt{m_2^s}}. \tag{12}$$

Here, $\mathrm{SNR}_{\rm CC}$, the MDL for the cross correlation detector, is interpreted as the SNR required to achieve the detection level of $P_d=0.5$ at the $P_{\rm fa}$ corresponding to z_n, the standardized threshold value. Since σ_s is proportional to $\sqrt{m_2^s}$, m_2^s dependence is canceled in the derivation, and $\mathrm{SNR}_{\rm CC}$ is constant for all signals. This is not to be interpreted as saying that all signals of equal duration and sampling are equally detectable using the cross correlation. The correct interpretation is that the cross correlation detector achieves $P_d=0.5$ for these signals at the same SNR, which requires varying levels of noise for different signals.

The bicorrelation s-p PDF has mean equal to m_3^s, which for simplicity is assumed to be positive. Evaluating the z_n gives

$$z_n=\frac{m_3^s-m_3^n}{\sqrt{\alpha_2^n}}=\frac{m_3^s}{\sigma_n^2\sqrt{\Delta t m_2^s}}. \tag{13}$$

Substituting to obtain the SNR yields the formula

$${\rm SNR}_{\rm BC}=\sigma_s\sqrt{\frac{z_n^{\rm BC}\sqrt{\Delta t m_2^s}}{m_3^s}}. \tag{14}$$

A similar analysis for the tricorrelation yields

$$z_n = \frac{m_4^s - m_4^n}{\sqrt{\alpha_2^n}} = \frac{m_4^s}{\sigma_n^3 \sqrt{\Delta t m_2^s}}, \tag{15}$$

and

$$\mathrm{SNR_{TC}} = \sigma_s \sqrt[3]{\frac{z_n^{\mathrm{TC}} \sqrt{\Delta t m_2^s}}{m_4^s}}. \tag{16}$$

For zero-mean energy signals, the SNR prediction formulas reduce to

$$\mathrm{SNR_{CC}} = \frac{z_n \sqrt{\Delta t}}{\sqrt{T_s}}, \tag{17}$$

$$\mathrm{SNR_{BC}} = \sqrt{\frac{(m_2^s)^{3/2} z_n \sqrt{\Delta t}}{m_3^s T_s}}, \tag{18}$$

and

$$\mathrm{SNR_{TC}} = \sqrt[3]{\frac{(m_2^s)^2 z_n \sqrt{\Delta t}}{m_4^s (T_s)^{3/2}}}. \tag{19}$$

For zero-mean signals, $\mathrm{SNR_{BC}}$ and $\mathrm{SNR_{TC}}$ can also be written as functions of signal skewness and kurtosis (Pflug *et al.*, 1995a).

To compare the three detection methods, $\mathrm{SNR_{CC}}$, $\mathrm{SNR_{BC}}$, and $\mathrm{SNR_{TC}}$ should be evaluated at the same P_{fa}. The difference in dB levels is referred to as the SNR gain of the bicorrelation or tricorrelation detector over the cross correlation detector. Using one source signal and p-1 distinct channels of received data, the bicorrelation SNR gain in dB is

$$\mathrm{BCG} = 20 \log_{10} \left[\frac{(m_3^s)^2 z_n^2 \Delta t}{(m_2^s)^3} \right]^{1/4}, \tag{20}$$

and the tricorrelation SNR gain in dB is

$$\mathrm{TCG} = 20 \log_{10} \left[\frac{(m_4^s)^2 z_n^4 (\Delta t)^2}{(m_2^s)^4} \right]^{1/6}. \tag{21}$$

The SNR gain formulas for zero-mean and nonzero-mean signals are the same, since the only dependence on the signal mean in the nonzero mean formulas is through the standard deviation, and the dependence of $\mathrm{SNR_{CC}}$, $\mathrm{SNR_{BC}}$, and $\mathrm{SNR_{TC}}$ on the standard deviation cancels in the gain ratios.

The gain formulas show no functional dependence on T_s. Known source detection, in contrast to unknown source detection (see Pflug *et al.*, 1995a, b) is independent of observation time, T, provided the entire signal is in the processing window. This is because the multiplication of the noise by the signal sets the noise to zero outside the signal window. Known source detection performance does, however, depend on Δt. In particular, for known source detection, $\mathrm{SNR_{CC}}$, $\mathrm{SNR_{BC}}$, and $\mathrm{SNR_{TC}}$ are proportional to the second, fourth, and sixth root of the sampling interval, respectively. Performance improves for all the correlation detectors with decreasing sampling interval. However, as sampling intervals increase, the bicorrelation and tricorrelation detectors degrade less quickly than the cross correlation detector. For a given signal, noise level, and P_{fa}, each of the three detectors is associated with a range of sampling intervals for which it will perform best. For example, if a zero-mean signal has moments $m_2^s = 0.0025$ s, $m_3^s = 0.0022$ s, and $m_4^s = 0.0015$ s, then the bicorrelation and tricorrelation SNR gains versus sampling interval will be as shown in Fig. 2 for $P_{\mathrm{fa}} = 0.001$ ($z_n = 3.09$). For a positive bicorrelation gain, the sampling interval must be 0.000 34 s or larger, and for a positive tricorrelation gain, the sampling interval must be 0.000 44 s or larger.

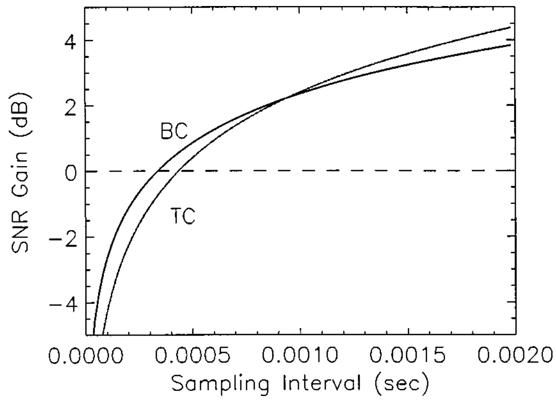


FIG. 2. Bicorrelation and tricorrelation SNR gain versus sampling interval for a zero-mean signal with moments $m_2^s = 0.0025$, $m_3^s = 0.0022$, $m_4^s = 0.0015$, and $P_{\mathrm{fa}} = 0.001$.

Detection performance improves (lower SNR) for the three detectors as z_n decreases, or the tolerance for false alarm increases. However, if the tolerance for false alarm is small, as it often is in practice, the bicorrelation and tricorrelation SNR formulas, which give increasing gain with decreasing P_{fa}, indicate that higher order detectors could outperform the cross correlation.

TABLE I. Moments m_p^s in amplitude units to the pth power times seconds for the eight test signals.

Signal	T_s (s)	m_1^s	m_2^s	m_3^s	m_4^s
Whale transient	1.00	-9.13×10^{-5}	2.78×10^{-1}	1.07×10^{-3}	1.75×10^{-1}
Low frequency whale	1.00	-1.42×10^{-2}	2.80×10^{-1}	2.31×10^{-2}	1.78×10^{-1}
5 Hz sinusoid	0.75	7.51×10^{-3}	7.31×10^{-2}	2.93×10^{-2}	4.05×10^{-2}
Narrow pulse	0.03	3.12×10^{-3}	2.68×10^{-3}	2.20×10^{-3}	1.93×10^{-3}
50 Hz sinusoid	1.72	-5.78×10^{-6}	1.77×10^{-1}	7.25×10^{-8}	9.38×10^{-2}
49–51 Hz sinusoid	1.21	3.25×10^{-3}	4.19×10^{-2}	2.25×10^{-3}	1.83×10^{-2}
FM linear sweep	1.72	7.42×10^{-6}	1.77×10^{-1}	-3.05×10^{-8}	9.38×10^{-2}
Nonlinear FM sweep	1.72	1.11×10^{-4}	1.77×10^{-1}	2.67×10^{-8}	9.39×10^{-2}

TABLE II. Known source detection SNR_{CC} in dB for $P_{fa}=0.001$ calculated from computer simulations and the prediction formulas, and the absolute dB difference between the two methods.

Signal	Simulated SNR_{CC}	Predicted SNR_{CC}	Absolute difference
Whale transient	−19.49 dB	−20.30 dB	0.81 dB
Low frequency whale	−19.47	−20.30	0.83
5 Hz sinusoid	−16.77	−17.29	0.52
Narrow pulse	−5.64	−5.67	0.03
50 Hz sinusoid	−17.96	−18.64	0.68
49–51 Hz sinusoid	−17.89	−17.99	0.10
FM linear sweep	−18.39	−18.64	0.25
Nonlinear FM sweep	−18.27	−18.63	0.36

TABLE IV. Known source detection SNR_{TC} in dB for $P_{fa}=0.001$ calculated from computer simulations and the prediction formulas, and the absolute dB difference between the two methods.

Signal	Simulated SNR_{TC}	Predicted SNR_{TC}	Absolute difference
Whale transient	−8.64 dB	−9.14 dB	0.50 dB
Low frequency whale	−8.81	−9.14	0.33
5 Hz sinusoid	−9.36	−9.62	0.26
Narrow pulse	−6.20	−8.31	2.11
50 Hz sinusoid	−9.98	−10.36	0.38
49–51 Hz sinusoid	−12.80	−13.31	0.51
FM linear sweep	−10.10	−10.36	0.26
Nonlinear FM sweep	−10.01	−10.35	0.34

III. COMPARISONS BETWEEN PREDICTIONS AND COMPUTER SIMULATIONS

Pflug *et al.* (1994b) present detection results using hypothesis testing and Monte Carlo simulations with 10 000 Gaussian noise realizations for eight energy signals. Note that the formulas do not require the noise to be Gaussian distributed, but they require the noise ensemble to be i.i.d. The signals are varied in nature, generally of low frequency, and they are described in detail in the above reference. Known source detection simulations are performed with the resulting P_d vs SNR curves interpolated at $P_d=0.5$. These simulations involve only one replica of the source signal in the bicorrelation and tricorrelation. While the focus of the paper by Pflug *et al.* (1994b) is on the advantage of prefiltering in higher order correlations, results for detection simulations with no prefiltering are also given. These results are used to corroborate the known source prediction formulas derived in this paper, which assume there is no signal-specific or situation-specific passband filtering.

Table I lists the signal duration and first through fourth order signal moments used in detection performance prediction for the eight test signals. The test signals include a model 20 Hz finback whale transient, the whale transient shifted to 12 Hz and called the low frequency whale, an amplitude modulated 5 Hz sinusoid, a narrow-time pulse which has a flat magnitude spectrum to approximately 80 Hz and a smooth rolloff to 256 Hz, an amplitude modulated 50 Hz sinusoid, a 49–51 Hz beating sinusoid, an FM linear sweep from 49 to 51 Hz, and a nonlinear FM sweep from 120 to 0 Hz. For the computer simulations, the first four signals are sampled with interval (1/1024) s, the last four with (1/500) s. The low frequency whale transient, 41–59 Hz FM linear sweep, and 120–0 Hz nonlinear FM sweep do not have bicorrelation maximum magnitudes at zero time lag. Also, the whale transient and 50 Hz sinusoid have approximately zero bispectra since the lowest frequency present in the signal is essentially half the highest frequency. This makes the bicorrelation value at zero time-lag inappropriate as a detection criterion (Ioup *et al.*, 1989; Pflug, 1990; Pflug *et al.*, 1992a).

Tables II–IV contain the results of the computer simulations and the theoretical predictions for known source detection at $P_{fa}=0.001$. Excepting the narrow pulse signal, the predictions and simulations agree reasonably well, with the absolute differences between the computer simulations and formula predictions of SNR ranging from 0.10 dB to 0.83 dB for the three detectors. For these seven test signals, the cross correlation detector performs better than the higher order correlation detectors, as shown by the predicted BCG and TCG values in Table V. For the narrow pulse, both the bicorrelation and tricorrelation detectors perform better than the cross correlation. However, the gain achieved in the simulations is significantly lower than predicted with the formulas. The reasons are discussed in detail in the following section.

The difference between the computer simulations and the predictions, which generally fluctuates between 0 and 0.5 dB for the test signals except the narrow pulse, is not removed by finer sampling, which implies that for those signals there is a sufficient number of signal samples for the finite summation to approximate the infinite summation in the formula derivations. For example, in Fig. 3, the computer

TABLE III. Known source detection SNR_{BC} in dB for $P_{fa}=0.001$ calculated from computer simulations and the prediction formulas, and the absolute dB difference between the two methods.

Signal	Simulated SNR_{BC}	Predicted SNR_{BC}	Absolute difference
Whale transient	⋯	⋯	⋯
Low frequency whale	⋯	⋯	⋯
5 Hz sinusoid	−8.41 dB	−8.84 dB	0.43 dB
Narrow pulse	−6.08	−7.51	1.43
50 Hz sinusoid	⋯	⋯	⋯
49–51 Hz sinusoid	−3.34	−3.59	0.25
FM linear sweep	⋯	⋯	⋯
Nonlinear FM sweep	⋯	⋯	⋯

TABLE V. Predicted bicorrelation and tricorrelation SNR gains over the cross correlation detector.

Signal	BCG (dB)	TCG (dB)
Whale transient	⋯	−11.16
Low frequency whale	⋯	−11.16
5 Hz sinusoid	−8.45	−7.67
Narrow pulse	1.84	2.64
50 Hz sinusoid	⋯	−8.28
49–51 Hz sinusoid	−14.40	−4.68
FM linear sweep	⋯	−8.28
Nonlinear FM sweep	⋯	−8.28

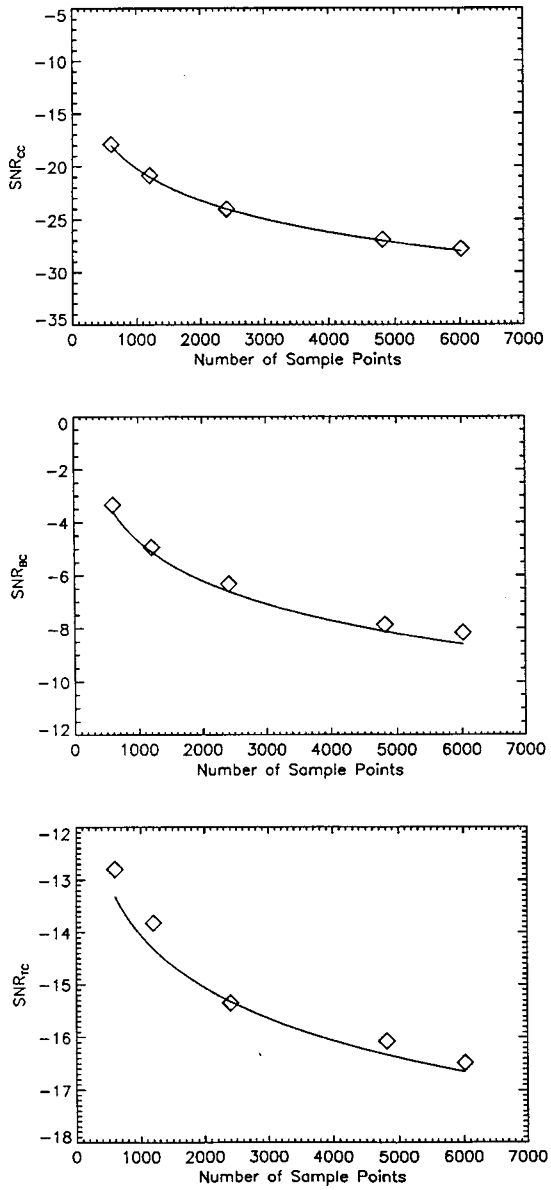


FIG. 3. The minimum detectable level as a function of the number of sample points for the 49–51 Hz sinusoid. The solid curves give the formula predictions and the symbols give the computer calculations.

simulated and predicted SNRs are shown versus number of sample points for the 49–51 Hz sinusoid. The differences fluctuate randomly with increased sampling, but remain less than 0.5 dB for each detector.

Another possible source of error in the computer simulations is that Gaussian noise generators may not perform well when very low levels of noise, corresponding to the tails of the noise distribution, are needed. A potential limitation at low P_{fa} occurs due to the finite number of realizations used in a computer simulation. If 10 000 realizations are used, as in this paper, the minimum nonzero P_{fa} value possible is 1×10^{-4}, or one false alarm out of 10 000 on average. Thus interpolation of the P_d vs P_{fa} curve for a fixed SNR at P_{fa} values lower than 1×10^{-3} is generally not statistically reliable. The number of realizations required to obtain a discrete point on the ROC curve at the desired P_{fa} increases with decreasing P_{fa}. For example, to obtain a P_d vs P_{fa} point at a P_{fa} value of 1×10^{-6}, at least one million realizations would be required.

IV. LIMITATIONS ON THE DESIGN OF SIGNALS FOR POSITIVE HIGHER ORDER GAIN

The physically motivated definition of the signal moments, which includes the time dimension, Δt, in the summation given in Eq. (1), has the intrinsic advantage that the signal moments do not change if the signal is interpolated or decimated in time. Of course, this will only be true if there is sufficient sampling (Pflug *et al.*, 1993), and the interpolation is bandlimited (Bracewell, 1986). The above holds whether or not the moment is normalized by dividing by T_s, and also for the variance, skewness, and kurtosis.

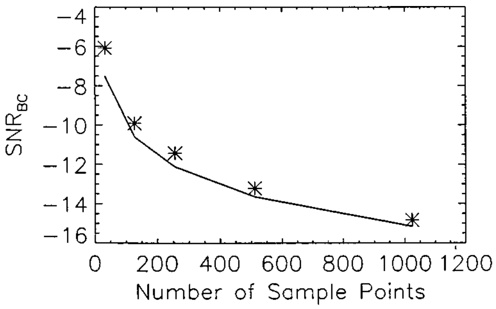
Since the moments do not change with interpolation or decimation, subject to the given restrictions, the entire effect of either of these two operations on the higher order detection gain is contained in the Δt factor of Eqs. (20) and (21). Larger Δt increases the higher order advantage, while smaller Δt does the opposite, favoring the cross correlation detector (matched filter). In fact, in the limit of the continuous-time signal (Δt goes to zero), the gain tends to negative infinity, and the cross correlation is best for any signal if the signal and noise satisfy the necessary assumptions. For a given signal, it is straightforward to determine whether there is any Δt large enough, within the sampling limitations, to give positive higher order gain. It should be noted that experimental conditions may also put limitations on Δt, and these are not considered here.

Another design question which may be asked of the gain formulas is how varying the width of a signal affects the gain. That is, if the signal samples are moved closer together or further apart (Δt is decreased or increased) without changing the signal ordinate values (but changing the moments), how does the gain change? The functions are related by similarity (Bracewell, 1986). As Δt is varied, an upper limit on Δt is determined by the sampling requirement (Pflug *et al.*, 1993). The Similarity Theorem (Bracewell, 1986) describes the effect of varying Δt on the Fourier transform. To analyze the effect of a width change, consider the tricorrelation gain formula

$$\mathrm{TCG}=20\log_{10}\left[\frac{(m_4^s)^2 z_n^4(\Delta t)^2}{(m_2^s)^4}\right]^{1/6}$$
$$=20\log_{10}\left[\frac{z_n^{2/3}(\Sigma_{k=0}^{N-1}s^4(t))^{1/3}}{(\Sigma_{k=0}^{N-1}s^2(t))^{2/3}}\right]. \tag{22}$$

All the Δt dependence cancels, leading to the conclusion, which agrees with intuition, that as long as the ordinate values of $s(t)$ are unchanged, the gain does not change with a similarity transformation. The same Δt cancellation occurs for the bicorrelation gain.

For many signals, there is a restriction on the use of these formulas for large Δt. First, $N_S=T_S/\Delta t$ must be large enough to ensure that the summation over the product of signal and noise is uncorrelated and that the finite sums approximate the infinite sums. Second, and more specifically, there must be a sufficient number of significant nonzero sig-



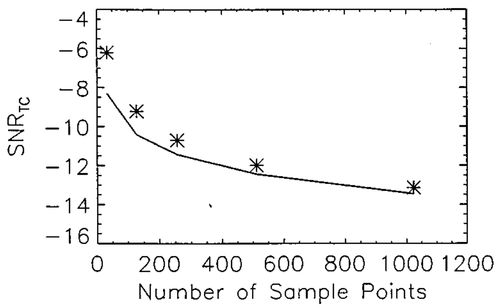


FIG. 4. Formula predictions (continuous curves) and computer simulations (asterisks) of SNR$_{BC}$ and SNR$_{TC}$ for the narrow pulse versus number of sample points within the original signal duration.

nal values, since these determine how many terms contribute significantly to the sums of Eqs. (7), (9a), and (10a). The gain prediction formulas may therefore fail to apply even for a Δt which is small enough to satisfy the sampling requirements.

Consider the case of the narrow pulse. The differences between the predicted SNR and the computer simulated SNR for SNR$_{CC}$, SNR$_{BC}$, and SNR$_{TC}$ are 0.03, 1.43, and 2.11 dB, respectively. Differences between the formula predictions and computer simulations grow larger with increasing detector order, and the differences for SNR$_{BC}$ and SNR$_{TC}$ are larger than the differences for the other seven signals in Tables II–IV. Investigation into the source of the difference for the narrow pulse reveals that the bicorrelation and tricorrelation s-a PDFs are noticeably nonGaussian, which violates the assumptions required for the prediction formula derivation. At its original sampling, there are only 32 nonzero signal points in the narrow pulse, causing the breakdown of the assumption that finite-time averages should approximate infinite-time averages, and resulting in a non-Gaussian s-a PDF. Additional computer simulations support this hypothesis. The original 32-point signal is interpolated to contain 128, 256, 512, and 1024 points within in the original 0.03-s signal duration, and the computer simulated values for SNR$_{BC}$ and SNR$_{TC}$ are calculated. The predicted and simulated values become closer as the number of signal samples increases (Fig. 4) and the s-a PDFs become increasingly Gaussian. With 1024 points, the difference between the predicted and simulated SNR values decreases to 0.32 dB for both the bicorrelation and tricorrelation detectors.

For a realistic P_{fa}, the narrow pulse signal does not show a positive SNR gain for the bicorrelation or tricorrelation detectors over the ordinary correlation detector, if Δt is chosen small enough for the prediction formulas to apply. With 128 points within the signal duration, fewer than necessary for the prediction formulas to apply, both the predictions and simulations show that the cross correlation detector performs best. While it is possible to select a large enough Δt to get a positive gain for this signal in simulations, it is simply not possible to predict that gain well with the gain formulas.

V. REPEATED SOURCE REPLICAS IN HIGHER ORDER CORRELATION DETECTION

Repeating the source signal when using bicorrelation and tricorrelation detectors can have beneficial or detrimental effects, depending on the source signal (Pflug *et al.*, 1992b). Derivation of bicorrelation and tricorrelation detector prediction formulas for repeated sources is analogous to the derivations for a single source. If the source is repeated once in the tricorrelation so that signals from only two sensors are used, then the s-p and s-a tricorrelations are

$$\text{s-p:}\quad \mathrm{TC}=\sum_{k=0}^{N_s-1} s^2(t)[s(t)+n_1(t)][s(t)+n_2(t)]\Delta t, \tag{23a}$$

$$\text{s-a:}\quad \mathrm{TC}=\sum_{k=0}^{N_s-1} s^2(t)n_1(t)n_2(t)\Delta t, \tag{23b}$$

and α_4^n, as given by Eq. (A5) in the Appendix, is $\alpha_4^n=\Delta t m_4^s\sigma_n^4$, and the predicted SNR is

$$\mathrm{SNR}_{\mathrm{TC}}^{(2,2)}=\sigma_s\sqrt{\frac{z_n^{\mathrm{TC}}\sqrt{\Delta t}}{\sqrt{m_4^s}}}, \tag{24}$$

where (2,2) denotes the power on the known source in the correlation and the number of channels used in the correlation [see Eq. (23)]. With this notation, all previous formulas would have superscripts of $(1,p-1)$ for the moment of order p. The corresponding SNR gain over the cross correlation detector is

$$\mathrm{TCG}^{(2,2)}=20\log_{10}\left[\frac{z_n^2 m_4^s\Delta t}{(m_2^s)^2}\right]^{1/4}. \tag{25}$$

This formula shows that for fixed Δt and P_{fa}, the gain depends explicitly on only the signal moments and that the simple calculation of these moments predicts whether repeating the source is advantageous or not.

Improved detection may result from repeating the source signal in the bicorrelation detector. However, at zero lag this higher order repeated source case, with noise from only one sensor, is equivalent to a correlation of a signal filter with one received signal. When only one received signal is used in the detection process, the cross correlation or matched filter is expected to be optimal for detection in white noise (Gardner, 1986). For the bicorrelation with a repeated source, the s-p and s-a bicorrelations are

$$\text{s-p:}\quad \mathrm{BC}=\sum_{k=0}^{N_s-1} s^2(t)[s(t)+n_1(t)]\Delta t, \tag{26a}$$

$$\text{s-a:}\quad \mathrm{BC}=\sum_{k=0}^{N_s-1} s^2(t)n_1(t)\Delta t, \tag{26b}$$

and $\alpha_3^n=\Delta t m_4^s \sigma_n^2$. This leads to

$$\mathrm{SNR}_{\mathrm{BC}}^{(2,1)}=\frac{z_n^{\mathrm{BC}}\sigma_s\sqrt{m_4^s\Delta t}}{m_3^s}, \tag{27}$$

and the corresponding gain over the cross correlation detector which is

$$\mathrm{BCG}^{(2,1)}=20\log_{10}\left[\frac{(m_3^s)^2}{m_2^s m_4^s}\right]^{1/2}. \tag{28}$$

If the source signal is repeated twice in the tricorrelation, then the s-a and s-p tricorrelations are

$$\text{s-p:}\quad \mathrm{TC}=\sum_{k=0}^{N_s-1} s^3(t)[s(t)+n_1(t)]\Delta t, \tag{29a}$$

$$\text{s-a:}\quad \mathrm{TC}=\sum_{k=0}^{N_s-1} s^3(t)n_1(t)\Delta t, \tag{29b}$$

and $\alpha_4^n=\Delta t m_6^s \sigma_n^2$. The predicted SNR is

$$\mathrm{SNR}_{\mathrm{TC}}^{(3,1)}=\frac{z_n^{\mathrm{TC}}\sigma_s\sqrt{m_6^s\Delta t}}{m_4^s}, \tag{30}$$

and the SNR gain for this case is

$$\mathrm{TCG}^{(3,1)}=20\log_{10}\left[\frac{(m_4^s)^2}{m_2^s m_6^s}\right]^{1/2}. \tag{31}$$

The SNR gains for the $\mathrm{SNR}_{\mathrm{BC}}^{(2,1)}$ and $\mathrm{SNR}_{\mathrm{TC}}^{(3,1)}$ are independent of P_{fa} and sampling interval, while the SNR gain for $\mathrm{SNR}_{\mathrm{TC}}^{(2,2)}$ is not. This means that $\mathrm{SNR}_{\mathrm{BC}}^{(2,1)}$ and $\mathrm{SNR}_{\mathrm{TC}}^{(3,1)}$ have the same dependence on the sampling interval and P_{fa} as $\mathrm{SNR}_{\mathrm{CC}}$, which is consistent with the fact that these higher order detectors are ordinary filters.

For the seven test signals other than the narrow pulse, the tricorrelation detector with the source repeated once shows a higher SNR gain than the tricorrelation detector without a repeated source for $P_{\mathrm{fa}}=0.001$. Repeating the source twice improves the SNR gain even more. Repeating the source in the bicorrelation detector for these signals results in improvement for some signals, but not others, but the cross correlation still performs best. The predicted bicorrelation gain for the narrow pulse is 1.83 dB without a repeated source, but drops to -0.29 dB with a repeated source. The tricorrelation gain for the narrow pulse is 2.64 dB without a repeated source, drops to 1.98 dB with the source is repeated once, and is -0.60 dB when the source is repeated twice.

VI. CONCLUSIONS

Formulas that predict cross correlation, bicorrelation, and tricorrelation known source detector performance at the minimum detectable level for any probability of false alarm are derived and corroborated with computer simulations. The prediction formulas indicate that there are circumstances under which the higher order correlation detectors could outperform the cross correlation detector for the known source case, and give the relationships to the variables upon which the detection performance depends. They show that the cross correlation, bicorrelation, and tricorrelation detectors are proportional to the second, fourth, and sixth roots of the sampling interval. The SNR gain formulas indicate that the higher order detectors are more likely to exhibit better detection performance than the cross correlation detector when the tolerance for false alarm is low. A simulation result is given in which there is positive SNR gain for the higher order correlations, but the prediction formula assumptions are not satisfied for this case. While it is theoretically possible for the higher order correlation detectors to perform better than the cross correlation detector, only a limited class of signals exists for which they do.

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APPENDIX: GAUSSIAN CHARACTER OF CORRELATION CENTRAL ORDINATE PDFS

Following Isserlis (1918) and Gardner (1986), the qth ensemble moments of the pth order correlation central ordinate PDFs are shown to be consistent with a zero-mean Gaussian density where the correlation consists of $(p\text{-}r)$ zero-mean noise sequences correlated with a known signal raised to the rth power. That is, all odd order ensemble moments are zero, and all even order ensemble moments greater than two are appropriately proportional to powers of the second moment, if the assumptions discussed in Sec. I are satisfied.

The qth order ensemble moment of the source signal and $p-r$ sequences drawn from an infinite noise ensemble is

$$\begin{aligned}
M_p^q&=E\left\{\left[\sum_{k=0}^{N_s-1} s^r(t)n_{i_1}(t)n_{i_2}(t)\cdots n_{i_{p-r}}(t)\Delta t\right]^q\right\}\\
&=(\Delta t)^q E\left\{\left[\sum_{k_1=0}^{N_s-1} s^r(t_1)n_{i_1}(t_1)n_{i_2}(t_1)\cdots n_{i_{p-r}}(t_1)\right]\right.\\
&\quad\times\left[\sum_{k_2=0}^{N_s-1} s^r(t_2)n_{i_1}(t_2)n_{i_2}(t_2)\cdots n_{i_{p-r}}(t_2)\right]\cdots\\
&\quad\left.\times\left[\sum_{k_q=0}^{N_s-1} s^r(t_q)n_{i_1}(t_q)n_{i_2}(t_q)\cdots n_{i_{p-r}}(t_q)\right]\right\}
\end{aligned} \tag{A1}$$

$$\begin{aligned}
&=(\Delta t)^q E\left\{\sum_{k_1=0}^{N_s-1}\sum_{k_2=0}^{N_s-1}\cdots\sum_{k_q=0}^{N_s-1} s^r(t_1)s^r(t_2)\cdots s^r(t_q)\right.\\
&\quad\times n_{i_1}(t_1)n_{i_1}(t_2)\cdots n_{i_1}(t_q)\cdot n_{i_2}(t_1)\\
&\quad\left.\times n_{i_2}(t_2)\cdots n_{i_2}(t_q)\cdots n_{i_{p-r}}(t_1)n_{i_{p-r}}(t_2)\cdots n_{i_{p-r}}(t_q)\right\},
\end{aligned} \tag{A2}$$

with the number of signal points N_s and $t_j=k_j\Delta t$. This expression is zero when q is odd, and whenever all t_j are distinct. It is only nonzero when q is even and times are equal in pairs. Using delta function notation, the moments are nonzero when

$$M_p^q=(\Delta t)^q E\left\{\sum_{k_1=0}^{N_s-1}\sum_{k_2=0}^{N_s-1}\cdots\sum_{k_q=0}^{N_s-1} s^r(t_1)s^r(t_2)\cdots s^r(t_q)\right.$$
$$\times n_{i_1}(t_1)n_{i_1}(t_2)\cdots n_{i_1}(t_q)n_{i_2}(t_1)n_{i_2}(t_2)\cdots n_{i_2}(t_q)\cdots$$
$$\times n_{i_{p-r}}(t_1)n_{i_{p-r}}(t_2)\cdots n_{i_{p-r}}(t_q)$$
$$\left.\times\left[\sum \delta(t_{j_1}-t_{j_2})\cdots\delta(t_{j_{q-1}}-t_{j_q})\right]\right\} \quad \text{(A3)}$$

is nonzero, where the summation over the product of delta functions is taken over all possible ways of dividing q integers into $q/2$ combinations of pairs. There are $(1)(3)(5)\cdots(q-3)(q-1)$ terms in the summation. Applying the delta summation, exchanging summations and expectations where appropriate, and using the uncorrelatedness assumptions results in

$$M_p^q=[(1)(3)(5)\cdots(q-3)(q-1)]$$
$$\times(\Delta t)^{q/2}(m_{2r}^s)^{q/2}E\{n_{i_1}^2(t_1)n_{i_1}^2(t_2)\cdots n_{i_1}^2(t_{q/2})\}$$
$$\times E\{n_{i_2}^2(t_1)n_{i_2}^2(t_2)\cdots n_{i_2}^2(t_{q/2})\}\cdots$$
$$\times E\{n_{i_{p-r}}^2(t_1)n_{i_{p-r}}^2(t_2)\cdots n_{i_{p-r}}^2(t_{q/2})\}. \quad \text{(A4)}$$

Since the square of the noise is uncorrelated in time, this is equal to

$$M_p^q=[(1)(3)(5)\cdots(q-3)(q-1)]$$
$$\times(\Delta t)^{q/2}(m_{2r}^s)^{q/2}E\{n_{i_1}^2(t_1)\}E\{n_{i_1}^2(t_2)\}\cdots E\{n_{i_1}^2(t_{q/2})\}$$
$$\times E\{n_{i_2}^2(t_1)\}E\{n_{i_2}^2(t_2)\}\cdots E\{n_{i_2}^2(t_{q/2})\}E\{n_{i_{p-r}}^2(t_1)\}$$
$$\times E\{n_{i_{p-r}}^2(t_2)\}\cdots E\{n_{i_{p-r}}^2(t_{q/2})\}$$
$$=[(1)(3)(5)\cdots(q-3)(q-1)]$$
$$\times(\Delta t)^{q/2}(m_{2r}^s)^{q/2}[E\{n^2(t)\}]^{(p-r)q/2}, \quad \text{(A5)}$$

which is the even qth order ensemble moment of the correlation of p-r zero-mean noise sequences and the signal raised to the rth power. These moments are consistent with the moment relationships for a Gaussian distributed density.

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